Advanced Studies in Pure Mathematics 65, 2015
Algebraic Geometry in East Asia — Taipei 2011
pp. 83–106

ADHM sheaf theory and wallcrossing

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Abstract.

In this article we survey the recent developments in ADHM sheaf theory on a smooth projective variety X. When X is a curve the theory is an alternative construction of stable pair theory of Pandharipande and Thomas or Gromov–Witten theory on local curve geometries. The construction relies on relative Beilinson spectral sequence and Fourier–Mukai transformation. We will present some applications of the theory, including the derivations of the wallcrossing formulas, higher rank Donaldson–Thomas invariants on local curves, and the coholomogies of the moduli of stable Hitchin pairs.

§1. Introduction

Recently we have seen much progress in the study of curve enumerations on Calabi–Yau 3-folds. Three different types of theories have been proposed, including Gromov–Witten (GW) theory, Donaldson–Thomas (DT) and Pandharipande–Thomas (PT) theory. They are conjectured to be equivalent at the level of generating functions after suitable changes of variables.

First let us briefly recall the definitions of these theories. Let X be smooth projective Calabi–Yau 3-fold over $\mathbb{C}$. For $g \geq 0$ and $\beta \in H_2(X,\mathbb{Z})$, the GW invariant $N_{g,\beta}$ is defined as the integration of the virtual class,

$$N_{g,\beta} = \int_{[\overline{\mathcal{M}}_g(X,\beta)]^{vir}} 1 \in \mathbb{Q},$$

Received December 27, 2011.
Revised August 6, 2012.
2010 *Mathematics Subject Classification.* Primary 14N35; Secondary 81T30.
Key words and phrases. Generalized Donaldson–Thomas invariants, ADHM sheaf theory, wallcrossing formulas.
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where $\overline{\mathcal{M}}_g(X,\beta)$ is the Deligne–Mumford stack of the stable map $f : C \to Z$ with C a genus g curve and $f_*[C] = \beta$. The reduced GW generating function is

$$Z_{GW} = \exp\Big(\sum_{g,\beta\neq 0} N_{g,\beta}\lambda^{2g-2}v^\beta\Big),$$

where we have omitted the contribution from the constant maps. The reduced GW generating function generates GW invariants with possibly disconnected domain curves, subject to the condition that no connected component be mapped to a point.

Now we define the DT theory. Consider the Hilbert scheme $I_n(X,\beta)$ of 1-dimensional subschemes $Z \subset X$ with $\chi(\mathcal{O}_Z) = n$ and $\beta \in H_2(X,\mathbb{Z})$. We regards $I_n(X,\beta)$ as the moduli parametrizing the ideal sheaves of 1-dimensional subschemes. The DT invariant $I_{n,\beta}$ is given by

$$I_{n,\beta} = \int_{[I_n(X,\beta)]^{vir}} 1 \in \mathbb{Z}.$$

The reduced DT generating function is given by

$$Z'_{DT} = \sum_{n,\beta} I_{n,\beta}q^n v^\beta / \sum_n I_{n,0}q^n.$$

and the MNOP conjecture [23][24] states that $Z_{GW} = Z'_{DT}$ after the change of variables $q = -e^{i\lambda}$.

The stable pair of Pandharipanda and Thomas [31] is, by definition, a pair (F,s), where s is a section of F,

$$s : \mathcal{O}_X \to F$$

such that F is a pure sheaf of dimension 1 and s has zero dimensional cokernel. The moduli $P_n(X,\beta)$ of PT stable pair (F,s) with $[F] = \beta$ and $\chi(F) = n$ is constructed in and also equipped with a symmetric perfect obstruction theory. The PT invariants and the generating function are given by

$$P_{n,\beta} = \int_{[P_n(X,\beta)]^{vir}} 1 \in \mathbb{Z} \quad , Z_{PT} = \sum_{n,\beta} P_{n,\beta}q^n v^\beta.$$

Pandharipande and Thomas conjectured that $Z'_{DT} = Z_{PT}$ as a wall-crossing formula where the denominator in Z'_{DT} is the wallcrossing difference. This conjecture has been proved in [34], [33], [2].

Motivated by string theoretical consideration, ADHM sheaf theory was first introduced by Diaconescu [7] and the theory has a natural variation of the stability conditions [6], [3]. In an asymptotic chamber of the stability condition space the ADHM sheaf theory on a projective curve X over $\mathbb{C}$ is equivalent to admissible pair theory on the projective plane bundle over X. When the twisting data (M_1, M_2) are chosen such that $M_1 \otimes_X M_2 \simeq K_X^{-1}$, this pair theory becomes a stable pair theory on the local Calabi–Yau 3-fold of the total space $M_1^{-1} \otimes_X M_2^{-1}$ over X. The key ingredients of the construction consist of a relative version of Beilinson spectral sequence and Fourier–Mukai transformation.

Using Joyce–Song theory of generalized Donaldson–Thomas invariants [20], explicit wallcrossing formulas for ADHM invariants on curves have been derived [3], which also give rise to a further generalization of higher rank ADHM invariants [4]. This part of higher rank generalization is also motivated by the work of Toda [35] and Stoppa [32].

Another interesting application of ADHM sheaf theory on curves is the computation of Betti and Hodge number of moduli spaces of stable Hitchin pairs [5]. The application is based on refined generalizations of wallcrossing formulas, generalized Donaldson–Thomas invariants, and multicover formulas.

The purpose of the article is to survey the aforementioned developments and related background material. The paper is organized as follows. In Section 2 we review the definition of ADHM sheaf theory and prove a correspondence between stable pair theory and ADHM sheaf theory on curves. In Section 3 we give a brief review of Joyce–Song theory of generalized Donaldson–Thomas invariants and present our results about wallcrossing formulas and higher rank invariants. A computational comparison between Joyce–Song and Kontsevich–Soibelman formulas is also presented. In the final section we give our conjectural recursive relations for the Poincaré polynomials of the Hitchin moduli space.

§2. ADHM sheaf theory

Let X be a smooth projective scheme over $\mathbb{C}$ equipped with a very ample line bundle $\mathcal{O}_X(1)$.

Definition 2.1. *Let M_1, M_2 be fixed line bundles on X. Set $M = M_1 \otimes_X M_2$. For fixed data $\mathcal{X} = (X, M_1, M_2)$, let $\mathcal{Q}_{\mathcal{X}}$ denote the abelian category of (M_1, M_2)-twisted coherent ADHM quiver sheaves. An object of $\mathcal{Q}_{\mathcal{X}}$ is given by a collection $\mathcal{E} = (E, E_\infty, \Phi_1, \Phi_2, \phi, \psi)$ where*

- *E, E_∞ are coherent $\mathcal{O}_X$-modules*
- *$\Phi_i : E \otimes_X M_i \to E$, $i = 1, 2$, $\phi : E \otimes_X M_1 \otimes_X M_2 \to E_\infty$, $\psi : E_\infty \to E$ are morphisms of $\mathcal{O}_X$-modules satisfying the*

ADHM relation

$$\Phi_1 \circ (\Phi_2 \otimes 1_{M_1}) - \Phi_2 \circ (\Phi_1 \otimes 1_{M_2}) + \psi \circ \phi = 0. \tag{2.1}$$

The morphisms are natural morphisms of quiver sheaves i.e. collections $(\xi, \xi_\infty) : (E, E_\infty) \to (E', E'_\infty)$ *of morphisms of* $\mathcal{O}_X$*-modules satisfying the obvious compatibility conditions with the ADHM data.*

Let $\mathcal{C}_\mathcal{X}$ *be the full abelian subcategory of* $\mathcal{Q}_\mathcal{X}$ *consisting of objects with* $E_\infty = V \otimes \mathcal{O}_X$*, where* V *is a finite dimensional vector spaces over* $\mathbb{C}$*. Note that given any two objects* $\mathcal{E}, \mathcal{E}'$ *of* $\mathcal{C}_\mathcal{X}$*, the morphisms* $\xi_\infty : V \otimes \mathcal{O}_X \to V' \otimes \mathcal{O}_X$ *must be of the form* $\xi_\infty = f \otimes 1_{\mathcal{O}_X}$*, where* $f : V \to V'$ *is a linear map.*

The numerical type of an object $\mathcal{E}$ *of* $\mathcal{C}_\mathcal{X}$ *is the collection*

$$(\mathrm{rank}(E), \deg(E), \dim(V)) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}.$$

An object of $\mathcal{C}_\mathcal{X}$ *is called an ADHM sheaf. An ADHM sheaf with* ψ *and* ϕ *identically 0 is called a Higgs sheaf.*

Definition 2.2. *Let* $\delta \in \mathbb{R}$ *be a stability parameter. The* δ*-degree of an object* $\mathcal{E}$ *of* $\mathcal{C}_\mathcal{X}$ *is* $\deg_\delta(\mathcal{E}) = d(\mathcal{E}) + \delta v(\mathcal{E})$*. If* $r(\mathcal{E}) \neq 0$*, the* δ*-slope of* $\mathcal{E}$ *is defined by* $\mu_\delta(\mathcal{E}) = \deg_\delta(\mathcal{E})/r(\mathcal{E})$*. A nontrivial object* $\mathcal{E}$ *of* $\mathcal{C}_\mathcal{X}$ *is* δ*-(semi)stable if*

$$r(E)\ \deg_\delta(\mathcal{E}')\ (\leq)\ r(E')\ \deg_\delta(\mathcal{E}) \tag{2.2}$$

for any proper nontrivial subobject $0 \subset \mathcal{E}' \subset \mathcal{E}$*.*

It was proved that the real parameter $\delta \in \mathbb{R}$ gives a stability condition in the abelian category $\mathcal{C}_\mathcal{X}$ of ADHM sheaves, *i.e.* it has see-saw property and every object has a unique Harder–Narasimhan filtration. For fixed numerical type (r, e, v) of an ADHM sheaf there are finitely many critical stability parameters dividing the real axis into chambers. The set of δ-semistable ADHM sheaves is constant within each chamber. When the numerical type is $(r, e, 1)$, the strictly semistable objects may exist only if δ takes a critical value and the origin $\delta = 0$ is a critical value for all $(r, e, 1) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times 1$.

