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Three dimensional divisorial contractions

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Abstract.

We give a brief survey of three dimensional divisorial contractions and provide many explicit examples.

§1. Introduction

In algebraic geometry, it is of fundamental importance to classify birational equivalent classes of algebraic varieties, find a good representative in each birational equivalent class, and investigate the maps between them. The minimal model program plays the central role in these goals. The purpose of minimal model program is to find the minimal model with mild singularities inside a birational equivalent class and to investigate those maps appeared in the process. This program is classical for dimension ≤ 2. However, a better understanding for higher dimensions in general appeared in only about 25 years ago, mainly due to Kawamata, Kollár, Mori, Reid, Shokurov, and some others.

We give a short tour of minimal model program. Given an algebraic variety X, we say that X is minimal if K_X is nef (whenever it makes sense). To obtain a minimal model, one can try to eliminate those non-nef curves by contracting them. In practice, one picks a curve class $[C]$ in an edge of the cone of numerically equivalent classes of curves of X. By using the Kawamata–Viehweg vanishing theorem, there is a base point free linear system $|H|$ producing a contraction map $\varphi : X \to W$ contracting the curve class $[C]$. If $\dim W < \dim X$, then φ is called a Mori fibered space and we stop. If $\dim W = \dim X$ then φ is birational. The case that the exceptional set is a divisor (resp. of smaller dimension) is called a divisorial contraction (resp. small contraction). The small contraction φ gives rise to wild singularities on W. It is conjectured that

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there exist a birational surgery $X \dashrightarrow X^+$ called a flip so that X^+ has only mild singularities. The minimal model conjecture predicts that one can have a finite sequence of divisorial contractions and flips that ends up with a minimal model or Mori fibered space.

Mori completes the minimal model program for threefolds in [24]. In his subsequent work with Kollár (cf. [20]), extremal neighborhood are classified and flips are studied in detail. However, since the proof of minimal model program does not rely on the understanding of divisorial contractions, the explicit studies of three dimensional divisorial contractions are available only quite recently.

The purpose of this note is to give an elementary introduction to the recent studies of divisorial contractions in dimension three. We feel that these kind of explicit studies will be helpful for various geometric problems in dimension three and higher. The detailed studies of flops and flips are not included in this note. For readers who are interesting in flops, we refer to Kollár's article [19]. There are many other interesting topics, especially the recent highlight of [1], are not covered in this note.

We work in the complex analytic category.

§2. Preliminaries

2.1. Classification of terminal singularities

One needs to allow mild singularities in minimal model program in dimension three or higher. In this note, all varieties are normal $\mathbb{Q}$-factorial and terminal, unless otherwise specified. In fact, the development of minimal model program in dimension three was built on the understanding of three dimensional terminal singularities. This can be dated back to 30 years ago (cf. [25], [26], [27]). Given a germ of three dimensional terminal singularity $(P \in X)$, there is a canonical cover $\mu : (Q \in Y) \to (P \in X)$ so that $Q \in Y$ is Gorenstein and terminal and $(P \in X)$ is the quotient by a cyclic group of order $\deg(\mu)$. The degree of μ is the index of $(P \in X)$. It is known that a Gorenstein terminal singularity is an isolated cDV hypersurface singularity, i.e. a singularity with local equation of the form

$$f(x, y, x) + ug(x, y, z, u) = 0,$$

for some $f(x, y, z)$ defining a Du Val (equivalently rational double point) singularity. If $(P \in X)$ is Gorenstein, then according to the type of $f(x, y, z)$, we have that $(P \in X) \cong (o \in (\varphi = 0) \subset \mathbb{C}^4)$ for some φ belongs to one of the following:

(1) type cA: $(xy + g(z, u) = 0) \subset \mathbb{C}^4$ and $g(z, u) \in \mathfrak{m}^2$.

(2) type cD: $(x^2+y^2z+g(y,z,u)=0)\subset\mathbb{C}^4$ and $g(y,z,u)\in\mathfrak{m}^3$.
(3) type cE: $(x^2+y^3+yg(z,u)+h(z,u)=0)\subset\mathbb{C}^4$ and $g(z,u)\in\mathfrak{m}^3, h(z,u)\in\mathfrak{m}^4$,

where $\mathfrak{m}$ denotes the maximal ideal of $o\in\mathbb{C}^4$. In the cE case, it is of type cE_6 (resp. cE_7, cE_8) if $h_4\neq 0$ (resp. $g_3\neq 0$, $h_5\neq 0$), where g_3, h_4, h_5 denotes of homogeneous part of g, h of degree $3,4,5$ respectively.

A three-dimensional terminal singularity $(P\in X)$ is therefore of the form of a cyclic quotient of isolated cDV singularity cDV/μ_r. Mori classified three dimensional terminal singularities with index $r>1$ explicitly (cf. [23]).

(1) type cA/r: $(xy+g(z,u)=0)\subset\mathbb{C}^4/\frac{1}{r}(a,-a,1,0)$ and $g(z,u)\in\mathfrak{m}$.
(2) type cAx/2: $(x^2+y^2+g(z,u)=0)\subset\mathbb{C}^4/\frac{1}{2}(0,1,1,1)$ and $g(z,u)\in\mathfrak{m}^3$.
(3) type cAx/4: $(x^2+y^2+g(z,u)=0)\subset\mathbb{C}^4/\frac{1}{4}(1,3,1,2)$ and $g(z,u)\in\mathfrak{m}^3$.
(4) type $cD/3$: $(\varphi=0)\subset\mathbb{C}^4/\frac{1}{3}(0,2,1,1)$, where φ is in one of the following forms:
  (a) $x^2+y^3+z^3+u^3$.
  (b) $x^2+y^3+z^2u+yg(z,u)+h(z,u)$ with $g\in\mathfrak{m}^4, h\in\mathfrak{m}^6$.
  (c) $x^2+y^3+z^3+yg(z,u)+h(z,u)$ with $g\in\mathfrak{m}^4, h\in\mathfrak{m}^6$.
(5) type $cD/2$: $(\varphi=0)\subset\mathbb{C}^4/\frac{1}{2}(1,0,1,1)$, where φ is in one of the following forms:
  (a) $x^2+y^3+yzu+g(z,u)$ with $g\in\mathfrak{m}^4$.
  (b) $x^2+yzu+y^n+g(z,u)$ with $n\geq 4$, $g\in\mathfrak{m}^4$.
  (c) $x^2+yz^2+y^n+g(z,u)$ with $n\geq 3$, $g\in\mathfrak{m}^4$.
(6) type $cE/2$: $(x^2+y^3+yg(z,u)+h(z,u)=0)\subset\mathbb{C}^4/\frac{1}{2}(1,0,1,1)$ and $g,h\in\mathfrak{m}^4$, $h_4\neq 0$.

2.2. Weighted blowups

Most of the examples are illustrated by weighted blowups. We recall the construction of weighted blowups by using the toric language.

Let $N=\mathbb{Z}^d$ be a free abelian group of rank d with standard basis $\{e_1,...,e_d\}$. Let $v=\frac{1}{r}(a_1,...,a_d)\in\mathbb{Q}^d$ be a vector. We may assume that $gcd(a_1,...,a_d)=1$. We consider $\overline{N}:=N+\mathbb{Z}v$. Clearly, $N\subset\overline{N}$. Let M (resp. $\overline{M}$) be the dual lattice of N (resp. $\overline{N}$).

Let σ be the cone generated by the standard basis $e_1,...,e_d$ and Σ be the fan consists of σ and all the subcones of σ. We consider

$$\mathcal{X}_{N,\Sigma}:=\mathrm{Spec}\mathbb{C}[\sigma^\vee\cap M]=\mathbb{C}^d,$$
$$\mathcal{X}_{\overline{N},\Sigma}:=\mathrm{Spec}\mathbb{C}[\sigma^\vee\cap\overline{M}].$$

Notice that $\mathcal{X}_0 := \mathcal{X}_{\overline{N},\Sigma}$ is a quotient variety of $\mathbb{C}^d$ by the cyclic group $\mathbb{Z}/r\mathbb{Z}$, which we denote it as $\mathbb{C}^d/v_0$ or $\mathbb{C}^d/\frac{1}{r}(a_1,...,a_d)$.

