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On the uniform perfectness of diffeomorphism groups

Takashi Tsuboi

Abstract.

We show that any element of the identity component of the group of C^r diffeomorphisms $\mathrm{Diff}^r_c(\boldsymbol{R}^n)_0$ of the n-dimensional Euclidean space $\boldsymbol{R}^n$ with compact support ($1 \leqq r \leqq \infty$, $r \neq n+1$) can be written as a product of two commutators. This statement holds for the interior M^n of a compact n-dimensional manifold which has a handle decomposition only with handles of indices not greater than $(n-1)/2$. For the group $\mathrm{Diff}^r(M)$ of C^r diffeomorphisms of a compact manifold M, we show the following for its identity component $\mathrm{Diff}^r(M)_0$. For an even-dimensional compact manifold M^{2m} with handle decomposition without handles of the middle index m, any element of $\mathrm{Diff}^r(M^{2m})_0$ ($1 \leqq r \leqq \infty$, $r \neq 2m+1$) can be written as a product of four commutators. For an odd-dimensional compact manifold M^{2m+1}, any element of $\mathrm{Diff}^r(M^{2m+1})_0$ ($1 \leqq r \leqq \infty$, $r \neq 2m+2$) can be written as a product of six commutators.

§1. Introduction

For a manifold M, let $\mathrm{Diff}^r_c(M)$ denote the group of C^r diffeomorphisms of M with compact support ($1 \leqq r \leqq \infty$). The *support* of a diffeomorphism f of M is defined to be the *closure* of $\{x \in M \mid f(x) \neq x\}$. Let $\mathrm{Diff}^r_c(M)_0$ denote the identity component of $\mathrm{Diff}^r_c(M)$. Here $\mathrm{Diff}^r_c(M)$ is equipped with the C^r topology. By the results of Mather and Thurston

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([7], [8], [12]), for an n-dimensional manifold M^n, $\mathrm{Diff}_c^r(M^n)_0$ is a perfect group if $r = 0$ or $1 \leqq r \leqq \infty$ and $r \neq n+1$. A group is perfect if it coincides with its commutator subgroup.

We study in this paper the uniform perfectness of $\mathrm{Diff}_c^r(M^n)_0$. A group is uniformly perfect if any element can be written as a product of a bounded number of commutators. In [7], Mather showed that any element of $\mathrm{Homeo}_c(\boldsymbol{R}^n)$ can be written as a commutator. Hence any element of $\mathrm{Homeo}(S^n)_0$ can be written as a product of two commutators. In [14], $\mathrm{Diff}_c^r(\boldsymbol{R}^n)_0$ $(1 \leqq r < n+1)$ is shown to be uniformly perfect. Hence $\mathrm{Diff}^r(S^n)_0$ $(1 \leqq r < n+1)$ is also uniformly perfect. By the result of Herman [5], any element of $\mathrm{Diff}^\infty(S^1)_0$ can be written as a product of two commutators.

We show in this paper that any element of $\mathrm{Diff}_c^r(\boldsymbol{R}^n)_0$ $(1 \leqq r \leqq \infty$, $r \neq n+1)$ can be written as a product of two commutators (Theorem 2.1). The same technique applies to showing that for the interior M^n of a compact n-dimensional manifold which has a handle decomposition only with handles of indices not greater than $(n-1)/2$, any element of $\mathrm{Diff}_c^r(M^n)_0$ $(1 \leqq r \leqq \infty$, $r \neq n+1)$ can be written as a product of two commutators (Theorem 4.1). The handle decomposition of a compact manifold is summarized in Section 3.

For compact manifolds M^n, we show (Theorem 5.1) that if M^n has a handle decomposition without handles of middle indices, then any element of $\mathrm{Diff}^r(M^n)_0$ can be written as a composition of elements to which Theorem 4.1 is applicable. Then we show that for an even-dimensional compact manifold M^{2m} which has a handle decomposition without handles of the middle index m, any element of $\mathrm{Diff}^r(M^{2m})_0$ $(1 \leqq r \leqq \infty$, $r \neq 2m+1)$ can be written as a product of four commutators (Theorem 5.2). For an odd-dimensional compact manifold M^{2m+1}, Theorem 5.2 asserts that if there are no handles of indices m and $m+1$, any element of $\mathrm{Diff}^r(M^{2m+1})_0$ $(1 \leqq r \leqq \infty$, $r \neq 2m+2)$ can be written as a product of four commutators, but we have a stronger result for odd-dimensional compact manifolds. By using the idea of the paper [2] by Burago, Ivanov and Polterovich, we can prove that for any odd-dimensional compact manifold M^{2m+1}, any element of $\mathrm{Diff}^r(M^{2m+1})_0$ $(1 \leqq r \leqq \infty$, $r \neq 2m+2)$ can be written as a product of six commutators (Theorem 6.1).

The topology of the manifold may prevent the group $\mathrm{Diff}^r(M)_0$ from being uniformly perfect. We thought that if an element of $\mathrm{Diff}^r(M)_0$ could be connected to the identity only by a very long isotopy, then the number of commutators to write this element would be long. What we show here is the following. Unless the manifold is even-dimensional and having a handle decomposition with handles of the middle index, we can replace the isotopy by a nicer one to write a diffeomorphisms as a product of bounded number of commutators.