Definition 2.3. *Let* X *be a projective curve and* Y *the total space of the projective bundle Proj*$(\mathcal{O}_X \oplus M_1 \oplus M_2)$*. Let* $d \in \mathbb{Z}_{\geq 1}$*,* $n \in \mathbb{Z}$*. An admissible pair of type* (d, n) *on* Y *is a pair* (Q, ρ)*, where* Q *is a coherent* $\mathcal{O}_Y$*-module and* $\rho \in H^0(Y, Q)$*, such that* ρ *is not identically zero,* Q *is flat over* X*,* $(\mathrm{ch}_0(Q), \mathrm{ch}_1(Q), \mathrm{ch}_2(Q)) = (0, 0, d[X])$*,* $\chi(Q) = n$ *and the cokernel of* $\rho : \mathcal{O}_Y \to Q$ *is of pure dimension 0.*

Let Π be the canonical projection $\Pi : Y \to X$. Let $z_0 \in H^0(Y, \mathcal{O}_Y(1))$ corresponding to $1 \in H^0(X, \mathcal{O}_X)$ under the canonical isomorphism

$$H^0(Y, \mathcal{O}_Y(1)) \simeq H^0(X, \mathcal{O}_X) \oplus H^0(X, M_1) \oplus H^0(X, M_2).$$

It can be shown that if (Q, ρ) is an admissible pair on Y, then Q is of pure dimension one and supp(Q) is disjoint from $D_\infty = \{z_0 = 0\}$. Notice that $Y \backslash D_\infty$ is the total space of $M_1^{-1} \oplus M_2^{-1}$ over X. Therefore the admissible pair on Y is equivalent to the stable pair theory on $Y \backslash D_\infty$. Next is the theorem relating the ADHM sheaf theory on X with the admissible pair theory on Y [7].

Theorem 2.4. *There exist a bijection between an S-family of admissible pairs on Y with certain support property and an S-family of $(\delta \gg 1)$ stable ADHM sheaves with $E_\infty = \mathcal{O}_X$ on X.*

Sketch of Proof. First we have the resolution of the diagonal $\Delta \in Y \times_X Y$. The Koszul resolution $\mathcal{K}_\Delta$ of Δ is given by

$$0 \to \mathcal{O}_Y(-2) \boxtimes \Omega^2_{Y/X}(2) \to \mathcal{O}_Y(-1) \boxtimes \Omega^1_{Y/X}(1) \to \mathcal{O}_Y \boxtimes \mathcal{O}_Y. \tag{2.3}$$

Secondly we have the identity Fourier–Mukai functor

$$\mathbf{R}p_{1*}(p_2^*(\text{.}) \otimes \mathcal{O}_\Delta). \tag{2.4}$$

From (2.3) and (2.4) we construct a spectral sequence with

$$E_1^{i,j} = R^i p_{1*}(p_2^*(Q) \otimes \mathcal{K}^j_\Delta), \tag{2.5}$$

converging to Q if $i + j = 0$ and 0 otherwise. In the end we obtain a three term complex centered at (-1) position with cohomology Q. We then take the cone construction for $(\mathcal{O}_Y \xrightarrow{\rho} Q)$, which turns out to give a $(\delta \gg 1)$ stable ADHM sheaf with $E_\infty = \mathcal{O}_X$ on X. The generalization to an S-family is straightforward and details can be found in [7, Sects. 6, 7]. Also note that this theorem does not need the condition on the deg(M_1) + deg(M_2). Q.E.D.

We also have the following results concerning moduli spaces of ADHM sheaf theory with $E_\infty = \mathcal{O}_X$:

- For fixed $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ and $\delta \in \mathbb{R}$ there is an algebraic moduli stack $\mathfrak{M}^{ss}_\delta(\mathcal{X}, r, e)$ of finite type over $\mathbb{C}$ of δ-semistable ADHM sheaves. If $\delta \in \mathbb{R}$ is noncritical, $\mathfrak{M}^{ss}_\delta(\mathcal{X}, r, e)$ is a quasi-projective scheme equipped with a perfect obstruction theory.

- For fixed $(r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ and $\delta \in \mathbb{R}$ there is a natural algebraic torus $\mathbf{S} = \mathbb{C}^{\times}$ action on the moduli stack $\mathfrak{M}_{\delta}^{ss}(\mathcal{X}, r, e)$ which acts on $\mathbb{C}$-valued points by scaling the morphisms $(\Phi_1, \Phi_2) \to (t^{-1}\Phi_1, t\Phi_2)$, $t \in \mathbf{S}$. If δ is noncritical the stack theoretic fixed locus $\mathfrak{M}_{\delta}^{ss}(\mathcal{X}, r, e)^{\mathbf{S}}$ is proper over $\mathbb{C}$. Therefore we define the ADHM invariants to be the residual ADHM invariants $A_{\delta}^{\mathbf{S}}(r,e)$ by equivariant virtual integration in each stability chamber.
- (Theorem 2.4) For $(r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ there exists a critical value $\delta_M \in \mathbb{R}_{>0}$ so that for any $\delta > \delta_M$, $\mathfrak{M}_{\delta}^{ss}(\mathcal{X}, r, e)$ is isomorphic to the moduli space of stable pairs of Pandharipande and Thomas on the total space of $M_1^{-1} \oplus M_2^{-1}$ over X. This identification includes the equivariant perfect obstruction theories establishing an equivalence between local stable pair theory and asymptotic ADHM theory.

§3. Wallcrossing and higher rank ADHM invariants

For completeness we include a very brief review of Joyce–Song theory. This review is far from self-contained and the interested readers are encouraged to refer to the original papers [20], [16], [17], [18], [19].

3.1. Joyce–Song theory of generalized Donaldson–Thomas invariants

Joyce–Song Theory is a virtual counting theory on an algebraic moduli stack $\mathfrak{M}_{\mathcal{A}}$ locally of finite type over $\mathbb{C}$, parametrizing all the objects in the abelian category $\mathcal{A}$. The central element in Joyce–Song theory is the stack function algebra $\mathsf{SF}(\mathfrak{M}_{\mathcal{A}})$, which is Grothendieck group generated over $\mathbb{Q}$ by isomorphism classes of pairs $[(\mathfrak{X}, \rho)]$ where $\mathfrak{X}$ is an algebraic stack of finite type over $\mathbb{C}$ and $\rho : \mathfrak{X} \to \mathfrak{M}_{\mathcal{A}}$ is a representable morphism of stacks.