Let $v_1 = \frac{1}{r_1}(b_1,...,b_d)$ be a primitive vector in $\overline{N}$. We assume that $b_i \in \mathbb{Z}_{\geq 0}$ and $gcd(b_1,...,b_d) = 1$. We are interested in the weighted blowup over $o \in \mathcal{X}_0$ with weights $v_1 = \frac{1}{r_1}(b_1,...,b_d)$ which we describe now.

Let $\overline{\Sigma}$ be the fan obtained by subdivision of Σ along v_1. One thus have a toric variety $\mathcal{X}_{\overline{N},\overline{\Sigma}}$ together with the natural map $\mathcal{X}_{\overline{N},\overline{\Sigma}} \to \mathcal{X}_{\overline{N},\Sigma}$. More concretely, for any $b_i > 0$, let σ_i be the cone generated by $\{e_1,...,e_{i-1},v_1,e_{i+1},...,e_d\}$, then

$$\mathcal{X}_1 := \mathcal{X}_{\overline{N},\overline{\Sigma}} = \cup_{b_i>0}\mathrm{Spec}\mathbb{C}[\sigma_i^\vee \cap \overline{M}].$$

Let $\mathcal{U}_i = \mathcal{X}_{\overline{N},\sigma_i} = \mathrm{Spec}\mathbb{C}[\sigma_i^\vee \cap \overline{M}] \cong \mathbb{C}^d/\frac{1}{b_i}(b_1,\ldots,b_{i-1},-r,b_{i+1},\ldots,b_d)$. We always denote the origin of $\mathcal{U}_i$ as Q_i. In each affine chart $\mathcal{U}_i$, the natural map $\mathcal{U}_i \to \mathcal{X}_0$ is given by

$$\begin{cases} x_j \mapsto x_j x_i^{b_j/r_1}, & \text{if } j \neq i; \\ x_i \mapsto x_i^{b_i/r_1}. \end{cases}$$

We denote the exceptional divisor $\mathcal{E} \cong \mathbb{P}(b_1,b_2,...,b_d)$ by $\mathbb{P}(v_1)$

Suppose that there is a primitive vector $v_2 = \frac{1}{r_2}(c_1,...,c_d) \in \overline{N}$ such that v_2 is contained in the cone σ_i. We can consider the second weighted blowup over $\mathcal{U}_i$ with vector v_2. To this purpose, we can write

$$\begin{aligned} v_2 &= \tfrac{1}{r_2}(c_1e_1 + ... + c_de_d) \\ &= \tfrac{1}{p}(q_1e_1 + \ldots + q_{i-1}e_{i-1} + q_iv_1 + q_{i+1}e_{i+1} + \ldots + q_de_d), \end{aligned}$$

for some $p \in \mathbb{Z}_{>0}$ and $q_i \in \mathbb{Z}_{\geq 0}$. We say that $w_2 = \frac{1}{p}(q_1,...,q_d)$ is the weights corresponding to the vector v_2 in the cone σ_i, or simply the weights corresponding to v_2 (in σ_i) and vice versa if no confusion is likely.

Indeed, let τ_{ij} be the cone generated by

$$\begin{cases} \{e_1,...,e_{j-1},v_2,e_{j+1},...,e_{i-1},v_1,e_{i+1},...,e_d\}, & \text{if } j \neq i, \\ \{e_1,...,e_{i-1},v_2,e_{i+1},..,e_d\}, & \text{if } j = i. \end{cases}$$

The map $\cup_{q_j>0}\mathrm{Spec}\mathbb{C}[\tau_{ij}^\vee \cap \overline{M}] \to \mathcal{U}_i$ is the weighted blowups of $\mathcal{U}_i$ of weights w_2 (or with vector v_2). Let $\overline{\overline{\Sigma}}$ be the fan obtained by subdivision along v_2 of all the cones containing v_2. One thus have a toric variety $\mathcal{X}_2 := \mathcal{X}_{\overline{N},\overline{\overline{\Sigma}}}$ and we say that the natural map $\mathcal{X}_2 \to \mathcal{X}_1$ is the weighted blowup with vector v_2.

Given a tower of weighted blowups $\mathcal{X}_2 \to \mathcal{X}_1 \to \mathcal{X}_0$ with vectors v_1 and then v_2, we may reverse the order of vectors and then obtain a tower of weighted blowups $\mathcal{X}_2' \to \mathcal{X}_1' \to \mathcal{X}_0$ with vectors v_2 and then v_1. We have the following diagram

$$\begin{tikzcd} \mathcal{X}_2 \arrow[rr, dashed] \arrow[d, "v_2"'] & & \mathcal{X}_2' \arrow[d, "v_1"] \\ \mathcal{X}_1 \arrow[dr, "v_1"'] & & \mathcal{X}_1' \arrow[dl, "v_2"] \\ & \mathcal{X}_0. & \end{tikzcd}$$

It is clear that $\mathcal{X}_2$ and $\mathcal{X}_2'$ are isomorphic in codimension one.

Given a semi-invariant $\varphi = \sum \alpha_{i_1,\ldots,i_d} x_1^{i_1} \ldots x_d^{i_d}$ on the quotient variety $\mathcal{X}_0$ and a vector $v = \frac{1}{r}(b_1, b_2, \ldots, b_d) \in \overline{N}$, we define

$$wt_v(\varphi) := \min\{\sum_{j=1}^{d} \frac{b_j}{r} i_j | \alpha_{i_1,\ldots,i_d} \neq 0\}.$$

Let $X \in \mathcal{X}_0$ be a complete intersection defined by semi-invariants $\varphi_1 = \ldots = \varphi_c = 0$. Let Y be its proper transform of X in $\mathcal{X}_1$. By abuse the notation, we also call the induced map $f : Y \to X$ the weighted blowups of X of weights v.

Notice that the local chart U_i of Y is defined by $\tilde{\varphi}_1 = \ldots = \tilde{\varphi}_c = 0$ with

$$\begin{aligned} \tilde{\varphi}_j := \varphi_j(x_1 x_i^{a_1/r_0}, \ldots, x_{i-1} x_i^{a_{i-1}/r_0}, x_i^{a_i/r_0}, x_{i+1} x_i^{a_{i+1}/r_0}, \\ \ldots, x_d x_i^{a_d/r_0}) x_i^{-wt_{v_0}(\varphi_j)}, \end{aligned}$$

for each i, j. We fix the notation that $E := \mathcal{E} \cap Y \subset \mathbb{P}(v)$ denotes the exceptional divisor and $U_i := \mathcal{U}_i \cap Y$. The adjunction formula yields that

$$K_Y = f^* K_X + a(v, X) E,$$

whenever E is irreducible and reduced. Where $a(v, X)$ can be computed as

$$a(v, X) = \sum_i wt_v(x_i) - \sum_j wt_v(\varphi_j) - 1.$$

For simplicity, we will use the notation

$$f = wBl_v : Y = wBl_v(X) \to X$$

(resp. $wBl_{wt=w}$) to denote the weighted blowup of X with vector v (resp. weights w). By construction, wBl_v is a divisorial contraction if the exceptional divisor E is reduced and irreducible, $a(v, X) > 0$ and Y is terminal.

§3. Classification of divisorial contractions to points

Among birational maps in minimal model program, divisorial contractions to points are most studied. Mori classified contractions $f : Y \to X \ni P$ when Y is nonsingular [22] and then Cutkosky extended it to the situation that Y allows Gorenstein singularities [7]. On the other hand, Kawamata classified the situation that $P \in X$ is a terminal quotient singularity [18] and Corti [6] studied the case that $P \in X$ is of cA_1 type. Markushevich [21] and Kawamata [17] showed the existence of divisorial contractions with discrepancy $\frac{1}{r}$ over a singular point of index $r = 1$ and $r > 1$ respectively. All their examples are weighted blowups. In [9], [10], Hayakawa classified all divisorial contractions to points of higher indices with minimal discrepancies. A recent highlight is Kawakita's series of work in which all divisorial contractions to points are classified in some sense.

3.1. Mori and Cutkosky's work

In [22], Mori studies extremal contractions from a nonsingular threefold. In [7], Cutkosky notices that the same proof is still valid if Y has only Gorenstein terminal singularities. We summarize their results.

