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A Willett type criterion with the best possible constant for linear dynamic equations

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Abstract.

We establish oscillation criteria for the linear dynamic equation $(r(t)y^{\Delta})^{\Delta} + p(t)y^{\sigma} = 0$. These criteria can be understood as an extension of the classical Willett criterion. What is special on these new results is that the constant involved in the criteria, which is equal to the "magic" 1/4 in the differential equations case, is in fact no more constant. In general case, it depends on the asymptotic behavior of the coefficients p, r, and primarily on the asymptotic behavior of graininess. In addition, we prove that the value of this new "constant" is the best possible.

§1. Introduction

Consider the linear dynamic equation

$$(r(t)y^{\Delta})^{\Delta} + p(t)y^{\sigma} = 0, \tag{1}$$

where $r(t) > 0$ and $p(t)$ are rd-continuous functions defined on a time scale interval $[a, \infty)$, $a \in \mathbb{T}$, and a time scale $\mathbb{T}$ is assumed to be unbounded from above.

If $\mathbb{T} = \mathbb{R}$, then (1) reduces to the Sturm-Liouville differential equation

$$(r(t)y')' + p(t)y = 0. \tag{2}$$

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It is known, see e.g. [14], that under the assumptions $\int_a^\infty 1/r(s)\,ds = \infty$ and $\int_t^\infty p(s)\,ds \geq 0$ ($\not\equiv 0$) for large t, equation (2) is oscillatory provided

$$\liminf_{t\to\infty} \left(\int_t^\infty p(s)\,ds\right)^{-1} \int_t^\infty \frac{1}{r(s)} \left(\int_s^\infty p(\tau)\,d\tau\right)^2 ds > \frac{1}{4}.$$

The constant $1/4$ on the right-hand side is the best possible. For historical reasons, we call this criterion as of Willett type.

The aim of this paper is to show how this criterion can be extended to linear dynamic equation (1). In particular, we will see that the constant $1/4$ is no more constant in general setting. In fact, in our criterion for (1), the new "constant" depends on the asymptotic behavior of the coefficients p, r and primarily on the asymptotic behavior of graininess. In addition, we will prove that this new constant is the best possible under quite mild assumptions. Thus a situation may happen (with $\mathbb{T} \neq \mathbb{R}$) where the constant is strictly greater than $1/4$, but still sharp.

Oscillatory properties of (1) or of closely related objects have been studied e.g. in [1, 2, 4, 5, 6, 7, 8, 9, 11, 12, 13]. Recall that (1) covers many of well-studied equations, like linear differential equations, linear difference equations, linear h-difference equations, and linear q-difference equations. Comparing our results with the existing ones, they are new even in the discrete case ($\mathbb{T} = \mathbb{Z}$) as well as in many other cases. There is a result in [12], which is very close to our new one, namely a Willett type criterion for half-linear dynamic equation. However, only the case which corresponds to the constant $1/4$ is discussed there. The newness of the presented result lies particularly in considering general case, where graininess becomes to play very important role, and this makes the problem quite complex. Thanks to this general setting, much wider class of equations can be examined.

The paper is organized as follows. In the next section we recall some important concepts and state preliminary results that are crucial to prove the main results. Generalized Willett type theorems are presented in the third section. Both cases are examined, $\int_a^\infty 1/r(s)\,\Delta s = \infty$ and $\int_a^\infty 1/r(s)\,\Delta s < \infty$.

§2. Basic concepts and preliminary results

We assume that the reader is familiar with the notion of time scales. Thus note just that $\mathbb{T}$, σ, f^σ, μ, f^Δ, and $\int_a^b f^\Delta(s)\,\Delta s$ stand for time scale, forward jump operator, $f \circ \sigma$, graininess, delta derivative of f, and delta integral of f from a to b, respectively. See [10], which is the initiating paper of the time scale theory written by Hilger, and the monographs [2, 3] by Bohner and Peterson containing a lot of information on time scale calculus.

