Advanced Studies in Pure Mathematics 64, 2015
Nonlinear Dynamics in Partial Differential Equations
pp. 151–162

Long time behavior of solutions to non-linear Schrödinger equations with higher order dispersion

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Abstract.

We give an asymptotic formulas in time of solutions to the linear Schrödinger equation with third and fourth order dispersions in one space dimension. Then we apply those formulas to drive the long time behavior of solution to the third and the fourth order Schrödinger type equations with the cubic nonlinearity.

§1. Introduction

We consider the long time behavior of solutions to some class of nonlinear dispersive equations. In this note, we focus on the construction of the solution to the non-linear Schrödinger equation with third order dispersion

$$(1)\quad \begin{cases} i\partial_t v + \partial_x^2 v + i\nu\partial_x^3 v = -\dfrac{1}{2}|v|^2 v - \dfrac{3}{2}i\nu|v|^2\partial_x v, \\ \hfill t > 0, x \in \mathbb{R}, \\ \lim\limits_{t\to+\infty}(v(t) - v_{ap}(t)) = 0 \quad in\ L^2(\mathbb{R}), \end{cases}$$

and the non-linear Schrödinger equation with fourth order dispersion

$$(2)\qquad \begin{cases} i\partial_t v + \partial_x^2 v + \mu\partial_x^4 v = \lambda|v|^2 v, \qquad t > 0, x \in \mathbb{R}, \\ \lim\limits_{t\to+\infty}(v(t) - v_{ap}(t)) = 0 \quad in\ L^2(\mathbb{R}), \end{cases}$$

where ν, μ and λ are non-zero constants, $v : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is an unknown function, and $v_{ap} : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is a given function.

Received December 1, 2011.
Revised February 3, 2012.
2010 *Mathematics Subject Classification.* 35Q55.
Key words and phrases. Higher order Schrödinger type equation, asymptotic behavior.

Those equations arise in the context of the motion of an isolated vortex filament embedded in an inviscid incompressible fluid filling an infinite region. More precisely, taking into account the effect from higher order corrections of equation, Fukumoto–Miyazaki [3] and Fukumoto–Moffatt [4] proposed (1) and

$$
\begin{aligned}
(3) \qquad & i\partial_t v + \partial_x^2 v + \nu \partial_x^4 v \\
&= \lambda_1 |v|^2 v + \lambda_2 |v|^4 v + \lambda_3 (\partial_x v)^2 \overline{v} + \lambda_4 |\partial_x v|^2 v \\
&\quad + \lambda_5 v^2 \partial_x^2 \overline{v} + \lambda_6 |v|^2 \partial_x^2 v, \qquad t > 0, x \in \mathbb{R}.
\end{aligned}
$$

Hence we can regard (2) as the simplified model of the Fukumoto–Moffatt equation (3). The equation (2) was also introduced by Karpman [9] to study the influence of higher order dispersion on stability of solitary waves and collapse phenomenon.

There exist several results on the well-posedness issues for the initial value problems (1), (2) and (3). Laurey [11] proved the global well-posedness of the initial value problem for (1) in Sobolev space $H^1(\mathbb{R})$. Her proof is based on the contraction mapping principle via the integral equation and the Kato local smoothing effect. For the Cauchy problem (2), the global well-posedness in $L^2(\mathbb{R})$ easily follows from the contraction principle with the Strichartz estimate. Concerning the initial value problem (3), the author [14], [15] and Huo–Jia [7], [8] proved that (3) is locally well-posed in H^s with $s > 1/2$ by applying the contraction principle in Bourgain's Fourier restriction space. Especially, when (3) is completely integrable ((3) with $\lambda_1 = -1/2$, $\lambda_2 = -3\nu/8, \lambda_3 = -3\nu/2$, $\lambda_4 = -\nu, \lambda_5 = -\nu/2$ and $\lambda_6 = -2\nu$), combining the local existence theory with the conservation laws and the Gagliardo–Nirenberg inequality, we have that (3) is globally well-posed in H^1.

There is a several literature on the long time behavior of the solutions to non-linear Schrödinger equations with the higher order dispersion in more general setting. See for instance, [1], [2], [5], [6], [13] and reference therein.