Theorem 3.1. *Let Y be a Gorenstein threefold and $f : Y \to X$ be a divisorial contraction to a point $P \in X$. Then f is one of the following:*

(1) *$P \in X$ is nonsingular, $f = Bl_P(X)$ the blowup over P.*
(2) *$P \in X$ is of type cA_1 with $\varphi = x^2 + y^2 + z^2 + u^n$, for some $n \geq 2$, $f = Bl_P(X)$ the blowup over P.*
(3) *$P \in X$ is a quotient singularity $\frac{1}{2}(1,1,1)$, $f = wBl_v(X)$ the weighted blowup with weight $v = \frac{1}{2}(1,1,1)$.*

We would like to remark that in the above cases, Y is singular if and only if it is in case (2) with $n \geq 4$. Even though it is not stated explicitly in [22], [7], it is not difficult to see that f is either a blowup or a weighted blowup.

Sketch of the proof. We will give a brief sketch of the proof, which is more or less a reproduction of Cutkosky's argument.

Let E be the exceptional divisor of f, which is Gorenstein. We may write $K_Y = f^*K_X + aE$ for some $a > 0 \in \mathbb{Q}$. Note that $-K_Y$

is f-ample and hence both $-K_E = -(a+1)E|_E$ and $-E|_E$ are ample. It follows from the Kawamata–Viehweg vanishing theorem that $h^i(E, \mathcal{O}_E) = 0, h^i(E, \mathcal{O}_E(-E)) = 0$ and $h^i(E, \mathcal{O}_E(-K_Y)) = 0$ for $i > 0$. In particular, $\chi(\mathcal{O}_E) = 1$.

Consider next the Δ-genus (cf. [8]) of the polarized variety $(E, \mathcal{O}_E(-K_Y))$, which is defined as

$$\Delta(E, \mathcal{O}_E(-K_Y)) := \dim E + d(E, \mathcal{O}_E(-K_Y)) - h^0(E, \mathcal{O}_E(-K_Y)).$$

In our situation, we have $d(E, \mathcal{O}_E(-K_Y)) = (-K_Y)^2 \cdot E$ and hence

$$\Delta(E, \mathcal{O}_E(-K_Y)) = 1 - \frac{1}{2} K_Y \cdot E^2$$

by Riemann–Roch formula.

Since $\Delta(E, \mathcal{O}_E(-K_Y)) \geq 0$ (cf. [8, Theorem 1.9]), one thus has $\Delta(E, \mathcal{O}_E(-K_Y)) = 0$ and $K_Y \cdot E^2 = aE^3 = 2$.

It follows that

$$4 = a^2(E^3)^2 = (K_Y^2 \cdot E) \cdot E^3.$$

Therefore, either $(K_Y^2 \cdot E) \leq 2$ or $E^3 \leq 2$. In the first case that $(K_Y^2 \cdot E) \leq 2$ then E is isomorphic to $\mathbb{P}^2$ or a quadric in $\mathbb{P}^3$ according to Fujita's work (cf. [8, Section 2]). In the latter case that $E^3 \leq 2$, we may consider the polarized variety $(E, \mathcal{O}_E(-E))$ and show that $\Delta(E, \mathcal{O}_E(-E)) = 0$, $d(E, \mathcal{O}_E(-E)) = E^3 \leq 2$. We thus conclude that in any case, E is isomorphic to $\mathbb{P}^2$ or a quadric in $\mathbb{P}^3$.

Next, one can compute direct images of $\mathcal{O}_Y(-jE)$ and obtained that

Lemma 3.2. *Keep the notation as above, one has*

(1) $R^i f_* \mathcal{O}_Y(-jE) = 0$ *for all* $i > 0$ *and* $j \geq 0$,
(2) $f_* \mathcal{O}_Y(-jE) = \mathfrak{m}^j$ *and* $\mathfrak{m}^j \mathcal{O}_X = \mathcal{O}(-jD)$ *for all* $j \geq 0$,
(3) $gr(\mathcal{O}_{X,P}) := \oplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1} \cong \oplus_{n \geq 0} H^0(E, \mathcal{O}(-nE))$ *as* $\mathbb{C}$-*algebra.*

Proof of the Lemma. The first statement follows from the relative version of Kawamata–Viehweg vanishing theorem.

Set $I_j := f_* \mathcal{O}_Y(-jE)$. By pushing forward the exact sequence

$$0 \to \mathcal{O}_Y(-(j+1)E) \to \mathcal{O}_Y(-jE) \to \mathcal{O}_E(-jE) \to 0,$$

there is an induced exact sequence

$$0 \to I_{j+1} \to I_j \to f_* \mathcal{O}_E(-jE) = H^0(E, \mathcal{O}_E(jE)) \to 0$$

for $j \geq 0$.

Hence $I_0/I_1 \cong H^0(\mathcal{O}_E) \cong \mathbb{C}$. It follows that $I_1 \cong \mathfrak{m}$, the maximal ideal of P, and there is an isomorphism of graded $\mathbb{C}$-algebra

$$\oplus_n I_n/I_{n+1} \cong \oplus_n H^0(E, \mathcal{O}_E(-nE)).$$

Since $\oplus_n H^0(E, \mathcal{O}_E(-nE))$ is generated by $H^0(E, \mathcal{O}_E(-E))$, one has

$$I_{n+1} = I_1 I_n + I_{n+2}, \qquad \dagger_1$$

for all $n \geq 0$. Inductively,

$$I_{n+1} = I_1 I_n + I_{n+m}, \qquad \dagger_2$$

for all $n \geq 0$, $m > 0$.

On the other hand, since $\mathcal{O}_Y(-E)$ is f-ample, one has that the graded $\mathcal{O}_X$-algebra $\oplus I_n$ is finitely generated. Hence there exists $m_0 \gg 0$ so that for all $n \geq 0$,

$$I_{n+m_0} = I_n I_{m_0}.$$

By $\dagger_1$ and $\dagger_2$, we have for all $n \geq 0$,

$$I_{n+1} = I_1 I_n + I_{n+m_0} = I_1 I_n + I_n I_{m_0} = I_1 I_n.$$

Therefore, $I_n = I_1^n = \mathfrak{m}^n$ for any $n \geq 0$.

Finally, one can verify that $\mathfrak{m}\mathcal{O}_Y = \mathcal{O}_Y(-D)$. Q.E.D.

With this Lemma, one can determine $gr(\widehat{\mathcal{O}_{X,P}}) \cong gr(\mathcal{O}_{X,P})$. By the classification of terminal singularities, one can thus determine $\widehat{\mathcal{O}_{X,P}}$, which is one of $\mathbb{C}[[x,y,z,u]]/(x^2+y^2+z^2+u^n)$ for some $n \geq 2$ or $\mathbb{C}[[x,y,z]]^{(2)}$, the invariant subring under the $\mathbb{Z}_2$ action $(x,y,z) \mapsto (-x, -y, -z)$.

Let $f' : Y' \to X \ni P$ be the blowup (resp. weighted blowup with weight $\frac{1}{2}(1,1,1)$) if $\widehat{\mathcal{O}_{X,P}} \cong \mathbb{C}[[x,y,z,u]]/(x^2+y^2+z^2+u^n)$ (resp. $\mathbb{C}[[x,y,z]]^{(2)}$) with exceptional divisor E'. It is easy to verify that $Rf'_*\mathcal{O}_{Y'}(-nE') = Rf_*\mathcal{O}_Y(-nE)$. By [14, Lemma 3.4], we have that f is isomorphic to f', which is a blowup or a weighted blowup. Q.E.D.

Since Y is Gorenstein and hence all the intersection numbers involved in the above computation are integers. If there are singularities of index > 1 on $E \subset Y$, the above computation of Δ does not work any more.

Example 3.3. Let $(P \in X)$ be a cA_1 singularity given by $(\varphi : xy + z^2 + u^n = 0) \subset \mathbb{C}^4$ and $n \geq 2$. Let $f = Bl_P : Y \to X$ be the blowup over P. Then $Sing(Y) = \{Q_4\}$ with local equation in U_4 given by $(\tilde{\varphi}: xy + z^2 + u^{n-2} = 0) \subset \mathbb{C}^4$, which is Gorenstein terminal of type cA_1.

