CHAPTER 4

# Fuchsian differential equation and generalized Riemann scheme

In this chapter we introduce generalized characteristic exponents at every singular point of a Fuchsian differential equation which are refinements of characteristic exponents and then we have the generalized Riemann scheme as the corresponding refinement of the Riemann scheme of the equation. We define the spectral type of the equation by the generalized Riemann scheme, which equals the multiplicity data of eigenvalues of the local monodromies when they are semisimple.

## 4.1. Generalized characteristic exponents

We examine the Fuchsian differential equations

$$P = a_n(x)\tfrac{d^n}{dx^n} + a_{n-1}(x)\tfrac{d^{n-1}}{dx^{n-1}} + \cdots + a_0(x) \tag{4.1}$$

with given local monodromies at regular singular points. For this purpose we first study the condition so that monodromy generators of the solutions of a Fuchsian differential equation is semisimple even when its exponents are not free of multiplicity.

Lemma 4.1. *Suppose that the operator* (4.1) *defined in a neighborhood of the origin has a regular singularity at the origin. We may assume $a_\nu(x)$ are holomorphic at* 0 *and $a_n(0) = a_n'(0) = \cdots = a_n^{(n-1)}(0) = 0$ and $a_n^{(n)}(0) \neq 0$. Then the following conditions are equivalent for a positive integer $k$.*

(4.2) $\quad P = x^k R \quad$ *with a suitable holomorphic differential operator $R$ at the origin,*

(4.3) $\quad Px^\nu = o(x^{k-1}) \quad$ *for* $\nu = 0, \dots, k-1$,

(4.4) $\quad Pu = 0 \quad$ *has a solution $x^\nu + o(x^{k-1})$ for $\nu = 0, \dots, k-1$,*

(4.5) $\quad P = \sum_{j\geq 0} x^j p_j(\vartheta) \quad$ *with polynomials $p_j$ satisfying $p_j(\nu) = 0$ for $0 \leq \nu < k-j$ and $j = 0, \dots, k-1$.*

Proof. (4.2) $\Rightarrow$ (4.3) $\Leftrightarrow$ (4.4) is clear.

Assume (4.3). Then $Px^\nu = o(x^{k-1})$ for $\nu = 0, \dots, k-1$ implies $a_j(x) = x^k b_j(x)$ for $j = 0, \dots, k-1$. Since $P$ has a regular singularity at the origin, $a_j(x) = x^j c_j(x)$ for $j = 0, \dots, n$. Hence we have (4.2).

Since $Px^\nu = \sum_{j=0}^\infty x^{\nu+j} p_j(\nu)$, the equivalence (4.3) $\Leftrightarrow$ (4.5) is clear. □

Definition 4.2. Suppose $P$ in (4.1) has a regular singularity at $x = 0$. Under the notation (1.57) we define that $P$ has a (*generalized*) *characteristic exponent* $[\lambda]_{(k)}$ at $x = 0$ if $x^{n-k}\,\mathrm{Ad}(x^{-\lambda})(a_n(x)^{-1}P) \in W[x]$.

Note that Lemma 4.1 shows that $P$ has a characteristic exponent $[\lambda]_{(k)}$ at $x = 0$ if and only if

$$x^n a_n(x)^{-1} P = \sum_{j\geq 0} x^j q_j(\vartheta) \prod_{0\leq i<k-j} (\vartheta - \lambda - i) \tag{4.6}$$

with polynomials $q_j(t)$. By a coordinate transformation we can define generalized characteristic exponents for any regular singular point as follows.

DEFINITION 4.3 (generalized characteristic exponents). Suppose $P$ in (4.1) has regular singularity at $x = c$. Let $n = m_1 + \cdots + m_N$ be a partition of the positive integer $n$ and let $\lambda_1, \dots, \lambda_N$ be complex numbers. We define that $P$ has the (*set of generalized*) *characteristic exponents* $\{[\lambda_1]_{(m_1)}, \dots, [\lambda_N]_{(m_N)}\}$ and the *spectral type* $\{m_1, \dots, m_N\}$ at $x = c \in \mathbb{C} \cup \{\infty\}$ if there exist polynomials $q_\ell(s)$ such that

$$(x-c)^n a_n(x)^{-1} P = \sum_{\ell \ge 0} (x-c)^\ell q_\ell\big((x-c)\partial\big) \prod_{\nu=1}^{N} \prod_{0 \le i < m_\nu - \ell} \big((x-c)\partial - \lambda_\nu - i\big) \tag{4.7}$$

in the case when $c \ne \infty$ and

$$x^{-n} a_n(x)^{-1} P = \sum_{\ell \ge 0} x^{-\ell} q_\ell(\vartheta) \prod_{\nu=1}^{N} \prod_{0 \le i < m_\nu - \ell} \big(\vartheta + \lambda_\nu + i\big) \tag{4.8}$$

in the case when $c = \infty$. Here if $m_j = 1$, $[\lambda_j]_{(m_j)}$ may be simply written as $\lambda_j$.

REMARK 4.4. i) In Definition 4.3 we may replace the left hand side of (4.7) by $\phi(x) a_n(x)^{-1} P$ where $\phi$ is analytic function in a neighborhood of $x = c$ such that $\phi(c) = \cdots = \phi^{(n-1)}(c) = 0$ and $\phi^{(n)}(c) \ne 0$. In particular when $a_n(c) = \cdots = a_n^{(n)}(c) = 0$ and $a_n(c) \ne 0$, $P$ is said to be *normalized* at the singular point $x = c$ and the left hand side of (4.7) can be replaced by $P$.

In particular when $c = 0$ and $P$ is normalized at the regular singular point $x = 0$, the condition (4.7) is equivalent to

$$\prod_{\nu=1}^{N} \prod_{0 \le i < m_\nu - \ell} (s - \lambda_\nu - i) \mid p_j(s) \qquad (\forall \ell = 0, 1, \dots, \max\{m_1, \dots, m_N\} - 1) \tag{4.9}$$

under the expression $P = \sum_{j=0}^{\infty} x^j p_j(\vartheta)$.

ii) In Definition 4.3 the condition that the operator $P$ has a set of generalized characteristic exponents $\{\lambda_1, \dots, \lambda_n\}$ is equivalent to the condition that it is the set of the usual characteristic exponents.

iii) Any one of $\{\lambda, \lambda+1, \lambda+2\}$, $\{[\lambda]_{(2)}, \lambda+2\}$ and $\{\lambda, [\lambda+1]_{(2)}\}$ is the set of characteristic exponents of

$$P = (\vartheta - \lambda)(\vartheta - \lambda - 1)(\vartheta - \lambda - 2 + x) + x^2(\vartheta - \lambda + 1)$$

at $x = 0$ but $\{[\lambda]_{(3)}\}$ is not.

iv) Suppose $P$ has a holomorphic parameter $t \in B_1(0)$ (cf. (2.7)) and $P$ has regular singularity at $x = c$. Suppose the set of the corresponding characteristic exponents is $\{[\lambda_1(t)]_{(m_1)}, \dots, [\lambda_N(t)]_{(m_N)}\}$ for $t \in B_1(0) \setminus \{0\}$ with $\lambda_\nu(t) \in \mathcal{O}\big(B_1(0)\big)$. Then this is also valid in the case $t = 0$, which clearly follows from the definition.

