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Weighted estimate of Stokes semigroup in unbounded domains

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Abstract.

We show $L^p - L^q$ estimates with the $\langle x \rangle^s$ type weight of Stokes semigroup in exterior domains and perturbed half-spaces. Moreover as application of these estimates, we obtain weighted estimates for global solution to the Navier–Stokes equations with small data in these domains.

§1. Introduction

Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be an exterior domain or a perturbed half-space with smooth boundary. We consider the following Navier–Stokes equations in these unbounded domains $\Omega \subset \mathbb{R}^n$:

$$\text{(NS)} \qquad \begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla \pi = 0 & \text{in } (0,\infty) \times \Omega, \\ \nabla \cdot u = 0 & \text{in } (0,\infty) \times \Omega, \\ \lim\limits_{|x| \to \infty} u(t,x) = 0, \quad u(t,x) = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ u(0,x) = a(x) & \text{in } \Omega. \end{cases}$$

Here $u(t,x) = (u_1(t,x), \ldots, u_n(t,x))$ and $\pi(t,x)$ denote unknown velocity field and scalar pressure and $a(x)$ is a given vector function.

Navier–Stokes equations (NS) have been already studied by many authors in some bounded domains and unbounded domains in L^p framework and weighted L^p framework. For example the following results are known: L^p space and L^p space with Muckenhoupt weight have Helmholtz decompostion in several unbounded domains ([1],[4],[5], [7],[13]). Moreover the Stokes operator generates an analytic semigroup in their

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solenoidal space. $L^p - L^q$ estimate of Stokes semigroup, which plays the important role when we analyze the Navier–Stokes equations, have been proved in L^p space and several domains (see [15] for example). On the other hand, in weighted L^p space, Bae [1] proved the certain $L^p - L^q$ estimate of Stokes semigroup with $w(x) = (1+|x|)^s$ type weight in exterior domains.

In this paper, we shall show the $L^p - L^q$ estimate with $w(x) = \langle x\rangle^s = (1+|x|^2)^{s/2}$ type weight of Stokes semigroup in some unbounded domains and its application to the Navier–Stokes equations.

§2. Main theorems and notations

In this paper, we shall consider the Navier–Stokes equations in exterior domains and perturbed half-spaces. For this purpose, we first introduce the definition of these domains.

Definition 1. Let $B_R = \{x \in \Omega \mid |x| < R\}$ and $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n \mid x_n > 0\}$.
(i) Ω is called an exterior domain if there exists $R > 0$ such that

$$\Omega \backslash B_R = \mathbb{R}^n \backslash B_R. \tag{1}$$

(ii) Ω is called a perturbed half-space if there exists $R > 0$ such that

$$\Omega \backslash B_R = \mathbb{R}^n_+ \backslash B_R. \tag{2}$$

In what follows, let Ω be an exterior domain or a perturbed half-space with smooth boundary. We next introduce the known results on the weighted L^q space. The weighted L^q space is defined as follows:

$$L^q_w(\Omega) = \left\{ u \in L^1_{\mathrm{loc}}(\overline{\Omega}) \mid \|u\|_{L^q_w} = \|uw^{\frac{1}{q}}\|_{L^q} = \left(\int_\Omega |u|^q w dx\right)^{\frac{1}{q}} < \infty \right\}$$

for $1 < q < \infty$. We set $w(x) = \langle x\rangle^s$ as the weight functions in this paper. If $-n < s < n(q-1)$, it is known that $w(x)$ belongs to Muckenhoupt class $\mathcal{A}_q$ (see [4] for definition of Muckenhoupt class). For $1 < q < \infty$ and $w \in \mathcal{A}_q$, Banach space $L^q_w(\Omega)$ has Helmholtz decomposition: $L^q_w(\Omega) = L^q_{w,\sigma}(\Omega) \oplus \nabla \dot{H}^{1,q}_w(\Omega)$, where

$$L^q_{w,\sigma}(\Omega) := \overline{\{u \in C_0^\infty(\Omega) \mid \nabla \cdot u = 0\}}^{L^q_w(\Omega)},$$
$$\dot{H}^{1,q}_w(\Omega) := \{\pi \in L^1_{\mathrm{loc}}(\overline{\Omega}) \mid \nabla \pi \in L^q_w(\Omega)\}.$$

(see [4] for exterior domains and [13] for perturbed half-spaces). By Helmholtz decomposition for $L^q_w(\Omega)$, we see that the Helmholtz projection P: $L^q_w(\Omega) \to L^q_{w,\sigma}(\Omega)$ is bounded. Moreover we know that the Stokes operator $-A$ generates an analytic semigroup e^{-tA} in $L^q_{w,\sigma}(\Omega)$.