We would like to remark that there exist some other divisorial contractions over $P \in X$ in this situation. For example, take $wBl_{v'}: Y' \to X$ of weights $v' = (1, 2n'-1, n', 1)$, where $n' = \lfloor \frac{n}{2} \rfloor$. It is straightforward to check that $Y' \to X$ is a divisorial contraction with discrepancy 1 and $Sing(Y') = \{Q_2\}$ which is a terminal quotient singularity of type $\frac{1}{2n'-1}(-1, n', 1)$.

3.2. Contractions with minimal discrepancies

Given a terminal singularity $P \in X$ of index $r > 1$, there is at least one divisor which has discrepancy $\frac{1}{r}$ over X by [18]. Similarly, if X is a terminal singularity of index 1, then there is at least one divisorial contraction with discrepancy one by [21]. One notices that above results are obtained by constructing weighted blowups with minimal discrepancy $\frac{1}{r}$.

By using similar construction, Hayakawa classified divisorial contractions $Y \to X \ni P$ to a point of index $r > 1$ with discrepancy $\frac{1}{r}$ in [9], [10]. In his recent work [12], [13], he tries to classify divisorial contractions $Y \to X \ni P$ to a point of index $r = 1$ with discrepancy 1. We briefly explain his method and give various examples for possible phenomena.

1. First, one starts with an explicit divisorial contraction $Y \to X$ to a point $P \in X$ of index $r > 1$ with minimal discrepancy $\frac{1}{r}$ which is a weighted blowup. Let E be its exceptional divisor.

2. Determine the number of valuations with minimal discrepancies. For a given valuation v with center in $P \in X$, one can consider a resolution $\mu: Z \to Y$ so that $v = v_F$ for some prime divisor $F \subset Z$.

Claim. If $a(F, X) \leq 1$, then $\mu(F)$ is a point of index > 1 in Y.

Proof. For any μ-exceptional divisor F with center in $P \in X$, one has that $\mu(F) \subset E$. It follows that

$$a(F, X) = a(F, Y) + v_F(E)a(E, X) > a(F, Y). \qquad \ddagger$$

If $\mu(F)$ is a curve, then $a(F, Y) = 1$. If $\mu(F) = Q \in Y$ is a point of index 1, then $a(F, Y) \geq 1 \in \mathbb{Z}$. Hence $\mu(F)$ must be a point of index > 1. The Claim now follows. Q.E.D.

Therefore, it suffices to search for points $Q \in Y$ of index $r' > r$ and valuations centered at Q satisfying $\ddagger$.

3. Find as many divisorial contractions as valuations with discrepancy $\frac{1}{r}$. Therefore, the divisorial contractions with minimal discrepancies are classified completely.

For any given explicit weighted blowup, it is easy to determine higher index points. Together with the explicit description of local equations, one can determine the number of valuations with minimal discrepancies $\frac{1}{r}$. Usually, these valuations corresponds to some other weighted blowups. In some rare situations, one needs to change the embedding and modify the weights. Hayakawa managed to determine the number of valuations with minimal discrepancy and to find as many divisorial contractions as valuations with discrepancy $\frac{1}{r}$. All of them are weighted blowups.

Example 3.4. Let $P \in X$ be a terminal singularity of cA/r type given by

$$(\varphi : xy + z^9 + u^3 = 0) \subset \mathbb{C}^4/\frac{1}{3}(2,1,1,0).$$

Let $f(z,u) = z^9 + u^3$. Since $wt_{\frac{1}{3}(1,3)} f(z,u) = 3$, following Hayakawa, we may consider $f = wBl_v : Y \to X$ of weights $v = \frac{1}{3}(2,7,1,3)$. It is straightforward to check that this is a divisorial contraction with discrepancy $\frac{1}{3}$.

The higher index points on Y consists of Q_1 and Q_2. We have

$$\begin{cases} U_1 = (y + z^9 + u^3 = 0) \subset \mathbb{C}^4/\frac{1}{2}(1,1,1,1) \overset{\psi_1}{\cong} \mathbb{C}^3/\frac{1}{2}(1,1,1) =: V_1, \\ U_2 = (x + z^9 + u^3 = 0) \subset \mathbb{C}^4/\frac{1}{7}(2,4,1,3) \overset{\psi_2}{\cong} \mathbb{C}^3/\frac{1}{7}(4,1,3) =: V_2. \end{cases}$$

Let $g : Z \to Y$ be the economic resolution over Q_1 and Q_2. Hence

$$K_Z = g^* K_Y + \frac{1}{2} F' + \sum_{i=1}^{6} \frac{i}{7} F_i,$$

for some exceptional divisors $F_1, ..., F_6$ and F'.

More explicitly, the economic resolution over Q_1 is isomorphic to the weighted blowup at $o \in V_1$ with weight $\frac{1}{2}(1,1,1)$. Therefore, it is a weighted blowup over Q_1 of weights $w' := \frac{1}{2}(1,3,1,1)$. The weight w' corresponds to the vector

$$v' := \frac{1}{2} v + \frac{1}{2} e_2 + \frac{1}{2} e_3 + \frac{1}{2} e_4 = \frac{1}{3}(1,8,2,3) \in \overline{N}.$$

The economic resolution over $o \in V_2$ is obtained by weighted blowup of weights $\frac{1}{7}(\overline{4i}, i, \overline{3i})$, where $\overline{4i}, \overline{3i}$ denotes the residue modulo 7. Hence the economic resolution over Q_2 is obtained by weighted blowups with weight $\frac{1}{7}(c_i, \overline{4i}, i, \overline{3i})$, where $c_i = \min\{9i, 3 \cdot \overline{3i}\} = 3 \cdot \overline{3i}$. Therefore, the corresponding vectors $v_i \in \overline{N}$ is given by

$$v_i = \frac{1}{21}(2 \cdot \overline{4i} + 9 \cdot \overline{3i}, 7 \cdot \overline{4i}, 3i + \overline{4i}, 21).$$

In total, one sees that there are three valuations v, v_1, v_2 with discrepancy $\frac{1}{3}$. In fact, the weighted blowup with weights v, v_1 or v_2 are divisorial contractions with discrepancy $\frac{1}{3}$. Hence we have identified all divisorial contractions with discrepancy $\frac{1}{3}$.

The above discussion also shows that there are exactly three valuation with discrepancy $\frac{2}{3}$ and each valuation corresponds to a divisorial contractions which is a weighted blowup with weight v_3, v_4 or v'.

Example 3.5. Let $P \in X$ be a terminal singularity of $cAx/2$ type given by

$$(\varphi : x^2 + y^2 - z^4 - 2z^2u^2 - u^4 + z^6 + u^6 = 0) \subset \mathbb{C}^4/\frac{1}{2}(0,1,1,1).$$

It is easy to see that there are only two vectors $v_1 = \frac{1}{2}(2,1,1,1,), v_2 = \frac{1}{2}(2,3,1,1)$ with $a(v_1, X) = a(v_2, X) = \frac{1}{2}$. However, the exceptional divisor of the weighted blowup of weights v_1 (resp. v_2) is non-reduced (resp. reducible). Hence none of these weighted blowups is a divisorial contraction.

One needs some modification in order to get divisorial contractions. We consider a coordinate change such as $x_+ := x+(z^2+u^2)$, then $P \in X$ is given by

$$(\varphi_+ : x_+^2 - 2x_+(z^2+u^2) + y^2 + z^6 + u^6 = 0) \subset \mathbb{C}^4/\frac{1}{2}(0,1,1,1).$$

One can check that the weighted blowup

$$f_+ : Y_+ \to X \subset \mathbb{C}^4_{\{x_+,y,z,u\}}/\frac{1}{2}(0,1,1,1)$$

with weight $\frac{1}{2}(4,3,1,1)$ is a divisorial contraction. Let E_+ be its exceptional divisor. Higher index point in Y consists Q_1, which is terminal quotient of type $\frac{1}{4}(3,1,1)$. Take the economic resolution $g_+ : Z \to Y_+$ over Q_1, one sees that

$$K_Z = g_+^* K_{Y_+} + \sum_{i=1}^{3} \tfrac{i}{4} F_i = g_+^* f_+^* K_X + \tfrac{1}{2} E_{+,Z} + \tfrac{1}{2} F_1 + F_2 + F_3,$$

where $E_{+,Z}$ denotes the proper transform of E_+ in Z. We thus conclude that there are two valuations with discrepancy $\frac{1}{2}$.