When

$$P = \sum_{\ell \ge 0} x^{-\ell} q_\ell\big((x-c)\partial\big) \prod_{\nu=1}^{N} \prod_{0 \le i < m_\nu - \ell} \big((x-c)\partial - \lambda_\nu - i\big),$$

we put

$$P_t = \sum_{\ell \ge 0} x^{-\ell} q_\ell\big((x-c)\partial\big) \prod_{\nu=1}^{N} \prod_{0 \le i < m_\nu - \ell} \big((x-c)\partial - \lambda_\nu - \nu t - i\big).$$

Here $\lambda_\nu \in \mathbb{C}$, $q_0 \ne 0$ and $\operatorname{ord} P = m_1 + \cdots + m_N$. Then the set of the characteristic exponents of $P_t$ is $\{[\tilde\lambda_1(t)]_{(m_1)}, \dots, [\tilde\lambda_N(t)]_{(m_N)}\}$ with $\tilde\lambda_j(t) = \lambda_j + jt$. Since $\tilde\lambda_i(t) - \tilde\lambda_j(t) \notin \mathbb{Z}$ for $0 < |t| \ll 1$, we can reduce certain claims to the case when the values of characteristic exponents are generic. Note that we can construct local independent solutions which holomorphically depend on $t$ (cf. [**O4**]).

LEMMA 4.5. i) *Let $\lambda$ be a complex number and let $p(t)$ be a polynomial such that $p(\lambda) \neq 0$. Then for non-negative integers $k$ and $m$ we have the exact sequence*

$$0 \longrightarrow \mathcal{O}_0(\lambda, k-1) \longrightarrow \mathcal{O}_0(\lambda, m+k-1) \xrightarrow{p(\vartheta)(\vartheta-\lambda)^k} \mathcal{O}_0(\lambda, m-1) \longrightarrow 0$$

*under the notation* (2.5).

ii) *Let $m_1, \dots, m_N$ be non-negative integers. Let $P$ be a differential operator of order $n$ whose coefficients are in $\mathcal{O}_0$ such that*

$$P = \sum_{\ell=0}^{\infty} x^\ell r_\ell(\vartheta) \prod_{\nu=1}^{N} \prod_{0 \le k < m_\nu - \ell} (\vartheta - k) \tag{4.10}$$

*with polynomials $r_\ell$. Put $m_{max} = \max\{m_1, \dots, m_N\}$ and suppose $r_0(\nu) \neq 0$ for $\nu = 0, \dots, m_{max} - 1$.*

*Let $\mathbf{m}^\vee = (m_1^\vee, \dots, m_{m_{max}}^\vee)$ be the dual partition of $\mathbf{m} := (m_1, \dots, m_N)$, namely,*

$$m_\nu^\vee = \#\{j\,;\, m_j \ge \nu\}. \tag{4.11}$$

*Then for $i = 0, \dots, m_{max} - 1$ and $j = 0, \dots, m_{i+1}^\vee - 1$ we have the functions*

$$u_{i,j}(x) = x^i \log^j x + \sum_{\mu=i+1}^{m_{max}-1} \sum_{\nu=0}^{j} c_{i,j}^{\mu,\nu} x^\mu \log^\nu x \tag{4.12}$$

*such that $c_{i,j}^{\mu,\nu} \in \mathbb{C}$ and $Pu_{i,j} \in \mathcal{O}_0(m_{max}, j)$.*

iii) *Let $m'_1, \dots, m'_N$ be non-negative integers and let $P'$ be a differential operator of order $n'$ whose coefficients are in $\mathcal{O}_0$ such that*

$$P' = \sum_{\ell=0}^{\infty} x^\ell r'_\ell(\vartheta) \prod_{\nu=1}^{N} \prod_{0 \le k < m'_\nu - \ell} \big(\vartheta - m_\nu - k\big) \tag{4.13}$$

*with polynomials $q'_\ell$. Then for a differential operator $P$ of the form* (4.10) *we have*

$$P'P = \sum_{\ell=0}^{\infty} x^\ell \Big(\sum_{\nu=0}^{\ell} r'_{\ell-\nu}(\vartheta+\nu) r_\nu(\vartheta)\Big) \prod_{\nu=1}^{N} \prod_{0 \le k < m_\nu + m'_\nu - \ell} \big(\vartheta - k\big). \tag{4.14}$$

PROOF. i) The claim is easy if $(p, k) = (1, 1)$ or $(\vartheta - \mu, 0)$ with $\mu \neq \lambda$. Then the general case follows from induction on $\deg p(t) + k$.

ii) Put $P = \sum_{\ell \ge 0} x^\ell p_\ell(\vartheta)$ and $m_\nu^\vee = 0$ if $\nu > m_{max}$. Then for a non-negative integer $\nu$, the multiplicity of the root $\nu$ of the equation $p_\ell(t) = 0$ is equal or larger than $m_{\nu+\ell+1}^\vee$ for $\ell = 1, 2, \dots$. If $0 \le \nu \le m_{max} - 1$, the multiplicity of the root $\nu$ of the equation $p_0(t) = 0$ equals $m_{\nu+1}^\vee$.

For non-negative integers $i$ and $j$, we have

$$x^\ell p_\ell(\vartheta) x^i \log^j x = x^{i+\ell} \sum_{0 \le \nu \le j - m_{i+\ell+1}^\vee} c_{i,j,\ell,\nu} \log^\nu x$$

with suitable $c_{i,j,\ell,\nu} \in \mathbb{C}$. In particular, $p_0(\vartheta) x^i \log^j x = 0$ if $j < m_i^\vee$. If $\ell > 0$ and $i + \ell < m_{\max}$, there exist functions

$$v_{i,j,\ell} = x^{i+\ell} \sum_{\nu=0}^{j} a_{i,j,\ell,\nu} \log^\nu x$$

with suitable $a_{i,j,\ell,\nu} \in \mathbb{C}$ such that $p_0(\vartheta)v_{i,j,\ell} = x^\ell p_\ell(\vartheta)x^i \log^j x$ and we define a $\mathbb{C}$-linear map $Q$ by

$$Qx^i \log^j x = -\sum_{\ell=1}^{m_{max}-i-1} v_{i,j,\ell} = -\sum_{\ell=1}^{m_{max}-i-1} \sum_{\nu=0}^{j} a_{i,j,\ell,\nu} x^{i+\ell} \log^\nu x,$$

which implies $p_0(\vartheta)Qx^i \log^j x = -\sum_{\ell=1}^{m_{max}-i-1} x^\ell p_\ell(\vartheta)x^i \log^j$ and $Q^{m_{max}} = 0$. Putting $Tu := \sum_{\nu=0}^{m_{max}-1} Q^\nu u$ for $u \in \sum_{i=0}^{m_{max}-1}\sum_{j=0}^{N-1} \mathbb{C}x^i \log^j x$, we have

$$\begin{aligned}
PTu &\equiv p_0(\vartheta)Tu + \sum_{\ell=1}^{m_{max}-1} x^\ell p_\ell(\vartheta)Tu && \mod \mathcal{O}_0(m_{max}, j)\\
&\equiv p_0(\vartheta)(1-Q)Tu && \mod \mathcal{O}_0(m_{max}, j)\\
&\equiv p_0(\vartheta)(1-Q)(1+Q+\cdots+Q^{m_{max}-1})u && \mod \mathcal{O}_0(m_{max}, j)\\
&= p_0(\vartheta)u.
\end{aligned}$$