We will proceed with some essentials of oscillation theory of (1). First note that we are interested only in nontrivial solutions of (1). We say that a solution y of (1) has a *generalized zero* at t in case $y(t) = 0$. If $\mu(t) > 0$, then we say that y has a *generalized zero* in $(t, \sigma(t))$ in case $y(t)y^\sigma(t) < 0$. A nontrivial *solution* y of (1) is called *oscillatory* if it has infinitely many generalized zeros; note that the uniqueness of IVP excludes the existence of a cluster point which is less than ∞. Otherwise it is said to be *nonoscillatory*. In view of the fact that the Sturm type separation theorem extends to (1) (see e.g. [11]), we have the following equivalence: One solution of (1) is oscillatory if and only if every solution of (1) is oscillatory. Hence we may speak about *oscillation* or *nonoscillation of equation* (1). Recall that the principal statements, like the Sturmian theory (Reid type roundabout theorem, Sturm type separation and comparison theorems) for (1), can be established under the mere assumption $r(t) \neq 0$ and the basic concepts, especially generalized zero, have to be adjusted, see e.g. [1] or [11].

The next lemma, called the function sequence technique, plays a crucial role in proving the main results. Its proof is based on the equivalence between nonoscillation of (1) and solvability of the Riccati type integral inequality

$$w(t) \geq \int_t^\infty p(s)\,\Delta s + \int_t^\infty w^2(s)/(r(s) + \mu(s)w(s))\,\Delta s.$$

Lemma 1 ([12]). *Assume that* $\int_a^\infty 1/r(s)\,\Delta s = \infty$, $\int_t^\infty p(s)\,\Delta s \geq 0$ *and nontrivial for large* t. *Define the function sequence* $\{\psi_k(t)\}$ *by*

$$\psi_0(t) = \int_t^\infty p(s)\,\Delta s, \quad \psi_1(t) = \int_t^\infty \frac{\psi_0^2(s)}{r(s) + \mu(s)\psi_0(s)}\,\Delta s,$$
$$\psi_{k+1}(t) = \int_t^\infty \frac{(\psi_0(s) + \psi_k(s))^2}{r(s) + \mu(s)(\psi_0(s) + \psi_k(s))}\,\Delta s, \quad k = 1, 2, \ldots.$$

Then equation (1) *is nonoscillatory if and only if there exists* $t_0 \in [a, \infty)$ *such that* $\lim_{k\to\infty} \psi_k(t) = \psi(t)$ *for* $t \geq t_0$, *i.e., the sequence* $\{\psi_k(t)\}$ *is well defined and pointwise convergent.*

The following lemma, which exploits the transformation of dependent variable, will be useful in the case when $\int_a^\infty 1/r(s)\,\Delta s$ converges.

Lemma 2 ([9]). *Assume that* h *is an rd-continuously delta differentiable function with* $h(t) \neq 0$. *Then* $y = hu$ *transforms equation* (1) *into the equation* $(\tilde{r}(t)u^\Delta)^\Delta + \tilde{p}(t)u^\sigma = 0$ *with* $\tilde{r} = rhh^\sigma$ *and*

$\tilde{p} = h^\sigma[(rh^\Delta)^\Delta + ph^\sigma]$. *This transformation preserves oscillatory properties.*

§3. Main results

Now we are ready to prove the main result, an extension of the Willett criterion.

Theorem 1. *Let*

$$\int_a^\infty \frac{1}{r(s)}\,\Delta s = \infty. \tag{3}$$

Assume that

$$\int_t^\infty p(s)\,\Delta s \geq 0 \quad \textit{and nontrivial for large } t. \tag{4}$$

Denote

$$N^* := \limsup_{t\to\infty} \frac{\mu(t)\int_t^\infty p(s)\,\Delta s}{r(t)} \tag{5}$$

and let $N^* < \infty$. *If*

(6)

$$\liminf_{t\to\infty}\left(\int_t^\infty p(s)\,\Delta s\right)^{-1}\int_t^\infty \frac{1}{r(s)}\left(\int_s^\infty p(\tau)\,\Delta\tau\right)^2 \Delta s > \frac{(N^*+1)^2}{4},$$

then equation (1) *is oscillatory.*

Moreover, if there exists the limit

$$N := \lim_{t\to\infty} \frac{\mu(t)\int_t^\infty p(s)\,\Delta s}{r(t)} < \infty, \tag{7}$$