Indeed, if we consider $x_- := x - (z^2+u^2)$ instead, then $P \in X$ is given by

$$(\varphi_- : x_-^2 + 2x_-(z^2+u^2) + y^2 + z^6 + u^6 = 0) \subset \mathbb{C}^4/\frac{1}{2}(0,1,1,1).$$

The weighted blowup

$$f_- : Y_- \to X \subset \mathbb{C}^4_{\{x_-,y,z,u\}}/\frac{1}{2}(0,1,1,1)$$

of weights $\frac{1}{2}(4,3,1,1)$ is a divisorial contraction.

We claim that $f_+ \ncong f_-$. To see this, consider the Weil divisor $D := div(x_+)$. It is clear that $f_+^*(D) = D_{X_+} + \frac{4}{2}E_+$ but $f_-^*(D) = D_{X_-} + \frac{2}{2}E_-$. Therefore, f_+ and f_- can not be isomorphic. Therefore, there are exactly two divisorial contractions with discrepancy $\frac{1}{2}$ in this situation.

With similar technique as in above examples, Hayakawa classified divisorial contractions to higher index points with minimal discrepancy $\frac{1}{r}$. Any one of such divisorial contractions can be realized as a weighted blowup in suitable embedding. However, there are several cases that one needs to embed into a 5-dimensional space as a quotient of complete intersection of hypersurfaces of degree 1 and 2.

Example 3.6. Let $P \in X$ be a terminal singularity of $cD/2$ type given by

$$(\varphi : x^2 + yzu + y^6 + z^4 + u^3 = 0) \subset \mathbb{C}^4/\frac{1}{2}(1,1,1,0).$$

One can consider the weighted blowup $f : Y \to X$ with weight $v = \frac{1}{2}(3,1,3,2)$. It is straightforward to see that f is a divisorial contractions with discrepancy $\frac{1}{2}$ and there are two valuations with discrepancy $\frac{1}{2}$. The other valuation corresponds to the vector $v_2 = \frac{1}{2}(3,1,1,2)$. However, the weighted blowup with weight v_2 is not a divisorial contraction for its exceptional divisor is reducible.

One can consider

$$(P \in X) \cong X' : \begin{cases} x^2 + zt + y^6 + u^3 = 0 \\ t = yu + z^3 \end{cases} \subset \mathbb{C}^5/\frac{1}{2}(1,1,1,0,1).$$

Under this embedding, the weighted blowup $Y' \to X'$ with weight $\frac{1}{2}(3,1,1,2,5)$ is a divisorial contraction.

3.3. Kawakita's classification

In [14], Kawakita proved the following results characterizing divisorial contractions to smooth points.

Theorem 3.7. *Let $Y \to X \ni P$ be a divisorial contraction to a smooth point $P \in X$. Then f is a weighted blowup of weights $(1,m,n)$ with $m,n \in \mathbb{Z}_{>0}$ and $(m,n) = 1$.*

To illustrate his work, we start by considering the weighted blowup of weights $(1, m, n)$.

Example 3.8. Let $f : Y \to X = \mathbb{C}^3$ be the weighted blowup of weight $v = (1, 2, 3)$ over $o \in \mathbb{C}^3$. Let E be the exceptional divisor. Y' is nothing but the toric variety obtained by subdivision along the vector $(1, 2, 3)$.

The vector $v = (1, 2, 3)$ defines a valuation. One can try to find a resolution $\mu : Z \to X$ so that $v = v_F$ for some exceptional divisor F start by usual blowups. To this end, we may start by considering $f_1 = Bl : Y_1 \to X$ the usual blowup by subdivision along the vector $v_1 = (1, 1, 1)$. In the cone $\sigma_3 = \langle v_1, e_2, e_3 \rangle$, we consider $f_2 = Bl_{Q_1} : Y_2 \to Y_1$ the blowup over Q_1, which is obtained by the subdivision along $v_2 = (1, 2, 2)$. Next, we consider $f_3 : Y_3 \to Y_2$ obtained by the subdivision along $v_3 = (1, 2, 3)$. This can be seen to be a blowup along a smooth curve. Let E_3 be the exceptional divisor of f_3. In total, we have a sequence

$$\mu : Z := Y_3 \to Y_2 \to Y_1 \to X, \qquad \natural$$

so that the valuation v_E is realized by the μ-exceptional divisor E_3. Indeed $v_E = v_{E_3}$.

On the other hand, we may consider a sequence of toric maps that

$$Z' = Y'_4 \to Y'_3 \to Y'_2 \to Y'_1 = Y \to X, \qquad \natural'$$

by subdivision along $v'_1 = v = (1, 2, 3)$, $v'_2 = (1, 1, 2)$, $v'_3 = (1, 1, 1)$ and $v'_4 = (1, 2, 2)$ successively. Notice that $Y'_i \to Y'_{i-1}$ are Kawamata blowups for $i = 2, 3, 4$. Indeed, Z' is smooth and $Z' \to Y'_1$ is the economic resolution of Y'_1.

We compare $\natural'$ with $\natural$. Let $\bar{Z} \to Z$ be the blowup along a smooth curve obtained by subdivision along $(1, 1, 2)$. It follows that $Z' \dashrightarrow \bar{Z}$ is a simple flop. This can easily be seen as replacing the edge connecting $(1, 0, 0), (1, 2, 3)$ by an edge connecting $(1, 1, 1), (1, 1, 2)$ in the toric language.

Sketch of the proof of Theorem 3.7. Given a divisorial contraction $f : Y \to X \ni P$ to a smooth point P with exceptional divisor E. One can construct a similar sequence

$$Z = Y_n \to \ldots \to Y_1 \to X$$

as in $\natural$ as following:

(1) Let $Z_0 = P$ and let $Y_1 \to X$ is the blowup over $P = Z_0$.
(2) For $i \geq 1$, let Z_i be the center of E in Y_i. Let $Y_{i+1} \to Y_i$ be a resolution of the blowup of Y_i along Z_i if Z_i is not a divisor.

(3) The construction stops at Y_n when Z_n is a divisor. Let m be the largest integer so that Z_{m-1} is a point.

A key observation is that $E = Z_n$ equals, as valuations, to an exceptional divisor of a weighted blowup of weights $(1, m, n)$ if and only if $f_*\mathcal{O}_Y(-2E) \neq \mathfrak{m}$ and $f_*\mathcal{O}_Y(-nE) \not\subset \mathfrak{m}^2$ (cf. [14, Proposition 3.6]). Therefore, it is essential to study $f_*\mathcal{O}_Y(-jE)$.

Let

$$D(i) := \dim_{\mathbb{C}} \mathcal{O}_X / f_*\mathcal{O}_Y(-iE),$$

for $i \geq 0$. By using the singular Riemann–Roch formula, Kawakita obtains numerical constraint on $D(i)$ and also on type of singularities on Y. The classification can be found in [14, Theorem 4.5] and details can be found in [14, Section 4]. Then one can verity that $f_*\mathcal{O}_Y(-2E) \neq \mathfrak{m}$ and $f_*\mathcal{O}_Y(-nE) \not\subset \mathfrak{m}^2$ are satisfied. Q.E.D.

In [15], Kawakita made great progress along the line. With a lot of elaborated studies, he classified all divisorial contractions to points in some sense. We summarize his result. A divisorial contraction $f : Y \to X$ is said to be of ordinary type or exceptional type depending on the singularities of Y. For its precise definition, please see [15, p.59].

Theorem 3.9 ([15], Theorem 1.2). *Suppose that f is of ordinary type.*

(1) *If $P \in X$ is of type cA or cA/r, then there exists an identification realizing f as a weighted blowup with weight $\frac{1}{r}(r_1, r_2, a, 1)$ for some r_1, r_2. The discrepancy of f is $\frac{a}{r}$.*

(2) *If $P \in X$ is not of type cA nor cA/r and the discrepancy is $\frac{a}{r} > \frac{1}{r}$, then $P \in X$ is of type cD or $cD/2$ and f can be realized as a weighted blowup explicitly.*

Theorem 3.10 ([15], Theorem 1.3). *Suppose that f is of exceptional type. Then $P \in X$ is not of type cA nor cA/r. The discrepancy of f is $\frac{1}{r}$, except for the cases listed in* [15, Table 3].

Notice also that by Hayakawa's work [9], [10], [11], together with Kawakita's work [16], the following cases are known to be weighted blowups.