Let $\mathfrak{Exact}_{\mathcal{A}}$ be the moduli stack locally of finite type over $\mathbb{C}$, parametrizing all the three term exact sequences $0 \to E_1 \to E_2 \to E_3 \to 0$ in $\mathcal{A}$. There are also three natural projections

$$\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3 : \mathfrak{Exact}_{\mathcal{A}} \to \mathfrak{M}_{\mathcal{A}}.$$

We define a $\mathbb{Q}$-bilinear operation $* : \mathsf{SF}(\mathfrak{M}_{\mathcal{A}}) \times \mathsf{SF}(\mathfrak{M}_{\mathcal{A}}) \to \mathsf{SF}(\mathfrak{M}_{\mathcal{A}})$ as follows. Given two stack functions $[(\mathfrak{X}_1, \rho_1)]$ and $[(\mathfrak{X}_3, \rho_3)]$, set the Ringel–Hall multiplication

$$[(\mathfrak{X}_1, \rho_1)] * [(\mathfrak{X}_3, \rho_3)] = [(\mathsf{p}_1, \mathsf{p}_3)^*(\mathfrak{X}_1 \times \mathfrak{X}_3), \mathsf{p}_2 \circ \mathsf{f})], \tag{3.1}$$

where f is determined by the following Cartesian diagram of stacks.

$$
\begin{tikzcd}
(\mathsf{p}_1,\mathsf{p}_3)^*(\mathfrak{X}_1\times\mathfrak{X}_3) \arrow[rr, "\mathsf{f}"] \arrow[d] & & \mathfrak{Exact}_{\mathcal{A}} \arrow[r, "\mathsf{p}_2"] \arrow[d, "(\mathsf{p}_1,\mathsf{p}_3)"'] & \mathfrak{M}_{\mathcal{A}} \\
\mathfrak{X}_1\times\mathfrak{X}_3 \arrow[rr, "\mathsf{f}_1\times\mathsf{f}_3"] & & \mathfrak{M}_{\mathcal{A}}\times\mathfrak{M}_{\mathcal{A}} &
\end{tikzcd} \tag{3.2}
$$

According to Joyce–Song, $(\mathsf{SF}(\mathfrak{M}_{\mathcal{A}}),*,\delta_{[0]})$ is an associative algebra with unity, where $\delta_{[0]}=[(\mathrm{Spec}(\mathbb{C}),0)]$ is the stack function of the zero object in $\mathfrak{M}_{\mathcal{A}}$. One then defines a Lie subalgebra $\mathsf{SF}^{\mathsf{ind}}_{\mathsf{alg}}(\mathfrak{M}_{\mathcal{A}})$ imposing certain conditions on the stabilizers of closed points $\mathfrak{x}$ of the stacks $\mathfrak{X}$. Namely, the subscript alg stands for 'algebra stabilizers', which requires each such stabilizer $Stab(\mathfrak{x})$ to be identified with the group of invertible elements in a certain subring of the endomorphism ring $\mathrm{End}_{\mathcal{A}}(\rho(\mathfrak{x}))$. The upperscript ind stands for 'virtually indecomposable' stack functions, which requires the closed points $\mathfrak{x}$ to have virtual rank one stabilizers. The definition of virtual rank is very technical and will not be reviewed here in detail. The important point here is the subspace $\mathsf{SF}^{\mathsf{ind}}_{\mathsf{alg}}(\mathfrak{M}_{\mathcal{A}})$ is closed under the Lie bracket determined by the product $*$. Therefore it has a Lie algebra structure.

We could look at a simple example of the $*$-multiplication for characteristic delta functions of two object $E_1,E_3\in\mathrm{obj}(\mathcal{A})$ to have a feel about it. We have

$$\delta_{E_i}=[(\mathrm{Spec}(\mathbb{C}),\rho_i)]\ ,\ \rho_i(\mathrm{Spec}(\mathbb{C}))=E_i\in\mathfrak{M}_{\mathcal{A}}.$$

Then $\delta_{E_1}*\delta_{E_2}$ is given by

$$\delta_{E_1}*\delta_{E_2}=\Big[\Big(\Big[\frac{\mathrm{Ext}^1(E_3,E_1)}{\mathrm{Hom}(E_3,E_1)}\Big],\rho\Big)\Big],$$

such that ρ is a 1-morphism sending the extension class $u\in\mathrm{Ext}^1(E_3,E_1)$ representing the exact sequence $0\to E_1\to E_2\to E_3\to 0$ to the object $E_2\in\mathrm{obj}(\mathcal{A})$ modulo the trivial action $\mathrm{Hom}(E_3,E_1)$. In the following theorem we consider the characteristic delta functions of τ-semistable objects in $\mathcal{A}$.

Lemma 3.1. [18, Thm. 8.7] *Let $(\tau,T,\leq)$ be a stability condition on $\mathcal{A}$. Define the stack function $\bar{\delta}^{\alpha}_{ss}(\tau)$ in $\mathsf{SF}_{\mathsf{alg}}(\mathfrak{M}_{\mathcal{A}})$ for $\alpha\in K(\mathcal{A})$ to be the characteristic function of the moduli substack $\mathfrak{M}^{\alpha}_{\mathcal{A}}(\tau)$ of τ-semistable objects with $\alpha\in K(\mathcal{A})$ in $\mathfrak{M}_{\mathcal{A}}$. We then define $\bar{\epsilon}^{\alpha}(\tau)$ in $\mathsf{SF}_{\mathsf{alg}}(\mathfrak{M}_{\mathcal{A}})$ to* *be*

$$(3.3)\ \bar{\epsilon}^{\alpha}(\tau) = \sum_{\substack{n\geq 1,\alpha_1,\cdots,\alpha_n\in\mathcal{C}(\mathcal{A})\\ \alpha_1+\cdots+\alpha_n=\alpha,\tau(\alpha_i)=\tau(\alpha)}} \frac{(-1)^n}{n}\bar{\delta}_{ss}^{\alpha_1}(\tau)*\bar{\delta}_{ss}^{\alpha_2}(\tau)*\cdots*\bar{\delta}_{ss}^{\alpha_n}(\tau),$$

where $*$ is the Ringel–Hall multiplication in $\mathsf{SF}_{\mathsf{alg}}(\mathfrak{M}_{\mathcal{A}})$. Moreover, $\bar{\epsilon}^{\alpha}(\tau)$ is in $\mathsf{SF}^{\mathsf{ind}}_{\mathsf{alg}}(\mathfrak{M}_{\mathcal{A}})$. We call $\bar{\epsilon}^{\alpha}(\tau)$ a log stack function.

Theorem 3.2. [19, Sec. 4, Sec. 5] *Under certain appropriate assumptions let $(\tau, T, \leq)$, $(\tilde{\tau}, \tilde{T}, \leq)$ and $(\tau', T', \leq)$ be three stability conditions on $\mathcal{A}$ with $(\tau', T', \leq)$ dominating $(\tau, T, \leq)$ and $(\tau, \tilde{T}, \leq)$. (i.e. $(\tau', T', \leq)$ is the critical stability condition on the wall.) Then for $\alpha \in K(\mathcal{A})$ we have the following wallcrossing formulas in terms of stack functions,*

$$(3.4)\quad \begin{aligned}&\bar{\delta}_{ss}^{\alpha}(\tilde{\tau})\\ &= \sum_{\substack{n\geq 1,\alpha_1,\cdots,\alpha_n\in\mathcal{C}(\mathcal{A})\\ \alpha_1+\cdots+\alpha_n=\alpha}} S(\alpha_1,\cdots,\alpha_n;\tau,\tilde{\tau})\bar{\delta}_{ss}^{\alpha_1}(\tau)*\bar{\delta}_{ss}^{\alpha_2}(\tau)*\cdots*\bar{\delta}_{ss}^{\alpha_n}(\tau),\\ &\bar{\epsilon}^{\alpha}(\tilde{\tau})\\ &= \sum_{\substack{n\geq 1,\alpha_1,\cdots,\alpha_n\in\mathcal{C}(\mathcal{A})\\ \alpha_1+\cdots+\alpha_n=\alpha}} U(\alpha_1,\cdots,\alpha_n;\tau,\tilde{\tau})\bar{\epsilon}^{\alpha_1}(\tau)*\bar{\epsilon}^{\alpha_2}(\tau)*\cdots*\bar{\epsilon}^{\alpha_n}(\tau),\end{aligned}$$

where the coefficients $S(*;\tau,\tilde{\tau}) \in \mathbb{Z}$ and $U(*;\tau,\tilde{\tau}) \in \mathbb{Q}$ are certain combinatorial coefficients which are complicated enough to be omitted here and there are only finitely many nonzero terms in the summation.

We now specialize to the case of the coherent sheaf $\mathcal{A} = \mathrm{coh}(M)$ on a Calabi–Yau 3-fold M or $\mathcal{A}$ is the abelian category of modules over a CY3 algebra. Using the Serre duality on Calabi–Yau spaces or Calabi–Yau algebra we have an antisymmetric Euler form $\chi([E],[F])$ for $E, F \in \mathrm{obj}(\mathcal{A})$. We define a Lie algebra $L(M)$ to be the $\mathbb{Q}$-vector space generated by the λ^{α} for each $\alpha \in K(\mathcal{A})$, with the Lie algebra

$$(3.5)\qquad [\lambda^{\alpha}, \lambda^{\beta}] = (-1)^{\chi(\alpha,\beta)}\chi(\alpha,\beta)\lambda^{\alpha+\beta}.$$

Theorem 3.3. [20, Thm. 5.14] *There exist a canonical Lie algebra morphism $\Psi : \mathsf{SF}^{\mathsf{ind}}_{\mathsf{alg}}(\mathfrak{M}_{\mathcal{A}}) \to L(M)$, which gives the $\mathbb{Q}$-valued generalized DT invariants $\mathrm{DT}^{\alpha}(\tau)$. Namely, $\Psi(\bar{\epsilon}^{\alpha}(\tau)) = \overline{\mathrm{DT}}^{\alpha}(\tau)\lambda^{\alpha}$. Moreover,* *conjecturally the multicover formula defines the $\mathbb{Z}$-valued DT invariants $\widehat{\mathrm{DT}}^{\alpha}(\tau)$ for generic τ by*

$$\overline{\mathrm{DT}}^{\alpha}(\tau) = \sum_{m\geq 1, m|\alpha} \frac{1}{m^2}\widehat{\mathrm{DT}}^{\alpha/m}(\tau). \tag{3.6}$$