Hence if $j < m_i^\vee$, $PTx^i \log^j x \equiv 0 \mod \mathcal{O}_0(m_{max}, j)$ and $u_{i,j}(x) := Tx^i \log^j x$ are required functions.

iii) Since

$$\begin{aligned}
&x^{\ell'} r'_{\ell'}(\vartheta) \prod_{\nu=1}^{N} \prod_{0\le k'<m'_\nu-\ell'} (\vartheta - m_\nu - k') \cdot x^\ell r_\ell(\vartheta) \prod_{\nu=1}^{N} \prod_{0\le k<m_\nu-\ell} (\vartheta - k)\\
&= x^{\ell+\ell'} r'_{\ell'}(\vartheta+\ell) r_\ell(\vartheta) \prod_{\nu=1}^{N} \prod_{0\le k'<m'_\nu-\ell'} (\vartheta - m_\nu - k' + \ell) \prod_{0\le k<m_\nu-\ell} (\vartheta - k)\\
&= x^{\ell+\ell'} r'_{\ell'}(\vartheta+\ell) r_\ell(\vartheta) \prod_{\nu=1}^{N} \prod_{0\le k<m_\nu+m_{\nu'}-\ell-\ell'} (\vartheta - k),
\end{aligned}$$

we have the claim. □

Definition 4.6 (generalized Riemann scheme). Let $P \in W[x]$. Then we call $P$ is *Fuchsian* in this paper when $P$ has at most regular singularities in $\mathbb{C} \cup \{\infty\}$. Suppose $P$ is Fuchsian with regular singularities at $x = c_0 = \infty$, $c_1, \ldots, c_p$ and the functions $\frac{a_j(x)}{a_n(x)}$ are holomorphic on $\mathbb{C} \setminus \{c_1, \ldots, c_p\}$ for $j = 0, \ldots, n$. Moreover suppose $P$ has the set of characteristic exponents $\{[\lambda_{j,1}]_{(m_{j,1})}, \ldots, [\lambda_{j,n_j}]_{(m_{j,n_j})}\}$ at $x = c_j$. Then we define the Riemann scheme of $P$ or the equation $Pu = 0$ by

$$\left\{\begin{matrix} x = c_0 = \infty & c_1 & \cdots & c_p\\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})}\\ \vdots & \vdots & \vdots & \vdots\\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{matrix}\right\}. \tag{4.15}$$

Remark 4.7. The Riemann scheme (4.15) always satisfies the Fuchs relation (cf. (2.21)):

$$\sum_{j=0}^{p}\sum_{\nu=1}^{n_j}\sum_{i=0}^{m_{j,\nu}-1} \bigl(\lambda_{j,\nu} + i\bigr) = \frac{(p-1)n(n-1)}{2}. \tag{4.16}$$

Definition 4.8 (spectral type). In Definition 4.6 we put

$$\mathbf{m} = (m_{0,1}, \ldots, m_{0,n_0}; m_{1,1}, \ldots; m_{p,1}, \ldots, m_{p,n_p}),$$

which will be also written as $m_{0,1}m_{0,2}\cdots m_{0,n_0}, m_{1,1}\cdots, m_{p,1}\cdots m_{p,n_p}$ for simplicity. Then $\mathbf{m}$ is a $(p+1)$-tuple of partitions of $n$ and we define that $\mathbf{m}$ is the *spectral type* of $P$.

If the set of (usual) characteristic exponents

$$\Lambda_j := \{\lambda_{j,\nu} + i\,;\ 0 \le i \le m_{j,\nu} - 1 \text{ and } \nu = 1, \dots, n_\nu\} \tag{4.17}$$

of the Fuchsian differential operator $P$ at every regular singular point $x = c_j$ are $n$ different complex numbers, $P$ is said to have *distinct exponents*.

REMARK 4.9. We remark that the Fuchsian differential equation $\mathcal{M} : Pu = 0$ is irreducible (cf. Definition 1.12) if and only if the monodromy of the equation is irreducible.

If $P = QR$ with $Q$ and $R \in W(x;\xi)$, the solution space of the equation $Qv = 0$ is a subspace of that of $\mathcal{M}$ and closed under the monodromy and therefore the monodromy is reducible. Suppose the space spanned by certain linearly independent solutions $u_1, \dots, u_m$ is invariant under the monodromy. We have a non-trivial simultaneous solution of the linear relations $b_m u_j^{(m)} + \cdots + b_1 u_j^{(1)} + b_0 u_j = 0$ for $j = 1, \dots, m$. Then $\frac{b_j}{b_m}$ are single-valued holomorphic functions on $\mathbb{C} \cup \{\infty\}$ excluding finite number of singular points. In view of the local behavior of solutions, the singularities of $\frac{b_j}{b_m}$ are at most poles and hence they are rational functions. Then we may assume $R = b_m \partial^m + \cdots + b_0 \in W(x;\xi)$ and $P \in W(x;\xi)R$.

Here we note that $R$ is Fuchsian but $R$ may have a singularity which is not a singularity of $P$ and is an *apparent singularity*. For example, we have

$$x(1-x)\partial^2+(\gamma-\alpha x)\partial+\alpha = \Bigl(\frac{\gamma}{\alpha}-x\Bigr)^{-1}\Bigl(x(1-x)\partial+(\gamma-\alpha x)\Bigr)\Bigl(\Bigl(\frac{\gamma}{\alpha}-x\Bigr)\partial+1\Bigr). \tag{4.18}$$

We also note that the equation $\partial^2 u = xu$ is irreducible and the monodromy of its solutions is reducible.

## 4.2. Tuples of partitions

For our purpose it will be better to allow some $m_{j,\nu}$ equal 0 and we generalize the notation of tuples of partitions as in [**O6**].

DEFINITION 4.10. Let $\mathbf{m} = \bigl(m_{j,\nu}\bigr)_{\substack{j=0,1,\dots\\ \nu=1,2,\dots}}$ be an ordered set of infinite number of non-negative integers indexed by non-negative integers $j$ and positive integers $\nu$. Then $\mathbf{m}$ is called a $(p+1)$-*tuple of partitions of* $n$ if the following two conditions are satisfied.