(1) $P \in x$ is a point of index $r > 1$ and discrepancy of f is $\frac{1}{r}$ (cf. [9], [10]).

(2) $P \in x$ is a point of index $r > 1$ and discrepancy of f is $\frac{r}{r}$ (cf. [11]).

(3) $P \in x$ is a point of type $cD/2$ and discrepancy of f is $\frac{4}{2}$ (cf. [16]).

As a consequence, one has the following:

Theorem 3.11 ([16]). *Any divisorial contraction $f : Y \to X$ to a point $P \in X$ of index $r > 1$ is a weighted blowup.*

We say that f "is" a weighted blowup means that there exists an embedding of the germ of $P \in X$ so that f is isomorphic to a weighted blowup of certain weights. Hayakawa started to work on a project to classify divisorial contractions to points of type cD, cE in [12], [13]. It is expected that such divisorial contractions are weighted blowups. In [14, Section 8], Kawakita gives some more examples of type e1, e2, e3, e9, which are weighted blowups. It is thus natural to ask the following

Question 3.12. Is every divisorial contraction a weighted blowup?

We give some more examples which are not known previously.

Example 3.13. Let $P \in X$ be defined as

$$\left\{ \begin{array}{l} \varphi_1 : x_1^2 + x_3^{2d+1} + x_4 x_5 = 0 \\ \varphi_2 : x_2^2 + x_4^{b-1} + x_3^{2d} x_4^2 - x_5 = 0 \end{array} \right\} \subset \mathbb{C}^5,$$

where $b \geq 8d+3$. Let $f : Y \to X$ be the weighted blowup of weights $v = (4d+2, 4d+1, 4, 1, 8d+3)$. One can check that P is terminal and f is a divisorial contraction with discrepancy 4 of type e1.

Example 3.14. Let $p \in X$ be defined as

$$\left\{ \begin{array}{l} \varphi_1 : x_1^2 + x_4 x_5 + x_2 x_3^{d+1} + x_3^a = 0 \\ \varphi_2 : x_2^2 + x_3^{2d+1} + x_4^{b-1} - x_5 = 0 \end{array} \right\} \subset \mathbb{C}^5,$$

where $a \geq 2d+2, b \geq 8d+5$. Let $Y \to X$ be the weighted blowup of weights $v = (4d+3, 4d+2, 4, 1, 8d+5)$. One can check that P is terminal and f is a divisorial contraction with discrepancy 4 of type e1.

§4. Resolution of terminal singularities

Given a germ of three-dimensional terminal singularity $P \in X$, it is expected that one can have a resolution by successive divisorial contractions. In [10], Hayakawa proved the following

Theorem 4.1. *For a terminal singularity $P \in X$ of index $r > 1$, there exists a partial resolution*

$$X_n \to \ldots \to X_1 \to X \ni P$$

such that X_n is Gorenstein and each $f_i : X_{i+1} \to X_i$ is a divisorial contraction to a point $P_i \in X_i$ of index $r_i > 1$ with minimal discrepancy $1/r_i$. All these maps f_i are weighted blowups.

It is natural to ask whether one can resolve Gorenstein terminal singularities in a similar manner.

Definition 4.1. Given a three-dimensional terminal singularity $P \in X$. We say that there exists a feasible resolution for $P \in X$ if there is a sequence

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X \ni P,$$

such that X_n is non-singular and each $X_{i+1} \to X_i$ is a divisorial contraction to a point with minimal discrepancy, i.e. a contraction to a point $P_i \in X_i$ of index r_i with discrepancy $1/r_i$.

In [3], we prove the existence of feasible resolution for any terminal singularity.

Theorem 4.2. *Given a three-dimensional terminal singularity $P \in X$. There exists a feasible resolution for $P \in X$.*

The proof is a straightforward but complicated inductive argument. The order of induction is as following: 1. quotient terminal singularities; 2. cA points; 3. cA/r points; 4. cD and $cAx/2$ poitns; 5. $cAx/4$, $cD/2$, and $cD/3$ points; 6. cE_6 points; 7. $cE/2$ points; 8. cE_7 points; 9. cE_8 points. For each singularity with given explicit local equation, we pick a convenient weighted blowup $Y \to X$ which is a divisorial contraction with minimal discrepancy. Keep track of singularities on Y, one finds that singularities upstairs are of milder type or of the same type but of smaller invariants. For details, please see [3].

§5. Divisorial contraction to a curve

In this section, we consider divisorial contraction $f : Y \to X$ contracting the exceptional divisor E to a curve $\Gamma \subset X$.

We first recall some well-known results of Mori, Cutkosky and Tziolas.

(1) If Y is smooth, then X is smooth (in the neighborhood of Γ) and f is the blowup along the smooth curve Γ (cf. [22]).
(2) If Y is Gorenstein, then X is smooth and f is the blowup along a locally complete intersection curve Γ (cf. [7]).
(3) Let $P \in \Gamma \subset X$ be a germ of Gorenstein threefold singularity and Γ a smooth curve. The general hyperplane section S containing Γ is Du Val of type A_n, D_n, E_6, E_7. Then such maps are classified. (cf. [28], [29], [30], [31]).

One of Tziolas approaches has the similar flavor as the 2-ray game. Let $\Gamma \subset X$ be a smooth curve that there is a singular point P of X lying on Γ. Take $f' : Y' \to X$ be the blowup over Γ. It produces singularities which is not $\mathbb{Q}$-factorial. Notice also that $f'^{-1}(\Gamma) = E_1 + mE_2$ for some $m \geq 2$, where $f'(E_1) = \Gamma$ and $f'(E_2) = P$. Let $Z' \to Y'$ be the $\mathbb{Q}$-factorialization of Y. Under certain conditions, one has a contraction $g : Z' \to Y$ contracting the proper transform of E_2 in Z and $f : Y \to X$ is a divisorial contraction to the curve Γ.

Another interesting result is a factorization of divisorial contractions into simpler birational maps (cf. [4]). The starting point is the following:

Theorem 5.1. [4, Theorem 3.1] *Let $g : Y \supset C \to X \ni R$ be a flipping contraction or a divisorial contraction contracting an irreducible curve C to a point $R \in X$. If Y is not Gorenstein, then there exists a divisorial contraction $g : Z \to Y$ to a point $Q \in Y$ of index r with minimal discrepancy $\frac{1}{r}$, such that $C_Z \cdot K_Z \leq 0$, where C_Z denotes the proper transform of C in Z.*

Therefore, one can play the 2-ray game and the run the minimal model program over X.

Theorem 5.2. [4, Theorem 3.3] *Let $f : Y \to X$ be a divisorial contraction to a curve Γ (resp. flipping map). If Y is not Gorenstein, then there is a diagram*

$$\begin{tikzcd}
Y_2 \arrow[rr, dashed] \arrow[d, "f_2"'] & & Y_2' \arrow[d, "f_2'"] \\
Y \arrow[dr, "f"'] & & Y' \arrow[dl, "f'"] \\
& X &
\end{tikzcd}$$

where $Y_2 \dashrightarrow Y_2'$ consists of flips and flops over X, f_2 is a divisorial contraction to a point $Q \in Y$ of index $r > 1$ with discrepancy $\frac{1}{r}$, f_2' is a divisorial contraction to a curve and f' is divisorial contraction to a point (resp. f_2' is a divisorial contraction and $Y' = Y^+$).

Recall from Theorem 4.1 that there is a partial resolution for terminal singularity of index $r > 1$ by a successive divisorial contractions over points of higher index with minimal discrepancies. In [4], the notion depth of Y, denoted $\operatorname{dep}(Y)$, is introduced as the minimal length of such partial resolution.

Proposition 5.3. [4, Proposition 2.15] *Let $f : Y \to X$ be a divisorial contraction to a point. Then* $\operatorname{dep}(X) \leq \operatorname{dep}(Y) + 1$.

By induction on $\operatorname{dep}(Y)$, one can prove the following facts on depth together with the factorization of flipping contractions divisorial contractions to curves.

Proposition 5.4. [4, Proposition 3.5, 3.6] *If $f : Y \to X$ is a divisorial contraction to a curve, then* $\operatorname{dep}(X) \leq \operatorname{dep}(Y)$. *If $f : X \dashrightarrow X^+$ is a flip, then* $\operatorname{dep}(X^+) < \operatorname{dep}(Y)$.