Roughly speaking, the temporal behavior of solutions of the nonlinear dispersive equations are determined by the balance between the dispersion and the nonlinearity. If the dispersion is dominant, then the solution to non-linear equation will converge to the solution of the linearized equation. Conversely, if the nonlinear effect is not negligible for long time, then the corresponding solution will not be approximated by the solution to the linear equation.

According to the L^∞ decay of solutions to the linearized equations and the linear scattering theory, it seems that the solutions to (1) and (2) do not converge to the solutions to their linearized equations as $t \to +\infty$. Therefore we have to take care to choose u_{ap}.

To simplify the equations (1) and (2), we make the change of variables. Putting $u(t,x) = \exp(-i\frac{t}{81\nu^3} - i\frac{x}{3\nu})v(\frac{t}{3\nu}, x + \frac{t}{9\nu^2})$ in (1), (1) is reduced to

$$(4) \qquad \begin{cases} \partial_t u + \frac{1}{3}\partial_x^3 u = -\frac{1}{2}|u|^2\partial_x u, & t > 0, x \in \mathbb{R}, \\ \lim_{(\mathrm{sgn}\nu)t \to +\infty} (u(t) - u_{ap}(t)) = 0 & in\ L^2(\mathbb{R}). \end{cases}$$

We put $u(t,x) = v(\frac{9}{2}|\nu|t, \sqrt{6|\nu|}x)$ in (2). Then (2) is rewritten as

$$(5) \qquad \begin{cases} i\partial_t u + \frac{3}{4}\partial_x^2 u + \frac{\mathrm{sgn}\mu}{8}\partial_x^4 u = \tilde{\lambda}|u|^2 u, & t > 0, x \in \mathbb{R}, \\ \lim_{t \to +\infty} (u(t) - u_{ap}(t)) = 0 \quad in\ L^2(\mathbb{R}), \end{cases}$$

where $\tilde{\lambda} = (9/2)|\nu|\lambda$.

Henceforth, we concentrate our attention to construct the solutions to the final state problems (4) and (5). We only consider (4) with $\mathrm{sgn}\nu = +1$ since the case $\mathrm{sgn}\nu = -1$ being similar. For (5), we only consider the case $\mathrm{sgn}\mu = -1$. Because, as we shall explain below, to study the long time behavior of solution to (5) with $\mathrm{sgn}\mu = +1$ is increasingly harder than the case $sgn\mu = +1$ even for the linear estimate.

Outline of this paper is as follows. In Section 2, we prepare the linear estimates needed to prove the existence of solutions to (4) and (5). In Section 3, we state our main results (see Theorem 1 and Theorem 2 below) and guarantee those theorems by using the fixed point theorem.

§2. Linear estimate

In this section, to prove the final state problems (4) and (5), we derive the asymptotic formulas for the solutions to the linearized equations associated to those equations:

$$(6) \qquad \begin{cases} \partial_t u + \frac{1}{3}\partial_x^3 u = 0, & t > 0, x \in \mathbb{R}, \\ u(0,x) = \phi(x), & x \in \mathbb{R}, \end{cases}$$

and

$$(7) \qquad \begin{cases} i\partial_t u + \frac{3}{4}\partial_x^2 u + \frac{sgn\mu}{8}\partial_x^4 u = 0, & t > 0, x \in \mathbb{R}, \\ u(0,x) = \phi(x), & x \in \mathbb{R}. \end{cases}$$

We drive the linear estimates for (6) and (7) by using the method of stationary phase. For the reader's convenience, we explain this method in the general framework. We consider the following dispersive equation

$$(8) \qquad \begin{cases} -i\partial_t u + P(-i\partial_x)u = 0, & t > 0, x \in \mathbb{R}, \\ u(0,x) = \phi(x) & x \in \mathbb{R}, \end{cases}$$

where $P(-i\partial_x)$ is the differential operator defined by $\widehat{P(-i\partial_x)u}(\xi) = P(\xi)\hat{u}(\xi)$with a symbol $P(\xi)$ being real polynomial. Then the solution to (8) is given by the oscillatory integral

$$(9) \qquad u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ix\xi - itP(\xi)} \hat{\phi}(\xi) d\xi.$$

Let $\{\chi_j\}_{j=1}^N$ be a stationary points of (9) which is solutions $\xi \in \mathbb{R}$ to the equation $x - t\frac{dP}{d\xi}(\xi) = 0$. We assume that $\hat{\phi}$ is supported in a sufficiently small neighborhood of the stationary points. Then the solution to (8) has the asymptotic formula

$$\begin{aligned} (10)\ u(t,x) &= t^{-1/2} \sum_{j=1}^{N} \underbrace{\frac{\hat{\phi}(\chi_j)}{\sqrt{|P''(\chi_j)|}}}_{Amplitude} \\ &\qquad \times \exp(i\underbrace{(t\chi_j P'(\chi_j) - tP(\chi_j) + \frac{\pi}{4} sgnP''(\chi_j))}_{Phase}) \\ &\quad +O(t^{-3/2}) \end{aligned}$$

as $t \to +\infty$, see for instance Stein [19, Section VIII].

