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Triple covers of algebraic surfaces and a generalization of Zariski's example

Dedicated to Professor Mutsuo Oka on his sixtieth birthday

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Abstract.

Let B be a reduced sextic curve in $\mathbb{P}^2$. In the case when singularities of B are only six cusps, Zariski proved that there exists a non-Galois triple cover branched at B if and only if B is given by an equation of the form $G_2^3 + G_3^2$, where G_i denotes a homogeneous polynomial of degree i. In this article, we generalize Zariski's statement to any reduced sextic curve with at worst simple singularities. To this purpose, we give formulae for numerical invariants of non-Galois triple covers by using Tan's canonical resolution.

§1. Introduction

In this article, all varieties are defined over the field of complex numbers, $\mathbb{C}$.

Let Σ be a smooth projective surface and let B be a reduced divisor on Σ. A normal projective surface X is called a triple cover of Σ with branch locus B if

- there exists a finite surjective morphism $\pi : X \to \Sigma$ of degree 3, and
- the branch locus $\Delta(\pi) = B$

Let X be a triple cover of Σ. We denote the rational function fields of X and Σ by $\mathbb{C}(X)$ and $\mathbb{C}(\Sigma)$, respectively. Under our circumstance, $\mathbb{C}(X)$ is a cubic extension of $\mathbb{C}(\Sigma)$. We say that X is a *non-Galois triple*

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cover (resp. *cyclic triple cover*) if the cubic extension $\mathbb{C}(X)/\mathbb{C}(\Sigma)$ is non-Galois (resp. cyclic). For a point $y \in \Sigma$, we call y a *total* (resp. *simple*) branch point if $\sharp\left(\pi^{-1}(y)\right) = 1$ (resp. $= 2$). We call a triple cover $\pi : X \to \Sigma$ *generic* if *its total branch points are finite* (see Definition 2.1 for detail). Note that a generic triple cover is always non-Galois (Remark 2.2).

The first systematic study on triple covers was done by Miranda [11]. Afterward, some have been done by [18, 19, 20], [4] and [16, 17]. Yet non-Galois triple covers are difficult to deal with. For example, a fundamental question as follows still remains as a subtle question:

Question 1.1. Let Σ and B be as above. Give a sufficient and necessary condition for B to be the branch locus of a non-Galois triple cover.

One can see the subtleness of Question 1.1 in Zariski's example ([23]) below.

Example 1.1. Let B be an irreducible plane sextic curve in $\mathbb{P}^2$ having only 6 cusps as its singularities. There exists a generic triple cover with branch locus B if and only if there exists a conic passing through all the 6 cusps.

Note that there exists no conic through assigned 6 points if these six points are in general position. In fact, it is known that there exists an irreducible sextic with only 6 cusps as its singularities such that no conic passes through all the six cusps ([12], [24]).

Remark 1.1. Zariski's example is a starting point of the study of so called "Zariski pairs" and there have been many results on it from various points of view (see [1] and its references for details).

Our goal of this article is to generalize Zariski's example to the case when B is a reduced sextic curve having only simple singularities as its singularities. For simple singularities, see [3, Theorem II, 8.1], page 64. To describe the type of singularities, we use the standard notations A_n, D_n and E_n. By abuse of notations, we also use the same notations to describe rational double points on surfaces (see [3], page 87). Let us state our result:

Theorem 1.1. *Let B be a reduced sextic curve in $\mathbb{P}^2$ with at worst simple singularities. There exists a generic triple cover $\pi : X \to \mathbb{P}^2$ with branch locus B if and only if B is given by an equation of the form*

$$G_2^3 + G_3^2 = 0,$$

where $G_i = G_i(X_0, X_1, X_2)$ $(i = 2, 3)$ *are homogeneous polynomials of degree* i, $[X_0 : X_1 : X_2]$ *being a homogeneous coordinate of* $\mathbb{P}^2$.

Corollary 1.1. *Let* B *be a reduced sextic curve in* $\mathbb{P}^2$. *Then the following two statements are equivalent:*

- B *is a* $(2,3)$*-torus curve (see Remark 1.2 below).*
- *There exists a surjective morphism from the fundamental group* $\pi_1(\mathbb{P}^2 \setminus B, *)$ *to the symmetric group of* 3 *letters such that all meridians around irreducible components of* B *are mapped to elements of order* 2

Remark 1.2. (i) A sextic curve given as in Theorem 1.1 is called a $(2,3)$-torus sextic (see [9]). Such curves are intensively studied by Oka ([13, 14, 15]).

(ii) In Example 1.1, the conic is given by $G_2 = 0$ as above. Hence Theorem 1.1 is a generalization of Example 1.1.

(iii) Note that Corollary 1.1 is a slight generalization of [5, Theorem 4.1.1], as we also consider the case when sextics are reducible.

In order to prove Theorem 1.1, our main tool are formulae for numerical invariants of the minimal resolution of a generic triple cover as follows:

Proposition 1.1. *Let* $\pi : X \to \Sigma$ *be a generic triple cover with* $\Delta(\pi) = B$, *where* B *is a reduced divisor on* Σ *with at worst simple singularities. We denote the set of total branch points by* T. *Then:*

(i) $T \subseteq \mathrm{Sing}(B)$ *and* T *consists of singular points of type either* A_{3k-1} $(k \in \mathbb{N})$ *or* E_6.