$$\sum_{\nu=1}^{\infty} m_{j,\nu} = n \qquad (j = 0, 1, \dots), \tag{4.19}$$

$$m_{j,1} = n \qquad (\forall j > p). \tag{4.20}$$

A $(p+1)$-tuple of partition $\mathbf{m}$ is called *monotone* if

$$m_{j,\nu} \ge m_{j,\nu+1} \quad (j = 0, 1, \dots,\ \nu = 1, 2, \dots) \tag{4.21}$$

and called *trivial* if $m_{j,\nu} = 0$ for $j = 0, 1, \dots$ and $\nu = 2, 3, \dots$. Moreover $\mathbf{m}$ is called *standard* if $\mathbf{m}$ is monotone and $m_{j,2} > 0$ for $j = 0, \dots, p$. The greatest common divisor of $\{m_{j,\nu}; j = 0, 1, \dots,\ \nu = 1, 2, \dots\}$ is denoted by $\gcd \mathbf{m}$ and $\mathbf{m}$ is called *divisible* (resp. *indivisible*) if $\gcd \mathbf{m} \ge 2$ (resp. $\gcd \mathbf{m} = 1$). The totality of $(p+1)$-tuples of partitions of $n$ are denoted by $\mathcal{P}_{p+1}^{(n)}$ and we put

$$\mathcal{P}_{p+1} := \bigcup_{n=0}^{\infty} \mathcal{P}_{p+1}^{(n)}, \quad \mathcal{P}^{(n)} := \bigcup_{p=0}^{\infty} \mathcal{P}_{p+1}^{(n)}, \quad \mathcal{P} := \bigcup_{p=0}^{\infty} \mathcal{P}_{p+1}, \tag{4.22}$$

$$\operatorname{ord} \mathbf{m} := n \quad \text{if} \quad \mathbf{m} \in \mathcal{P}^{(n)}, \tag{4.23}$$

$$\mathbf{1} := (1, 1, \dots) = \bigl(m_{j,\nu} = \delta_{\nu,1}\bigr)_{\substack{j=0,1,\dots\\ \nu=1,2,\dots}} \in \mathcal{P}^{(1)}, \tag{4.24}$$

$$\operatorname{idx}(\mathbf{m},\mathbf{m}') := \sum_{j=0}^{p}\sum_{\nu=1}^{\infty} m_{j,\nu}m'_{j,\nu} - (p-1)\operatorname{ord}\mathbf{m}\cdot\operatorname{ord}\mathbf{m}', \tag{4.25}$$

$$\operatorname{idx}\mathbf{m} := \operatorname{idx}(\mathbf{m},\mathbf{m}) = \sum_{j=0}^{p}\sum_{\nu=1}^{\infty} m_{j,\nu}^2 - (p-1)\operatorname{ord}\mathbf{m}^2, \tag{4.26}$$

$$\operatorname{Pidx}\mathbf{m} := 1 - \frac{\operatorname{idx}\mathbf{m}}{2}. \tag{4.27}$$

Here $\operatorname{ord}\mathbf{m}$ is called the *order* of $\mathbf{m}$. For $\mathbf{m}$, $\mathbf{m}' \in \mathcal{P}$ and a non-negative integer $k$, $\mathbf{m} + k\mathbf{m}' \in \mathcal{P}$ is naturally defined. Note that

$$\operatorname{idx}(\mathbf{m}+\mathbf{m}') = \operatorname{idx}\mathbf{m} + \operatorname{idx}\mathbf{m}' + 2\operatorname{idx}(\mathbf{m},\mathbf{m}'), \tag{4.28}$$

$$\operatorname{Pidx}(\mathbf{m}+\mathbf{m}') = \operatorname{Pidx}\mathbf{m} + \operatorname{Pidx}\mathbf{m}' - \operatorname{idx}(\mathbf{m},\mathbf{m}') - 1. \tag{4.29}$$

For $\mathbf{m} \in \mathcal{P}_{p+1}^{(n)}$ we choose integers $n_0, \dots, n_p$ so that $m_{j,\nu} = 0$ for $\nu > n_j$ and $j = 0, \dots, p$ and we will sometimes express $\mathbf{m}$ as

$$\begin{aligned}\mathbf{m} &= (\mathbf{m}_0, \mathbf{m}_1, \dots, \mathbf{m}_p)\\ &= m_{0,1}, \dots, m_{0,n_0}; \dots; m_{k,1}, \dots, m_{p,n_p}\\ &= m_{0,1}\cdots m_{0,n_0}, m_{1,1}\cdots m_{1,n_1}, \dots, m_{k,1}\cdots m_{p,n_p}\end{aligned}$$

if there is no confusion. Similarly $\mathbf{m} = (m_{0,1}, \dots, m_{0,n_0})$ if $\mathbf{m} \in \mathcal{P}_1$. Here

$$\mathbf{m}_j = (m_{j,1}, \dots, m_{j,n_j}) \quad\text{and}\quad \operatorname{ord}\mathbf{m} = m_{j,1} + \cdots + m_{j,n_j} \quad (0 \le j \le p).$$

For example $\mathbf{m} = (m_{j,\nu}) \in \mathcal{P}_3^{(4)}$ with $m_{1,1} = 3$ and $m_{0,\nu} = m_{2,\nu} = m_{1,2} = 1$ for $\nu = 1, \dots, 4$ will be expressed by

$$\mathbf{m} = 1,1,1,1;3,1;1,1,1,1 = 1111,31,1111 = 1^4,31,1^4$$

and mostly we use the notation $1111, 31, 1111$ in the above. To avoid the confusion for the number larger than 10, we sometimes use the convention given in §13.1.3.

Let $\mathfrak{S}_\infty$ be the restricted permutation group of the set of indices $\mathbb{Z}_{\ge 0} = \{0,1,2,3,\dots\}$, which is generated by the transpositions $(j, j+1)$ with $j \in \mathbb{Z}_{\ge 0}$. Put $\mathfrak{S}'_\infty = \{\sigma \in \mathfrak{S}_\infty\,;\, \sigma(0) = 0\}$, which is isomorphic to $\mathfrak{S}_\infty$.

Definition 4.11. The transformation groups $S_\infty$ and $S'_\infty$ of $\mathcal{P}$ are defined by

$$\begin{aligned}&S_\infty := H \ltimes S'_\infty,\\ &S'_\infty := \{(\sigma_i)_{i=0,1,\dots}\,;\, \sigma_i \in \mathfrak{S}'_\infty,\ \sigma_i = 1 \ \ (i \gg 1)\}, \quad H \simeq \mathfrak{S}_\infty,\\ &m'_{j,\nu} = m_{\sigma(j),\sigma_j(\nu)} \qquad (j = 0,1,\dots,\ \nu = 1,2,\dots)\end{aligned} \tag{4.30}$$

for $g = (\sigma, \sigma_1, \dots) \in S_\infty$, $\mathbf{m} = (m_{j,\nu}) \in \mathcal{P}$ and $\mathbf{m}' = g\mathbf{m}$. A tuple $\mathbf{m} \in \mathcal{P}$ is *isomorphic* to a tuple $\mathbf{m}' \in \mathcal{P}$ if there exists $g \in S_\infty$ such that $\mathbf{m}' = g\mathbf{m}$. We denote by $s\mathbf{m}$ the unique monotone element in $S'_\infty\mathbf{m}$.