then the constant $(N+1)^2/4$ *in* (6) *is the best possible.*

Proof. We will apply Lemma 1 and use its notation. Condition (6) can be rewritten as

$$\int_t^\infty \psi_0^2(s)/r(s)\,\Delta s \geq \delta\psi_0(t)$$

for large t, say $t \geq t_0 \geq a$, where $\delta > (N^*+1)^2/4$. Then

$$\begin{aligned}\psi_1(t) &= \int_t^\infty \frac{\psi_0^2(s)}{r(s)} \cdot \frac{r(s)}{r(s)+\mu(s)\psi_0(s)}\,\Delta s\\ &\geq \inf_{s\geq t_0} \frac{r(s)}{r(s)+\mu(s)\psi_0(s)} \int_t^\infty \frac{\psi_0^2(s)}{r(s)}\,\Delta s\\ &\geq \Gamma(t_0,1)\delta\psi_0(t) = \delta_1\psi_0(t),\end{aligned}$$

where $\delta_1 = \Gamma(t_0, 1)\delta$ and

$$\Gamma(t_0, x) = \inf_{t \geq t_0} \frac{1}{1 + x\mu(t)\psi_0(t)/r(t)}.$$

Further, since $x \mapsto x^2/(y + zx)$ is increasing for $x > 0, y > 0, z > 0$, similarly as above we have

$$\begin{aligned}\psi_2(t) &\geq (\delta_1 + 1)^2 \int_t^\infty \frac{\psi_0^2(s)}{r(s) + (\delta_1 + 1)\mu(s)\psi_0(s)} \Delta s \\ &\geq (\delta_1 + 1)^2 \Gamma(t_0, \delta_1 + 1)\delta\psi_0(t) = \delta_2\psi_0(t),\end{aligned}$$

where $\delta_2 = (\delta_1 + 1)^2\Gamma(t_0, \delta_1 + 1)\delta$. Similarly, by induction,

$$\psi_k(t) \geq \delta_k\psi_0(t), \quad k = 1, 2, \ldots, \tag{8}$$

where $\delta_1 = \Gamma(t_0, 1)\delta$ and

$$\delta_{k+1} = (\delta_k + 1)^2\Gamma(t_0, \delta_k + 1)\delta, \quad k = 1, 2, \ldots. \tag{9}$$

First assume that there exists the limit $N = \lim_{t\to\infty} \mu(t) \int_t^\infty p(s)\Delta s/r(t)$. Then taking the limit as $t_0 \to \infty$, (9) yields

$$\delta_{k+1} = \frac{(\delta_k + 1)^2}{1 + (\delta_k + 1)N}\delta. \tag{10}$$

Since $x \mapsto (x + 1)^2/[1 + (x + 1)N]$ is increasing, $\delta_k \leq \delta_{k+1}$ $[\delta_k \geq \delta_{k+1}]$ implies $\delta_{k+1} \leq \delta_{k+2}$ $[\delta_{k+1} \geq \delta_{k+2}]$. Hence, $\{\delta_k\}$ is monotone. We claim that $\delta_k \to \infty$ as $k \to \infty$. If not, let $\delta_k \to L < \infty$ as $k \to \infty$. Clearly, we cannot have $L = 0$, otherwise, from (10), $0 = 1/(1 + N) > 0$. Thus $L > 0$. From (10) we obtain

$$L = \frac{(L + 1)^2}{1 + (L + 1)N}\delta. \tag{11}$$

Next we show that (11) has no real solution. Indeed, (11) can be rewritten as

$$(N - \delta)L^2 + (1 + N - 2\delta)L - \delta = 0. \tag{12}$$

Note that $N \neq \delta$, otherwise, in view of $\delta > (N+1)^2/4$, we get $(N-1)^2 < 0$, contradiction. The discriminant of (12) is equal to $(N + 1)^2 - 4\delta$, and so it is negative since we have assumed $\delta > (N + 1)^2/4$. This proves that $\delta_k \to \infty$ as $k \to \infty$, which implies $\psi_k(t) \to \infty$ as $k \to \infty$ for $t \geq t_0$, by (8). Consequently, (1) is oscillatory, in view of Lemma 1. Next we examine the case when

$$\liminf_{t\to\infty} \frac{\mu(t)\int_t^\infty p(s)\Delta s}{r(t)} < \limsup_{t\to\infty} \frac{\mu(t)\int_t^\infty p(s)\Delta s}{r(t)} = N^*.$$