(ii) Put $T = \{p_1, \ldots, p_m, p_{m+1}, \ldots, p_{m+n}\}$ *in such a way that* p_i *is of type* A_{3k_i-1} *for* $1 \leq i \leq m$, *and* p_i *is of type* E_6 *for* $m+1 \leq i \leq m+n$. *Let* $\delta := \sum_{i=1}^{m} k_i + 2n$ *and we denote the minimal resolution of* X *by* $\tilde{X}$. *Then we have*

$$\begin{aligned} K_{\tilde{X}}^2 &= 3K_\Sigma^2 + 2K_\Sigma B + \frac{1}{2}B^2 - \delta, \\ e(\tilde{X}) &= 3e(\Sigma) + K_\Sigma B + B^2 - 3\delta \quad \textit{and} \\ \chi(\mathcal{O}_{\tilde{X}}) &= 3\chi(\mathcal{O}_\Sigma) + \frac{1}{4}K_\Sigma B + \frac{1}{8}B^2 - \frac{1}{3}\delta. \end{aligned}$$

Here $K_\bullet, e(\bullet)$ *and* $\chi(\mathcal{O}_\bullet)$ *denote a canonical divisor, the topological Euler number, and the Euler characteristic of a surface* $\bullet$.

We apply the formulae in Proposition 1.1 to the case when $\Sigma = \mathbb{P}^2$ and B is a reduced sextic curve, and we obtain $K_{\tilde{X}}^2$ and $e(\tilde{X})$. These values play important roles to prove Theorem 1.1.

This article consists of 4 sections. In §1, we review a theory of triple covers developed in [19]. In §2, we summarize generic triple covers and their canonical resolutions based on [17]. We prove Proposition 1.1 in §3 and Theorem 1.1 in §4.

§2. Non-Galois triple covers over smooth varieties

In this section, we first review the method to deal with non-Galois triple covers developed in [19].

Let Y be a smooth projective variety. Let X be a normal projective variety with a finite morphism $\pi : X \to Y$. We call X *a triple cover* of Y if $\deg \pi = 3$. Let $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ denote the rational function fields of X and Y, respectively. For a triple cover $\pi : X \to Y$, $\mathbb{C}(X)$ is a cubic extension of $\mathbb{C}(Y)$, and it is either a 3-cyclic extension or a non-Galois cubic extension. Let θ be an element of $\mathbb{C}(X)$ such that (i) $\mathbb{C}(X) = \mathbb{C}(Y)(\theta)$ and (ii) the minimal equation of θ is $z^3 + 3az + 2b$, $a, b \in \mathbb{C}(Y)$. Put $L = \mathbb{C}(Y)(\sqrt{a^3 + b^2})$ and let K be the Galois closure of $\mathbb{C}(X)$. The following facts are well-known:

- If $\mathbb{C}(X)/\mathbb{C}(Y)$ is cyclic, $K = \mathbb{C}(X)$ and $L = \mathbb{C}(Y)$.
- If $\mathbb{C}(X)/\mathbb{C}(Y)$ is non-Galois, K is a $\mathcal{D}_6$-extension of $\mathbb{C}(Y)$, $\mathcal{D}_6$ being the dihedral group of oder 6 given by $\langle \sigma, \tau \mid \sigma^2 = \tau^3 = (\sigma\tau)^2 = 1 \rangle$. L is a quadratic extension of $\mathbb{C}(Y)$ and $L = K^{\tau}$, the fixed field of τ.

Define a normal varieties $\hat{X}$ and $D(X/Y)$ to be the K- and L-normalizations of Y, respectively, and we denote the induced morphisms by $\hat{\pi} : \hat{X} \to Y$, $\alpha(\pi) : \hat{X} \to X$, $\beta_1(\pi) : D(X/Y) \to Y$ and $\beta_2(\pi) : \hat{X} \to D(X/Y)$. Note that $\hat{\pi} = \pi \circ \alpha(\pi) = \beta_1(\pi) \circ \beta_2(\pi)$. Also (i) $\alpha(\pi)$ and $\beta_1(\pi)$ are identities if $\mathbb{C}(X)/\mathbb{C}(Y)$ is Galois, while (ii) if $\mathbb{C}(X)/\mathbb{C}(Y)$ is non-Galois, $\alpha(\pi)$ and $\beta_1(\pi)$ are degree 2 finite morphisms; and $\beta_2(\pi)$ is a degree 3 morphism so that $\mathbb{C}(\hat{X})/\mathbb{C}(D(X/Y))$ is a cyclic extension.

We call $\pi : X \to Y$ *cyclic* for the case (i) and *non-Galois* for the case (ii) respectively.

For any finite morphism $f : X \to Y$, we define the branch locus of f, denoted by $\Delta(f)$ or $\Delta(X/Y)$, as follows:

$$\Delta(f) := \{y \in Y \mid f \text{ is not locally isomorphic over } y\}.$$

By the purity of the branch locus [24], $\Delta(f)$ is a reduced divisor on Y if Y is smooth.

Remark 2.1. Since all varieties are projective and defined over $\mathbb{C}$, varieties can be considered as analytic ones and we do not have to distinguish "algebraic" and "analytic" (see [7]). When we look into the local structures of covering morphisms, e.g., covering morphisms, resolutions of singularities and so on, we consider them analytically.

Lemma 2.1. *Let $\pi : X \to Y$ be a triple cover. Then $\Delta(X/Y) = \Delta(\hat{X}/Y)$.*

For a proof, see [19, Lemma1.4].

Definition 2.1. (i) Let $\pi : X \to Y$ be a triple cover and let y be a point on Y. We say that π is *totally* (resp. *simply*) *ramified* over y if $\sharp(\pi^{-1}(y)) = 1$ (resp. $= 2$). We call such a point y a *total* (resp. *simple*) *branch point.*

(ii) We call a triple cover $\pi : X \to Y$ *"generic"* if the set of total branch points has codimension at least 2.

Let $\pi : X \to Y$ be a non-Galois triple cover and let $\Delta(\pi) = D_1 + \ldots + D_r$ be the irreducible decomposition of $\Delta(\pi)$. We say that π is simply ramified along D_i if there exists a Zariski open set U_{D_i} of D_i such that π is simply ramified over y, $y \in U_{D_i}$. We say that π is totally ramified along D_i if any point in D_i is a total branch point of π. We decompose $\Delta(\pi) = \Delta_1(\pi) + \Delta_2(\pi)$ in such a way that π is simply ramified along irreducible components of $\Delta_1(\pi)$ and is totally ramified along those of $\Delta_2(\pi)$.

