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Wave front set defined by wave packet transform and its application

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Abstract.

We introduce the wave front set $WF_s^{p,q}$ by using the wave packet transform. This is another characterization of the Fourier–Lebesgue type wave front set $WF_{\mathcal{F}L_s^q}$. We apply this to the propagation of singularities for the wave equation.

§1. Introduction

In this talk, we introduce the wave front set $WF_s^{p,q}$ (Definition 1.1) by using the wave packet transform.

The wave packet transform has been introduced by Córdoba–Fefferman [1]. For $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\phi(0) \neq 0$, the wave packet transform $W_\phi u$ is defined by

$$W_\phi u(x,\xi) = \int_{\mathbb{R}^n} \overline{\phi(y-x)} u(y) e^{-iy\cdot\xi} dy, \tag{1}$$

which has the information of frequency of u around x.

Definition 1.1. *Let $1 \le p, q \le \infty$, $s \in \mathbb{R}$, $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Then $(x_0, \xi_0) \notin WF_s^{p,q}(u)$ means that there exists a neighborhood K of x_0, a conic neighborhood Γ of ξ_0 and a function $\phi \in C_0^\infty(\mathbb{R})$ with $\phi(0) \neq 0$ satisfying that*

$$\| \| \chi_K(x) \chi_\Gamma(\xi) \langle \xi \rangle^s W_\phi u(x,\xi) \|_{L_x^p} \|_{L_\xi^q} < \infty, \tag{2}$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$, χ_K and χ_Γ are characteristic functions of K and Γ, respectively.

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As an application of $WF_s^{p,q}$, we give the following theorem on propagation of singularities.

Theorem 1. *Let* $1 \leq p, q \leq \infty$ *and* $r \in \mathbb{R}$. *Suppose that* $u \in C(\mathbb{R}; \mathcal{S}'(\mathbb{R}^n))$ *satisfies*

$$\begin{cases} (\partial_t \pm i|D|)u(t,x) = 0, & (t,x) \in \mathbb{R}^{n+1}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{3}$$

where $i = \sqrt{-1}$ *and* $|D| = \mathcal{F}^{-1}|\xi|\mathcal{F}$. *If* $(x_0, \xi_0) \notin WF_r^{p,q}(u_0)$ *then* $(x_0 \pm \frac{\xi_0}{|\xi_0|}t, \xi_0) \notin WF_r^{p,q}(u(t,\cdot))$ *for all* $t \in \mathbb{R}$.

We briefly review some background on the wave front sets and propagation of singularities. The notion of wave front set, introduced by Hörmander [3] is a main tool of microlocal analysis. There are many kind of wave front sets. For example, C^∞ type, analytic type, Sobolev type, Fourier–Lebesgue type and so on (see Hörmander [4], Sato–Kawai–Kashiwara [8], Pilipović–Teofanov–Toft [6]). Here, we focus on the Fourier–Lebesgue type wave front sets. For $1 \leq q \leq \infty$ and $s \in \mathbb{R}$, the Fourier–Lebesgue space $\mathcal{F}L_s^q(\mathbb{R}^n)$ is the set of all distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $\widehat{u}(\xi) = \int_{\mathbb{R}^n} u(x)e^{-ix\cdot\xi}$ is a function and $\|\langle\xi\rangle^s\widehat{u}(\xi)\|_{L_\xi^q}$. We note that $\mathcal{F}L_s^2(\mathbb{R}^n)$ is the sobolev space $H^s(\mathbb{R}^n)$. While, the Fourier–Lebesgue type wave front set $WF_{\mathcal{F}L_s^q}(u)$ defined by [6] is defined as follows. For $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$, $(x_0, \xi_0) \notin WF_{\mathcal{F}L_s^q}(u)$ means that there exist a conic neighborhood Γ of ξ_0 and a function $a \in C_0^\infty(\mathbb{R}^n)$ with $a(x_0) \neq 0$ satisfying that

$$\|\chi_\Gamma(\xi)\langle\xi\rangle^s\widehat{au}(\xi)\|_{L_\xi^q} < \infty. \tag{4}$$

We note that $WF_{\mathcal{F}L_s^2}$ is the Sobolev type wave front set WF_{H^s}. Although a considerable number of studies have been done on the propagation of singularity in the framework of Sobolev type wave front set (see Beals [2]), a few works have been done in the framework of Fourier–Lebesgue type wave front set ([6], [7]).

