CHAPTER 3

Series expansion and Contiguity relation

In this chapter we examine the transformation of series expansions and contiguity relations of the solutions of Fuchsian differential equations under our operations, which will be used in Chapter 8 and Chapter 11.

3.1. Series expansion

In this section we review the Euler transformation and remark on its relation to middle convolutions.

First we note the following which will be frequently used:

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{3.1}$$

$$\begin{aligned}(1-t)^{-\gamma} &= \sum_{\nu=0}^{\infty} \frac{(-\gamma)(-\gamma-1)\cdots(-\gamma-\nu+1)}{\nu!}(-t)^{\nu} \\ &= \sum_{\nu=0}^{\infty} \frac{\Gamma(\gamma+\nu)}{\Gamma(\gamma)\nu!}t^{\nu} = \sum_{\nu=0}^{\infty} \frac{(\gamma)_\nu}{\nu!}t^{\nu}.\end{aligned} \tag{3.2}$$

The integral (3.1) converges if $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\beta > 0$ and the right hand side is meromorphically continued to $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$. If the integral in (3.1) is interpreted in the sense of generalized functions, (3.1) is valid if $\alpha \notin \{0, -1, -2, \ldots\}$ and $\beta \notin \{0, -1, -2, \ldots\}$.

Euler transformation I_c^μ is sometimes expressed by $\partial^{-\mu}$ and as is shown in ([**Kh**, §5.1]), we have

$$\begin{aligned}I_c^\mu u(x) &:= \frac{1}{\Gamma(\mu)}\int_c^x (x-t)^{\mu-1}u(t)dt \\ &= \frac{(x-c)^\mu}{\Gamma(\mu)}\int_0^1 (1-s)^{\mu-1}u((x-c)s+c)ds,\end{aligned} \tag{3.3}$$

$$I_c^\mu \circ I_c^{\mu'} = I_c^{\mu+\mu'}, \tag{3.4}$$

$$I_c^{-n}u(x) = \frac{d^n}{dx^n}u(x), \tag{3.5}$$

$$\begin{aligned}I_c^\mu \sum_{n=0}^{\infty} c_n(x-c)^{\lambda+n} &= \sum_{n=0}^{\infty} \frac{\Gamma(\lambda+n+1)}{\Gamma(\lambda+\mu+n+1)}c_n(x-c)^{\lambda+\mu+n} \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)}\sum_{n=0}^{\infty} \frac{(\lambda+1)_n c_n}{(\lambda+\mu+1)_n}(x-c)^{\lambda+\mu+n},\end{aligned} \tag{3.6}$$

$$I_\infty^\mu \sum_{n=0}^{\infty} c_n x^{\lambda-n} = e^{\pi\sqrt{-1}\mu}\sum_{n=0}^{\infty} \frac{\Gamma(-\lambda-\mu+n)}{\Gamma(-\lambda+n)}c_n x^{\lambda+\mu-n}. \tag{3.7}$$

Moreover the following equalities which follow from (1.47) are also useful.

$$\begin{aligned}
&I_0^\mu \sum_{n=0}^\infty c_n x^{\lambda+n}(1-x)^\beta \\
(3.8)\qquad &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)} \sum_{m,\,n=0}^\infty \frac{(\lambda+1)_{m+n}(-\beta)_m c_n}{(\lambda+\mu+1)_{m+n} m!} x^{\lambda+\mu+m+n} \\
&= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)} (1-x)^{-\beta} \sum_{m,\,n=0}^\infty \frac{(\lambda+1)_n(\mu)_m(-\beta)_m c_n}{(\lambda+\mu+1)_{m+n} m!} x^{\lambda+\mu+n} \Bigl(\frac{x}{x-1}\Bigr)^m.
\end{aligned}$$

If $\lambda \notin \mathbb{Z}_{<0}$ (resp. $\lambda+\mu \notin \mathbb{Z}_{\geq 0}$) and moreover the power series $\sum_{n=0}^\infty c_n t^n$ has a positive radius of convergence, the equalities (3.6) (resp. (3.7)) is valid since I_c^μ (resp. I_∞^μ) can be defined through analytic continuations with respect to the parameters λ and μ. Note that I_c^μ is an invertible map of $\mathcal{O}_c(x-c)^\lambda$ onto $\mathcal{O}_c(x-c)^{\lambda+\mu}$ if $\lambda \notin \{-1,-2,-3,\dots\}$ and $\lambda+\mu \notin \{-1,-2,-3,\dots\}$.