Definition 4.12. For a tuple of partitions $\mathbf{m} = \Big(m_{j,\nu}\Big)_{\substack{1\le\nu\le n_j\\ 0\le j\le p}} \in \mathcal{P}_{p+1}$ and $\lambda = \Big(\lambda_{j,\nu}\Big)_{\substack{1\le\nu\le n_j\\ 0\le j\le p}}$ with $\lambda_{j,\nu} \in \mathbb{C}$, we define

$$\big|\{\lambda_{\mathbf{m}}\}\big| := \sum_{j=0}^{p}\sum_{\nu=1}^{n_j} m_{j,\nu}\lambda_{j,\nu} - \operatorname{ord}\mathbf{m} + \frac{\operatorname{idx}\mathbf{m}}{2}. \tag{4.31}$$

We note that the Fuchs relation (4.16) is equivalent to

$$|\{\lambda_{\mathbf{m}}\}| = 0 \tag{4.32}$$

because

$$\begin{aligned}\sum_{j=0}^{p}\sum_{\nu=1}^{n_j}\sum_{i=0}^{m_{j,\nu}-1} i &= \frac{1}{2}\sum_{j=0}^{p}\sum_{\nu=1}^{n_j} m_{j,\nu}(m_{j,\nu}-1) = \frac{1}{2}\sum_{j=0}^{p}\sum_{\nu=1}^{n_j} m_{j,\nu}^2 - \frac{1}{2}(p+1)n \\ &= \frac{1}{2}\Big(\operatorname{idx}\mathbf{m} + (p-1)n^2\Big) - \frac{1}{2}(p+1)n \\ &= \frac{1}{2}\operatorname{idx}\mathbf{m} - n + \frac{(p-1)n(n-1)}{2}.\end{aligned}$$

## 4.3. Conjugacy classes of matrices

Now we review on the conjugacy classes of matrices. For $\mathbf{m} = (m_1, \dots, m_N) \in \mathcal{P}_1^{(n)}$ and $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$ we define a matrix $L(\mathbf{m}; \lambda) \in M(n, \mathbb{C})$ as follows, which is introduced and effectively used by [**O2**] and [**O6**]:

If $\mathbf{m}$ is monotone, then

$$L(\mathbf{m};\lambda) := \Big(A_{ij}\Big)_{\substack{1\le i\le N\\ 1\le j\le N}}, \quad A_{i,j} \in M(m_i, m_j, \mathbb{C}),$$

$$A_{ij} = \begin{cases} \lambda_i I_{m_i} & (i=j), \\ I_{m_i,m_j} := \Big(\delta_{\mu\nu}\Big)_{\substack{1\le\mu\le m_i\\ 1\le\nu\le m_j}} = \begin{pmatrix} I_{m_j} \\ 0 \end{pmatrix} & (i=j-1), \\ 0 & (i\ne j,\ j-1). \end{cases} \tag{4.33}$$

Here $I_{m_i}$ denote the identity matrix of size $m_i$ and $M(m_i, m_j, \mathbb{C})$ means the set of matrices of size $m_i \times m_j$ with components in $\mathbb{C}$ and $M(m, \mathbb{C}) := M(m, m, \mathbb{C})$.

For example

$$L(2,1,1;\lambda_1,\lambda_2,\lambda_3) := \begin{pmatrix} \lambda_1 & 0 & 1 & \\ 0 & \lambda_1 & 0 & \\ & & \lambda_2 & 1 \\ & & & \lambda_3 \end{pmatrix}.$$

Suppose $\mathbf{m}$ is not monotone. Then we fix a permutation $\sigma$ of $\{1, \dots, N\}$ so that $(m_{\sigma(1)}, \dots, m_{\sigma(N)})$ is monotone and put

$$L(\mathbf{m};\lambda) = L(m_{\sigma(1)}, \dots, m_{\sigma(N)}; \lambda_{\sigma(1)}, \dots, \lambda_{\sigma(N)}).$$

When $\lambda_1 = \cdots = \lambda_N = \mu$, $L(\mathbf{m}; \lambda)$ may be simply denoted by $L(\mathbf{m}, \mu)$.

We denote $A \sim B$ for $A$, $B \in M(n, \mathbb{C})$ if and only if there exists $g \in GL(n, \mathbb{C})$ with $B = gAg^{-1}$.

When $A \sim L(\mathbf{m}; \lambda)$, $\mathbf{m}$ is called the *spectral type* of $A$ and denoted by $\operatorname{spc} A$ with a monotone $\mathbf{m}$.

Remark 4.13. i) If $\mathbf{m} = (m_1, \dots, m_N) \in \mathcal{P}_1^{(n)}$ is monotone, we have

$$A \sim L(\mathbf{m};\lambda) \ \Leftrightarrow\ \operatorname{rank}\prod_{\nu=1}^{j}(A - \lambda_\nu) = n - (m_1 + \cdots + m_j) \quad (j = 0, 1, \dots, N).$$

ii) For $\mu \in \mathbb{C}$, put

$$(\mathbf{m};\lambda)_\mu = (m_{i_1}, \dots, m_{i_N}; \mu) \ \text{ with } \ \{i_1, \dots, i_N\} = \{i\,;\, \lambda_i = \mu\}. \tag{4.34}$$

Then we have

$$L(\mathbf{m};\lambda) \sim \bigoplus_{\mu\in\mathbb{C}} L\big((\mathbf{m};\lambda)_\mu\big). \tag{4.35}$$

iii) Suppose $\mathbf{m}$ is monotone. Then for $\mu \in \mathbb{C}$

$$\begin{aligned} L(\mathbf{m},\mu) &\sim \bigoplus_{j=1}^{m_1} J\bigl(\max\{\nu\,;\,m_\nu \ge j\},\mu\bigr), \\ J(k,\mu) &:= L(1^k,\mu) \in M(k,\mathbb{C}). \end{aligned} \tag{4.36}$$

iv) For $A \in M(n,\mathbb{C})$, we put $Z(A) = Z_{M(n,\mathbb{C})}(A) := \{X \in M(n,\mathbb{C})\,;\, AX = XA\}$. Then

$$\dim Z_{M(n,\mathbb{C})}\bigl(L(\mathbf{m},\lambda)\bigr) = m_1^2 + m_2^2 + \cdots$$

v) (cf. [**O8**, Lemma 3.1]). Let $\mathbf{A}(t) : [0,1) \to M(n,\mathbb{C})$ be a continuous function. Suppose there exist a continuous function $\lambda = (\lambda_1,\ldots,\lambda_N) : [0,1) \to \mathbb{C}^N$ such that $A(t) \sim L(\mathbf{m};\lambda(t))$ for $t \in (0,1)$. Then

$$A(0) \sim L\bigl(\mathbf{m};\lambda(0)\bigr) \quad \text{if and only if} \quad \dim Z\bigl(A(0)\bigr) = m_1^2 + \cdots + m_N^2. \tag{4.37}$$

Note that the Jordan canonical form of $L(\mathbf{m};\lambda)$ is easily obtained by (4.35) and (4.36). For example, $L(2,1,1;\mu) \simeq J(3,\mu) \oplus J(1,\mu)$.