In Theorem 2, we show $WF_s^{p,q}$ coincides with $WF_{\mathcal{F}L_s^q}$. Thus, using Theorem 1 and Theorem 2, we obtain the result concerning the propagation of singularity in the framework of the Fourier–Lebesgue type wave front set.

Theorem 2. *For* $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$ *and* $u \in \mathcal{S}'(\mathbb{R}^n)$, *we have*

$$WF_s^{p,q}(u) = WF_{\mathcal{F}L_s^q}(u). \tag{5}$$

Notation. For $x \in \mathbb{R}^n$ and $r > 0$, $B_r(x)$ stands $\{y \in \mathbb{R}^n; |y-x| \le r\}$. $\mathcal{F}[f](\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx$ is the Fourier transform of f. For a subset A of $\mathbb{R}^n$, we denote the complement of A by A^c, the set of all interior points of A by A° and the closure of A by $\overline{A}$. Throughout this paper, C and C_i $(i = 1, 2, 3, \ldots)$ serve as positive constants, if the precise value of which is not needed and C_N denote positive constants depending on N.

§2. Sketch of the proof of Theorem 2

To show Theorem 2 we use the following lemma.

Lemma 1. (*Kato–Kobayashi–Ito* [5]) *Let ζ be a measurable function on $\mathbb{R}^n$ such that $\langle\cdot\rangle^k\zeta \in L^1(\mathbb{R}^n)$ for all $k \in \mathbb{R}$, $F \in \mathcal{S}'(\mathbb{R}^n)$, $1 \le q \le \infty$, and Γ, Γ' be open conic sets satisfying $\overline{\Gamma'} \subset \Gamma \subset \mathbb{R}^n$. Assume that $\|\chi_\Gamma(\xi)\langle\xi\rangle^s F(\xi)\|_{L^q_\xi} < \infty$ and $\|\langle\xi\rangle^{-N}F(\xi)\|_{L^q_\xi} < \infty$ for some $s \in \mathbb{R}$ and $N \in \mathbb{N}$. Then we have*

$$\|\chi_{\Gamma'}(\xi)\langle\xi\rangle^s(\zeta * F)(\xi)\|_{L^q_\xi} \le C_{s,N,\zeta}\left(\|\chi_\Gamma(\xi)\langle\xi\rangle^s F(\xi)\|_{L^q_\xi} + \left\|\frac{F(\xi)}{\langle\xi\rangle^N}\right\|_{L^q_\xi}\right)$$

for some positive constant $C_{s,N,\zeta}$.

Suppose that $(x_0, \xi_0) \notin WF_{\mathcal{F}L^q_s}(u)$. Then there exist a conic neighborhood Γ of ξ_0 and a function $a \in C_0^\infty(\mathbb{R}^n)$ with $a(x_0) \ne 0$ satisfying $\|\chi_\Gamma(\xi)\langle\xi\rangle^s\widehat{au}(\xi)\|_{L^q_\xi} < \infty$. For $r > 0$ and $b \in C_0^\infty(\mathbb{R}^n)$ satisfying $\operatorname{supp} b \subset B_{4r}(x_0) \subset \operatorname{supp} a$ and $b \equiv 1$ in $B_{2r}(x_0)$, simple calculation yields $\|\chi_\Gamma(\xi)\langle\xi\rangle^s\widehat{bu}(\xi)\|_{L^q_\xi} < \infty$. Take a neighborhood K of x_0 and a function $\phi \in C_0^\infty(\mathbb{R}^n)$ satisfying $K \subset B_r(x_0)$, $\phi(0) \ne 0$ and $\operatorname{supp}\phi \subset B_r(0)$. Note that $x \in K$ and $y - x \in B_r(0)$ imply $y \in B_{2r}(x_0)$. So $\chi_K(x)\overline{\phi(y-x)}u(y) = \chi_K(x)\overline{\phi(y-x)}b(y)u(y)$. Let Γ' be a conic neighborhood of ξ_0 such that $\overline{\Gamma'} \subset \Gamma$. Since $W_\phi(bu)(x,\xi) = \mathcal{F}[\overline{\phi(\cdot - x)}] * \mathcal{F}[bu](\xi)$ we have by Lemma 1

$$\begin{aligned}&\|\|\chi_K(x)\chi_{\Gamma'}(\xi)\langle\xi\rangle^s W_\phi u(x,\xi)\|_{L^p_x}\|_{L^q_\xi}\\ &\qquad\le C_{s,N,\phi,K}\left(\left\|\chi_\Gamma(\xi)\langle\xi\rangle^s\widehat{bu}(\xi)\right\|_{L^q_\xi} + \left\|\frac{\widehat{bu}(\xi)}{\langle\xi\rangle^N}\right\|_{L^q_\xi}\right).\end{aligned}$$