Then we have

$$\lim_{t_0\to\infty} \Gamma(t_0, \delta_k + 1) = \frac{1}{1 + (\delta_k + 1)N^*}$$

and hence, using the same arguments as above, we come to the conclusion that $\delta_k \to \infty$ as $k \to \infty$. This implies again oscillation of (1).

In the second part we prove that the constant $(N+1)^2/4$ in (6) is the best possible provided (7) holds. To show this, consider the Euler type dynamic equation

$$y^{\Delta\Delta} + \frac{\lambda}{t\sigma(t)} y^\sigma = 0, \tag{13}$$

where λ is a positive real parameter, i.e., equation (1) where $r(t) \equiv 1$ and $p(t) = \lambda/(t\sigma(t))$. Assume that there exists the limit $M := \lim_{t\to\infty} \mu(t)/t < \infty$. Then, for N defined by (7), we have $N = \lambda M$. Further,

$$\begin{aligned}
&\left(\int_t^\infty p(s)\,\Delta s\right)^{-1} \int_t^\infty \frac{1}{r(s)} \left(\int_s^\infty p(\tau)\,\Delta\tau\right)^2 \Delta s \\
&= \frac{t}{\lambda} \int_t^\infty \frac{\lambda^2}{s^2}\,\Delta s \\
&= \lambda t \int_t^\infty \frac{1}{s\sigma(s)} \cdot \frac{\sigma(s)}{s}\,\Delta s \\
&\geq \lambda \inf_{s\geq t} \frac{\sigma(s)}{s}.
\end{aligned}$$

In order to be (6) fulfilled, it suffices to take λ such that

$$\lambda \liminf_{t\to\infty} \frac{\sigma(t)}{t} > \frac{(N+1)^2}{4}. \tag{14}$$

Since $\sigma(t)/t = 1 + \mu(t)/t \to 1 + M$ as $t \to \infty$ and $N = \lambda M$, (14) is equivalent to

$$\lambda(M+1) > \frac{(\lambda M + 1)^2}{4}. \tag{15}$$

If $M = 0$, then clearly $\lambda > 1/4$ implies oscillation of (13) by the first part of Theorem 1. Solving quadratic inequality (15) when $M > 0$ and using the first part of Theorem 1 combined with the Sturm type comparison theorem, we find that (13) is oscillatory provided

$$\lambda > \frac{M + 2 - 2\sqrt{M+1}}{M^2} = \frac{\left(\sqrt{M+1} - 1\right)^2}{M^2} = \frac{1}{\left(\sqrt{M+1} + 1\right)^2}.$$

Altogether, for any $M \in [0, \infty)$ we have that $\lambda > \left(\sqrt{M+1} + 1\right)^{-2}$ implies oscillation of (13). From [13] we know that $\lambda_0 = \left(\sqrt{M+1} + 1\right)^{-2}$ is the oscillation constant for (13), i.e., (13) is oscillatory for all $\lambda > \lambda_0$ and nonoscillatory for all $\lambda < \lambda_0$. Now, since $N = \lambda M$, we see that N cannot be lowered, and this proves that $(N+1)^2/4$ in (14) and so in (6) is the best possible. Q.E.D.

Using the transformation of dependent variable and Theorem 1 we can easily treat the complementary case to (3), namely $\int_a^\infty 1/r(s)\,\Delta s$ converges.