Remark 3.4. Consider an element in $\mathsf{SF}^{\mathsf{ind}}_{\mathsf{alg}}(\mathfrak{M}_{\mathcal{A}})$ of the form $[(U \times [\mathrm{Spec}(\mathbb{C})/\mathbb{C}^*], \rho)]$, where U is a quasi-projective variety and ρ a representable 1-morphism $\rho : U \times [\mathrm{Spec}(\mathbb{C})/\mathbb{C}^*] \to \mathfrak{M}^{\alpha}_{\mathcal{A}} \subset \mathfrak{M}_{\mathcal{A}}$. Then the canonical Lie algebra morphism will send this to

$$\Psi\big([(U \times [\mathrm{Spec}(\mathbb{C})/\mathbb{C}^*], \rho)]\big) = \chi(U, \rho^*(\nu_{\mathfrak{M}_{\mathcal{A}}}))\lambda^{\alpha},$$

where $\rho^*(\nu_{\mathfrak{M}_{\mathcal{A}}})$ is the pullback of the Behrend function $\nu_{\mathfrak{M}_{\mathcal{A}}}$ to a constructible function on $U \times [\mathrm{Spec}(\mathbb{C})/\mathbb{C}^*]$.

3.2. Wallcrossing formulas in ADHM sheaf theory

We now consider ADHM sheaf theory on curves with line bundles M_1 and M_2 such that $M_1 \otimes_X M_2 \simeq K_X^{-1}$. The following definition is used to take care of the summation over all the possible Harder–Narasimhan filtrations in the wallcrossing formulas.

Definition 3.5. *Let $\alpha = (r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. For any integer $l \geq 1$ and $v = 1, 2$ let $\mathcal{HN}_-(\alpha, v, \delta_c, l, l-1)$ denote the set of order sequence $(\alpha_i)_{1\leq i\leq l}$, $\alpha_i = (r_i, e_i) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ satisfying $\alpha_1 + \cdots + \alpha_l = \alpha$ and $e_1/r_1 = \cdots = e_{l-1}/r_{l-1} = (e_l + v\delta_c)/r_l = (e + v\delta_c)/r$.*
For any integer $l \geq 2$ let $\mathcal{HN}_-(\alpha, 2, \delta_c, l, l-2)$ denote the set of order sequence $(\alpha_i)_{1\leq i\leq l}$, $\alpha_i = (r_i, e_i) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ satisfying $\alpha_1 + \cdots + \alpha_l = \alpha$ and $e_1/r_1 = \cdots = e_{l-2}/r_{l-2} = (e_{l-1} + \delta_c)/r_{l-1} = (e_l + \delta_c)/r_l = (e + 2\delta_c)/r$.

In the ADHM sheaf theory with $E_\infty = \mathcal{O}_X$ we have proved the following ADHM rank one wallcrossing formula [3].

Theorem 3.6. *Let δ_c be a critical stability parameter and $(\alpha, 1) = (r, e, 1)$ be the numerical types of ADHM sheaves.*
(i) The following wallcrossing formula holds for $\delta_c > 0$

$$\begin{aligned} & A_+(\alpha,1) - A_-(\alpha,1) = \\ & \sum_{l\geq 2} \frac{1}{(l-1)!} \sum_{(\alpha_i)\in\mathcal{HN}_-(\alpha,1,\delta_c,l,l-1)} A_-(\alpha_l,1)\prod_{i=1}^{l-1} f_1(\alpha_i)H(\alpha_i). \end{aligned} \tag{3.7}$$

(ii) The following wallcrossing formula holds for $\delta_c = 0$.

$$
\begin{aligned}
&A_+(\alpha,1) - A_-(\alpha,1) = \\
(3.8)\qquad &\sum_{l\geq 2}\frac{1}{(l-1)!}\sum_{(\alpha_i)\in\mathcal{HN}_-(\alpha,1,\delta_c,l,l-1)} A_-(\alpha_l,1)\prod_{i=1}^{l-1} f_1(\alpha_i)H(\alpha_i) \\
&+\sum_{l\geq 1}\frac{1}{l!}\sum_{(\alpha_i)\in\mathcal{HN}_-(\alpha,1,0,l,l-1)}\prod_{i=1}^{l} f_1(\alpha_i)H(\alpha_i),
\end{aligned}
$$

where $f_1(\alpha) = (-1)^{v(e-r(g-1))}v(e-r(g-1))$ *for* $\alpha = (r,e)$ *and* $A_\pm(\alpha,1)$ *and* $H(\alpha)$ *are generalized DT invariants with numerical invariants* $(r,e,1)$ *and* $(r,e,0)$ *respectively for* $\alpha = (r,e)$.

Strategy of Proof. The first step to derive the wallcrossing formulas is usually to write the formulas in terms of characteristic delta stack function. Take (i) for example. First we have a relation in $\mathsf{SF}(\mathfrak{M}_{\mathcal{A}})$

$$
\mathfrak{d}_+^\alpha - \mathfrak{d}_-^\alpha = \sum_{l\geq 2}(-1)^l \sum_{(\alpha_i)\in\mathcal{HN}_-(\alpha,1,\delta_c,l,l-1)} \mathfrak{h}^{\alpha_1} * \mathfrak{h}^{\alpha_2} * \cdots * [\mathfrak{d}_-^{\alpha_l}, \mathfrak{h}^{\alpha_{l-1}}],
$$

where $\mathfrak{d}_\pm^\alpha$ and $\mathfrak{h}^{\alpha_i}$ are the characteristic stack functions for ADHM sheaf moduli and Higgs sheaf moduli.

The second step is to transform all the characteristic stack functions in the formula to the log stack function. Assume that the log stack functions for the stack functions $\mathfrak{h}^{\alpha_i}$ is $\mathfrak{g}^{\alpha_i}$. After the transformation we have

(3.9)

$$
\mathfrak{d}_+^\alpha - \mathfrak{d}_-^\alpha = \sum_{l\geq 2}\frac{(-1)^l}{(l-1)!}\sum_{(\alpha_i)\in\mathcal{HN}_-(\alpha,1,\delta_c,l,l-1)} [\mathfrak{g}^{\alpha_1},[\mathfrak{g}^{\alpha_2},[\cdots[\mathfrak{g}_-^{\alpha_{l-1}},\mathfrak{d}^{\alpha_l}]\cdots].
$$

Notice that the automorphism group of all ADHM sheaves with $E_\infty = \mathcal{O}_X$ is isomorphic to $\mathbb{C}^\times$. Therefore all the stack functions in (3.9) belong to the Lie algebra $\mathsf{SF}^{\mathsf{ind}}_{\mathsf{alg}}(\mathfrak{M}_{\mathcal{A}})$. We arrange the formula as a sequence of Lie brackets such that when applying the canonical Lie algebra homomorphism Ψ to (3.9) we extract the invariants directly. Q.E.D.

For fixed $r \in \mathbb{Z}_{\geq 1}$, and fixed $\delta \in \mathbb{R}_{>0}\setminus\mathbb{Q}$ let

$$
Z_\delta(q)_r = \sum_{e\in\mathbb{Z}} q^{e-r(g-1)}A_\delta(r,e), \qquad Z_\infty(q)_r = \sum_{e\in\mathbb{Z}} q^{e-r(g-1)}A_\infty(r,e)\ .
$$

$Z_\infty(q)_r$ is the generating function of degree r local stable pair invariants of the data $\mathcal{X} = (X, M_1, M_2)$. Using the above theorem the following rationality result is proven, which is a consequence of the BPS expansion conjectured by Gopakumar–Vafa.

Theorem 3.7. *For any $r \in \mathbb{Z}_{\geq 1}$, and any $\delta \in \mathbb{R}_{>0} \setminus \mathbb{Q}$, $Z_\delta(q)_r$, $Z_\infty(q)_r$ are Laurent expansions of rational functions of q. Moreover, the rational function corresponding to $Z_\infty(q)_r$ is invariant under $q \leftrightarrow q^{-1}$. If $g \geq 1$, $Z_\delta(q)_r = Z_\infty(q)_r$ is a polynomial in q, q^{-1} invariant under $q \leftrightarrow q^{-1}$.*

3.3. Higher rank ADHM invariants

In this section we consider ADHM sheaves with $E_\infty = V \otimes \mathcal{O}_X$, where V is a finite dimensional vector space over $\mathbb{C}$. We proved the following rank 2 wallcrossing formulas for ADHM sheaf theory [4].