Since $|\widehat{bu}(\xi)|$ has at most polynomial growth we obtain $(x_0, \xi_0) \notin WF^{p,q}_s$ if we take an integer N sufficiently large.

Conversely, if $(x_0, \xi_0) \notin WF^{p,q}_s$ then we can choose Γ being a conic neighborhood of ξ_0, $R \in \mathbb{R}$ and $\phi \in C_0^\infty(\mathbb{R}^n)$ which satisfy $\phi \equiv 1$ in

$B_{2R}(0)$ and $\|\|\chi_{B_R(x_0)}(x)\chi_\Gamma(\xi)\langle\xi\rangle^s W_\phi u(x,\xi)\|_{L^p_x}\|_{L^q_\xi} < \infty$. Put $K = B_R(x_0)$ and take $a \in C_0^\infty(\mathbb{R}^n)$ satisfying $a(x_0) \neq 0$ and $\operatorname{supp} a \subset B_R(x_0)$. Since $\phi(y-x) \equiv 1$ for $x \in K$ and $y \in \operatorname{supp} a$, we have $\chi_K(x)\widehat{au}(\xi) = \chi_K(x)\int_{\mathbb{R}^n} \widehat{a}(\xi-\eta)W_\phi(x,\eta)d\eta$. So we have by Lemma 1

$$\begin{aligned}
&\|\chi_K(x)\|_{L^p_x}\|\chi_{\Gamma'}(\xi)\langle\xi\rangle^s\widehat{au}(\xi)\|_{L^q_\xi} \\
&\qquad \leq C_{s,N,a}\Bigg(\|\|\chi_K(x)\chi_\Gamma(\xi)\langle\xi\rangle^s W_\phi u(x,\xi)\|_{L^p_x}\|_{L^q_\xi} \\
&\qquad\qquad + \left\|\frac{1}{\langle\xi\rangle^N}\|\chi_K(x)W_\phi u(x,\xi)\|_{L^p_x}\right\|_{L^q_\xi}\Bigg)
\end{aligned}$$

for a conic neighborhood Γ' of ξ_0 satisfying $\overline{\Gamma'} \subset \Gamma$. Since χ_K has compact support and $|W_\phi u(x,\xi)|$ is majored by a constant times $\langle\xi\rangle^{N_0}$ for sufficiently large N_0, we obtain $(x_0,\xi_0) \notin WF_{\mathcal{F}L^q_s}(u)$ if we take an integer $N > N_0$ sufficiently large.

§3. Sketch of the proof of Theorem 1

In the sequel, for a function $f(t,x)$ on $\mathbb{R}\times\mathbb{R}^n$, we denote $\widehat{f}(t,\xi) = \int_{\mathbb{R}^n} f(t,x)e^{-ix\cdot\xi}dx$ and $W_\phi f(t,x,\xi) = W_\phi(f(t,\cdot))(x,\xi)$. Here, we only treat the initial value problem

$$(6)\qquad \begin{cases} (\partial_t - i|D|)u(t,x) = 0, & (t,x) \in \mathbb{R}^{n+1}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

since we can treat the case $(\partial_t + i|D|)u(t,x) = 0$ in the same way. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\phi(0) \neq 0$. The initial value problem (6) is transformed by the wave packet transform to

$$(7)\qquad \begin{cases} \left(\partial_t - \dfrac{\xi}{|\xi|}\cdot\nabla_x - i|\xi|\right)W_\phi u(t,x,\xi) = iR_\phi(u;t,x,\xi), \\ W_\phi u(0,x,\xi) = W_\phi u_0(x,\xi), \end{cases}$$

where $đ\eta = (2\pi)^{-n}d\eta$ and

$$R_\phi(u;t,x,\xi) = \iint_{\mathbb{R}^{2n}} \overline{\phi(y-x)}\left(|\eta| - \frac{\xi\cdot\eta}{|\xi|}\right)\widehat{u}(t,\eta)e^{iy\cdot(\eta-\xi)}đ\eta dy.$$