Applying the formula (10) to (6) and (7), we obtain the following asymptotic formulas. The solution to (6) satisfies

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{2}} t^{-1/2} \mathbf{1}_{\{x<0\}} \sum_{\pm} |\chi_\pm|^{-1/2} \hat{\phi}(\chi_\pm) \\ &\qquad \times \exp\left(i\left(-\frac{2}{3}t\chi_\pm^3 \pm \frac{\pi}{4}\right)\right) + O(t^{-3/2}) \end{aligned}$$

as $t \to +\infty$, where $\chi_\pm = \pm\sqrt{-x/t}$. For the solution to (7) with $\mathrm{sgn}\nu = -1$, we have

$$u(t,x) = \sqrt{\frac{2}{3}} t^{-1/2} \frac{\hat{\phi}(\chi)}{\sqrt{\chi^2+1}} \exp\left(i\left(t\frac{3}{8}\chi^4 + t\frac{3}{4}\chi^2 - \frac{\pi}{4}\right)\right) + O(t^{-3/2})$$

as $t \to +\infty$, where $\chi = |\Lambda_+|^{-2/3}\Lambda_+ + |\Lambda_-|^{-2/3}\Lambda_-$ with $\Lambda_\pm = t^{-1}(x \pm \sqrt{x^2+t^2})$, and for the solution to (7) with $\mathrm{sgn}\nu = +1$,

$$
\begin{aligned}
u(t,x) \;&=\; t^{-1/2}\mathbf{1}_{\{|x|<t\}}\sum_{j=1}^{3}\sqrt{\frac{2}{3}}\frac{\hat{\phi}(\mu_j)}{\sqrt{(-1)^j(1-\mu_j^2)}} \\
&\qquad \times \exp\left(i\left(-\frac{3}{8}t\chi_j^4+\frac{3}{4}t\chi_j^2-(-1)^j\frac{\pi}{4}\right)\right) \\
&\quad + t^{-1/2}\mathbf{1}_{\{|x|>t\}}\sqrt{\frac{2}{3}}\frac{\hat{\phi}(\mu_4)}{\sqrt{\mu_4^2-1}} \\
&\qquad \times \exp\left(i\left(-\frac{3}{8}it\chi_4^4+\frac{3}{4}t\chi_4^2+\frac{\pi}{4}\right)\right) + O(t^{-3/2})
\end{aligned}
$$

as $t \to \infty$, where $\chi_1 = \cos(\frac{\theta}{3}+\frac{2\pi}{3})$, $\chi_2 = \cos(\frac{\theta}{3}-\frac{2\pi}{3})$, $\chi_3 = \cos\frac{\theta}{3}$ with $\cos\theta = -\frac{x}{t}$, $0<\theta<\pi$, and $\chi_4 = |\Lambda_+|^{-2/3}\Lambda_+ + |\Lambda_-|^{-2/3}\Lambda_-$ with $\Lambda_\pm = t^{-1}(-x \pm \sqrt{x^2-t^2})$.

We gave the point-wise asymptotic formulas in L^∞ for the solutions to the (6) and (7) under the assumption that support of $\hat{\phi}$ is compact. To solve the final state problems (4) and (5) for general final data, we need to drive the asymptotic formulas for the solutions to the linearized equations in L^p framework and relax the assumption on the data ϕ.