Theorem 3.8. *The ADHM invariants with numerical types $(\alpha, 2) = (r, e, 2)$ satisfy the following wallcrossing formula*

$$
\begin{aligned}
&A_-(\alpha,2) - A_+(\alpha,2) = \\
&\sum_{l\geq 2} \frac{1}{(l-1)!} \sum_{(\alpha_i)\in \mathcal{HN}_-(\alpha,2,\delta_c,l,l-1)} A_+(\alpha_l,2)\prod_{i=1}^{l-1} f_2(\alpha_i)H(\alpha_i) \\
&-\frac{1}{2}\sum_{l\geq 1}\frac{1}{(l-1)!}\sum_{(\alpha_i)\in\mathcal{HN}_-(\alpha,2,\delta_c,l+1,l-1)} g(\alpha_{l+1},\alpha_l)\times \\
&\qquad A_+(\alpha_l,1)A_+(\alpha_{l+1},1)\prod_{i=1}^{l-1} f_2(\alpha_i)H(\alpha_i) \\
&+\frac{1}{2}\sum_{(\alpha_1,\alpha_2)\in\mathcal{HN}_-(\alpha,2,\delta_c,2,0)}\sum_{l_1\geq 1}\sum_{l_2\geq 1}\frac{1}{(l_1-1)!}\frac{1}{(l_2-1)!} \\
&\sum_{\substack{(\alpha_{1,i})\in\mathcal{HN}_-(\alpha_1,1,\delta_c,l_1,l_1-1)\\(\alpha_{2,i})\in\mathcal{HN}_-(\alpha_2,1,\delta_c,l_2,l_2-1)}} g(\alpha_1,\alpha_2)A_+(\alpha_{1,l_1},1)A_+(\alpha_{2,l_2},1)\times \\
&\qquad \prod_{i=1}^{l_1-1} f_1(\alpha_{1,i})H(\alpha_{1,i})\prod_{i=1}^{l_2-1} f_1(\alpha_{2,i})H(\alpha_{2,i})
\end{aligned}
\tag{3.10}
$$

where $A_\pm(\alpha,2)$ are the generalized DT invariants of the numerical invariants $(\alpha,2)$ and $f_v(\alpha)$ and $g(\alpha_1,\alpha_2)$ are given by

$$\begin{aligned} f_v(\alpha) &= (-1)^{v(e-r(g-1))}v(e-r(g-1)), \qquad v=1,2 \\ g(\alpha_1,\alpha_2) &= (-1)^{e_1-e_2-(r_1-r_2)(g-1)}(e_1-e_2-(r_1-r_2)(g-1)). \end{aligned}$$

Strategy of Proof. The proof strategy of this theorem is almost the same as $v=1$ case. But since the numerical invariants are of the form $(\alpha,2)$, we need to tranform the characteristic stack function $\mathfrak{d}_\pm^{(\alpha,2)}$ to its corresponding log stack functions $\mathfrak{e}_\pm^{(\alpha,2)}$ by (3.3) when α is divisible by 2. Q.E.D.

An application of this theorem gives rank 2 genus zero invariants. Consider the following generating function

$$Z_{\mathcal{X},v}(u,q) = \sum_{r\geq 1}\sum_{n\in\mathbb{Z}} u^r q^n A_\infty(r,n-r,v) \tag{3.11}$$

where $v=1,2$. Using the wallcrossing formula the following closed formulas are proven [4].

Corollary 3.9. *Suppose X is a genus 0 curve and $M_1 \simeq \mathcal{O}_X(d_1)$, $M_2\simeq\mathcal{O}_X(d_2)$ where $(d_1,d_2)=(1,1)$ or $(0,2)$. Then*
(3.12)

$$\begin{aligned} Z_{\mathcal{X},1}(u,q) &= \prod_{n=1}^{\infty}(1-u(-q)^n)^{(-1)^{d_1-1}n} \\ Z_{\mathcal{X},2}(u,q) &= \frac{1}{4}\prod_{n=1}^{\infty}(1-uq^n)^{2(-1)^{d_1-1}n} \\ &\quad -\frac{1}{2}\sum_{\substack{r_1>r_2\geq 1,\ n_1,n_2\in\mathbb{Z} \\ \text{or } r_1=r_2\geq 1,\ n_2>n_1 \\ \text{or } r_1\geq 1,\ n_1\in\mathbb{Z},\ r_2=n_2=0}} (n_1-n_2)(-1)^{(n_1-n_2)}\times \\ &\qquad A_\infty(r_1,n_1-r_1,1)A_\infty(r_2,n_2-r_2,1)u^{r_1+r_2}q^{n_1+n_2}. \end{aligned}$$

Proof. Let $\mathcal{C}_{\mathcal{X}}^0$ be the full abelian subcategory of $\mathcal{C}_{\mathcal{X}}$ consisting of ADHM sheaves $\mathcal{E}$ with $\phi=0$. For any $\delta\in\mathbb{R}$, an object $\mathcal{E}$ of $\mathcal{C}_{\mathcal{X}}^0$ will be called δ-semistable if it is δ-semistable as an object of $\mathcal{C}_{\mathcal{X}}$. One can see that the properties of δ-stability and moduli stacks of semistable objects in $\mathcal{C}_{\mathcal{X}}^0$ are analogous to those of $\mathcal{C}_{\mathcal{X}}$.

Given an ADHM sheaf $\mathcal{E}=(E,V,\Phi_i,\psi)\in\mathcal{C}_{\mathcal{X}}^0$ of type (r,e,v), it cab be checked that for $\delta<0$ the proper nontrivial object $(E,0,\Phi_i,0)$ is always destabilizing. Therefore the main difference between $\mathcal{C}_{\mathcal{X}}^0$ and $\mathcal{C}_{\mathcal{X}}$

is that for any $(r,e,v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 1}$ the moduli stack of δ-semistable objects of $\mathcal{C}^0_{\mathcal{X}}$ of type (r,e,v) is empty if $\delta < 0$.

Let $\mathcal{E} = (E, 0, \Phi_i, 0)$ be a semistable Higgs sheaf of $\mathcal{C}^0_{\mathcal{X}}$ of type $(r,e,0)$, $(r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. If $(d_1, d_2) = (1,1)$, E must be isomorphic to $\mathcal{O}_X(n)^{\oplus r}$ for some $n \in \mathbb{Z}$, and $\Phi_i = 0$ for $i = 1,2$. If $(d_1,d_2) = (0,2)$, E must be isomorphic to $\mathcal{O}_X(n)^{\oplus r}$ for some $n \in \mathbb{Z}$, and $\Phi_2 = 0$.

This implies that for $(d_1,d_2) = (1,1)$ the moduli stack $\mathfrak{M}^{ss}(r,e,0)$ is isomorphic to the quotient stack $[*/GL(r)]$ if $e = rn$ for some $n \in \mathbb{Z}$, and empty otherwise. For $(d_1,d_2) = (0,2)$ the moduli stack $\mathfrak{M}^{ss}(r,rn,0)$, $n \in \mathbb{Z}$, is isomorphic to the moduli stack of trivially semistable representations of dimension r of a quiver consisting of one vertex and one arrow joining the unique vertex with itself. If e is not a multiple of r, the moduli stack $\mathfrak{M}^{ss}(\mathcal{X}, r, e, 0)$ is empty.

Performing a computation similar to [20, Sect. 7.5.1] we obtain the Higgs sheaf invariant $H(r,e)$ and the only invariants in $\delta < 0$ chamber of $\mathcal{C}^0_{\mathcal{X}}$ are given by

$$H(r,e) = \begin{cases} \frac{(-1)^{d_1-1}}{r^2} & \text{if } e = rn,\ n \in \mathbb{Z} \\ 0 & \text{otherwise,} \end{cases} \tag{3.13}$$

$$A_\delta(0,0,1) = 1\ ,\ A_\delta(0,0,2) = \frac{1}{4}. \tag{3.14}$$

Then we could apply (3.10) or the analogous wallcrossing formulas of Kontsevich and Soibelman to compute the invariants in the asymptotic $\delta \gg 0$ chamber. We leave out the remaning details and refer the interested readers to [4]. Q.E.D.

3.4. Comparison with Kontsevich–Soibelman formula

The goal of this section is to illustrate that the ADHM wallcrossing formulas (3.7), (3.10) are in agreement with the wallcrossing formulas of Kontsevich and Soibelman [21], which will be referred to as the KS formula in the following.

Numerical types of ADHM sheaves will be denoted by $\gamma = (\alpha, v)$, $\alpha = (r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, $v \in \mathbb{Z}_{\geq 0}$. In order to streamline the computations, let $\mathsf{L}(\mathcal{X})_{\leq 2}$ denote the truncation of the Lie algebra $\mathsf{L}(\mathcal{X})$ defined by

$$[\lambda(\alpha_1,v_1), \lambda(\alpha_2,v_2)]_{\leq 2} = \begin{cases} [\lambda(\alpha_1,v_1), \lambda(\alpha_2,v_2)] & \text{if } v_1+v_2 \leq 2 \\ 0 & \text{otherwise.} \end{cases} \tag{3.15}$$

Furthermore, it will be more convenient to use the alternative notation $\mathsf{e}_\alpha = \lambda(\alpha,0)$, $\mathsf{f}_\alpha = \lambda(\alpha,1)$, and $\mathsf{g}_\alpha = \lambda(\alpha,2)$.