It is easy to see that (7) is equivalent to the integral equation

$$
\begin{aligned}
(8)\quad W_\phi u(t,x,\xi) &= e^{it|\xi|}W_\phi u_0\left(x+\frac{\xi}{|\xi|}t,\xi\right)\\
&\quad + i\int_0^t e^{i(t-\theta)|\xi|}R_\phi\left(u;\theta,x+\frac{\xi}{|\xi|}(t-\theta),\xi\right)d\theta.
\end{aligned}
$$

Let $T>0$. For $t\in[-T,T]$, we show $(x_0-\frac{\xi_0}{|\xi_0|}t,\xi_0)\notin WF^{p,q}_r(u(t,\cdot))$ by induction.

Since $u(t,\cdot)\in\mathcal{S}'(\mathbb{R}^n)$, there exists $s\in\mathbb{R}$ satisfying $\|\langle\cdot\rangle^s\widehat{au}(t,\cdot)\|_{L^q}<\infty$ for all $a\in C_0^\infty(\mathbb{R}^n)$ and $t\in[-T,T]$. Thus we have $(x_0-\frac{\xi_0}{|\xi_0|}t,\xi_0)\notin WF^{p,q}_s(u(t,\cdot))$ for all $t\in[-T,T]$ by Theorem 2.

Next we show $(x_0-\frac{\xi_0}{|\xi_0|}t,\xi_0)\notin WF^{p,q}_{\sigma+1}(u(t,\cdot))$ for all $t\in[-T,T]$ and $s\le\sigma\le r-1$ under the assumption $(x_0-\frac{\xi_0}{|\xi_0|}t,\xi_0)\notin WF^{p,q}_\sigma(u(t,\cdot))$ for all $t\in[-T,T]$. Let K be a neighborhood of $x_0-\frac{\xi_0}{|\xi_0|}t$, Γ be a conic neighborhood of ξ_0 and $\widetilde{\Gamma}=\Gamma\cap\{|\xi|\ge1\}$. From the equation (8), it is enough to show that

$$
(9)\quad I^{(1)}_{K,\widetilde{\Gamma},\phi}\equiv\left\|\left\|\chi_K(x)\chi_{\widetilde{\Gamma}}(\xi)\langle\xi\rangle^\sigma|\xi|W_\phi u_0\left(x+\frac{\xi}{|\xi|}t,\xi\right)\right\|_{L^p_x}\right\|_{L^q_\xi}<\infty,
$$

$$
\begin{aligned}
(10)\quad I^{(2)}_{K,\widetilde{\Gamma},\phi,\psi}\equiv\Bigg\|\Bigg\|&\chi_K(x)\chi_{\widetilde{\Gamma}}(\xi)\langle\xi\rangle^\sigma|\xi|\\
&\times\int_0^t\left|R_\phi\left(\psi u;\theta,x+\frac{\xi}{|\xi|}(t-\theta),\xi\right)\right|d\theta\Bigg\|_{L^p_x}\Bigg\|_{L^q_\xi}<\infty
\end{aligned}
$$

and

$$
\begin{aligned}
(11)\quad I^{(3)}_{K,\widetilde{\Gamma},\phi,\psi}\equiv\Bigg\|\Bigg\|&\chi_K(x)\chi_{\widetilde{\Gamma}}(\xi)\langle\xi\rangle^\sigma|\xi|\\
&\times\int_0^t\left|R_\phi\left((1-\psi)u;\theta,x+\frac{\xi}{|\xi|}(t-\theta),\xi\right)\right|d\theta\Bigg\|_{L^p_x}\Bigg\|_{L^q_\xi}<\infty
\end{aligned}
$$

for some $\psi\in C_0^\infty(\mathbb{R}^n)$ and all $t\in[-T,T]$. From the assumption $(x_0,\xi_0)\notin WF^{p,q}_s(u_0)$, there exist a constant $\varepsilon>0$, a function $\phi_1\in C_0^\infty(\mathbb{R}^n)$ with $\phi_1(0)\ne0$ and a conic neighborhood Γ' of ξ_0 such that $\|\|\chi_{B_{2\varepsilon}(x_0)}(x)\chi_{\Gamma'}(\xi)\langle\xi\rangle^r W_{\phi_1}u_0(x,\xi)\|_{L^p_x}\|_{L^q_\xi}<\infty$. Let $K_1=B_\varepsilon(x_0-\frac{\xi_0}{|\xi_0|}t)$ and Γ_1 be a conic neighborhood of ξ_0 satisfying $\varepsilon T^{-1}>d_1=$