To give the linear estimate for (6) in L^p, we define the following function space: For $0<\alpha<1$

$$
\begin{aligned}
\mathcal{A}^\alpha \;&=\; \{\phi \in L^2(\mathbb{R}); \|\phi\|_{\mathcal{A}^\alpha} < \infty\}, \\
\|\phi\|_{\mathcal{A}^\alpha} \;&=\; \left\|\frac{\langle\xi\rangle^\alpha}{|\xi|^\alpha}\hat{\phi}(\xi)\right\|_{L^\infty_\xi} + \left\|\frac{|\xi|^{1-\alpha}}{\langle\xi\rangle^{1-\alpha}}\hat{\phi}'(\xi)\right\|_{L^\infty_\xi} + \||\xi|^{1-\alpha}\hat{\phi}'(\xi)\|_{L^\infty_\xi},
\end{aligned}
$$

where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$.

Lemma 1. *Let $\phi \in \mathcal{A}^\alpha$ with $0<\alpha<1$ and let u be a solution to (6). Then u satisfies*

$$
\begin{aligned}
u(t,x) \;&=\; \frac{1}{\sqrt{2}}t^{-1/2}\mathbf{1}_{\{x;x<0\}}(t,x)\sum_{\pm}|\chi_\pm|^{-1/2}\hat{\phi}(\chi_\pm) \\
&\qquad \times \exp\left(-\frac{2}{3}it\chi_\pm^3 \pm i\frac{\pi}{4}\right) + R_A(t,x), \quad \textit{for } t \geq 1,
\end{aligned}
$$

where $\chi_{\pm} = \pm\sqrt{-\dfrac{x}{t}}$ *and* R_A *satisfies*

$$\|R_A(t)\|_{L^p_x} \le Ct^{-\alpha/3+2/(3p)-1/3}\|\phi\|_{\mathcal{A}^\alpha}$$

for $2 \le p \le \infty$.

From Lemma 1 we see that pointwise decay of the solution to (6) is $\mathcal{O}(t^{-1/2})$ as $t \to +\infty$ under the mean zero assumption on the initial data. In general, the pointwise decay of the solution to (6) is $\mathcal{O}(t^{-1/3})$ as $t \to +\infty$ without the mean zero condition on the initial data. Indeed it is known that

$$\begin{aligned} \|e^{-t\partial_x^3/3}\phi\|_{L^\infty_x} &\le Ct^{-1/2}\||\partial_x|^{-1/2}\phi\|_{L^1_x}, \\ \|e^{-t\partial_x^3/3}\phi\|_{L^\infty_x} &\le Ct^{-1/3}\|\phi\|_{L^1_x}. \end{aligned}$$

To state the linear estimate associated to (7), we introduce the function space

$$\begin{aligned} \mathcal{B} &= \{\phi \in L^2(\mathbb{R}); \|\phi\|_{\mathcal{B}} < \infty\}, \\ \|\phi\|_{\mathcal{B}} &= \left\|\langle\xi\rangle^{-1}\hat{\phi}(\xi)\right\|_{L^\infty_\xi} + \left\|\hat{\phi}'(\xi)\right\|_{L^\infty_\xi}. \end{aligned}$$

Lemma 2. *Let* $\phi \in \mathcal{B}$ *and let* u *be a solution to* (7) *with* $sgn\mu = -1$. *Then* u *satisfies*

$$u(t,x) = \sqrt{\frac{2}{3}}t^{-1/2}\frac{\hat{\phi}(\chi)}{\sqrt{\chi^2+1}}\exp\left(\frac{3}{8}it\chi^4 + \frac{3}{4}it\chi^2 - i\frac{\pi}{4}\right) + R_B(t,x),$$

where $\chi = |\Lambda_+|^{-2/3}\Lambda_+ + |\Lambda_-|^{-2/3}\Lambda_-$ *with* $\Lambda_\pm = t^{-1}(x \pm \sqrt{x^2+t^2})$ *and* R_B *satisfies*

$$\|R_B(t)\|_{L^p_x} \le Ct^{-\beta+1/p}\|\phi\|_{\mathcal{B}}$$

for $2 \le p \le \infty$ *and* $7/8 < \beta < 1$.

For the proofs for Lemma 1 and Lemma 2, see [17, Lemma 2.1] and [18, Theorem 1.1], respectively.

§3. Final state problems

In this section, we state main theorems and give outlines of their proof. According to the L^∞ decay of solutions to the linearized equations and the linear scattering theory, it seems that the solutions to (4) and (5) do not converge to the solutions to their linearized equations as $t \to +\infty$. Therefore we have to take care to choose u_{ap}.