Remark 2.2. (i) Our terminology for "generic" is different from those in Miranda [11] and Kulikov–Kulikov [10]. In those article, total branch points are only ordinary cusps, while other kind of singularity are allowed in this article (see Lemma 4.1).

(ii) If $\pi : X \to Y$ is cyclic (i.e., $\mathbb{C}(X)/\mathbb{C}(Y)$ is cyclic), then the set of total branch points coincides with $\Delta(\pi)$ (Note that the converse of this is not true ([20])). In particular, a generic triple cover is non-Galois.

§3. Generic triple covers of smooth projective surfaces and Tan's canonical resolution

In this section, we give a summary on Tan's canonical resolution of a triple cover. The canonical resolution was first studied by Horikawa in [8] for double covers. For triple covers, it was studied by Ashikaga in [2] for certain special triple covers and by Tan in [17] for general case. We explain Tan's method briefly.

Let $\pi : X \to \Sigma$ be a triple cover. In [17], Tan shows that there exists a resolution of singularities of $\mu : X^{(n)} \to X$ given by the following commutative diagrams,

$$\begin{array}{ccccccccc}
X = X^{(0)} & \xleftarrow{\mu_0} & X^{(1)} & \xleftarrow{\mu_1} & X^{(2)} & \xleftarrow{\mu_2} & \cdots & \xleftarrow{\mu_{n-1}} & X^{(n)} \\
\downarrow{\scriptstyle \pi=\pi^{(0)}} & & \downarrow{\scriptstyle \pi^{(1)}} & & \downarrow{\scriptstyle \pi^{(2)}} & & & & \downarrow{\scriptstyle \pi^{(n)}} \\
\Sigma = \Sigma^{(0)} & \xleftarrow{q_0} & \Sigma^{(1)} & \xleftarrow{q_1} & \Sigma^{(2)} & \xleftarrow{q_2} & \cdots & \xleftarrow{q_{n-1}} & \Sigma^{(n)}
\end{array}$$

where q_i is the blowing-up at a singular point p_i of the branch locus of $\pi^{(i)}$, $X^{(i)}$ is the normalization of $X^{(i)} \times_{\Sigma^{(i-1)}} \Sigma^{(i)}$ and $\pi^{(i)}$ the natural morphism to $\Sigma^{(i)}$. Let $\Delta_1(\pi)$ (resp. $\Delta_2(\pi)$) be the divisors as in §1. Let E_i be the exceptional curve of q_{i-1} and $\mathcal{E}_i$ the total transform of E_i in $\Sigma^{(n)}$. Set $q = q_0 \circ q_1 \circ \cdots \circ q_{n-1}$. For a divisor D, we denote the multiplicity of D at p by $m_p(D)$. With these notations, $\chi(\mathcal{O}_{X^{(n)}})$ and $K^2_{X^{(n)}}$ are given as follows:

Theorem 3.1. (Tan [17, Theorem 6.3]) *Let $\pi \colon X \longrightarrow \Sigma$ be a normal triple cover of a smooth projective surface Σ and let $\mu \colon X^{(n)} \longrightarrow X$ be the resolution of singularities as above. Let m_i and n_i be integers given by Remark 3.1 below. Then*

$$\Delta_1(\pi^{(n)}) = q^*\Delta_1(\pi) - 2\sum_{i=0}^{n-1} m_i\mathcal{E}_{i+1}, \ \Delta_2(\pi^{(n)}) = q^*\Delta_2(\pi) - \sum_{i=0}^{n-1} n_i\mathcal{E}_{i+1},$$

and

$$\begin{aligned}
\chi(\mathcal{O}_{X^{(n)}}) &= 3\chi(\mathcal{O}_\Sigma) + \frac{1}{8}\Delta_1(\pi)^2 + \frac{1}{4}\Delta_1(\pi)K_\Sigma + \frac{5}{18}\Delta_2(\pi)^2 + \\
&\quad + \frac{1}{2}\Delta_2(\pi)K_\Sigma - \sum_{i=0}^{n-1}\frac{m_i(m_i-1)}{2} - \sum_{i=0}^{n-1}\frac{n_i(5n_i-9)}{18}, \\
K^2_{X^{(n)}} &= 3K^2_\Sigma + \frac{1}{2}\Delta_1(\pi)^2 + 2\Delta_1(\pi)K_\Sigma + \frac{4}{3}\Delta_2(\pi)^2 + 4\Delta_2(\pi)K_\Sigma \\
&\quad - \sum_{i=0}^{n-1} 2(m_i-1)^2 - \sum_{i=0}^{n-1}\frac{4n_i(n_i-3)}{3} - n.
\end{aligned}$$

Remark 3.1. The above integer m_i is the greatest integer not exceeding $(m_{p_i}(\Delta_1(\pi^{(i)})))/2$. Furthermore, n_i is computed as follows:

$$n_i = \begin{cases} m_{p_i}(\Delta_2(\pi^{(i)})) - 1 & \text{if } E_{i+1} \subset \operatorname{Supp}(\Delta_2(\pi^{(i+1)})) \\ m_{p_i}(\Delta_2(\pi^{(i)})) & \text{if } E_{i+1} \not\subset \operatorname{Supp}(\Delta_2(\pi^{(i+1)})). \end{cases}$$

We now assume that π is a generic triple cover. In this case, we have $\Delta_1(\pi) = B$ and $\Delta_2(\pi) = 0$. For a point $p \in \operatorname{Sing}(B)$, we set integers $\delta(p, \pi)$ and $\kappa(p, \pi)$ as follows:

$$\delta(p,\pi) = \sum_{p_i \in N(p,\pi)} \frac{m_i(m_i-1)}{2} + \sum_{p_i \in N(p,\pi)} \frac{n_i(5n_i-9)}{18},$$

$$\kappa(p,\pi) = \sum_{p_i \in N(p,\pi)} 2(m_i-1)^2 + \sum_{p_i \in N(p,\pi)} \frac{4n_i(n_i-3)}{3} + \sharp N(p,\pi),$$

where $N(p,\pi)$ is the set of points, $p_0 = p, p_1, \dots$, which are infinitely near points lying over p.