$\sup_{\xi\in\Gamma_1}\mathrm{dist}(\frac{\xi}{|\xi|},\frac{\xi_0}{|\xi_0|})$ and $\overline{\Gamma}_1\subset\Gamma'$. If $x\in K_1$ and $\xi\in\Gamma_1$ then $x+\frac{\xi}{|\xi|}t\in B_{2\varepsilon}(x_0)$. Thus we have

$$I^{(1)}_{K_1,\widetilde{\Gamma}_1,\phi_1}\le\left\|\left\|\chi_{B_{2\varepsilon}(x_0)}(x)\chi_{\Gamma'}(\xi)\langle\xi\rangle^rW_{\phi_1}u_0(x,\xi)\right\|_{L^p_x}\right\|_{L^q_\xi}<\infty,$$

where $\widetilde{\Gamma}_1=\Gamma_1\cap\{|\xi|\ge1\}$

Next we show (10). By the assumption of induction and Theorem 2 we can take a conic neighborhood Γ'' of ξ_0 and $\psi_t\in C_0^\infty(\mathbb{R}^n)$ such that $\psi_t\equiv1$ near $x_0-\frac{\xi_0}{|\xi_0|}t$ and $\|\chi_{\Gamma''}(\xi)\langle\xi\rangle^\sigma\widehat{\psi_tu}(t,\xi)\|_{L^q_\xi}<\infty$ for all $t\in[-T,T]$. Take $\varepsilon'>0$ satisfying $\psi_t\equiv1$ on $B_{6\varepsilon'}(x_0-\frac{\xi_0}{|\xi_0|}t)$. Let $\phi_2\in C_0^\infty(\mathbb{R}^n)$ with $\phi_2(0)\ne0$ and $\operatorname{supp}\phi_2\subset B_{2\varepsilon'}(0)$, $K_2=B_{\varepsilon'}(x_0-\frac{\xi_0}{|\xi_0|}t)$ and Γ_2 be a conic neighborhood of ξ_0 satisfying $\overline{\Gamma}_2\subset\Gamma''$ and $\varepsilon'T^{-1}>d_2=\sup_{\xi\in\Gamma_2}\mathrm{dist}(\frac{\xi}{|\xi|},\frac{\xi_0}{|\xi_0|})$. Put $\widetilde{\Gamma}_2=\Gamma_2\cap\{|\xi|\ge1\}$. By integration by parts and an inequality

$$\left(|\eta|-\frac{\xi\cdot\eta}{|\xi|}\right)\langle\eta-\xi\rangle^{-2}\le\frac{|\xi||\eta|-\xi\cdot\eta}{|\xi|(2|\xi||\eta|-2\xi\cdot\eta)}=\frac{1}{2|\xi|},$$

we have

$$\begin{aligned}I^{(2)}_{K_2,\widetilde{\Gamma}_2,\phi_2,\psi_\theta}&\le C_{K_2,\phi_2}\int_0^T\left\|\int_{\mathbb{R}^n}\frac{\chi_{\widetilde{\Gamma}_2}(\xi)\langle\xi\rangle^\sigma}{\langle\eta-\xi\rangle^{2N}}|\widehat{\psi_\theta u}(\theta,\eta)|đ\eta\right\|_{L^q_\xi}d\theta\\&\le C_{K_2,\phi_2}(J_{\Gamma''}+J_{(\Gamma'')^c}),\end{aligned}$$

where $J_A=\int_0^T\|\int_A\chi_{\widetilde{\Gamma}_2}(\xi)\langle\xi\rangle^\sigma\langle\eta-\xi\rangle^{-2N}|\widehat{\psi_\theta u}(\theta,\eta)|d\eta\|_{L^q_\xi}d\theta$ and $N\in\mathbb{N}$. Since $\langle\xi\rangle\le2\langle\eta-\xi\rangle$ or $\langle\xi\rangle\le2\langle\eta\rangle$ hold, we have