Proposition 3.1. *Let λ and μ be complex numbers satisfying $\lambda \notin \mathbb{Z}_{<0}$. Differentiating the equality* (3.6) *with respect to λ, we have the linear map*

$$I_c^\mu : \mathcal{O}_c(\lambda, m) \to \mathcal{O}_c(\lambda+\mu, m) \tag{3.9}$$

under the notation (2.5)*, which is also defined by* (3.3) *if* $\operatorname{Re}\lambda > -1$ *and* $\operatorname{Re}\mu > 0$. *Here m is a non-negative integer. Then we have*

$$I_c^\mu\Bigl(\sum_{j=0}^m \phi_j \log^j(x-c)\Bigr) - I_c^\mu(\phi_m)\log^m(x-c) \in \mathcal{O}(\lambda+\mu, m-1) \tag{3.10}$$

for $\phi_j \in \mathcal{O}_c$ and I_c^μ satisfies (1.43)*. The map* (3.9) *is bijective if $\lambda+\mu \notin \mathbb{Z}_{<0}$. In particular for $k \in \mathbb{Z}_{\geq 0}$ we have $I_c^\mu \partial^k = \partial^k I_c^\mu = I_c^{\mu-k}$ on $\mathcal{O}_c(\lambda, m)$ if $\lambda - k \notin \{-1,-2,-3,\dots\}$.*

Suppose that $P \in W[x]$ and $\phi \in \mathcal{O}_c(\lambda, m)$ satisfy $P\phi = 0$, $P \neq 0$ and $\phi \neq 0$. Let k and N be non-negative integers such that

$$\partial^k P = \sum_{i=0}^N \sum_{j\geq 0} a_{i,j} \partial^i \bigl((x-c)\partial\bigr)^j \tag{3.11}$$

with suitable $a_{j,j} \in \mathbb{C}$ and put $Q = \sum_{i=0}^N \sum_{j\geq 0} c_{i,j} \partial^i \bigl((x-c)\partial - \mu\bigr)^j$. Then if $\lambda \notin \{N-1, N-2, \dots, 0, -1, \dots\}$, we have

$$I_c^\mu \partial^k P u = Q I_c^\mu(u) \quad \text{for} \quad u \in \mathcal{O}_c(\lambda, m) \tag{3.12}$$

and in particular $QI_c^\mu(\phi) = 0$.

Fix $\ell \in \mathbb{Z}$. For $u(x) = \sum_{i=\ell}^\infty \sum_{j=0}^m c_{i,j}(x-c)^i \log^j(x-c) \in \mathcal{O}_c(\ell, m)$ we put $(\Gamma_N u)(x) = \sum_{\nu=\max\{\ell,N-1\}}^\infty \sum_{j=0}^m c_{i,j}(x-c)^i \log^j(x-c)$. Then

$$\Bigl(\prod_{\ell-N\leq\nu\leq N-1} \bigl((x-c)\partial - \nu\bigr)^{m+1}\Bigr)\partial^k P\bigl(u(x) - (\Gamma_N u)(x)\bigr) = 0$$

ane therefore

$$\begin{aligned}
&\Bigl(\prod_{\ell-N\leq\nu\leq N-1} \bigl((x-c)\partial - \mu - \nu\bigr)^{m+1}\Bigr) Q I_c^\mu(\Gamma_N u) \\
&\quad = I_c^\mu\Bigl(\prod_{\ell-N\leq\nu\leq N-1} \bigl((x-c)\partial - \nu\bigr)^{m+1}\Bigr)\partial^k P u.
\end{aligned} \tag{3.13}$$

In particular, $\prod_{\ell-N\leq\nu\leq N-1} \bigl((x-c)\partial - \mu - \nu\bigr)^{m+1} \cdot QI_c^\mu\bigl(\Gamma_N(u)\bigr) = 0$ if $Pu = 0$.