## 4.4. Realizable tuples of partitions

PROPOSITION 4.14. *Let $Pu = 0$ be a differential equation of order $n$ which has a regular singularity at $0$. Let $\{[\lambda_1]_{(m_1)},\ldots,[\lambda_N]_{(m_N)}\}$ be the corresponding set of the characteristic exponents. Here $\mathbf{m} = (m_1,\ldots,m_N)$ a partition of $n$.*

i) *Suppose there exists $k$ such that*

$$\begin{gathered} \lambda_1 = \lambda_2 = \cdots = \lambda_k, \\ m_1 \ge m_2 \ge \cdots \ge m_k, \\ \lambda_j - \lambda_1 \notin \mathbb{Z} \qquad (j = k+1,\ldots,N). \end{gathered}$$

*Let $\mathbf{m}^\vee = (m_1^\vee,\ldots,m_r^\vee)$ be the dual partition of $(m_1,\ldots,m_k)$* (cf. (4.11)). *Then for $i = 0,\ldots,m_1-1$ and $j = 0,\ldots,m_{i+1}^\vee - 1$ the equation has the solutions*

$$u_{i,j}(x) = \sum_{\nu=0}^{j} x^{\lambda_1+i}\log^\nu x \cdot \phi_{i,j,\nu}(x). \tag{4.38}$$

*Here $\phi_{i,j,\nu}(x) \in \mathcal{O}_0$ and $\phi_{i,\nu,j}(0) = \delta_{\nu,j}$ for $\nu = 0,\ldots,j-1$.*

ii) *Suppose*

$$\lambda_i - \lambda_j \neq \mathbb{Z}\setminus\{0\} \qquad (0 \le i < j \le N). \tag{4.39}$$

*In this case we say that the set of characteristic exponents $\{[\lambda_1]_{(m_1)},\ldots,[\lambda_N]_{(m_N)}\}$ is distinguished. Then the monodromy generator of the solutions of the equation at $0$ is conjugate to*

$$L\bigl(\mathbf{m};(e^{2\pi\sqrt{-1}\lambda_1},\ldots,e^{2\pi\sqrt{-1}\lambda_N})\bigr).$$

PROOF. Lemma 4.5 ii) shows that there exist $u_{i,j}(x)$ of the form stated in i) which satisfy $Pu_{i,j}(x) \in \mathcal{O}_0(\lambda_1+m_1,j)$ and then we have $v_{i,j}(x) \in \mathcal{O}_0(\lambda_1+m_1,j)$ such that $Pu_{i,j}(x) = Pv_{i,j}(x)$ because of (2.6). Thus we have only to replace $u_{i,j}(x)$ by $u_{i,j}(x) - v_{i,j}(x)$ to get the claim in i). The claim in ii) follows from that of i). $\square$

REMARK 4.15. i) Suppose $P$ is a Fuchsian differential operator with regular singularities at $x = c_0 = \infty, c_1,\ldots,c_p$ and moreover suppose $P$ has distinct exponents. Then the Riemann scheme of $P$ is (4.15) if and only if $Pu = 0$ has local solutions $u_{j,\nu,i}(x)$ of the form

$$u_{j,\nu,i}(x)=\begin{cases}(x-c_j)^{\lambda_{j,\nu}+i}\bigl(1+o(|x-c_j|^{m_{j,\nu}-i-1})\bigr)\\ \qquad\qquad (x\to c_j,\ i=0,\dots,m_{j,\nu}-1,\ j=1,\dots,p),\\ x^{-\lambda_{0,\nu}-i}\bigl(1+o(x^{-m_{0,\nu}+i+1})\bigr)\\ \qquad\qquad (x\to\infty,\ i=0,\dots,m_{0,\nu}).\end{cases}\tag{4.40}$$

Moreover suppose $\lambda_{j,\nu}-\lambda_{j,\nu'}\notin\mathbb{Z}$ for $1\le\nu<\nu'\le n_j$ and $j=0,\dots,p$. Then

$$u_{j,\nu,i}(x)=\begin{cases}(x-c_j)^{\lambda_{j,\nu}+i}\phi_{j,\nu,i}(x) & (1\le j\le p)\\ x^{-\lambda_{0,\nu}-i}\phi_{0,\nu,i}(x) & (j=0)\end{cases}\tag{4.41}$$

with $\phi_{j,\nu,i}(x)\in\mathcal{O}_{c_j}$ satisfying $\phi_{j,\nu,i}(c_j)=1$. In this case $P$ has the Riemann scheme (4.15) if and only if at the each singular point $x=c_j$, the set of characteristic exponents of the equation $Pu=0$ equals $\Lambda_j$ in (4.17) and the monodromy generator of its solutions is semisimple.

ii) Suppose $P$ has the Riemann scheme (4.15) and $\lambda_{1,1}=\cdots=\lambda_{1,n_1}$. Then the monodromy generator of the solutions of $Pu=0$ at $x=c_1$ has the eigenvalue $e^{2\pi\sqrt{-1}\lambda_{1,1}}$ with multiplicity $n$. Moreover the monodromy generator is conjugate to the matrix $L\bigl((m_{1,1},\dots,m_{1,n_1}),e^{2\pi\sqrt{-1}\lambda_{1,1}}\bigr)$, which is also conjugate to

$$J(m_{1,1}^\vee,e^{2\pi\sqrt{-1}\lambda_{1,1}})\oplus\cdots\oplus J(m_{1,n_1'}^\vee,e^{2\pi\sqrt{-1}\lambda_{1,1}}).$$

Here $(m_{1,1}^\vee,\dots,m_{1,n_1^\vee}^\vee)$ is the dual partition of $(m_{1,1},\dots,m_{1,n_1})$. A little weaker condition for $\lambda_{j,\nu}$ assuring the same conclusion is given in Proposition 9.9.

Definition 4.16 (realizable spectral type). Let $\mathbf{m}=(\mathbf{m}_0,\dots,\mathbf{m}_p)$ be a $(p+1)$-tuple of partitions of a positive integer $n$. Here $\mathbf{m}_j=(m_{j,1},\dots,m_{j,n_j})$ and $n=m_{j,1}+\cdots+m_{j,n_j}$ for $j=0,\dots,p$ and $m_{j,\nu}$ are non-negative numbers. Fix $p$ different points $c_j$ $(j=1,\dots,p)$ in $\mathbb{C}$ and put $c_0=\infty$.

Then $\mathbf{m}$ is a *realizable spectral type* if there exists a Fuchsian operator $P$ with the Riemann scheme (4.15) for generic $\lambda_{j,\nu}$ satisfying the Fuchs relation (4.16). Moreover in this case if there exists such $P$ so that the equation $Pu=0$ is irreducible, which is equivalent to say that the monodromy of the equation is irreducible, then $\mathbf{m}$ is *irreducibly realizable*.

Remark 4.17. i) In the above definition $\{\lambda_{j,\nu}\}$ are generic if, for example, $0<m_{0,1}<\operatorname{ord}\mathbf{m}$ and $\{\lambda_{j,\nu}\,;\,(j,\nu)\neq(0,1),\ j=0,\dots,p,\ 1\le\nu\le n_j\}\cup\{1\}$ are linearly independent over $\mathbb{Q}$.

ii) It follows from the facts (cf. (2.22)) in §2.1 that if $\mathbf{m}\in\mathcal{P}$ satisfies

$$\begin{gathered}|\{\lambda_{\mathbf{m}'}\}|\notin\mathbb{Z}_{\le 0}=\{0,-1,-2,\dots\}\text{ for any }\mathbf{m}',\,\mathbf{m}''\in\mathcal{P}\\ \text{satisfying }\mathbf{m}=\mathbf{m}'+\mathbf{m}''\text{ and }0<\operatorname{ord}\mathbf{m}'<\operatorname{ord}\mathbf{m},\end{gathered}\tag{4.42}$$

the Fuchsian differential equation with the Riemann scheme (4.15) is irreducible. Hence if $\mathbf{m}$ is indivisible and realizable, $\mathbf{m}$ is irreducibly realizable.