Let $\widetilde{X}$ be the minimal resolution of X. There exists a birational morphism $\gamma\colon X^{(n)} \to \widetilde{X}$. For any point $p \in \mathrm{Sing}(B)$, let $\epsilon(p,\pi)$ be the number of exceptional curves in $(\pi^{(n)})^{-1}q^{-1}(p)$ contracted by γ. Then, we have $\chi(\mathcal{O}_{\widetilde{X}}) = \chi(\mathcal{O}_{X^{(n)}})$ and $K^2_{\widetilde{X}} = K^2_{X^{(n)}} + \sum_{p\in B}\epsilon(p,\pi)$. By Theorem 3.1, we obtain

$$\text{(1)} \quad \chi(\mathcal{O}_{\widetilde{X}}) = 3\chi(\mathcal{O}_\Sigma) + \frac{1}{8}B^2 + \frac{1}{4}BK_\Sigma - \sum_{p\,\in \mathrm{Sing} B} \delta(p,\pi),$$

$$\text{(2)} \quad K^2_{\widetilde{X}} = 3K^2_\Sigma + \frac{1}{2}B^2 + 2BK_\Sigma - \sum_{p\,\in \mathrm{Sing} B} (\kappa(p,\pi) - \epsilon(p,\pi))$$

§4. Proof of Proposition 1.1

Let Σ be a smooth projective surface and let B be a reduced divisor on Σ with at worst simple singularities. Let $\pi : X \to \Sigma$ be a generic triple cover branched at B. Let $D(X/\Sigma)$ and $\hat{X}$ be the double cover and the $\mathcal{D}_6$-cover, respectively, determined by X as in the previous section.

Let us start with the following lemma:

Lemma 4.1. *(i) The branch locus of $\beta_1(\pi)$ is B.*

(ii) The branch locus $\Delta(\beta_2(\pi))$ of $\beta_2(\pi)$ is contained in $\mathrm{Sing}(D(X/\Sigma))$.

(iii) Suppose that $\Delta(\beta_2(\pi)) \neq \emptyset$. For any $x \in \Delta(\beta_2(\pi))$, $\beta_1(\pi)(x)$ is a singular point of B whose type is either A_{3k-1} or E_6.

(iv) Let T be the set of total branch points. Then $T = \beta_1(\pi)(\Delta(\beta_2(\pi)))$

Proof. (*i*) Since π is generic, $\beta_1(\pi) : D(X/\Sigma) \to \Sigma$ is branched along B.

(*ii*) By Lemma 2.1, $\beta_1(\pi)(\Delta(\beta_2(\pi))) \subset B$. Suppose that $\beta_2(\pi)$ is ramified along some irreducible component D of $\beta_1(\pi)^{-1}(B)$. Then the ramification index along $\hat{\pi}^{-1}(D)$ is equal to 6, and we infer that the stabilizer group at a smooth point of $\hat{\pi}^{-1}(D)$ is a cyclic group of order 6. This contradicts our assumption. Hence $\beta_2(\pi)$ is branched at some points, and this implies that $\beta_2(\pi)$ is not ramified over any smooth point of $D(X/\Sigma)$ by the purity of the branch locus. Hence $\Delta(\beta_2(\pi)) \subset \mathrm{Sing}(D(X/\Sigma))$.

(iii) Suppose that $\Delta(\beta_2(\pi)) \neq \emptyset$. Choose any $p \in \Delta(\beta_2(\pi))$. $\beta_2(\pi)$ is unramified over a small neighborhood except p. Hence the local fundamental group at p contains a normal subgroup of index 3. Under our assumption for singularities of B, singularities of $D(X/\Sigma)$ are all rational double points. Hence the type of $\beta_1(\pi)(p)$ is either A_{3k-1} or E_6.

(iv) Our statement is immediately from the observation:

$$x \in T \Leftrightarrow \sharp(\hat{\pi}^{-1}(x)) \leq 2 \Leftrightarrow \beta_1(\pi)^{-1}(x) \subset \Delta(\beta_2(\pi))$$

Q.E.D.

By Lemma 4.1, we have Proposition 1.1 *(i)*. In what follows, we always assume that

$$\Delta(\beta_2(\pi)) \neq \emptyset.$$

We put $NT = \mathrm{Sing}(\Delta(\pi)) \setminus \mathrm{T}$.

Now we compute $\delta(p,\pi)$, $\kappa(p,\pi)$ and $\epsilon(p,\pi)$ in the previous section for each $p \in \mathrm{Sing}(B)$. Here are some of facts on the canonical resolution, which we need to compute $\delta(p,\pi)$, $\kappa(p,\pi)$ and $\epsilon(p,\pi)$. For their proof, see [17]

Lemma 4.2. (Tan [17, Corollary 5.3]) *The triple cover π is totally ramified over p if and only if there exists an integer i satisfying $p_i \in N(p,\pi)$ and $E_{i+1} \subset \mathrm{Supp}(\Delta_2(\pi^{(n)}))$.*

Lemma 4.3. (Tan [17, Theorem 4.1, Lemma 6.1]) *Let $\pi\colon X \to \Sigma$ be a triple cover of a smooth algebraic surface Σ. Then:*