For the final state problem (4), we define

$$\begin{aligned} u_{ap}^j(t,x) &\equiv \frac{1}{\sqrt{2}} t^{-1/2} \mathbf{1}_{\{x;x<0\}} \sum_{\pm} \frac{(i\chi_\pm)^j}{|\chi_\pm|^{1/2}} \hat{\phi}(\chi_\pm) \\ &\qquad \times \exp\left(-\frac{2}{3} it\chi_\pm^3 + iS_\pm(t,\chi_\pm) \pm i\frac{\pi}{4}\right), \\ S_\pm(t,\xi) &\equiv \mp\frac{1}{4}|\hat{\phi}(\xi)|^2 \log t, \end{aligned}$$

where $j = 0, 1, 2$ and $\chi_\pm = \pm\sqrt{-\frac{x}{t}}$.

Theorem 1. *Let* $1/2 < \alpha < 1$. *There exists* $\epsilon > 0$ *with the following properties: For any* $\langle\xi\rangle^3\phi \in \mathcal{A}^\alpha$ *with* $\|\langle\xi\rangle^3\phi\|_{\mathcal{A}^\alpha} < \epsilon$, *there exsits a unique solution* $u \in L^\infty(\mathbb{R}; H^2(\mathbb{R})) \cap C(\mathbb{R}; H^1(\mathbb{R}))$ *to* (4) *satisfying*

$$\sum_{j=0}^{2} \|\partial_x^j u - u_{ap}^j\|_{L^\infty(t,\infty;L_x^2)} \le Ct^{-\gamma},$$

where $1/3 < \gamma < \alpha/3 + 1/6$.

Next we consider (5). We introduce the asymptotic profile

$$\begin{aligned} u_{ap}(t,x) &\equiv \sqrt{\frac{2}{3}} t^{-1/2} \frac{\hat{\phi}(\chi)}{\sqrt{\chi^2+1}} \\ &\qquad \times \exp\left(\frac{3}{8} it\chi^4 + \frac{3}{4} it\chi^2 + iS(t,\chi) - i\frac{\pi}{4}\right), \\ S(t,\xi) &\equiv -\frac{2}{3}\tilde{\lambda}\frac{|\hat{\phi}(\xi)|^2}{\xi^2+1} \log t, \end{aligned}$$

where χ is given in Lemma 2.

Theorem 2. *There exists* $\epsilon > 0$ *with the following properties: For any* $\phi \in \mathcal{B}$ *with* $\|\phi\|_{\mathcal{B}} < \epsilon$, *there exsits a unique solution* $u \in C(\mathbb{R}; L^2(\mathbb{R})) \cap L_{loc}^4(\mathbb{R}; L^\infty(\mathbb{R}))$ *to* (5) *with* $sgn\mu = -1$ *satisfying*

$$\|u - u_{ap}\|_{L^\infty(t,\infty;L_x^2)} + \|u - u_{ap}\|_{L^4(t,\infty;L_x^\infty)} \le Ct^{-\delta},$$

where $1/4 < \delta < 1/2$.

To prove Theorem 1 and Theorem 2, we give the two different approaches, one is due to Ozawa [12] and another is due to Hayashi–Naumkin [5], [6]. Although both of those approaches are based on the contraction mapping principle via the integral equation, the derivation of the integral equation are slightly different. In Ozawa's approach, we rewrite the final state problems into the integral equation of Yang–Ferdman type, see (12) below. Since this reduction is very simple, we may see how to choose the asymptotic profile u_{ap}. However, we have to impose slightly tight assumption for the final data ϕ. To relax the assumption for the final data, we next employ the Hayashi–Naumkin method. In this approach we reduce the final state problem to some integral form which is more complicated than that of Yang–Ferdman, see (16) below. This reduction enables us to relax the assumption on the final data.

Let us explain the method of Ozawa. Firstly, we reduce the final state problems to the integral equation of Yang–Ferdman type. For the simplicity, we rewrite (4) and (5) to the following abstract form

$$
\begin{cases} Lu = N(u), & t, x \in \mathbb{R} \\ \lim_{t \to +\infty} (u(t) - u_{ap}(t)) = 0 & in\ L^2(\mathbb{R}) \end{cases} \tag{11}
$$

where $L = -i\partial_t + P(-i\partial_x)$ is a linear operator and $N(u)$ is a non-linear term. Let $w \equiv u - u_{ap}$. Then the final state problem (11) is rewritten as the integral equation of Yang–Ferdman type

$$
w(t) = -i \int_t^{+\infty} e^{i(t-s)P(-i\partial_x)} \{N(w + u_{ap}) - Lu_{ap}\}(s) ds. \tag{12}
$$