Suppose moreover $\lambda \notin \mathbb{Z}$ and $\lambda + \mu \notin \mathbb{Z}$ and $Q = ST$ with $S,\ T \in W[x]$ such that $x = c$ is not a singular point of the operator S. Then $TI_c^\mu(\phi) = 0$. In particular,

$$\big(\mathrm{RAd}(\partial^{-\mu})P\big)I_c^\mu(\phi) = 0. \tag{3.14}$$

Hence if the differential equation $\big(\mathrm{RAd}(\partial^{-\mu})P\big)v = 0$ is irreducible, we have

$$W(x)\big(\mathrm{RAd}(\partial^{-\mu})P\big) = \{T \in W(x)\,;\, TI_c^\mu(\phi) = 0\}. \tag{3.15}$$

The statements above are also valid even if we replace $x - c$, I_c^μ by $\frac{1}{x}$, I_∞^μ, respectively.

Proof. It is clear that (3.9) is well-defined and (3.10) is valid. Then (3.9) is bijective because of (3.6) and (3.10). Since (1.43) is valid when $m = 0$, it is also valid when $m = 1, 2, \ldots$ by the definition of (3.9).

The equalities (3.6) and (3.7) assure that $QI_c^\mu(\phi) = 0$. Note that $TI_c^\mu(\phi) \in \mathcal{O}(\lambda + \mu - N, m)$ with a suitable positive integer N. Since $\lambda + \mu - N \notin \mathbb{Z}$ and any solution of the equation $Sv = 0$ is holomorphic at $x = c$, the equality $S\big(TI_c^\mu(\phi)\big) = 0$ implies $TI_c^\mu(\phi) = 0$.

The remaining claims in the theorem are similarly clear. □

Remark 3.2. i) Let $\gamma : [0,1] \to \mathbb{C}$ be a path such that $\gamma(0) = c$ and $\gamma(1) = x$. Suppose $u(x)$ is holomorphic along the path $\gamma(t)$ for $0 < t \le 1$ and $u(\gamma(t)) = \phi(\gamma(t))$ for $0 < t \ll 1$ with a suitable function $\phi \in \mathcal{O}_c(\lambda, m)$. Then $I_c^\mu(u)$ is defined by the integration along the path γ. In fact, if the path $\gamma(t)$ with $t \in [0,1]$ splits into the three paths corresponding to the decomposition $[0,1] = [0,\epsilon] \cup [\epsilon, 1-\epsilon] \cup [1-\epsilon, 1]$ with $0 < \epsilon \ll 1$. Let $c_1 = c, \ldots, c_p$ be points in $\mathbb{C}^n$ and suppose moreover $u(x)$ is extended to a multi-valued holomorphic function on $\mathbb{C} \setminus \{c_1, \ldots, c_p\}$. Then $I_c^x(u)$ also defines a multi-valued holomorphic function on $\mathbb{C} \setminus \{c_1, \ldots, c_p\}$.

ii) Proposition 3.1 is also valid if we replace $\mathcal{O}_c(\lambda, m)$ by the space of functions given in Remark 1.7 ii). In fact the above proof also works in this case.

3.2. Contiguity relation

The following proposition is clear from Proposition 3.1.