(1) *The intersection multiplicities between $\Delta_1(\pi)$ and $\Delta_2(\pi)$ at their intersection points are ≥ 2.*

(2) *If X is smooth, then the self-intersection numbers of irreducible components of $\Delta_2(\pi)$ are multiples of three.*

Lemma 4.4. (Tan [17, Theorem 4.1]) *Let D_1 and D_2 be two distinct irreducible components of $\Delta_2(\pi)$ and i_p an integer satisfying that q_{i_p} is a blowing-up at p. We assume that D_1 meets D_2 transversely at $p \notin \Delta_1(\pi)$. Then, p satisfies either;*

(i) $E_{i_p+1} \not\subset \mathrm{Supp}(\Delta_2(\pi^{(i_p+1)}))$, *or*

(ii) $E_{i_p+1} \subset \mathrm{Supp}(\Delta_2(\pi^{(i_p+1)}))$ *and the infinitely near points of D_1 and D_2 lying over p satisfy the property* (i).

First, we consider a singular point p not contained in T. Let i_p be an integer as in Lemma 4.4. We may assume that $i_p = 0$.

Proposition 4.1. *If $p \in \mathrm{Sing}(B) \setminus \mathrm{T}$, then $\delta(p,\pi) = 0$, $\kappa(p,\pi) = \sharp N(p,\pi)$ and $\epsilon(p,\pi) = \sharp N(p,\pi)$.*

Proof. By Lemma 4.2, we have $E_{i+1} \not\subset \mathrm{Supp}(\Delta_2(\pi^{(i+1)}))$ for $p_i \in N(p,\pi)$, i.e., $n_i = 0$. Let $D(X/\Sigma)$ be the double cover introduced in §1 and let Z be its canonical resolution. Then, $\delta(p,\pi)$ and $\kappa(p,\pi) - \sharp N(p,\pi)$ coincide with $\chi(\mathcal{O}_Z) - \chi(\mathcal{O}_{D(X/\Sigma)})$ and $K_Z^2 - K^2_{D(X/\Sigma)}$, respectively. (See [4].) Since $D(X/\Sigma)$ is a double cover of Σ branched along B and B has at worst simple singularities, we have $\chi(\mathcal{O}_Z) = \chi(\mathcal{O}_{D(X/\Sigma)})$ and $K_Z^2 = K^2_{D(X/\Sigma)}$. Thus, we have $\delta(p,\pi) = 0$ and $\kappa(p,\pi) = \sharp N(p,\pi)$.

It is obvious that $(\pi^{(n-1)})^*(q^{-1}(p))$ contains $\sharp N(p,\pi)$ exceptional curves contracted by γ. Thus, we obtain $\epsilon(p,\pi) = \sharp N(p,\pi)$. Q.E.D.

Next we consider a singular point $p \in T$. By Lemma 4.1, p is either of type A_{3k-1} or of type E_6. We may assume that q_0 is a blowing up at p.

Lemma 4.5. *Let $p \in T$ be of type A_{3k-1}. Then, the exceptional curve E_1 of q_0 is contained in $\Delta_2(\pi^{(1)})$.*

Proof. Suppose that $E_1 \not\subset \mathrm{Supp}(\Delta_2(\pi^{(1)}))$. In the case of $k = 1$, the singular point $p \in B$ is resolved by blowing up at p. Hence we infer that $N(p,\pi) = \{p\}$ and $p \notin T$ by Lemma 4.2, but this contradicts to our assumption. We next consider the cases of $k > 1$. Since $E_1 \not\subset \mathrm{Supp}(\Delta_2(\pi^{(1)}))$, $\pi^{(1)}\colon X^{(1)} \to \Sigma^{(1)}$ is also a generic triple cover. Let p' be the infinitely near point of B lying over p. By Lemma 4.2, $\pi^{(1)}$ is totally ramified over p'. On the other hand, by the property $m_p(B) = 2$, we have $E_1 \not\subset \mathrm{Supp}(\Delta_1(\pi^{(1)}))$. Hence, p' is a singular point of $\Delta_1(\pi^{(1)})$ whose type is A_{3k-3}. This contradicts to Lemma 4.1. Hence we have $E_1 \subset \mathrm{Supp}(\Delta_2(\pi^{(1)}))$. Q.E.D.

The figures in this section show exceptional curves of q and inverse image of these by $\pi^{(n)}$. Thick lines denote exceptional curves in $\mathrm{Supp}(\Delta_2(\pi^{(n)}))$ and thin lines denote those in $\mathrm{Supp}(\Delta_1(\pi^{(n)}))$. Also broken lines denote exceptional curves not contained in $\mathrm{Supp}(\Delta_1(\pi^{(n)}) + \Delta_2(\pi^{(n)}))$. Lines with numbers mean preimages of exceptional curves of q by $\pi^{(n)}$ and the self-intersection numbers of them.

Proposition 4.2. *Let $p \in T$ be a singular point of type A_2. Then, $\delta(p,\pi) = 1/3$ and $\kappa(p,\pi) = 5$*

Proof. By Lemma 4.5, we have $(m_0, n_0) = (1, -1)$. Since $N(p,\pi)$ contains the infinitely near point of B lying over p, we may assume that p_1 is this point. By the property $m_{p_1}(\Delta_1(\pi^{(1)})) = 1$, we have $E_2 \subset \mathrm{Supp}(\Delta_1(\pi^{(2)}))$, i.e., $(m_1, n_1) = (0, 1)$.

Since $N(p, \pi)$ contains the infinitely near point of E_1 lying over p_1, we may assume that p_2 is this point. By the property $m_{p_2}(\Delta_1(\pi^{(2)})) = 2$ and Lemma 4.3 (1), $E^{(3)} \not\subset \mathrm{Supp}(\Delta_1(\pi^{(3)}) + \Delta_2(\pi^{(3)}))$, i.e., $(m_2, n_2) = (1, 1)$. Since there exist no singular points of the branch locus of $\pi^{(3)}$ which are infinitely near points lying over p, we have $N(p, \pi) = \{p, p_1, p_2\}$ (See Figure 1).