$$\frac{\langle\xi\rangle^\sigma}{\langle\eta-\xi\rangle^{2N}\langle\eta\rangle^\sigma}\le\frac{C}{\langle\eta-\xi\rangle^{2N-|\sigma|}}\tag{12}$$

for $2N>|\sigma|$. Thus if we take an integer N sufficiently large, then Young's inequality, (12) and the assumption of induction yield

$$J_{\Gamma''}\le C\left\|\frac{1}{\langle\cdot\rangle^{2N-|\sigma|}}\right\|_{L^1}\int_0^T\left\|\chi_{\Gamma''}(\xi)\langle\xi\rangle^\sigma\widehat{\psi_\theta u}(\theta,\xi)\right\|_{L^q_\xi}d\theta<\infty.$$

On the other hand, if $\eta\notin\Gamma''$, $\xi\in\widetilde{\Gamma}_2$ and $2N>|\sigma|$ then we have

$$\frac{\langle\xi\rangle^\sigma}{\langle\eta-\xi\rangle^{2N}}\le\frac{C}{\langle\eta-\xi\rangle^{2N-|\sigma|}}\le\frac{C}{\langle\eta-\xi\rangle^{N_1}\langle\eta\rangle^{N_2}},\tag{13}$$

where $N_1 + N_2 = 2N - |\sigma|$. Since $|\widehat{\psi_\theta u}(\theta,\xi)|$ has at most polynomial growth with respect to ξ, Young's inequality and (13) yield

$$J_{(\Gamma'')^c} \le C \left\| \frac{1}{\langle\cdot\rangle^{N_1}} \right\|_{L^1} \int_0^T \left\| \frac{\widehat{\psi_\theta u}(\theta,\xi)}{\langle\xi\rangle^{N_2}} \right\|_{L^q_\xi} d\theta < \infty,$$

if we take N_1 and N_2 sufficiently large. Thus we have $I^{(2)}_{K_2,\widetilde{\Gamma}_2,\phi_2,\psi_\theta} < \infty$.

Finally we show (11). Let $\zeta_1 \in C^\infty(\mathbb{R}^n)$ equal to 0 for $|\eta| \le 1$ and equal to 1 for $|\eta| \ge 2$ and put $\zeta_2(\eta) = 1 - \zeta_1(\eta)$. It suffices to show that

$$I^{(3)}_{K_2,\widetilde{\Gamma}_2,\phi_2,\psi_\theta} \le \sum_{j=1,2} \left\| \left\| \chi_{K_2}(x)\chi_{\widetilde{\Gamma}_2}(\xi)\langle\xi\rangle^\sigma |\xi| \int_0^t |R_j| d\theta \right\|_{L^p_x} \right\|_{L^q_\xi} < \infty,$$

where

$$R_j = \lim_{h_1,h_2 \to 0} \iiint_{\mathbb{R}^{3n}} \overline{\phi_2\Big(y - x - \frac{\xi}{|\xi|}(t-\theta)\Big)} \Big(|\eta| - \frac{\xi\cdot\eta}{|\xi|}\Big) b(h_1\eta)\zeta_j(\eta) \times (1 - \psi_\theta(\tilde{x}))u(\theta,\tilde{x})b(h_2\tilde{x})e^{-i(\tilde{x}\cdot\eta - y\cdot\eta + y\cdot\xi)} d\tilde{x}\,đ\eta\,dy$$

for $b \in \mathcal{S}(\mathbb{R}^n)$ with $b(0) = 1$. From the structure theorem of $\mathcal{S}'(\mathbb{R}^n)$, there exist $l, m \ge 0$ and $f_\alpha(\theta,\cdot) \in L^2(\mathbb{R}^n)$ for multi-indices α such that

$$u(\theta,\tilde{x}) = \langle\tilde{x}\rangle^l \sum_{|\alpha| \le m} D^\alpha f_\alpha(\theta,\tilde{x}). \tag{14}$$