Main results are the following two theorem. First theorem says the weighted $L^p - L^q$ estimate for small t. Second theorem gives the weighted $L^p - L^q$ estimate for large t.

Theorem 1. *Let $n \geq 2$ and $1 < p \leq q < \infty$. Let $-\frac{n}{q} < s < n(1-\frac{1}{p})$ if Ω is an exterior domain and let $-\frac{n-1}{q} < s < (n-1)(1-\frac{1}{p})$ if Ω is a perturbed half-space. Moreover let $w(x) = \langle x \rangle^{sp}$. Then for $a \in L^p_w(\Omega)$, we have*

$$(3) \qquad \|\langle x \rangle^s \nabla^\alpha e^{-tA} Pa\|_{L^q} \leq Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{|\alpha|}{2}} \|\langle x \rangle^s a\|_{L^p}$$

for $0 < t < 2$ and $|\alpha| = 0, 1$.

Theorem 2. (i) *Let $n \geq 3$ and let Ω be an exterior domain. For $1 < p \leq q < \infty$, let $-\frac{n}{q} < s' \leq s < n(1-\frac{1}{p})$ and $s \geq 0$ and let $w = \langle x \rangle^{sp}$. For $a \in L^p_w(\Omega)$, the following estimate holds:*

$$(4) \qquad \|\langle x \rangle^{s'} \nabla^\alpha e^{-tA} Pa\|_{L^q} \leq Ct^{-\sigma_E} \|\langle x \rangle^s a\|_{L^p}$$

for $t > 2$ and $|\alpha| = 0, 1$, where σ_E is defined by

$$(5) \qquad \sigma_E = \min\left[\frac{n}{2p}, \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{|\alpha|}{2} + \frac{s - s'}{2}\right].$$

(ii) *Let $n \geq 2$ and let Ω be a perturbed half-space. For $1 < p \leq q < \infty$, let $-\frac{n-1}{q} < s' \leq s < (n-1)(1-\frac{1}{p})$ and $s \geq 0$ and let $w = \langle x \rangle^{sp}$. For $a \in L^p_w(\Omega)$, the following estimate holds:*

$$(6) \qquad \|\langle x \rangle^{s'} \nabla^\alpha e^{-tA} Pa\|_{L^q} \leq Ct^{-\sigma_P} \|\langle x \rangle^s a\|_{L^p}$$

for $t > 2$ and $|\alpha| = 0, 1$, where σ_P is defined by

$$(7) \qquad \sigma_P = \min\left[\frac{n}{2p} + \frac{1}{2}, \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{|\alpha|}{2} + \frac{s - s'}{2}\right].$$

Remark 2. For the proof, we refer [14]. Our proof is based on cut-off techinique with local energy decay estimate obtained by Iwashita [11] and Kubo and Shibata [15]. We remark that the estimate for $\Omega_R = \Omega \cap B_R$ implies the restriction on the decay rate in (4) and (6).

We next consider the application of the weighted $L^p - L^q$ estimates to the Navier–Stokes equations. Following Kato's argument [12], we shall prove the unique existence of global solution to (NS) with small initial data. By applying the Helmholtz projection P to (NS), we can rewrite (NS) as follows:

$$\text{(P-NS)} \qquad \partial_t u + Au + P[(u\cdot\nabla)u] = 0, \quad u(0) = a.$$

By Duhamel's principle, we obtain the integral equation:

$$\text{(IE)} \qquad u(t) = e^{-tA}a - \int_0^t e^{-(t-\tau)A}P[(u\cdot\nabla)u](\tau)d\tau.$$

By using the weighted $L^p - L^q$ estimates and constructing mapping principle, we can show the following theorem:

Theorem 3. (i) *Let $n \geq 3$ and Ω be an exterior domain. Let $0 \leq s < n-1$ and let $w = \langle x\rangle^{sn}$. Then there exists $\delta > 0$ such that if $a \in L^n_{w,\sigma}(\Omega)$ satisfied $\|\langle x\rangle^s a\|_{L^n} \leq \delta$, then* (IE) *admits a unique solution $u \in BC([0,\infty) : L^n_{w,\sigma}(\Omega))$ satisfying*

$$\begin{aligned}
&\|\langle x\rangle^s u(t)\|_{L^q} \leq Ct^{-\frac{1}{2}+\frac{n}{2q}} && \text{for } n \leq q < \infty,\\
&\|\langle x\rangle^s \nabla u(t)\|_{L^q} \leq Ct^{-\frac{1}{2}} && \text{for } n \leq q < \infty,\\
&\|u(t)\|_{L^q} \leq Ct^{-\sigma_E} && \text{for } n \leq q < \infty,\\
&\|\nabla u(t)\|_{L^q} \leq Ct^{-\frac{1}{2}} && \text{for } n \leq q < \infty
\end{aligned}$$

for $t > 2$, where $\sigma_E = \min[\frac{1}{2}, \frac{1}{2} - \frac{n}{2p} + \frac{s}{2}]$.
(ii) *Let $n \geq 2$ and Ω be a perturbed half-space. Let $0 \leq s < n-2+1/n$ and let $w = \langle x\rangle^{sn}$. Then there exists $\delta > 0$ such that if $a \in L^n_{w,\sigma}(\Omega)$ satisfied $\|\langle x\rangle^s a\|_{L^n} \leq \delta$, then* (IE) *admits a unique solution $u \in BC([0,\infty) : L^n_{w,\sigma}(\Omega))$ satisfying*

$$\begin{aligned}
&\|\langle x\rangle^s u(t)\|_{L^q} \leq Ct^{-\frac{1}{2}+\frac{n}{2q}} && \text{for } n \leq q < \infty,\\
&\|\langle x\rangle^s \nabla u(t)\|_{L^q} \leq Ct^{-1+\frac{n}{2q}} && \text{for } n \leq q < \infty,\\
&\|u(t)\|_{L^q} \leq Ct^{-\sigma_{P1}} && \text{for } n \leq q < \infty,\\
&\|\nabla u(t)\|_{L^q} \leq Ct^{-\sigma_{P2}} && \text{for } n \leq q < \infty
\end{aligned}$$

for $t > 2$, where $\sigma_{P1} = \min[\frac{1}{2} - \frac{n}{2p} + \frac{s}{2}, 1]$ and $\sigma_{P2} = \min[1 - \frac{n}{2p} + \frac{s}{2}, 1]$.

Remark 3. Weighted estimates for global solution to the Navier–Stokes equations with small data are recently studied by Bae and Roh [2] and so on. In [2], they considered exterior domains $\Omega \subset \mathbb{R}^n (n = 2, 3)$

case and they proved the following asymptotic behavieor as $t \to \infty$: if $a \in L^n_\sigma \cap L^r$ for $1 < r < n$ and $|x|^\alpha a \in L^r$ for $0 < \alpha < n$

$$\||x|^\alpha u(t)\|_{L^p(\Omega)} = O(t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})+\frac{\alpha}{2}})$$

for $nr/(n - r\alpha) < p$. Advantage of our results is to have the decay rate without $\alpha/2$ in the weighted asymptotic behavior as $t \to \infty$.

§3. Outline of proof

Theorem 1 is proved by real interpolation theory and resolvent estimate (see [13] for detail) and Theorem 3 is proved by standard argument (Kato's method) with Theorem 2. In this paper, we shall focus on proof of Theorem 2.

We shall first show exterior domains case. Our proof is based on the cut-off technique. To this end, we have to obtain certain estimate on Ω_R and $\mathbb{R}^n$. The estimate on Ω_R has been already proved by Iwashita [11]:

Lemma 4 (Iwashita [11]). *Let $n \geq 3$, $1 < p < \infty$ and Let Ω be an exterior domain. Let R_0 be a number satisfied* (1). *Then there exists positive number C such that*

$$\|\partial_t e^{-tA} f\|_{W^{1,p}(\Omega_R)} + \|e^{-tA} f\|_{W^{1,p}(\Omega_R)} \leq C t^{-\frac{n}{2p}} \|f\|_{L^p(\Omega)}$$

for $R > R_0$, $t \geq 2$ and $f \in L^p_\sigma(\Omega)$.