As the second step, we apply the Banach fixed point theorem into the integral equation (12). To this end, for given u_{ap} we shall show that the map

$$
(\Phi w)(t) = -i \int_t^{+\infty} e^{i(t-s)P(-i\partial_x)} \{N(w + u_{ap}) - Lu_{ap}\}(s) ds
$$

is a contraction on function space $X_{r,T}$ for some $r, T > 0$, where $X_{r,T}$ is given by

$$
\begin{aligned}
X_{r,T} &= \{w \in \mathcal{S}'(\mathbb{R}^2) | \|w\|_{X_T} \le r\}, \\
\|w\|_{X_T} &= \begin{cases} \sup_{t \ge T} t^{\gamma} \{\|w\|_{L^\infty(t,\infty;H_x^2)} + \|w\|_{L^6(t,\infty;W_x^{1,\infty})}\} & for\ (4), \\ \sup_{t \ge T} t^{\delta} \{\|w\|_{L^\infty(t,\infty;L_x^2)} + \|w\|_{L^4(t,\infty;L_x^\infty)}\} & for\ (5). \end{cases}
\end{aligned}
$$

To prove that Φ is the contraction map on $X_{r,T}$, we split Φw into a two pieces.

$$\begin{aligned}(\Phi w)(t) &= -i\int_t^{+\infty} e^{i(t-s)P(-i\partial_x)}\{N(w+u_{ap})-N(u_{ap})\}(s)ds \\ &\quad +i\int_t^{+\infty} e^{i(t-s)P(-i\partial_x)}\{Lu_{ap}-N(u_{ap})\}(s)ds \\ &\equiv I_1+I_2.\end{aligned}$$

We evaluate the first term I_1 by using the classical energy method and Strichartz estimate (see for instance Kenig–Ponce–Vega [10]). To evaluate I_2, we need to modify the phase function in the final profile. Since how to choose the phase function is slightly different for (4) and (5), we explain the choice of u_{ap} for (4) and (5) separately.

Firstly we consider the simpler case (5). In this case $L = i\partial_t + (3/4)\partial_x^2 -(1/8)\partial_x^4$ and $N(u) = \tilde{\lambda}|u|^2u$. We simply write $u_{ap}(t,x) = t^{-1/2}A(t,\chi)\exp(iB(t,\chi)+iS(t,\chi))$, where χ is defined by Lemma 1. A simple calculation yields

$$\begin{aligned}Lu_{ap} &= -t^{-\frac{1}{2}}\partial_t SA\exp(iB+iS) + (\textit{remainder terms}), \\ N(u_{ap}) &= \tilde{\lambda}t^{-\frac{3}{2}}|A|^2A\exp(iB+iS).\end{aligned}$$

We notice that the non-linear term $N(u_{ap})$ resonances with the leading term of the linear term Lu_{ap}. Hence choosing

$$S(t,\xi) = -\frac{2}{3}\tilde{\lambda}\frac{|\hat{\phi}(\xi)|^2}{\xi^2+1}\log t,$$

we can cancel those terms each other.

Next we consider (4). As in the case for (5), we simply write

$$u_{ap}(t,x) = t^{-1/2}\sum_{\pm} A_\pm \exp(iB_\pm + iS_\pm).$$

Let $L = \partial_t + (1/3)\partial_x^3$ and $N(u) = (1/2)|u|^2\partial_x u$. Then we see that

$$\begin{aligned}Lu_{ap} &= it^{-1/2}\sum_{\pm}\partial_t S_\pm A_\pm\exp(iB_\pm+iS_\pm) + (\textit{remainder terms}), \\ N(u_{ap}) &= it^{-3/2}\sum_{\pm}\chi_\pm|A_\pm|^2A_\pm\exp(iB_\pm+iS_\pm) \\ &\quad +it^{-3/2}\sum_{\pm}\chi_\pm A_\pm^2\overline{A}_\mp\exp(2iB_\pm - iB_\mp + 2iS_\pm - iS_\mp).\end{aligned}$$