Theorem 5.5. [4, Theorem 1.1] *Let $g : X \to W$ be a $\mathbb{Q}$-factorial flipping contraction and $\phi : X \dashrightarrow X^+$ be the corresponding flip, then ϕ can be factored as*

$$X = X_0 \overset{f_0}{\dashrightarrow} X_1 \dashrightarrow \ldots \dashrightarrow X_n \overset{f_n}{\dashrightarrow} X^+,$$

such that each f_i is a flop, a blow-down to a LCI curve, a divisorial contraction to a point or the inverse of a divisorial contraction to a point od index $r > 1$ with minimal discrepancy $\frac{1}{r}$.

Let $g : X \to W$ be a $\mathbb{Q}$-factorial divisorial contraction to a curve, then g can be factored as

$$X = X_0 \overset{f_0}{\dashrightarrow} X_1 \dashrightarrow \ldots \dashrightarrow X_n \overset{f_n}{\dashrightarrow} W,$$

such that each f_i is a flop, a blow-down to a LCI curve, a divisorial contraction to a point or the inverse of a divisorial contraction to a point od index $r > 1$ with minimal discrepancy $\frac{1}{r}$.

Example 5.6. Let $\Gamma = (z = g(x, y) = 0) \subset \mathbb{C}^3 = X$ be a complete intersection curve which is singular at the origin. Let $f : Y \to X$ be blowup along Γ and let $\tau = wt_{(1,1)}g(x, y) \geq 2$. It is easy to easy that Y has only one singular point, which is given by $zu - g(x, y) = 0$ in local chart. This is a singularity of cA type.

In fact, we may consider an embedding $X \hookrightarrow X_0 \subset \mathbb{C}^4$ that $X_0 = (u - g(x, y)) = 0$ and $\Gamma = X_0 \cap Z$, where $Z = (z = u = 0)$. Let $Y_1 \to X_0$ be the weighted blowup with weights $(0, 0, 1, 1)$ and $Y_2 \to Y_1$ be the weighted blowup with vector $(1, 1, 1, \tau)$, i.e of weights $(1, 1, 1, \tau - 1)$ over Q_3. One sees that $Y_1 \to X_0$ is isomorphic to f, the blowup along Γ and $Y_2 \to Y_1$ is the weighted blowup over a singularity Q_3 of cA type of vector $(1, 1, 1, \tau)$.

On the other hand, we may consider $Y_2' \to Y_1' \to X_0$ by weighted blowup with vector $(1, 1, 1, \tau)$ and then $(0, 0, 1, 1)$. Then $Y_1' \to X_0$ is isomorphic to $Bl(X) \to X = \mathbb{C}^3$, and $Y_2' \to Y_1'$ is the blowup along a curve Γ_1' which is the proper transform of Γ. The equation of Γ_1' is given by $z = g(x, xy)x^{-\tau} = 0$ and $z = g(xy, y)y^{-\tau}$ in the chart U_1, U_2 respectively. It is clear that Γ_1' has milder singularities than that of Γ.

Moreover, one sees that $Y_2 \dashrightarrow Y_2'$ is isomorphic in codimension 1. Indeed, by the same trick as in [4], this fits into the diagram as in Theorem 5.2 and one has that $Y_2 \dashrightarrow Y_2'$ consists of a sequence of flips and flops.

Example 5.7. Let $P \in X$ be a germ of terminal quotient singularity of index $r \geq 2$. Let $C \subset X$ be any curve passing through P. Kawamata [18] shows that there is no divisorial contraction $f : Y \to X$ such that $f(E) = C$. Therefore, there are more restriction on the existence of divisorial contractions to a given curve.

Example 5.8. Let $P \in X$ be a cD singularity given by

$$(\varphi : x^2 + y^2 z + yz^2 + u^3 = 0) \subset \mathbb{C}^4$$

and $\Gamma = (x = y = u = 0)$.

By Tziolas's construction, we may start with a blowup $f = Bl_\Gamma : Y = Bl_\Gamma(X) \to X$. The exceptional set consists of E_1 and E_2 such that $f(E_1) = \Gamma$ and $f(E_2) = P$. In the chart U_4, the exceptional set $E_1 = (y = u = 0)$ and $E_2 = (z = u = 0)$.

Notice that Y can be realized as the proper transform of X in the weighted blowup $\mathcal{X}_1 \to \mathcal{X}_0$ of weights $v_1 = (1, 1, 0, 1)$.

Let $g : Z \to Y$ be the blowup of Y along E_1, which can be realized as weighted blowup of U_4 of weights $w_2 = (0, 1, 0, 1)$, which corresponds to the vector $v_2 = (1, 2, 0, 1)$. Therefore, these maps fit into the following diagram.

$$\begin{tikzcd}
Z \arrow[rr, dashed] \arrow[d, "g"'] & & Z' \arrow[d, "g'"] \\
U_4 \subset Y \arrow[dr, "f"'] & & Y' \supset U_2' \arrow[dl, "f'"] \\
& X &
\end{tikzcd}$$

Where $f' : Y' \to X$ is the weighted blowup with weight $v_2 = (1, 2, 0, 1)$ and $g' : Z' \to Y'$ is the weighted blowup with weight $\frac{1}{2}(1, 1, 0, 1)$ (with vector $v_1 = (1, 1, 0, 1)$) over U_2'. One can check that $Y' \to X$ is a divisorial contraction to a curve Γ. Notice that there is a singularity of index 2 which is Q_2 of type $cAx/2$ in Y' with the local equation

$$(x^2 + y^2 z + z^2 + u^3 y = 0) \subset \mathbb{C}^4 / \frac{1}{2}(1, 1, 0, 1).$$

Notice also that g' is the weighted blowup of weights $w_2 = \frac{1}{2}(1, 1, 0, 1)$ over U_2'.

Another way to realized this contraction is to follow the diagram in Theorem 5.2. Take a weighted blowup $Z^\star \to Y'$ of weights $w_3 = \frac{1}{2}(3,1,2,1)$ over $U_3' \subset Y'$, which is a divisorial contraction with minimal discrepancy $\frac{1}{2}$. This weights correspond to the vector $v_3 = (2,1,1,1)$. Let $Y'' \to X$ be the weighted blowup of weights v_3. One can check that the weighted blowup $Y'' \to X$ is a divisorial contraction with discrepancy 1. These maps fit into the following diagram.

$$\begin{tikzcd}
Z^\star \arrow[r, dashed] \arrow[d, "g^\star"', "wt=\frac{1}{2}(3,1,2,1)"] & Z'' \arrow[d, "g''"', "wt=(1,2,0,1)"] \\
U_3' \subset Y' \arrow[d, "f'"', "wt=(1,2,0,1)"] & Y'' \supset U_3'' \arrow[d, "f''"', "wt=(2,1,1,1)"] \\
X \arrow[r, "="] & X
\end{tikzcd}$$

One can check that U_3'' is given by $(x^2z + y^2 + y + u^3 = 0) \subset \mathbb{C}^4$ and $g'' : Z'' \to U_3''$ is isomorphic to $wBl_v(\mathbb{C}^3) \to \mathbb{C}^3$ with $v = (1,0,1)$. In other words, g'' is isomorphic to the blowup along a smooth curve.

§6. Factoring divisorial contractions with non-minimal discrepancy

As we have seen in the previous section, divisorial contractions with minimal discrepancies play a very interesting role. First of all, for any terminal singularity $P \in X$ of index $r > 1$, there exists a partial resolution $X_n \to \ldots \to X_0 := X$ such that X_n has only terminal Gorenstein singularities and each $X_{i+i} \to X_i$ is a divisorial contraction to a point with minimal discrepancy (cf. [10]). In fact, for any terminal singularity $P \in X$, there exists a feasible resolution by a sequence of divisorial contractions with minimal discrepancies.

Moreover, for any flipping contraction or divisorial contraction to a curve, by taking a divisorial extraction over the highest index point with minimal discrepancy, one gets a factorization into simpler birational maps. It is thus natural to ask whether one can factorize divisorial contractions to points with non-minimal discrepancies into simpler ones. In [2], we work on the factorization of divisorial contraction to a point of index $r > 1$ with discrepancy $\frac{a}{r} > \frac{1}{r}$.