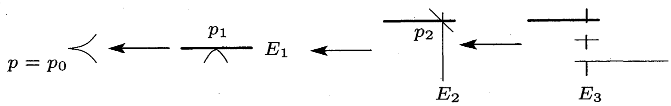


Fig. 1. the branch locus of $\widetilde{\pi}$ (p is of type A_2)

Thus, we have $\delta(p, \pi) = 1/3$ and $\kappa(p, \pi) = 5$. Q.E.D.

Proposition 4.3. *Let k be a positive integer and $p \in T$ a singular point of type A_{3k-1}. Then,*
(1) if $k = 2l - 1$, then $\delta(p, \pi) = (2l - 1)/3$, $\kappa(p, \pi) = 6l - 1$ and $\epsilon(p, \pi) = 4l$.
(2) if $k = 2l$, then $\delta(p, \pi) = 2l/3$, $\kappa(p, \pi) = 6l$ and $\epsilon(p, \pi) = 4l$.

Proof. We have $(m_0, n_0) = (1, -1)$ as in the proof of Lemma 4.2. We may assume that p_1 is the infinitely near point of B lying over p. By the property $m_{p_1}(\Delta_1(\pi^{(1)})) = 2$, we have $E_2 \not\subset \Delta_1(\pi^{(2)})$. If we assume that E_2 is not contained in $\Delta_2(\pi^{(2)})$, the proper transform of E_1 in $\Sigma^{(n)}$ is a curve contained in $\Delta_2(\pi^{(n)})$ with self-intersection -2. It contradicts to Lemma 4.3 (2). Therefore, we obtain $E_2 \subset \Delta_2(\pi^{(2)})$, i.e., $(m_1, n_1) = (1, 0)$.

We may assume that p_2 is the infinitely near point of E_1 lying over p_1. If p_2 satisfies the property (ii) in Lemma 4.4, then the self-intersection number of the proper transform of E_1 in $\Sigma^{(n)}$ is -4. It contradicts to Lemma 4.3 (2). Hence, p_2 satisfies the property (i) in Lemma 4.4. Therefore, E_3 is not contained in $\Delta_2(\pi^{(3)})$, i.e., $(m_2, n_2) = (0, 2)$.

We may assume that p_3 is the infinitely near point of $\Delta_1(\pi^{(2)})$. By the property $m_{p_3}(\Delta_1(\pi^{(3)})) = 2$, we have $E_4 \not\subset \Delta_1(\pi^{(4)})$. Let p'' be the intersection point of the proper transform of E_2 in $\Sigma^{(4)}$ and E_4. If E_4 is contained in $\Delta_2(\pi^{(4)})$, then we have $p'' \in N(p, \pi)$. By Lemma 4.4, p'' satisfies either the property (i) or (ii). In both cases, the self-intersection number of the proper transform of E_2 in $\Sigma^{(n)}$ is not divisible by three. It contradicts to Lemma 4.3 (2). Therefore, we have $E_4 \not\subset \Delta_2(\pi^{(4)})$, i.e., $(m_3, n_3) = (1, 1)$.

In the case that $k = 2$, we have $N(p, \pi) = \{p, p_1, p_2, p_3\}$. (See Figure 2).

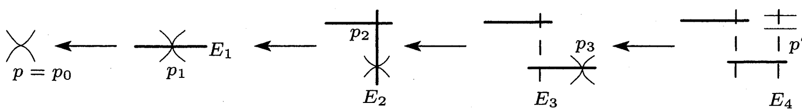


Fig. 2. the branch locus of $\widetilde{\pi}$ (p is of type A_5)

Thus, we obtain

$$\delta(p, \pi) = \frac{2}{3}, \tag{3}$$

$$\kappa(p, \pi) = 6. \tag{4}$$

In the case that k is greater than two, let p' be the infinitely near point of $\Delta_1(\pi^{(3)})$ lying over p_3. (See Figure 3). Then, we have $N(p, \pi) = \{p, p_1, p_2, p_3\} \cup N(p', \pi^{(4)})$ and

$$\delta(p, \pi) = \frac{2}{3} + \delta(p', \pi^{(4)}), \tag{5}$$

$$\kappa(p, \pi) = 6 + \kappa(p', \pi^{(4)}). \tag{6}$$

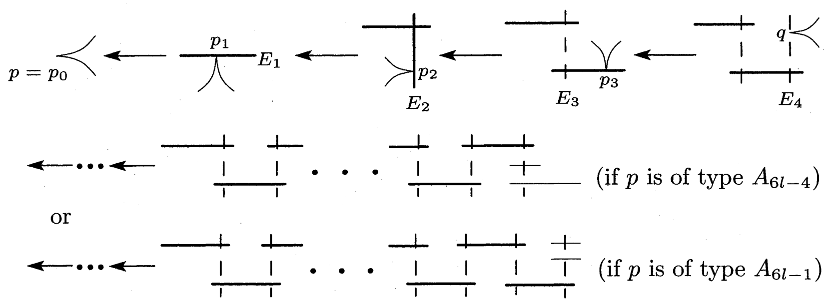


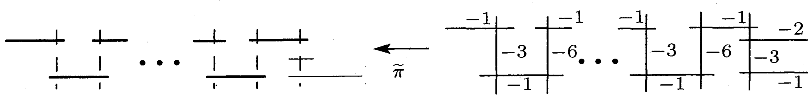
Fig. 3. the branch locus of $\widetilde{\pi}$

By applying the Hurwitz formula to $\pi^{(4)}|_{(\pi^{(4)})^{-1}}(E_4)$, we see that $\pi^{(4)}$ is totally branched at p'. Hence, by applying the equations (5), (6) to p' and by using Lemma 4.2 and the equations (3), (4), we have the following equations:

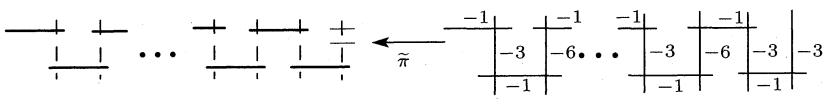
$$\delta(p, \pi) = \begin{cases} \frac{2l-1}{3} & \text{if } k = 2l - 1 \\ \frac{2l}{3} & \text{if } k = 2l \end{cases}$$

$$\kappa(p, \pi) = \begin{cases} 6l - 5 & \text{if } k = 2l - 1 \\ 6l & \text{if } k = 2l \end{cases}$$

Furthermore, it is obvious that all curves in $(\pi^{(n)})^{-1}(q^{-1}(p))$ are contracted by γ. (See Figure 4.)



(p is of type A_{6l-4})



(p is of type A_{6l-1})

Fig. 4. Inverse image of exceptional curves

Therefore, we have

$$\epsilon(p, \pi) = \begin{cases} 4l & \text{if } k = 2l - 1 \\ 4l & \text{if } k = 2l \end{cases}$$

Q.E.D.

Proposition 4.4. *Let $p \in T$ is a singular point of type E_6. Then, we have $\delta(p, \pi) = 2/3$, $\kappa(p, \pi) = 7$ and $\epsilon_p = 5$.*

Proof. By the property $m_p(B) = 3$, E_1 is contained in $\Delta_1(\pi^{(1)})$, i.e., $m_0 = 1$ and $n_0 = 0$. We may assume that p_1 is the infinitely near point of B lying over p. By [17, Corollary 5.3], $\pi^{(1)}$ is totally ramified over p_1. Since p_1 is of type A_5, by equations (3) and (4), we have $N(p, \pi) = \{p\} \cup N(p_1, \pi^{(1)})$, $\delta(p, \pi) = 2/3$ and $\kappa(p, \pi) = 7$. Furthermore, all curves in $(\pi^{(n)})^{-1}(q^{-1}(p))$ are contracted by φ, i.e., $\epsilon(p, \pi) = 5$. (See Figure 5.)

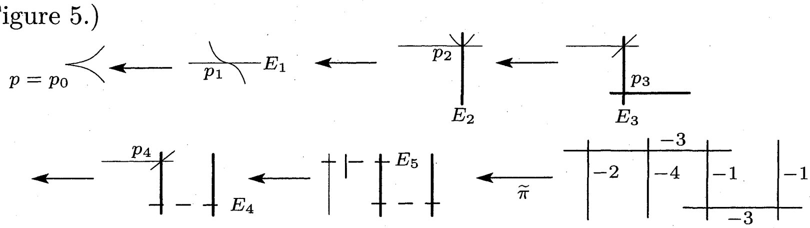


Fig. 5. The branch locus of $\widetilde{\pi}$ and the inverse image of exceptional curves (p is of type E_6)

Q.E.D.

By equations (1), (2), Propositions 4.3, 4.4 and Noether's formula, we have Proposition 1.1 (*ii*).

Example 4.1. Let B be a reduced plane curve of degree n in $\mathbb{P}^2$ with at worst simple singularities. Suppose that there exists a generic triple cover $\pi : X \to \mathbb{P}^2$ with $\Delta(\pi) = B$. Note that n is necessarily even in this case. Let $\tilde{X}$ be the minimal resolution of X. Then:

$$\begin{aligned} K_{\tilde{X}}^2 &= \tfrac{1}{2}n^2 - 6n + 27 - \delta, \\ e(\tilde{X}) &= n^2 - 3n + 9 - 3\delta, \\ \chi(\mathcal{O}_{\tilde{X}}) &= \tfrac{1}{8}n^2 - \tfrac{3}{4}n + 3 - \tfrac{1}{3}\delta. \end{aligned}$$

Remark 4.1. Let $\pi\colon X \to \Sigma$ be a generic triple cover as before. Since B has at worst simple singularities, $D(X/\Sigma)$ has only rational double points as its singularities. Let

$$\begin{array}{ccc} D(X/\Sigma) & \xleftarrow{\nu} & Z \\ {\scriptstyle \beta_1(\pi)}\downarrow & & \downarrow{\scriptstyle f} \\ \Sigma & \xleftarrow{q_0} & \widetilde{\Sigma} \end{array}$$

be the canonical resolution, let $\hat{X}_Z$ be the $\mathbb{C}(\hat{X})$-normalization of Z and let $g : \hat{X}_Z \to Z$ be the induced cyclic triple cover. By what we have seen in this section, we infer the following:

- Irreducible components of $\Delta(g)$ are those in the exceptional curves for singularities in $\beta_1(\pi)^{-1}(T)$.
- $\Delta(g)$ is a disjoint union of $\mathbb{A}_2$-configurations, where $\mathbb{A}_2$- configuration means a divisor consisting of two irreducible components C_1 and C_2 such that $C_i \cong \mathbb{P}^1, C_i^2 = -2 (i = 1, 2)$ and $C_1C_2 = 1$.
- For each $p \in T$, the number of $\mathbb{A}_2$-configurations arising from p in $\Delta(g)$ is k (resp. 2) if p is of type A_{3k-1} (resp. E_6). In particular, the number of $\mathbb{A}_2$-configurations in $\Delta(g)$ is equal to δ in Proposition 1.1.

§5. Proof of Theorem 1.1

We apply our previous results to the case when $\Sigma = \mathbb{P}^2$ and B is a reduced sextic curve with at worst simple singularities. Note that we keep the notations as before. Let us start with the following lemma.