Therefore main task of our proof is the estimate on $\mathbb{R}^n$, namely, to show the following weighed $L^p - L^q$ estimate on $\mathbb{R}^n$,

Theorem 4. *Let $n \geq 2$ and $1 < p \leq q < \infty$. Let $-\frac{n}{q} < s' \leq s < n(1 - \frac{1}{p})$ and let $w = \langle x \rangle^{sp}$. Then the following estimate holds*:

$$(8) \qquad \|\langle x \rangle^{s'} \nabla^\alpha e^{-tA} Pa\|_{L^q} \leq C(1+t)^{-\frac{s-s'}{2}} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{|\alpha|}{2}} \|\langle x \rangle^s a\|_{L^p}$$

for $a \in L^p_w(\mathbb{R}^n)$, $t > 0$ and $|\alpha| = 0, 1$.

Proof. Here we shall prove $0 \leq s' \leq s < n(1 - 1/q)$ and $|\alpha| = 0$ case (see [13] for general case). We first introduce some properties and inequality which used in this proof.

$$(9) \qquad \|\langle x \rangle^{-\alpha}\|_{L^{n/\alpha,\infty}} \leq C$$

$$(10) \quad \|f * g\|_{L^r} \leq \|f\|_{L^{q,\infty}} \|g\|_{L^p}, \quad 1 < p, r < \infty, \qquad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q},$$

$$(11) \quad \|fg\|_{L^{r,\infty}} \leq \|f\|_{L^{q,\infty}} \|g\|_{L^p}, \qquad 1 < p, q, r < \infty, \qquad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

We shall prove (8) by using (9)–(11). Taking $\langle x\rangle^{s'} \leq \langle x-y\rangle^{s'} + \langle y\rangle^{s'}$ into account, we have

$$
\begin{aligned}
|\langle x\rangle^{s'} e^{-tA} Pa| \leq \frac{C}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \langle x-y\rangle^{s'} e^{-|x-y|^2/4t} Pa(y)dy \\
+ \frac{C}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \langle y\rangle^{s'} e^{-|x-y|^2/4t} Pa(y)dy \\
=: G_1 * Pa + G_2 * Pa
\end{aligned}
$$

We first consider the term G_1. By Young's inequality, we have

$$
\|G_1 * Pa\|_{L^q} \leq \|G_1\|_{L^r} \|Pa\|_{L^{\frac{np}{sp+n}}} \leq \|G_1\|_{L^r} \|\langle x\rangle^{-s}\|_{L^{\frac{n}{s},\infty}} \|\langle x\rangle^s Pa\|_{L^p}
$$

for $1+1/q = 1/r + (sp+n)/np = 1/r + 1/p + s/n$. Since G_1 is estimated by $|G_1(t,x-y)| \leq Ct^{-\frac{n}{2}+\frac{s'}{2}} e^{-\frac{|x-y|^2}{8t}}$, we obtain $\|G_1\|_{L^r} \leq Ct^{-\frac{n}{2}(1-\frac{1}{r})+\frac{s'}{2}}$ for $1 < r < \infty$. By using (9), we see

$$
\|G_1 * Pa\|_{L^q} \leq Ct^{-\sigma - \frac{s-s'}{2}} \|\langle x\rangle^s Pa\|_{L^p} \tag{12}
$$

with $\sigma = \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$.

We next consider the second term G_2. By Young's inequality for L^p space, we have

$$
\|G_2 * Pa\|_{L^q} \leq \|G_2\|_{L^r} \|\langle x\rangle^{-(s-s')}\|_{L^{\frac{n}{s-s'},\infty}} \|\langle x\rangle^s Pa\|_{L^p}
$$

for $1 + 1/q = 1/r + (s-s')/n + 1/p$. By (9), we see

$$
\|G_2 * Pa\|_{L^q} \leq Ct^{-\frac{n}{2}(1-\frac{1}{r})} \|\langle x\rangle^s Pa\|_{L^p} = Ct^{-\sigma-\frac{s-s'}{2}} \|\langle x\rangle^s Pa\|_{L^p}.
$$

Summing up, we obtain the desired result. Q.E.D.

By using the cut-off technique with Theorem 4 and the result of Iwashita [11], we can prove Theorm 2 for exterior domains case (see [13] for detail).