Proposition 3.3. *Let $\phi(x)$ be a non-zero solution of an ordinary differential equation $Pu = 0$ with an operator $P \in W[x]$. Let P_j and $S_j \in W[x]$ for $j = 1, \ldots, N$ so that $\sum_{j=1}^N P_j S_j \in W[x]P$. Then for a suitable $\ell \in \mathbb{Z}$ we have*

$$\sum Q_j\big(I_c^\mu(\phi_j)\big) = 0 \tag{3.16}$$

by putting

$$\begin{aligned} \phi_j &= S_j\phi, \\ Q_j &= \partial^{\ell-\mu} \circ P_j \circ \partial^\mu \in W[x], \end{aligned} \qquad (j = 1, \ldots, N) \tag{3.17}$$

if $\phi(x) \in \mathcal{O}(\lambda, m)$ with a non-negative integer m and a complex number λ satisfying $\lambda \notin \mathbb{Z}$ and $\lambda + \mu \notin \mathbb{Z}$ or $\phi(x)$ is a function given in Remark 1.7 ii). *If $P_j = \sum_{k\ge 0,\ \ell \ge 0} c_{j,k,\ell}\partial^k\vartheta^\ell$ with $c_{j,k,\ell} \in \mathbb{C}$, then we can assume $\ell \le 0$ in the above. Moreover we have*

$$\partial\big(I_c^{\mu+1}(\phi_1)\big) = I_c^\mu(\phi_1). \tag{3.18}$$

Proof. Fix an integer k such that $\partial^k P_j = \tilde{P}_j(\partial, \vartheta) = \sum_{i_1,i_2} c_{i_1,i_2}\partial^{i_1}\vartheta^{i_2}$ with $c_{i_1,i_2} \in \mathbb{C}$. Since $0 = \sum_{j=1}^N \partial^k P_j S_j \phi$, Proposition 3.1 proves

$$0 = \sum_{j=1}^N I_c^\mu(\tilde{P}_j(\partial, \vartheta)S_j\phi) = \sum_{j=1}^N \tilde{P}_j(\partial, \vartheta - \mu)I_c^\mu(S_j\phi),$$

which implies the first claim of the proposition.

The last claim is clear from (3.4) and (3.5). □

Corollary 3.4. *Let $P(\xi)$ and $K(\xi)$ be non-zero elements of $W[x;\xi]$. If we substitute ξ and μ by generic complex numbers, we assume that there exists a solution $\phi_\xi(x)$ satisfying the assumption in the preceding proposition and that $I_c^\mu(\phi_\xi)$ and $I_c^\mu(K(\xi)\phi_\xi)$ satisfy irreducible differential equations $T_1(\xi,\mu)v_1 = 0$ and $T_2(\xi,\mu)v_2 = 0$ with $T_1(\xi,\mu)$ and $T_2(\xi,\mu) \in W(x;\xi,\mu)$, respectively. Then the differential equation $T_1(\xi,\mu)v_1 = 0$ is isomorphic to $T_2(\xi,\mu)v_2 = 0$ as $W(x;\xi,\mu)$-modules.*

Proof. Since $K(\xi)\cdot 1 - 1\cdot K(\xi) = 0$, we have $Q(\xi,\mu)I_c^\mu(\phi_\xi) = \partial^\ell I_c^\mu(K(\xi)\phi_\xi)$ with $Q(\xi,\mu) = \partial^{\ell-\mu}\circ K(\xi)\circ\partial^\mu$. Since $\partial^\ell I_c^\mu(\phi_\xi) \neq 0$ and the equations $T_j(\xi,\mu)v_j = 0$ are irreducible for $j = 1$ and 2, there exist $R_1(\xi,\mu)$ and $R_2(\xi,\mu) \in W(x;\xi,\mu)$ such that $I_c^\mu(\phi_\xi) = R_1(\xi,\mu)Q(\xi,\mu)I_c^\mu(\phi_\xi) = R_1(\xi,\mu)\partial^\ell I_c^\mu(K(\xi)\phi_\xi)$ and $I_c^\mu(K(\xi)\phi_\xi) = R_2(\xi,\mu)\partial^\ell I_c^\mu(K(\xi)\phi_\xi) = R_2(\xi,\mu)Q(\xi,\mu)I_c^\mu(\phi_\xi)$. Hence we have the corollary. □

Using the proposition, we get the contiguity relations with respect to the parameters corresponding to powers of linear functions defining additions and the middle convolutions.