Lemma 5.1. *Let $\pi : X \to \mathbb{P}^2$ be a generic triple cover branched at a reduced plane sextic curve B as above. Then δ is either* 6 *or* 9. *If* $\delta = 6$ (resp. $= 9$), *the minimal resolution S of $\hat{X}$ is a $K3$* (resp. an Abelian) *surface.*

Proof. If $\deg B = 6$, then $D(X/\mathbb{P}^2)$ is a $K3$ surface with rational double points and Z is its minimal resolution. Let $\hat{X}_Z$ be the $\mathbb{C}(\hat{X})$-normalization of Z and $g : \hat{X}_Z \to Z$ be the induced cyclic triple cover as in Remark 4.1. Since Z is simply connected, the branch locus of g is non-empty and consists of disjoint union of $\mathbb{A}_2$-configurations by Remark 4.1. Now our statement follows from [22, Lemma 8.8]. Q.E.D.

Lemma 5.2. *Under the assumption of Lemma 5.1, B is a sextic curve with $9A_2$ singularities if $\delta = 9$.*

Proof. If $NT \neq \emptyset$, then $\hat{X}$ has a rational double point as $\beta_2(\pi)$ is unramified over $\beta_1(\pi)^{-1}(NT)$. This means that S in Lemma 5.1 contains a rational curve, but this impossible as S is an Abelian surface. Now we show that T consists of $9A_2$ points. Let $x \in T$ be any non A_2 point. Then by [22, Lemma 9.1], $\hat{X}$ has a rational double point; and S again contains a rational curve. As S is an Abelian surface, this is impossible. Q.E.D.

By [21], a sextic with $9A_2$ singularities is a $(2,3)$ torus curve. Hence, throughout the rest of this section, we always assume that $\delta = 6$. Therefore, by Example 4.1,

$$K_{\tilde{X}}^2 = 3, \quad e(\tilde{X}) = 9.$$

Lemma 5.3.

$$-K_{\tilde{X}} \sim \gamma^*\pi^*l,$$

where l is a line in $\mathbb{P}^2$.

Proof. We simply repeat the argument in the proof of [1, Lemma 3.15]. Let x be a general point in $\mathbb{P}^2 \setminus B$ and let $\gamma_1 : \tilde{X}_1 \to \tilde{X}$ be blowing ups at $(\pi \circ \gamma)^{-1}(x)$. The pencil of lines through x induces an elliptic fibration $\varphi_x : \tilde{X}_1 \to \mathbb{P}^1$ with a section. Since $e(\hat{X}_1) = 12$ and $K_{\tilde{X}_1}^2 = 0$, we infer that $\tilde{X}_1$ is rational surface and φ_x is relatively minimal. Hence $K_{\tilde{X}_1} \sim -F$, where F is a fiber of φ_x. As $\gamma_1^*((\pi\circ\gamma)^*l) \sim F+E_1+E_2+E_3$, where E_i $(i = 1,2,3)$ denote the exceptional curves of ρ, we have

$$\gamma_1^*((\pi \circ \gamma)^*l) \sim F + E_1 + E_2 + E_3 \sim -K_{\tilde{X}_1} + E_1 + E_2 + E_3 \sim -\gamma_1^*(K_{\tilde{X}}).$$

Therefore $(\pi \circ \gamma)^*l \sim -K_{\tilde{X}}$. Q.E.D.

By Lemma 5.3, it follows that $\tilde{X}$ is a smooth rational surface such that $K_{\tilde{X}}$ is big and numerically effective. Put

$$\overline{X} = \mathrm{Proj}(\oplus_{n\geq 0}\mathrm{H}^0(\tilde{X}, -nK_{\tilde{X}})).$$

Then by [6] p. 61–66, we have

Proposition 5.1. *Let $\varphi_{|-K_{\tilde{X}}|} : \tilde{X} \to \mathbb{P}^3$ be a morphism given by $|-K_{\tilde{X}}|$. Then $\varphi_{|-K_{\tilde{X}}|}(\tilde{X})$ is a normal cubic surface with rational double point isomorphic to $\overline{X}$.*

We are now in a position to prove Theorem 1.1. Our proof is almost the same as in [1, Proposition 3.17].

Suppose that B is given by the equation $G_2^3 + G_3^2 = 0$ as in Theorem 1.1. Consider the cubic surface X in $\mathbb{P}^3$ given by

$$X : X_3^3 + 3G_2(X_0, X_1, X_2)X_3 + 2G_3(X_0, X_1, X_2) = 0,$$

where $[X_0 : X_1 : X_2 : X_3]$ denotes a homogeneous coordinate system of $\mathbb{P}^3$. By [11, Lemma 5.1], X is smooth in codimension one, and therefore is normal. Let $P = [0 : 0 : 0 : 1]$ and let $pr_P : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be the projection centered at P. The restriction pr_P to X gives a non-Galois triple cover $pr_{P|X} : X \to \mathbb{P}^2$. By its defining equation, $\Delta(X/\mathbb{P}^2) = B$. Hence it is a generic triple cover branched at B.

Conversely, if there exists a generic triple cover $\pi : X \to \mathbb{P}^2$ branched at B, we have a normal cubic surface $\overline{X}$ as above. $\overline{X}$ is an image of $\tilde{X}$ by $\phi_{|-K_{\tilde{X}}|}$. Moreover, by Lemma 5.3, one has the following commutative diagram:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\phi_{|-K_{\tilde{X}}|}} & \mathbb{P}^3 \setminus \{P_0\} \\ {\scriptstyle \pi\circ\gamma}\downarrow & & \downarrow{\scriptstyle pr} \\ \mathbb{P}^2 & = & \mathbb{P}^2, \end{array}$$

where pr denotes the projection centered at a suitable point $P_0 \in \mathbb{P}^3 \setminus \overline{X})$. The remaining part of our proof is the same as [1, Proposition 3.17], and we omit it.

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