Given a critical stability parameter δ_c of type $(r, e, 2)$, $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, there exist two pairs $\alpha = (r_\alpha, e_\alpha)$ and $\beta = (r_\beta, e_\beta)$ with

$$\frac{e_\alpha + \delta_c}{r_\alpha} = \frac{e_\beta}{r_\beta} = \mu_{\delta_c}(\gamma)$$

so that any $\eta \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ with $\mu_{\delta_c}(\eta) = \mu_{\delta_c}(\gamma)$ can be uniquely written as $\eta = (q\beta, 0), (\alpha + q\beta, 1)$, or $(2\alpha + q\beta, 2)$, with $q \in \mathbb{Z}_{\geq 0}$.

For any $q \in \mathbb{Z}_{\geq 0}$ the following formal expressions will be needed in the KS formula,

$$\begin{aligned} U_{\alpha+q\beta} &= \exp(\mathsf{f}_{\alpha+q\beta} + \frac{1}{4}\mathsf{g}_{2\alpha+2q\beta}), \\ U_{2\alpha+q\beta} &= \exp(\mathsf{g}_{2\alpha+q\beta}), \quad U_{q\beta} = \exp(\sum_{m\geq 1} \frac{\mathsf{e}_{mq\beta}}{m^2}). \end{aligned} \tag{3.16}$$

Moreover, let

$$\mathbb{H} = \sum_{q\geq 0} H(q\beta)\mathsf{e}_{q\beta},$$

where the invariants $H(\alpha)$ are the Higgs sheaf invariants. Then the wallcrossing formula of Kontsevich and Soibelman reads

$$\begin{aligned} &\exp(\mathbb{H}) \prod_{q\geq 0,\ q\downarrow} U_{2\alpha+q\beta}^{\widehat{A}_+(2\alpha+q\beta,2)} \prod_{q\geq 0,\ q\downarrow} U_{\alpha+q\beta}^{A_+(\alpha+q\beta,1)} \\ &= \prod_{q\geq 0,\ q\uparrow} U_{\alpha+q\beta}^{A_-(\alpha+q\beta,1)} \prod_{q\geq 0,\ q\uparrow} U_{2\alpha+q\beta}^{\widehat{A}_-(2\alpha+q\beta,2)} \exp(\mathbb{H}) \end{aligned} \tag{3.17}$$

where an up, respectively down arrow means that the factors in the corresponding product are taken in increasing, respectively decreasing order of q. Note that the hatted invariants in the exponents are the integral invariants corresponding to $\widehat{DT}^{(\alpha,v)}(\tau_\pm)$ in (3.6) while the unhatted invariants correspond to the rational $\overline{DT}^{(\alpha,v)}(\tau_\pm)$. In this case we have

$$\begin{aligned} &A_\pm(\alpha + q\beta, 1) = \widehat{A}_\pm(\alpha + q\beta, 1), \\ &A_\pm(2\alpha + q\beta, 2) = \widehat{A}_\pm(2\alpha + q\beta, 2) + \frac{1}{4}A_\pm(\alpha + q\beta/2, 1) \text{ , when } q \mid 2, \\ &A_\pm(2\alpha + q\beta, 2) = \widehat{A}_\pm(2\alpha + q\beta, 2), \text{when } q \nmid 2. \end{aligned}$$

Expanding the right hand side, equation (3.17) yields

(3.18)
$$\begin{aligned}&\exp(\sum_{q\geq 0} A_-(2\alpha+q\beta,2)\mathfrak{g}_{2\alpha+q\beta}+\\ &\sum_{q_2>q_1\geq 0}\frac{1}{2}g(q_1\beta,q_2\beta)A_-(\alpha+q_1\beta,1)A_-(\alpha+q_2\beta,1)\mathfrak{g}_{2\alpha+(q_1+q_2)\beta})=\\ &\exp(\mathbb{H})\exp(\sum_{q\geq 0}A_+(2\alpha+q\beta,2)\mathfrak{g}_{2\alpha+q\beta}\\ &+\sum_{q_1>q_2\geq 0}\frac{1}{2}g(q_1\beta,q_2\beta)A_+(\alpha+q_1\beta,1)A_+(\alpha+q_2\beta,1)\mathfrak{g}_{2\alpha+(q_1+q_2)\beta})\\ &\times\exp(-\mathbb{H}),\end{aligned}$$

modulo terms involving $\mathfrak{f}_\gamma$. In fact the terms involving $\mathfrak{f}_\gamma$ precisely give us the $v=1$ wallcrossing formula (3.7).

The BCH formula

$$\begin{aligned}\exp(A)\exp(B)\exp(-A) &= \exp(\sum_{n=0}\frac{1}{n!}(Ad(A))^nB)\\ &=\exp(B+[A,B]+\frac{1}{2}[A,[A,B]]+\cdots),\end{aligned} \tag{3.19}$$

yields

(3.20)
$$\begin{aligned}&\exp(\mathbb{H})\exp(\mathfrak{g}_{2\alpha+q\beta})\exp(-\mathbb{H})=\\ &\exp(\mathfrak{g}_{2\alpha+q\beta}+\sum_{q_1>0}f_2(q_1\beta)H(q_1\beta)\mathfrak{g}_{2\alpha+(q+q_1)\beta}\\ &+\frac{1}{2!}\sum_{q_1>0,q_2>0}f_2(q_1\beta)H(q_1\beta)f_2(q_2\beta)H(q_2\beta)\mathfrak{g}_{2\alpha+(q+q_1+q_2)\beta}+\cdots)\\ &=\exp\Big(\sum_{l\geq 0,q_i>0}\frac{1}{l!}(\prod_{i=1}^{l}f_2(q_i\beta)H(q_i\beta))\mathfrak{g}_{2\alpha+(q+q_1+\cdots+q_l)\beta}\Big).\end{aligned}$$

Substituting (3.20) in (3.18) results in
(3.21)

$$
\begin{aligned}
&\exp\Big(\sum_{q\geq 0} A_-(2\alpha+q\beta,2)\mathfrak{g}_{2\alpha+q\beta} \\
&+\sum_{q_2>q_1\geq 0}\frac{1}{2}g(q_1\beta,q_2\beta)A_-(\alpha+q_1\beta,1)A_-(\alpha+q_2\beta,1)\mathfrak{g}_{2\alpha+(q_1+q_2)\beta}\Big) \\
&=\exp\Big(\sum_{\substack{q\geq 0,l\geq 0\\ q_i>0}} A_+(2\alpha+q\beta,2)\frac{1}{l!}\big(\prod_{i=1}^{l} f_2(q_i\beta)H(q_i\beta)\big)\mathfrak{g}_{2\alpha+(q+q_1+\cdots+q_l)\beta} \\
&\quad+\sum_{\substack{q_1'>q_2'\geq 0\\ l\geq 0,q_i>0}}\frac{1}{2}g(q_1'\beta,q_2'\beta)A_+(\alpha+q_1'\beta,1)A_+(\alpha+q_2'\beta,1)\times \\
&\quad\frac{1}{l!}\big(\prod_{i=1}^{l} f_2(q_i\beta)H(q_i\beta)\big)\mathfrak{g}_{2\alpha+(q_1'+q_2'+q_1+\cdots+q_l)\beta}\Big).
\end{aligned}
$$

In order to further simplify the notation, let

$$A_\pm(v\alpha+q\beta,v)\equiv A_\pm(q,v),\qquad \mathfrak{g}_{2\alpha+q\beta}\equiv\mathfrak{g}_q.$$

Comparing the coefficients of $\mathfrak{g}_Q$ in (3.18), yields
(3.22)

$$
\begin{aligned}
A_-(Q,2)=&\sum_{\substack{q'\geq 0,\ l\geq 0,\ q_i>0\\ q'+q_1+\cdots+q_l=Q}} A_+(q',2)\frac{1}{l!}\big(\prod_{i=1}^{l} f_2(q_i\beta)H(q_i\beta)\big) \\
&+\frac{1}{2}\sum_{\substack{q_1'>q_2'\geq 0\\ l\geq 0,\ q_i>0\\ q_1'+q_2'+q_1+\cdots+q_l=Q}} g(q_1'\beta,q_2'\beta)A_+(q_1',1)A_+(q_2',1)\frac{1}{l!}\big(\prod_{i=1}^{l} f_2(q_i\beta)H(q_i\beta)\big) \\
&-\frac{1}{2}\sum_{q_2'>q_1'\geq 0,\ q_1'+q_2'=Q} g(q_1'\beta,q_2'\beta)A_-(q_1',1)A_-(q_2',1).
\end{aligned}
$$

Using the $v=1$ wallcrossing formula (3.7) to transform the last term in (3.22) we finally obtain the $v=2$ wallcrossing formula.