Theorem 2. *Let*

$$\int_a^\infty \frac{1}{r(s)}\,\Delta s < \infty.$$

Assume that

$$\int_t^\infty \left(\int_{\sigma(s)}^\infty \frac{1}{r(\tau)}\,\Delta\tau\right)^2 p(s)\,\Delta s \geq 0 \quad \textit{and nontrivial for large } t.$$

Denote $R(t) := \int_t^\infty 1/r(s)\,\Delta s$. *Let*

$$\tilde{N}^* := \limsup_{t\to\infty} \frac{\mu(t)\int_t^\infty (R^\sigma(s))^2 p(s)\,\Delta s}{r(t)R(t)R^\sigma(t)} < \infty.$$

If

$$(16)\quad \liminf_{t\to\infty} \left(\int_t^\infty (R^\sigma(s))^2 p(s)\,\Delta s\right)^{-1} \int_t^\infty \frac{\left(\int_s^\infty (R^\sigma(\tau))^2 p(\tau)\,\Delta\tau\right)^2}{r(s)R(s)R^\sigma(s)}\,\Delta s > \frac{\tilde{N}^* + 1}{4},$$

then equation (1) *is oscillatory.*

Moreover, if there exists the limit

$$\tilde{N} := \lim_{t\to\infty} \frac{\mu(t)\int_t^\infty (R^\sigma(s))^2 p(s)\,\Delta s}{r(t)R(t)R^\sigma(t)} < \infty,$$

then the constant $(\tilde{N}+1)^2/4$ *in* (16) *is the best possible.*

Proof. Denote $R(t) := \int_t^\infty 1/r(s)\,\Delta s$. First note that by Lemma 2, the transformation $y = hu$ with $h(t) = R(t)$ transforms (1) into the equation $(\tilde{r}(t)u^\Delta)^\Delta + \tilde{p}(t)u^\sigma = 0$, where $\tilde{r}(t) = R(t)R^\sigma(t)r(t)$ and $\tilde{p}(t) = (R^\sigma(t))^2 p(t)$. Since $(1/R(t))^\Delta = 1/\tilde{r}(t)$, we get that $\int_a^\infty 1/\tilde{r}(s)\,\Delta s = \infty$. Applying now Theorem 1 to the transformed equation and using the fact that oscillatory properties are preserved after the transformation, we get the statement. Q.E.D.

References

[1] R. P. Agarwal and M. Bohner, Quadratic functionals for second order matrix equations on time scales, Nonlinear Anal., **33** (1998), 675–692.
[2] M. Bohner and A. C. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2001.
[3] M. Bohner and A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.
[4] M. Bohner and S. H. Saker, Oscillation of second order nonlinear dynamic equations on time scales, Rocky Mountain J. Math., **34** (2004), 1239–1254.
[5] M. Bohner and M. Ünal, Kneser's theorem in q-calculus, J. Phys. A, **38** (2005), 6729–6739.
[6] O. Došlý and S. Hilger, A necessary and sufficient condition for oscillation of the Sturm-Liouville dynamic equation on time scales, J. Comput. Appl. Math., **141** (2002), 147–158.
[7] L. H. Erbe and S. Hilger, Sturmian theory on measure chains, Differential Equations Dynam. Systems, **1** (1993), 223–244.
[8] L. H. Erbe and A. C. Peterson, Some recent results in linear and nonlinear oscillation, Dynam. Systems Appl., **13** (2004), 381–395.
[9] L. H. Erbe, A. C. Peterson and P. Řehák, Integral comparison theorems for second order linear dynamic equations, In: Difference Equations, Special Functions and Orthogonal Polynomials, Proceedings of the International Conference, Munich, Germany, 25-30 July 2005, (eds. S. Elaydi, J. Cushing, R. Lasser, A. Ruffing, V. Papageorgiou and W. Van Assche), World Scientific, 2007, pp. 497–506.
[10] S. Hilger, Analysis on measure chains — a unified approach to continuous and discrete calculus, Results Math., **18** (1990), 18–56.
[11] P. Řehák, Half-linear dynamic equations on time scales: IVP and oscillatory properties, Nonlinear Funct. Anal. Appl., **7** (2002), 361–404.
[12] P. Řehák, Function sequence technique for half-linear dynamic equations on time scales, Panamer. Math. J., **16** (2006), 31–56.
[13] P. Řehák, How the constants in Hille–Nehari theorems depend on time scales, Adv. Difference Equ., **2006** (2006), 1–15.

[14] D. Willett, On the oscillatory behavior of the solutions of second order linear differential equations, Ann. Polon. Math., **21** (1969), 175–194.

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