Finally we consider proof of Theorem 2 for perturbed half-spaces. In this case, we need the estimate on Ω_R and $\mathbb{R}^n_+$. The following estimate on Ω_R has been already proved by Kubo and Shibata [15]:

Lemma 5 (Kubo and Shibata [15]). *Let $n \geq 2$, $1 < p < \infty$ and Let Ω be a perturbed half-space. Let R_0 be a number satisfied* (2). *Then there exists positive number C such that*

$$
\|\partial_t e^{-tA} f\|_{W^{1,p}(\Omega_R)} + \|e^{-tA} f\|_{W^{1,p}(\Omega_R)} \leq Ct^{-\frac{n}{2p}-\frac{1}{2}} \|f\|_{L^p(\Omega)}
$$

for $R > R_0$, $t \geq 2$ and $f \in L^p_\sigma(\Omega)$.

Therefore we have to consider the weighted $L^p - L^q$ estimate on $\mathbb{R}^n_+$. For this purpose, we use the solution formula obtained by Ukai [17]: let R_j and S_j be the Riesz transform and the partial Riesz transform defined as follows:

$$(13) \qquad R_j f(x) := \mathcal{F}_\xi^{-1}\left[\frac{i\xi_j}{|\xi|}\mathcal{F}_x[f](\xi)\right] \qquad j = 1, \ldots, n,$$

$$(14) \qquad S_j f(x) := \mathcal{F}_{\xi'}^{-1}\left[\frac{i\xi_j}{|\xi'|}\mathcal{F}_{x'}[f](\xi', x_n)\right] \qquad j = 1, \ldots, n-1.$$

And let $\gamma f = f|_{\mathbb{R}^n_+}$ and e_0 zero extension operator from $\mathbb{R}^n_+$ to $\mathbb{R}^n$. Finally, let $E(t)$ be the solution operator for the heat equation in $\mathbb{R}^n_+$, which is derived from $E_0(t)$ by odd extension from $\mathbb{R}^n_+$ to $\mathbb{R}^n$. Then the solution $(u(t), \pi(t))$ of the non-stationary Stokes equations in $\mathbb{R}^n_+$ is

$$u(t) = WE(t)Va, \qquad \pi(t) = -D\gamma\partial_n E(t)V_1 a,$$

where

$$W = \begin{pmatrix} I & -SU \\ 0 & U \end{pmatrix}, \qquad V = \begin{pmatrix} V_2 \\ V_1 \end{pmatrix}$$

with $S = {}^t(S_1, \ldots, S_{n-1})$ and $R' = {}^t(R_1, \ldots, R_{n-1})$,

$$U = \gamma R' \cdot S(R' \cdot S + R_n)e_0, \quad V_1 a = -S \cdot a' + a^n, \quad V_2 a = a' + Sa^n$$

and D is the Poisson operator for the Dirichlet problem of the Laplace equation in $\mathbb{R}^n_+$. Taking the fact that R_j and S_j is bounded operator on $L^q_w(\mathbb{R}^n)$ and $L^q_w(\mathbb{R}^{n-1})$ to themselves respectively into account, we can reduce to the whole space case, so that we can obtain the following estimate for the half-space:

Theorem 5. *Let $n \geq 2$, $1 < p \leq q < \infty$, $-\frac{n-1}{q} < s' \leq s < (n-1)(1 - \frac{1}{p})$ and let $w = \langle x\rangle^{sp}$. Then the following estimate holds:*

$$(15) \qquad \|\langle x\rangle^{s'}\nabla^\alpha e^{-tA}Pa\|_{L^q} \leq C(1+t)^{-\frac{s-s'}{2}} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{|\alpha|}{2}}\|\langle x\rangle^s a\|_{L^p}$$

for $a \in L^p_w(\Omega)$, $t > 0$ and $|\alpha| = 0, 1$.

Remark 6. The restriction on the weight $-(n-1)/q < s < (n-1)(1 - 1/q)$ comes from the weighted L^p boundedness of $S_j(x)$. In fact, S_j is bounded operator on $L^q_w(\mathbb{R}^{n-1})$ to themself. Therefore the weight $w(x) = \langle x\rangle^s$ considered for fixed x_n as weight in $\mathbb{R}^{n-1}$ is in the class $\mathcal{A}_q$ only if $-(n-1)/q < s < (n-1)(1 - 1/q)$.

By using Theorem 5 with Local energy decay estimate proved by Kubo and Shibata [15], we can prove Theorem 2 for perturbed half-space in a similar way to exterior domains case.

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