(3.23)

$$A_-(Q,2) = \sum_{\substack{q'\geq 0,\ l\geq 0,\ q_i>0\\ q'+q_1+\cdots+q_l=Q}} A_+(q',2)\frac{1}{l!}(\prod_{i=1}^{l} f_2(q_i\beta)H(q_i\beta))$$

$$+\frac{1}{2}\sum_{\substack{q_1'>q_2'\geq 0\\ l\geq 0,\ q_i>0\\ q_1'+q_2'+q_1+\cdots+q_l=Q}} \frac{1}{2}g(q_1'\beta,q_2'\beta)A_+(q_1',1)A_+(q_2',1)\frac{1}{l!}(\prod_{i=1}^{l} f_2(q_i\beta)H(q_i\beta))$$

$$-\frac{1}{2}\sum_{\substack{q_2>q_1\geq 0\\ q_1+q_2=Q\\ l\geq 0,\ \tilde{l}\geq 0\\ q_1'\geq 0,\ q_2'\geq 0\\ n_i>0,\tilde{n}_i>0\\ q_1'+n_1+\cdots+n_l=q_1\\ q_2'+\tilde{n}_1+\cdots+\tilde{n}_{\tilde{l}}=q_2}} g(q_1\beta,q_2\beta)A_+(q_1',1)A_+(q_2',1)\times$$

$$\frac{1}{l!}(\prod_{i=1}^{l} f_1(n_i\beta)H(n_i\beta))\frac{1}{\tilde{l}!}(\prod_{i=1}^{\tilde{l}} f_1(\tilde{n}_i\beta)H(\tilde{n}_i\beta)).$$

This formula agrees with (3.10).

§4. Cohomology of the moduli space of Hitchin pairs

In this section we present a conjectural formalism to determine the Poincaré (Hodge) polynomial of Hitchin moduli space [5]. More precisely refined wallcrossing formulas and the refined ADHM invariants with $E_\infty = \mathcal{O}_X$ in the asymptotic chamber are conjectured on local curves. It is shown that these formulas yield a recursive relation which correctly determines the Poincaré (Hodge) polynomial of the moduli space of Hitchin pairs with coprime rank and degree on a smooth projective curve of genus at least two.

We recall the definition of a Hitchin pair. Let X be a smooth projective curve and L is an invertible sheaf on X. A Hitchin pair is a pair (E,Φ) where E is a coherent sheaf on X and $\Phi : E \to E\otimes_X L$ a morphism of coherent sheaves. A Hitchin pair is semistable if for any proper subsheaf $0\subset E'\subset E$ such that $\Phi(E')\subset E'\otimes_X L$, we have

$$\frac{\deg(E')}{\mathrm{rank}(E')} \leq \frac{\deg(E)}{\mathrm{rank}(E)}. \tag{4.1}$$

Note that if $\mathrm{rank}(E)>0$, semistability of the Hitchin pair implies that E is locally free. Assume that $\deg(L)\geq 2g-2$. According to

[12], [30], [1] we have an algebraic moduli stack $\mathfrak{Hit}(X, L, r, e)$ locally of finite type parametrizing the semistable Hitchin pairs with numerical type (r, e). If (r, e) are coprime, this stack is a $C^\times$-gerbes over a smooth quasi-projective variety $\mathrm{Hit}(X, L, r, e)$.

One of the major observations, which make the enumerations possible, is the relation between semistable Higgs sheaves and semistable Hitchin pairs as follows.

- Suppose $M_1 = \mathcal{O}_X$, $M_2 = K_X^{-1}$ and let $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ be coprime. Then there is an isomorphism

$$\mathfrak{Higgs}(\mathcal{X}, r, e) \simeq \mathbb{C} \times \mathfrak{Hit}(X, K_X, r, e). \tag{4.2}$$

- Suppose M_2 is a line bundle of degree $2-2g-p$, where $p \in \mathbb{Z}_{>0}$. Then there is an isomorphism

$$\mathfrak{Higgs}(\mathcal{X}, r, e) \simeq \mathfrak{Hit}(X, M_2^{-1}, r, e). \tag{4.3}$$

Both statements rely on the fact that for coprime (r, e) slope semistability is equivalent to slope stability. Therefore the endomorphism ring of any semistable ADHM sheaf $\mathcal{E}$ is canonically isomorphic to $\mathbb{C}$.

Then note that in the first case, given any semistable object $\mathcal{E} = (E, \Phi_1, \Phi_2)$ the relation $\Phi_1 \circ (\Phi_2 \otimes 1_{M_1}) - \Phi_2 \circ (\Phi_1 \otimes 1_{M_2}) = 0$ implies that $\Phi_1 : E \to E$ is an endomorphism of $\mathcal{E}$ since it obviously commutes with itself. Therefore it must be of the form $\Phi_1 = \lambda 1_E$ for some $\lambda \in \mathbb{C}$. In particular, it preserves any subsheaf $E' \subset E$. Generalizing this observation to flat families it follows that there is an forgetful morphism

$$\mathfrak{Higgs}(\mathcal{X}, r, e) \to \mathfrak{Hit}(X, K_X, r, e)$$

projecting (E, Φ_1, Φ_2) to $(E, \Phi_2 \otimes 1_{K_X})$. The isomorphism (4.2) then follows easily.

In the second case, note that given a semistable Higgs sheaf (E, Φ_1, Φ_2), of type (r, e), the data

$$\mathcal{E}' = \left(E \otimes_X M_1^{-1}, \Phi_1 \otimes 1_{M_1^{-1}}, \Phi_2 \otimes 1_{M_1^{-1}}\right)$$

determines a semistable Higgs sheaf of type $(r, e - r\deg(M_1)) = (r, e - rp)$. The relation

$$(\Phi_1 \otimes 1_{M_1^{-1}}) \circ ((\Phi_2 \otimes 1_{M_1^{-1}}) \otimes 1_{M_1}) - (\Phi_2 \otimes 1_{M_1^{-1}}) \circ ((\Phi_1 \otimes 1_{M_1^{-1}}) \otimes 1_{M_2}) = 0$$

implies that $\Phi_1 \otimes 1_{M_1^{-1}}$ is a morphism of (semistable) Higgs sheaves. However $\mu(\mathcal{E}) > \mu(\mathcal{E}')$ since $p > 0$, therefore any such morphism must vanish. Then (4.3) follows.

Besides these observations we make the following conjectures.

Conjecture 4.1. *Let $\alpha = (r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. Then for δ not critical there exist refined equivariant ADHM invariants $A_\delta(r,e,1)(y) \in \mathbb{Z}[y,y^{-1}]$, for any $\delta \in \mathbb{R}$, and refined equivariant Higgs sheaf invariants $H(r,e)(y) \in \mathbb{Q}(y)$ so that $A_\delta(r,e,1)(1) = A_\delta(r,e,1)$, $H(r,e)(1) = H(r,e)$ and refined wallcrossing formulas hold. The conjectural refined wallcrossing formulas are obtained by the following direct substitution: $A_\delta(r,e,1) \to A_\delta(r,e,1)(y)$, $H(r,e) \to H(r,e)(y)$, and $(-1)^{(e-r(g-1))}v(e-r(g-1)) \to (-1)^{(e-r(g-1))}[(e-r(g-1))]_y$ in the rank one wallcrossing formulas, where $[n]_y = \frac{y^n - y^{-n}}{y-y^{-1}}$. Moreover $H(r,e)(y) \in \mathbb{Z}[y,y^{-1}]$ if (r,e) are coprime.*

Conjecture 4.2. *The following refined multicover relation holds for any $(r,e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$*

$$H(r,e)(y) = \sum_{\substack{k\in\mathbb{Z},\ k\geq 1 \\ k|r,\ k|e}} \frac{1}{k\,[k]_y} \overline{H}\left(\frac{r}{k}, \frac{e}{k}\right)(y^k), \tag{4.4}$$

where $\overline{H}(r,e)(y) \in \mathbb{Z}[y,y^{-1}]$.

Remark 4.3. The invariants $A_\delta(r,e,1)(y) \in \mathbb{Z}[y,y^{-1}]$, $H(r,e)(y)$ are refined generalizations of Joyce–Song invariants of ADHM sheaves. They are conjecturally related to Kontsevich–Soibelman invariants $\overline{A}_\delta(r,e,1)(y) \in \mathbb{Z}[y,y^{-1}]$, $\overline{H}(r,e)(y) \in \mathbb{Z}[y,y^{-1}]$ by a refined multicover formula in Conjecture 4.2. For $v = 1$ invariants this formula states simply that $A_\delta(r,e,1)(y) = \overline{A}_\delta(r,e,1)(y)$.

The last essential piece in the construction is the Nekrasov partition function of instanton counting. Physically we can geometrically engineer a five dimensional $SU(2)$ gauge theory with g adjoint hypermultiplets by putting M-theory on the total space Z of canonical bundle K_S on the ruled surface $S = \mathbb{P}(\mathcal{O}_X \oplus M_1)$. Inside S there are two sections X_1 and X_2. The normal bundle $N_{X_1}|Z$ of X_1 inside Z is given by

$$N_{X_1}|Z \simeq M_1^{-1} \oplus M_2^{-1}\ .$$

This construction involving ruled surfaces is needed because other constructions are problematic physically, *i.e.* other geometries do not admit geometric engineering of five dimensional gauge theory. However the subtlety in ruled surface construction is that we need to separate the contributions coming from the other section X_2.