For example, in the case of Gauss hypergeometric functions, we have

$$\begin{aligned}
u_{\lambda_1,\lambda_2,\mu}(x) &:= I_0^\mu(x^{\lambda_1}(1-x)^{\lambda_2}),\\
u_{\lambda_1,\lambda_2,\mu-1}(x) &= \partial u_{\lambda_1,\lambda_2,\mu}(x),\\
\partial u_{\lambda_1+1,\lambda_2,\mu}(x) &= (x\partial + 1 - \mu)u_{\lambda_1,\lambda_2,\mu}(x),\\
\partial u_{\lambda_1,\lambda_2+1,\mu}(x) &= ((1-x)\partial + \mu - 1)u_{\lambda_1,\lambda_2,\mu}(x).
\end{aligned}$$

Here Proposition 3.3 with $\phi = x^{\lambda_1}(1-x)^{\lambda_2}$, $(P_1,S_1,P_2,S_2) = (1,x,-x,1)$ and $\ell = 1$ gives the above third identity.

Since $P_{\lambda_1,\lambda_2,\mu}u_{\lambda_1,\lambda_2,\mu}(x) = 0$ with

$$\begin{aligned}
P_{\lambda_1,\lambda_2,\mu} = \bigl(&x(1-x)\partial + (1-\lambda_1-\mu-(2-\lambda_1-\lambda_2-2\mu)x\bigr)\partial\\
&-(\mu-1)(\lambda_1+\lambda_2+\mu)
\end{aligned}$$

as is given in Example 1.8, the inverse of the relation $u_{\lambda_1,\lambda_2,\mu-1}(x) = \partial u_{\lambda_1,\lambda_2,\mu}(x)$ is

$$u_{\lambda_1,\lambda_2,\mu}(x) = -\frac{x(1-x)\partial + (1-\lambda_1-\mu-(2-\lambda_1-\lambda_2-2\mu)x\bigr)}{(\mu-1)(\lambda_1+\lambda_2+\mu)}u_{\lambda_1,\lambda_2,\mu-1}(x).$$

The equalities $u_{\lambda_1,\lambda_2,\mu-1}(x) = \partial u_{\lambda_1,\lambda_2,\mu}(x)$ and (1.47) mean

$$\begin{aligned}
&\frac{\Gamma(\lambda_1+1)x^{\lambda_1+\mu-1}}{\Gamma(\lambda_1+\mu)}F(-\lambda_2,\lambda_1+1,\lambda_1+\mu;x)\\
&\quad= \frac{\Gamma(\lambda_1+1)x^{\lambda_1+\mu-1}}{\Gamma(\lambda_1+\mu)}F(-\lambda_2,\lambda_1+1,\lambda_1+\mu+1;x)\\
&\qquad+ \frac{\Gamma(\lambda_1+1)x^{\lambda_1+\mu}}{\Gamma(\lambda_1+\mu+1)}\frac{d}{dx}F(-\lambda_2,\lambda_1+1,\lambda_1+\mu+1;x)
\end{aligned}$$

and therefore $u_{\lambda_1,\lambda_2,\mu-1}(x) = \partial u_{\lambda_1,\lambda_2,\mu}(x)$ is equivalent to

$$(\gamma-1)F(\alpha,\beta,\gamma-1;x) = (\vartheta+\gamma-1)F(\alpha,\beta,\gamma;x).$$

The contiguity relations are very important for the study of differential equations. For example the author's original proof of the connection formula (0.24) announced in [**O6**] is based on the relations (cf. §12.3).

Some results related to contiguity relations will be given in Chapter 11 but we will not go further in this subject and it will be discussed in another paper.