Fix distinct $p$ points $c_1,\dots,c_p$ in $\mathbb{C}$ and put $c_0=\infty$. The Fuchsian differential operator $P$ with regular singularities at $x=c_j$ for $j=1,\dots,n$ has the *normal form*

$$P=\Bigl(\prod_{j=1}^p(x-c_j)^n\Bigr)\partial^n+a_{n-1}(x)\partial^{n-1}+\cdots+a_1(x)\partial+a_0(x),\tag{4.43}$$

where $a_i(x)\in\mathbb{C}[x]$ satisfy

$$\deg a_i(x)\le(p-1)n+i,\tag{4.44}$$

$$(\partial^\nu a_i)(c_j)=0\quad(0\le\nu\le i-1)\tag{4.45}$$

for $i = 0, \dots, n-1$.

Note that the condition (4.44) (resp. (4.45)) corresponds to the fact that $P$ has regular singularities at $x = c_j$ for $j = 1, \dots, p$ (resp. at $x = \infty$).

Since $a_i(x) = b_i(x) \prod_{j=1}^p (x - c_j)^i$ with $b_i(x) = \sum_{r=0}^{(p-1)(n-i)} b_{i,r} x^r \in W[x]$ satisfying $\deg b_i(x) \le (p-1)n + i - pi = (p-1)(n-i)$, the operator $P$ has the parameters $\{b_{i,r}\}$. The numbers of the parameters equals

$$\sum_{i=0}^{n-1} \bigl((p-1)(n-i) + 1\bigr) = \frac{(pn + p - n + 1)n}{2},$$

The condition $(x - c_j)^{-k} P \in W[x]$ implies $(\partial^\ell a_i)(c_j) = 0$ for $0 \le \ell \le k-1$ and $0 \le i \le n$, which equals $(\partial^\ell b_i)(c_j) = 0$ for $0 \le \ell \le k - 1 - i$ and $0 \le i \le k-1$. Therefore the condition

$$(x - c_j)^{-m_{j,\nu}} \operatorname{Ad}\bigl((x - c_j)^{-\lambda_{j,\nu}}\bigr) P \in W[x] \tag{4.46}$$

gives $\frac{(m_{j,\nu}+1)m_{j,\nu}}{2}$ independent linear equations for $\{b_{\nu,r}\}$ since $\sum_{i=0}^{m_{j,\nu}-1} (m_{j,\nu} - i) = \frac{(m_{j,\nu}+1)m_{j,\nu}}{2}$. If all these equations have a simultaneous solution and they are independent except for the relation caused by the Fuchs relation, the number of the parameters of the solution equals

$$\begin{aligned}
&\frac{(pn + p - n + 1)n}{2} - \sum_{j=0}^{p} \sum_{\nu=1}^{n_j} \frac{m_{j,\nu}(m_{j,\nu}+1)}{2} + 1 \\
&= \frac{(pn + p - n + 1)n}{2} - \sum_{j=0}^{p} \sum_{\nu=1}^{n_j} \frac{m_{j,\nu}^2}{2} - (p+1)\frac{n}{2} + 1 \\
&= \frac{1}{2}\Bigl((p-1)n^2 - \sum_{j=0}^{p} \sum_{\nu=1}^{n_j} m_{j,\nu}^2 + 1\Bigr) = \operatorname{Pidx} \mathbf{m}.
\end{aligned} \tag{4.47}$$

Remark 4.18 (cf. [**O6**, §5]). Katz [**Kz**] introduced the *index of rigidity* of an irreducible local system by the number $\operatorname{idx} \mathbf{m}$ whose spectral type equals $\mathbf{m} = (m_{j,\nu})_{\substack{j=0,\dots,p \\ \nu=1,\dots,n_j}}$ and proves $\operatorname{idx} \mathbf{m} \le 2$, if the local system is irreducible.

Assume the local system is irreducible. Then Katz [**Kz**] shows that the local system is uniquely determined by the local monodromies if and only if $\operatorname{idx} \mathbf{m} = 2$ and in this case the local system and the tuple of partition $\mathbf{m}$ are called *rigid*. If $\operatorname{idx} \mathbf{m} > 2$, the corresponding system of differential equations of *Schleginger normal form*

$$\frac{du}{dx} = \sum_{j=1}^{p} \frac{A_j}{x - a_j} u \tag{4.48}$$

has $2 \operatorname{Pidx} \mathbf{m}$ parameters which are independent from the characteristic exponents and local monodromies. They are called *accessory parameters*. Here $A_j$ are constant square matrices of size $n$. The number of accessory parameters of the single Fuchsian differential operator without apparent singularities will be the half of this number $2 \operatorname{Pidx} \mathbf{m}$ (cf. Theorem 6.14 and [**Sz**]).

Lastly in this section we calculate the Riemann scheme of the products and the dual of Fuchsian differential operators.

Theorem 4.19. *Let $P$ be a Fuchsian differential operator with the Riemann scheme* (4.15). *Suppose $P$ has the normal form* (4.43).

i) *Let $P'$ be a Fuchsian differential operator with regular singularities also at $x = c_0 = \infty, c_1, \dots, c_p$. Then if $P'$ has the Riemann scheme*

$$\left\{\begin{matrix} x=c_0=\infty & c_j \quad (j=1,\dots,p) \\ [\lambda_{0,1}+m_{0,1}-(p-1)\operatorname{ord}\mathbf{m}]_{(m'_{0,1})} & [\lambda_{j,1}+m_{j,1}]_{(m'_{j,1})} \\ \vdots & \vdots \\ [\lambda_{0,n_0}+m_{0,n_0}-(p-1)\operatorname{ord}\mathbf{m}]_{(m'_{0,n_0})} & [\lambda_{j,n_j}+m_{j,n_j}]_{(m'_{j,n_j})} \end{matrix}\right\}, \tag{4.49}$$

*the Fuchsian operator $P'P$ has the spectral type $\mathbf{m}+\mathbf{m}'$ and the Riemann scheme*

$$\left\{\begin{matrix} x=c_0=\infty & c_1 & \cdots & c_p \\ [\lambda_{0,1}]_{(m_{0,1}+m'_{0,1})} & [\lambda_{1,1}]_{(m_{1,1}+m'_{1,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1}+m'_{p,1})} \\ \vdots & \vdots & \vdots & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0}+m'_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1}+m'_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p}+m'_{1,n_p})} \end{matrix}\right\}. \tag{4.50}$$

*Suppose the Fuchs relation* (4.32) *for* (4.15)*. Then the Fuchs relation for* (4.49) *is valid if and only if so is the Fuchs relation for* (4.50)*.*

ii) *For $Q = \sum_{k\ge 0} q_k(x)\partial^k \in W(x)$, we define the formal adjoint $Q^*$ of $Q$ by*

$$Q^* := \sum_{k\ge 0}(-\partial)^k q_k(x) \tag{4.51}$$

*and the dual operator $P^\vee$ of $P$ by*

$$P^\vee := a_n(x)(a_n(x)^{-1}P)^* \tag{4.52}$$

*when $P = \sum_{k=0}^n a_k(x)\partial^k$. Then the Riemann scheme of $P^\vee$ equals*

$$\left\{\begin{matrix} x=c_0=\infty & c_j \quad (j=1,\dots,p) \\ [2-n-m_{0,1}-\lambda_{0,1}]_{(m_{0,1})} & [n-m_{j,1}-\lambda_{j,1}]_{(m_{j,1})} \\ \vdots & \vdots \\ [2-n-m_{0,n_0}-\lambda_{0,n_0}]_{(m_{0,n_0})} & [n-m_{j,n_j}-\lambda_{j,n_j}]_{(m_{j,n_j})} \end{matrix}\right\}. \tag{4.53}$$

PROOF. i) It is clear that $P'P$ is a Fuchsian differential operator of the normal form if so is $P'$ and Lemma 4.5 iii) shows that the characteristic exponents of $P'P$ at $x = c_j$ for $j = 1,\dots,p$ are just as given in the Riemann scheme (4.50). Put $n = \operatorname{ord}\mathbf{m}$ and $n' = \mathbf{m}'$. We can also apply Lemma 4.5 iii) to $x^{-(p-1)n}P$ and $x^{-(p-1)n'}P'$ under the coordinate transformation $x \mapsto \frac{1}{x}$, we have the set of characteristic exponents as is given in (4.50) because $x^{-(p-1)(n+n')}P'P = \big(\operatorname{Ad}(x^{-(p-1)n})x^{-(p-1)n'}P'\big)(x^{-(p-1)n})P$.