In this case the first term in $N(u_{ap})$ have same oscillations as the linear terms and the second term in $N(u_{ap})$ have the oscillations different from the linear one. Therefore we call the first term "resonance term" and the second term "non-resonance term". For the non-resonance term thanks to the difference of the oscillations we are able to apply the integration by parts in time variable and we see that the non-resonance terms decay faster than the resonance terms. Therefore we can regard the non-resonance terms as the negligible terms. Hence we choose the phase function $S_\pm$ so that the leading term of the linear term Lu_{ap} and the resonance term in nonlinear term $N(u_{ap})$ are canceled each other. This is accomplished by taking

$$S_\pm(t,\xi) = \mp\frac{1}{4}|\hat{\phi}(\xi)|^2 \log t.$$

Choosing S and $S_\pm$ as above, we can regard the second term I_2 is the remainder term and we can close the estimate. In this step we have to impose the strong assumption on the final data, see [16] for the detail for (5). This is the outline of the proof by Ozawa's approach.

To improve our previous results, we employ Hayashi–Naumkin's approach. We only give the outline of the proof for Theorem 2 since the proof of Theorem 1 being similar. Put $w(t,\xi) \equiv \hat{\phi}(\xi)e^{iS(t,\xi)}$. By Lemma 2,

$$e^{it(3\partial_x^2/4-\partial_x^4/8)}\mathcal{F}^{-1}[w] = u_{ap} + R_1, \tag{13}$$

where R_1 satisfies

$$\begin{aligned} t^{\beta-\frac{1}{2}}\|R_1\|_{L^\infty(t,\infty;L_x^2)} &+ t^{\beta-\frac{1}{4}}\|R_1\|_{L^4(t,\infty;L_x^\infty)} \\ &\le C(\|\phi\|_{\mathcal{A}^\alpha} + \|\phi\|_{\mathcal{B}^\alpha}^3). \end{aligned}$$

The equation (5) is rewritten as

$$i\partial_t\mathcal{F}\left[e^{it(3\partial_x^2/4-\partial_x^4/8)}u\right] = \tilde{\lambda}\mathcal{F}\left[e^{-it(\frac{3}{4}\partial_x^2-\frac{1}{8}\partial_x^4)}|u|^2u\right]. \tag{14}$$

From the final condition for (5) and Lemma 2, we obtain

$$\begin{aligned} i\partial_t w &= \frac{2}{3}\tilde{\lambda}t^{-1}\frac{|\hat{\phi}(\xi)|^2}{\xi^2+1}e^{iS(t,\xi)} \\ &= \mathcal{F}\left[e^{-it(3\partial_x^2/4-\partial_x^4/8)}(\tilde{\lambda}|u_{ap}|^2u_{ap} + R_2)\right], \end{aligned} \tag{15}$$

where R_2 satisfies

$$\|R_2\|_{L^1(t,\infty;L_x^2)} \le Ct^{-\beta+1/2}(\|\phi\|_{\mathcal{B}^\alpha}^3 + \|\phi\|_{\mathcal{B}^\alpha}^5).$$

From (13),(14) and (15), we have

$$(16)\quad \begin{aligned} & u(t)-u_{ap}(t) \\ = \ & i\tilde{\lambda}\int_t^{+\infty} e^{-i(t-\tau)(3\partial_x^2/4-\partial_x^4/8)}[|u|^2u-|u_{ap}|^2u_{ap}](\tau)d\tau \\ & +R_1(t)-i\int_t^{+\infty} e^{-i(t-\tau)(3\partial_x^2/4-\partial_x^4/8)}R_2(\tau)d\tau. \end{aligned}$$

Applying the contraction principle for the above integral equation, we can construct the solution to (5). See [17, Theorem 1.1] and [18, Theorem 1.2] for the proofs of Theorem 1 and Theorem 2, respectively.

Finally, we give several open problems related to this note.

So far, we do not have any result on the long time behavior of solution to (5) with $\mathrm{sgn}\mu=+1$. This is because we do not succeed to justify the asymptotic formula for the solution to (7) with $\mathrm{sgn}\mu=+1$ in L^p framework. The asymptotic behavior of solution to (5) with $\mathrm{sgn}\mu=+1$ is still an open problem even for the linear equation.

In this note we focused on the one dimensional dispersive equation. It is also interesting to study the large time behavior of the solution to multi-dimensional dispersive equations such as Kadomtsev–Petvishvili equation which is third order dispersive equation with quadratic nonlinearity in two dimension.

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