Given such a five dimensional theory, Nekrasov has constructed an equivariant instanton partition function $\mathcal{Z}^{(p)}_{inst}(Q, \epsilon_1, \epsilon_2, a_1, a_2, y)$, where ϵ_1, ϵ_2, a_1, a_2 are equivariant parameters for a natural torus action, Q is a formal variable counting instanton charges, and y is another formal variable.

It has been verified by string theorists in other local geometries that the instanton partition function could be identified with the (refined) topological string after suitable changes of variables [8], [29], [13], [14], [9], [15].

Mathematically $\mathcal{Z}^{(p)}_{inst}(Q, \epsilon_1, \epsilon_2, a_1, a_2, y)$ is the generating function for the χ_y-genus of a certain holomorphic bundle on a partial compactification of the instanton moduli space $M(r,k)$ as described in the following.

Hirzebruch genus Let $M(r,k)$ denote the moduli space of rank r framed torsion-free sheaves (F, f) on $\mathbb{P}^2$ with second Chern class $k \in \mathbb{Z}_{\geq 0}$. The framing data is an isomorphism

$$f : F|_{l_\infty} \to \mathcal{O}^{\oplus r}_{l_\infty}. \tag{4.5}$$

along the line l_{infty} at infinity defined by $[0, z_1, z_2]$ in terms of the homogeneous coordinates of $\mathbb{P}^2$.

$M(r,k)$ is a smooth quasi-projective fine moduli space i.e. there is an universal framed sheaf (F, f) on $M(r,k) \times \mathbb{P}^2$. Let $\mathsf{V} = R^1 p_{1*}\mathsf{F} \otimes p_2^* \mathcal{O}_{\mathbb{P}^2}(-1)$ where $p_1, p_2 : M(r,k) \times \mathbb{P}^2 \to M(r,k), \mathbb{P}^2$ denote the canonical projections. It follows from [26] that V is a locally free sheaf of rank k on $M(r,k)$.

There is a torus $\mathbf{T} = \mathbb{C}^\times \times \mathbb{C}^\times \times (\mathbb{C}^\times)^{\times r}$ action on acting on $M(r,k)$, where the action of the first two factors is induced by the canonical action on $\mathbb{C}^\times \times \mathbb{C}^\times$ on $\mathbb{P}^2$, and the last r factors act linearly on the framing. According to [27] the fixed points of the $\mathbf{T}$-action on $M(r,k)$ are isolated and classified by collections of Young diagrams $\underline{Y} = (Y_1, \ldots, Y_r)$ so that the total number of boxes in all diagrams is $|\underline{Y}| = |Y_1| + \cdots |Y_r| = k$. Let $\mathcal{Y}_{r,k}$ denote the set of all such r-tuples of Young diagrams. Note also that both the holomorphic cotangent bundle $T^\vee_{M(r,k)}$ and the bundle V constructed in the previous paragraph carry canonical equivariant structures.

The K-theoretic instanton partition function of an $SU(2)$ theory with g adjoint hypermultiplets and a level p Chern–Simons term is given by the equivariant residual Hirzebruch genus of the holomorphic $\mathbf{T}$-equivariant bundle

$$(T^\vee_{M(2,k)})^{\oplus g} \otimes (\det \mathsf{V})^{-p}.$$

This is defined by equivariant localization as follows [28], [22]. Let $(\epsilon_1, \epsilon_2, a_1, a_2)$ be equivariant parameters associated to the torus $\mathbf{T}$. Then the localization formula yields [28], [22]

$$\mathcal{Z}_{inst}^{(g,p)}(Q, \epsilon_1, \epsilon_2, a_1, a_2, y) = \sum_{k=0}^{\infty} Q^k \mathcal{Z}_k^{(g,p)}(\epsilon_1, \epsilon_2, a_1, a_2; y) \tag{4.6}$$

where $\mathcal{Z}_0^{(g,p)}(\epsilon_1, \epsilon_2, a_1, a_2; y) = 1$ and
(4.7)

$$\begin{aligned}
\mathcal{Z}_k^{(g,p)}(\epsilon_1, \epsilon_2, a_1, a_2; y) &= \sum_{(Y_2,Y_2)\in \mathcal{Y}_{2,k}} \mathcal{Z}_{(Y_2,Y_2)}^{(g,p)}(\epsilon_1, \epsilon_2, a_1, a_2; y) \\
&= \sum_{\underline{Y}\in\mathcal{Y}_{2,k}} \prod_{\alpha=1}^{2} \left(e^{-|Y_\alpha| a_\alpha} \prod_{(i,j)\in Y_\alpha} e^{(i-1)\epsilon_1 + (j-1)\epsilon_2} \right)^p \\
&\quad \prod_{\alpha,\beta=1}^{2} \prod_{(i,j)\in Y_\alpha} \frac{\left(1 - y e^{(Y^t_{\beta,j} - i)\epsilon_1 - (Y_{\alpha,i} - j + 1)\epsilon_2 + a_{\alpha\beta}}\right)^g}{\left(1 - e^{(Y^t_{\beta,j} - i)\epsilon_1 - (Y_{\alpha,i} - j + 1)\epsilon_2 + a_{\alpha\beta}}\right)} \\
&\quad \prod_{(i,j)\in Y_\beta} \frac{\left(1 - y e^{-(Y^t_{\alpha,j} - i + 1)\epsilon_1 + (Y_{\beta,i} - j)\epsilon_2 + a_{\alpha\beta}}\right)^g}{\left(1 - e^{-(Y^t_{\alpha,j} - i + 1)\epsilon_1 + (Y_{\beta,i} - j)\epsilon_2 + a_{\alpha\beta}}\right)}
\end{aligned}$$

where for any Young tableau Y, Y_i, $i \in \mathbb{Z}_{\geq 1}$ denotes the length of the i-th column and Y^t denotes the transpose of Y. If i is greater than the number of columns of Y, $Y_i = 0$. Moreover $a_{\alpha\beta} = a_\alpha - a_\beta$ for any $\alpha, \beta = 1, 2$.

Let $\mathcal{Z}_{(Y_1,Y_2)}^{(g,p)}(q_1, q_2, Q_f, y)$ be the expression obtained by setting $q_1 = e^{-\epsilon_1}$, $q_2 = e^{-\epsilon_2}$ and $Q_f = e^{a_{12}}$, and $e^{a_1} = -1$ in $\mathcal{Z}_{(Y_1,Y_2)}^{(g,p)}(\epsilon_1, \epsilon_2, a_1, a_2; y)$. Let

$$\Omega_Y^{(g,p)}(\lambda, y) = y^{2|Y|} \lambda^{(g-1)|Y|} \mathcal{Z}_{(Y,\emptyset)}^{(g,p)}(\lambda^{-1} y, \lambda y, Q_f = 0, y^{-1})^{(0)}. \tag{4.8}$$

Conjecture 4.4. *The generating function of asymptotic refined ADHM invariants is given by*

$$\mathcal{Z}_{+\infty}(\mathcal{X}, r; \lambda, y) = \sum_{e\in\mathbb{Z}} \lambda^e A_{+\infty}(r, e)(y) = \sum_{|Y|=r} \Omega_Y^{(g,p)}(\lambda, y). \tag{4.9}$$

Combining (4.2), (4.3), Conjecture 4.1, 4.2, 4.4 we have a complete recursive relation determining the cohomologies of the Hitchin moduli. The physics reason behind (4.8) and (4.9) is that we think of $\mathcal{Z}_{inst}^{(p)}(Q, \epsilon_1, \epsilon_2, a_1, a_2, y)$ as a refined GW theory or refined topological string theory and by the correspondences $Z_{GW} = Z'_{DT} = Z_{PT}$ in the introduction, properly generalized to the refined case, it is also a refined PT theory conjecturally (*i.e.* refined ADHM theory in the asymptotic chamber.) Then the refined wallcrossing formula can determine the Poincaré (Hodge) polynomials of Higgs moduli, which in turn determines the Poincaré (Hodge) polynomials of Hitchin moduli.

After this work was completed, it was shown by Mozgovoy [25] that the H-polynomials and E-polynomials introduced in a series of conjectures by Hausel and Rodriguez-Villegas [11], [10] are consistent with our computational results determined in the recursive way.

Acknowledgements. The author is grateful to the organizers of the conference "Algebraic Geometry in East Asia" held in National Taiwan University in November 2011 for giving him an opportunity to participate in such a fruitful event. He would also like to thank Duiliu-Emanuel Diaconescu and Guang Pan for the collaboration on the topics presented in this article and the referees for pointing out some false statements. He is supported by NSC grant 99-2115-M-002-013-MY2.

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