The Fuchs relation for (4.49) equals

$$\sum_{j=0}^p\sum_{\nu=1}^{n_j} m'_{j,\nu}\big(\lambda_{j,\nu}+m_{j,\nu}-\delta_{j,0}(p-1)\operatorname{ord}\mathbf{m}\big) = \operatorname{ord}\mathbf{m}' - \frac{\operatorname{idx}\mathbf{m}'}{2}.$$

Since

$$\sum_{j=0}^p\sum_{\nu=1}^{n_j} m'_{j,\nu}\big(m_{j,\nu}-\delta_{j,0}(p-1)\operatorname{ord}\mathbf{m}\big) = \operatorname{idx}(\mathbf{m},\mathbf{m}'),$$

the condition is equivalent to

$$\sum_{j=0}^p\sum_{\nu=1}^{n_j} m'_{j,\nu}\lambda_{j,\nu} = \operatorname{ord}\mathbf{m}' - \frac{\operatorname{idx}\mathbf{m}}{2} - \operatorname{idx}(\mathbf{m},\mathbf{m}') \tag{4.54}$$

and also to

$$\sum_{j=0}^p\sum_{\nu=1}^{n_j}(m_{j,\nu}+m'_{j,\nu})\lambda_{j,\nu} = \operatorname{ord}(\mathbf{m}+\mathbf{m}') - \frac{\operatorname{idx}(\mathbf{m}+\mathbf{m}')}{2} \tag{4.55}$$

under the condition (4.32).

ii) We may suppose $c_1 = 0$. Then

$$\begin{aligned}
a_n(x)^{-1}P &= \sum_{\ell\ge 0} x^{\ell-n} q_\ell(\vartheta) \prod_{\substack{1\le\nu\le n_1\\ 0\le i<m_{1,\nu}-\ell}} (\vartheta-\lambda_{1,\nu}-i),\\
a_n(x)^{-1}P^\vee &= \sum_{\ell\ge 0} q_\ell(-\vartheta-1) \prod_{\substack{1\le\nu\le n_1\\ 0\le i<m_{1,\nu}-\ell}} (-\vartheta-\lambda_{1,\nu}-i-1)x^{\ell-n}\\
&= \sum_{\ell\ge 0} x^{\ell-n} s_\ell(\vartheta) \prod_{\substack{1\le\nu\le n_1\\ 0\le i<m_{1,\nu}-\ell}} (\vartheta+\lambda_{1,\nu}+i+1+\ell-n)\\
&= \sum_{\ell\ge 0} x^{\ell-n} s_\ell(\vartheta) \prod_{\substack{1\le\nu\le n_1\\ 0\le j<m_{1,\nu}-\ell}} (\vartheta+\lambda_{1,\nu}-j+m_{1,\nu}-n)
\end{aligned}$$

with suitable polynomials $q_\ell$ and $s_\ell$ such that $q_0,\, s_0 \in \mathbb{C}^\times$. Hence the set of characteristic exponents of $P^\vee$ at $c_1$ is $\{[n-m_{1,\nu}-\lambda_{1,\nu}]_{(m_{1,\nu})}\,;\,\nu=1,\dots,n_1\}$.

At infinity we have

$$\begin{aligned}
a_n(x)^{-1}P &= \sum_{\ell\ge 0} x^{-\ell-n} q_\ell(\vartheta) \prod_{\substack{1\le\nu\le n_1\\ 0\le i<m_{0,\nu}-\ell}} (\vartheta+\lambda_{0,\nu}+i),\\
(a_n(x)^{-1}P)^* &= \sum_{\ell\ge 0} x^{-\ell-n} s_\ell(\vartheta) \prod_{\substack{1\le\nu\le n_0\\ 0\le i<m_{0,\nu}-\ell}} (\vartheta-\lambda_{0,\nu}-i+1-\ell-n)\\
&= \sum_{\ell\ge 0} x^{-\ell-n} s_\ell(\vartheta) \prod_{\substack{1\le\nu\le n_1\\ 0\le j<m_{0,\nu}-\ell}} (\vartheta-\lambda_{0,\nu}+j+2-n-m_{0,\nu})
\end{aligned}$$

with suitable polynomials $q_\ell$ and $s_\ell$ with $q_0,\, s_0 \in \mathbb{C}^\times$ and the set of characteristic exponents of $P^\vee$ at $c_1$ is $\{[2-n-m_{0,\nu}-\lambda_{0,\nu}]_{(m_{0,\nu})}\,;\,\nu=1,\dots,n_0\}$ $\square$

EXAMPLE 4.20. i) The Riemann scheme of the dual $P^\vee_{\lambda_1,\dots,\lambda_p,\mu}$ of the Jordan-Pochhammer operator $P_{\lambda_1,\dots,\lambda_p,\mu}$ given in Example 1.8 iii) is

$$\begin{Bmatrix}
\frac{1}{c_1} & \cdots & \frac{1}{c_p} & \infty\\
[1]_{(p-1)} & \cdots & [1]_{(p-1)} & [2-2p+\mu]_{(p-1)}\\
\lambda_1-\mu+p-1 & \cdots & -\lambda_p-\mu+p-1 & \lambda_1+\cdots+\lambda_p+\mu-p+1
\end{Bmatrix}.$$

ii) (Okubo type) Suppose $\bar P_{\mathbf{m}}(\lambda)\in W[x]$ is of the form (11.34). Moreover suppose $\bar P_{\mathbf{m}}(\lambda)$ has the the Riemann scheme (11.34) with (11.33). Then the Riemann scheme of $\bar P_{\mathbf{m}}(\lambda)^*$ equals

$$\begin{Bmatrix}
x=\infty & x=c_j\ (j=1,\dots,p)\\
[2-m_{0,1}-\lambda_{0,1}]_{(m_{0,1})} & [0]_{(m_{j,1})}\\
[2-m_{0,2}-\lambda_{0,2}]_{(m_{0,2})} & [m_{j,1}-m_{j,2}-\lambda_{j,2}]_{(m_{j,2})}\\
\vdots & \vdots\\
[2-m_{0,n_0}-\lambda_{0,n_0}]_{(m_{0,n_0})} & [m_{j,1}-m_{j,n_j}-\lambda_{j,n_j}]_{(m_{j,n_j})}
\end{Bmatrix}. \tag{4.56}$$