Theorem 6.1. *Let $f : Y \to X$ be a divisorial contraction to a point $P \in X$ of index $r > 1$ with discrepancy $\frac{a}{r} > \frac{1}{r}$. Let $E \subset Y$ be the exceptional divisor and and $g : Z \to Y$ be a divisorial contraction over a* *point of highest index p in $E \subset Y$ with discrepancy $\frac{1}{p}$. Then the relative canonical divisor $-K_{Z/X}$ is nef.*

Notice that the relative Picard number $\rho(Z/X) = 2$. Therefore, we are able to play the so called 2-ray game. As a consequence, there is a flip or flop $Z \dashrightarrow Z^+$. By running the minimal model program of Z^+/X, we have $Z \dashrightarrow Z' \xrightarrow{g'} Y' \xrightarrow{f'} X$, where $Z \dashrightarrow Z'$ consists of a sequence of flips and flops, $Z' \to Y'$ is a divisorial contraction. This can be summarize into the following diagram.

$$\begin{tikzcd}
Z \arrow[rr, dashed] \arrow[d, "g"'] & & Z' \arrow[d, "g'"] \\
Y \arrow[dr, "f"'] & & Y' \arrow[dl, "f'"] \\
& X &
\end{tikzcd}$$

We have the following more precise description.

Theorem 6.2. *Let $f : Y \to X \ni P$ be a divisorial contraction to a point $P \in X$ of index r with discrepancy $\frac{a}{r} > \frac{1}{r}$. Keep the notation as in the above diagram. We have that f' is a divisorial contraction to $P \in X$ with discrepancy $\frac{a'}{r} < \frac{a}{r}$. Moreover, g' is a divisorial contraction to a singular point $Q' \in Y'$ and exactly one of the following holds.*

(1) *If $P \in X$ is of type other than $cE/2$, then Q' is a point of index r, and g' has discrepancy $\frac{a''}{r}$ with $a' + a'' = a$.*
(2) *If $P \in X$ is of type $cE/2$, then Q' is a point of index 3, and g' has minimal discrepancy $\frac{1}{3}$.*

As an immediate corollary by induction on discrepancy a, we have:

Corollary 6.3. *For any divisorial contraction $Y \to X$ to a point $P \in X$ of index $r > 1$ with discrepancy $\frac{a}{r} > \frac{1}{r}$. There exists a sequence of birational maps*

$$Y =: X_n \dashrightarrow \ldots \dashrightarrow X_0 =: X$$

such that each map $X_{i+1} \dashrightarrow X_i$ is one of the following:

(1) *a divisorial extraction over a point of index $r_i > 1$ with minimal discrepancy $\frac{1}{r_i}$ or its inverse;*
(2) *a flip or a flop.*

Example 6.4. We consider a divisorial contraction over a cA/r point with discrepancy $\frac{a}{r} > \frac{1}{r}$. This case is described in [15, Theorem 1.1.i], and its local equation is given by

$$\varphi : x_1x_2 + g(x_3^r, x_4) = 0 \subset \mathbb{C}^4/v,$$

where $v = \frac{1}{r}(1, -1, b, 0)$.

The map f is given by weighted blowup with weight $v_1 = \frac{1}{r}(r_1, r_2, a, r)$. We may write $r_1 + r_2 = dar$ for some $d > 0$ with the term $x_3^{dr} \in \varphi$. We also have that $s_1 := \frac{a-br_1}{r}$ is relatively prime to r_1 and $s_2 := \frac{a+br_2}{r}$ is relatively prime to r_2 (cf. [15, Lemma 6.6]). We thus have the following:

$$\begin{cases} a = br_1 + rs_1, \\ 1 = q_1r_1 + s_1^*s_1, \\ a = -br_2 + rs_2, \\ 1 = q_2r_2 + s_2^*s_2, \end{cases}$$

for some $0 \le s_i^* < r_i$ and some q_i.

We set

$$\delta_1 := -nq_1 + bs_1^*, \quad \delta_2 := -nq_2 - bs_2^*.$$

One sees easily that

$$\begin{cases} \delta_1r_1 + r = as_1^*, \\ \delta_2r_2 + r = as_2^*. \end{cases}$$

It is easy to see that $a > \delta_i \neq 0$ for $i = 1, 2$ and $\delta_i > 0$ for some i. One can also check that if both $\delta_1, \delta_2 > 0$ and $(a, r_1) = 1$, then $\delta_1 + \delta_2 = a$.

Case 1. Suppose that $\delta_1 > 0$.
Since Q_1 which is a quotient singularity of type $\frac{1}{r_1}(r_1 - s_1^*, 1, s_1^*)$. Let $g = wBl_{wt=w_2} : Z \to Y$ be Kawamata blowup over Q_1 with $w_2 = \frac{1}{r_1}(r_1 - s_1^*, dr, 1, s_1^*)$.

The diagram ‡ is as following.

$$\begin{tikzcd} Z \arrow[r, dashed] \arrow[d, "\frac{1}{r_1}"', "wt=w_2"] & Z' \arrow[d, "\frac{\delta_1}{n}"', "wt=w_2'"] \\ Q_1 \in Y \arrow[d, "\frac{a}{n}"', "wt=w_1"] & Y' \ni Q_4' \arrow[d, "\frac{a-\delta_1}{n}"', "wt=w_1'"] \\ X \arrow[r, "="] & X \end{tikzcd}$$

Where

$$\begin{array}{ll} w_1 = \frac{1}{r}(r_1, r_2, a, r), & w_1' = \frac{1}{r}(r_1 - s_1^*, r_2 - \delta_1 dr + s_1^*, a - \delta_1, r); \\ w_2 = \frac{1}{r_1}(r_1 - s_1^*, dr, 1, s_1^*), & w_2' = \frac{1}{r}(s_1^*, \delta_1 dr - s_1^*, \delta_1, r). \end{array}$$

Note that $0 < a' := a - \delta_1 < a$ and both f', g' are extremal contractions with discrepancies $< \frac{a}{r}$.
Case 2. Suppose that $\delta_2 > 0$.
Since Q_2 is a quotient singularity of type $\frac{1}{r_2}(r_2 - s_2^*, 1, s_2^*)$, we take $g = wBl_{wt=w_2} : Z \to Y$ the Kawamata blowup over Q_2 with $w_2 = \frac{1}{r_2}(dr, r_2 - s_2^*, 1, s_2^*)$.

The diagram ‡ is as following.

$$\begin{tikzcd}
Z \arrow[r, dashed] \arrow[d, "\frac{1}{r_1}"', "wt=w_2"] & Z' \arrow[d, "\frac{\delta_2}{r}"', "wt=w_2'"] \\
Q_2 \in Y \arrow[d, "\frac{a}{r}"', "wt=w_1"] & Y' \ni Q_4' \arrow[d, "\frac{a-\delta_2}{r}"', "wt=w_1'"] \\
X \arrow[r, "="] & X
\end{tikzcd}$$

Where

$$\begin{aligned}
w_1 &= \tfrac{1}{r_1}(r_1, r_2, a, r), & w_1' &= \tfrac{1}{r}(r_1 + s_2^* - \delta_2 dr, r_2 - s_2^*, a - \delta_2, r);\\
w_2 &= \tfrac{1}{r_2}(dr, r_2 - s_2^*, 1, s_2^*), & w_2' &= \tfrac{1}{r}(\delta_2 dr - s_2^*, s_2^*, \delta_2, r).
\end{aligned}$$

It is easy to see that if $r_1 \geq r_2$, then $\delta_1 > 0$. Hence extracting over Q_1 provides the desired factorization. Similar argument holds if $r_2 \geq r_1$. Therefore, one obtain a factorization by extracting over the point of highest index.

§7. Further remarks

It is interesting and useful to find a set of simple and explicit birational maps so that each birational maps can be factored into a composition of these simple maps. According to our discussion above, one can expected that a divisorial contraction to a point, to a curve, or a flip can be factored into a sequence of birational maps such that each map is one of the following:

(1) a divisorial contraction to a point with minimal discrepancies (or its inverse);
(2) a divisorial contraction to a curve which is the blowup over a smooth curve in a smooth threefold;
(3) a flop.

The reader might also find that the technique and results of factorizations are very similar to that of Sarkisov's program (cf. [5]). It would be very interesting if there exists a unified program which realizes the factorization of birational maps together with Sarkisov's program.

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