We note that $x \in K_2$, $\xi \in \widetilde{\Gamma}_2$, $y - x - (t-\theta)\xi/|\xi| \in \operatorname{supp}\phi_2$ and $\tilde{x} \in \operatorname{supp}(1 - \psi_\theta(\tilde{x}))$ imply $|\tilde{x} - y| \ge \varepsilon' > 0$ and, hence, $|\tilde{x} - y| \ge C\langle\tilde{x}\rangle$. Since

$$e^{-i(\tilde{x}-y)\cdot\eta} = \frac{(-\Delta_\eta)^{N_3} e^{-i(\tilde{x}-y)\cdot\eta}}{|\tilde{x}-y|^{2N_3}} \quad \text{and} \quad e^{iy\cdot(\eta-\xi)} = \frac{(1-\Delta_y)^{N_4} e^{iy\cdot(\eta-\xi)}}{\langle\eta-\xi\rangle^{2N_4}}$$

for positive integers N_3 and N_4, (14) and integration by parts imply

$$|R_1| \le C \int_{\mathbb{R}^n} \frac{\|\widehat{f}_\alpha(\theta,\cdot)\|_{L^2}}{\langle\eta\rangle^{2N_3-1-|\alpha|}\langle\xi-\eta\rangle^{2N_4}} đ\eta.$$

On the other hand, since $\zeta_2 \in C_0^\infty(\mathbb{R}^n)$ we have

$$(1-\Delta_\eta)^N \left\{ \Big(\eta - \frac{\eta\cdot\xi}{|\xi|}\Big)\zeta_2(\eta) \right\} \le \frac{C}{\langle\eta\rangle^{2N-1}}. \tag{15}$$

Since

$$e^{-i(\tilde{x}-y)\cdot\eta} = \frac{(1-\Delta_\eta)^{N_3} e^{-i(\tilde{x}-y)\cdot\eta}}{\langle \tilde{x}-y\rangle^{2N_3}} \quad \text{and} \quad e^{iy\cdot(\eta-\xi)} = \frac{(1-\Delta_y)^{N_4} e^{iy\cdot(\eta-\xi)}}{\langle \eta-\xi\rangle^{2N_4}}$$

for positive integers N_3 and N_4, (14), (15) and integration by parts imply

$$|R_2| \le C \int_{\mathbb{R}^n} \frac{\|\widehat{f}_\alpha(\theta,\cdot)\|_{L^2}}{\langle\eta\rangle^{2N_3-1-|\alpha|}\langle\xi-\eta\rangle^{2N_4}} \, đ\eta.$$

Since

$$\frac{\langle\xi\rangle^\sigma|\xi|}{\langle\eta\rangle^{2N_3-1-|\alpha|}\langle\xi-\eta\rangle^{2N_4}} \le \frac{C}{\langle\eta\rangle^{2N_3-2-|\alpha|-\sigma}\langle\xi-\eta\rangle^{2N_4-\sigma-1}}$$

for $N_3 \ge (2+|\alpha|+\sigma)/2$ and $N_4 \ge (\sigma+1)/2$, we have by Young's inequality

$$\left\|\int_{\mathbb{R}^n} \frac{\langle\xi\rangle^\sigma|\xi|}{\langle\eta\rangle^{2N_3-1-|\alpha|}\langle\xi-\eta\rangle^{2N_4}} \, đ\eta\right\|_{L^q_\xi} \le \left\|\frac{1}{\langle\cdot\rangle^{2N_3-2-|\alpha|-\sigma}}\right\|_{L^1} \left\|\frac{1}{\langle\cdot\rangle^{2N_4-\sigma-1}}\right\|_{L^q}.$$

Thus if we take N_3 and N_4 sufficiently large, we obtain

$$I^{(3)}_{K_2,\tilde{\Gamma}_2,\phi_2,\psi_\theta} \le C_{K_2,N_3,N_4} \int_0^T \left\|\widehat{f}_\alpha(\theta,\cdot)\right\|_{L^2} d\theta < \infty.$$

Hence we get the inequality (11). Taking $K \subset K_1 \cap K_2$, $\Gamma \subset \Gamma_1 \cap \Gamma_2$ and $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\phi(0) \ne 0$ and $\operatorname{supp}\phi \subset \operatorname{supp}\phi_1 \cap \operatorname{supp}\phi_2$, we obtain $(x_0 - \xi_0 t/|\xi_0|, \xi_0) \notin WF^q_{\sigma+1}(u)$ for $t \in [-T,T]$. Since T is an arbitrary positive number, we obtain the desired result. Q.E.D.

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