In the proof of Theorem 2.1, we use the result on the perfectness of the group $\mathrm{Diff}_c^r(\boldsymbol{R}^n)_0$ ($1 \leqq r \leqq \infty$, $r \neq n+1$) by Mather and Thurston ([8], [12]) and construct necessary diffeomorphisms. The author got the idea of this construction when he was studying the paper [6] of Dieter Kotschick remembering some discussion with him during his stay at the University of Tokyo in 2006. The author is very grateful to him. The author also thank Shigenori Matsumoto for several valuable comments.

While the author was preparing a preliminary version of this paper, Danny Calegari informed him the existence of the paper [2] by Burago, Ivanov and Polterovich, which the author overlooked. In [2] they proved the results corresponding to our Theorems 2.1, 4.1, and 5.2 in the case of spheres. Moreover they made an excellent observation of tracing the isotopy of a graph after intersecting another graph and showed the uniform perfectness of $\mathrm{Diff}^r(M^3)_0$ for closed 3-dimensional manifolds M^3. The proof of the uniform perfectness of $\mathrm{Diff}^r(M^{2m+1})_0$ for odd-dimensional manifolds M^{2m+1} is rather straight forward after the idea of their paper and our Theorem 5.2. Leonid Polterovich pointed out to the author that these groups treated in this paper are meager in their terminology (Remark 6.6). The author is very grateful to Danny Calegari and Leonid Polterovich for their valuable comments. The author is also grateful to the referee for the suggestions for improving the exposition of this paper.

§2. Diffeomorphisms of the Euclidean space

First we give the proof of the following theorem.

Theorem 2.1. *Let* $\mathrm{Diff}_c^r(\boldsymbol{R}^n)$ *be the group of diffeomorphisms of the n-dimensional Euclidean space* $\boldsymbol{R}^n$ *with compact support and let* $\mathrm{Diff}_c^r(\boldsymbol{R}^n)_0$ *be its identity component. If* $1 \leqq r \leqq \infty$ *and* $r \neq n+1$, *then any element of* $\mathrm{Diff}_c^r(\boldsymbol{R}^n)_0$ *can be written as a product of two commutators.*

Proof. Take an element $f \in \mathrm{Diff}_c^r(\boldsymbol{R}^n)_0$ ($r \neq n+1$). By the result of Mather and Thurston ([7], [8], [12]), f can be written as a product of commutators.

$$f = [a_1, b_1] \cdots [a_k, b_k], \quad a_1,\, b_1,\, \ldots,\, a_k,\, b_k \in \mathrm{Diff}_c^r(\boldsymbol{R}^n)_0,$$

where $[a_i, b_i] = a_i b_i {a_i}^{-1} {b_i}^{-1}$. Let U be an open ball in $\boldsymbol{R}^n$ such that the supports of a_i, b_i as well as the supports of the isotopies $\{a_{it}\}_{t\in[0,1]}$ ($a_{i0} = \mathrm{id}$ and $a_{i1} = a_i$), $\{b_{it}\}_{t\in[0,1]}$ ($b_{i0} = \mathrm{id}$ and $b_{i1} = b_i$) are contained in U. Let $g \in \mathrm{Diff}^r_c(\boldsymbol{R}^n)_0$ be an element such that $g^i(U)$ $(i \in \boldsymbol{Z})$ are disjoint. Put

$$F = \prod_{i=1}^{k} g^{k-i}([a_1, b_1] \cdots [a_i, b_i]) g^{i-k}.$$

Then F is an element of $\mathrm{Diff}^r_c(\boldsymbol{R}^n)_0$. Now the conjugate of F by g is as follows:

$$\begin{aligned} gFg^{-1} &= \prod_{i=1}^{k} g^{k-i+1}([a_1, b_1] \cdots [a_i, b_i]) g^{i-k-1} \\ &= \prod_{i=0}^{k-1} g^{k-i}([a_1, b_1] \cdots [a_{i+1}, b_{i+1}]) g^{i-k}. \end{aligned}$$

Hence

$$\begin{aligned} F^{-1}gFg^{-1} &= ([a_1, b_1] \cdots [a_k, b_k])^{-1} \prod_{i=0}^{k-1} g^{k-i}[a_{i+1}, b_{i+1}] g^{i-k} \\ &= f^{-1} \prod_{i=0}^{k-1} g^{k-i}[a_{i+1}, b_{i+1}] g^{i-k} \\ &= f^{-1} \Big[\prod_{i=0}^{k-1} g^{k-i} a_{i+1} g^{i-k}, \prod_{i=0}^{k-1} g^{k-i} b_{i+1} g^{i-k} \Big]. \end{aligned}$$

Put

$$A = \prod_{i=0}^{k-1} g^{k-i} a_{i+1} g^{i-k} \quad \text{and} \quad B = \prod_{i=0}^{k-1} g^{k-i} b_{i+1} g^{i-k},$$

then A and B are elements of $\mathrm{Diff}^r_c(\boldsymbol{R}^n)_0$. Thus f can be written as a product of two commutators: $f = [A, B][g, F^{-1}]$. Q.E.D.

The proof uses only the fact that there is an open set U which contains the support of given finitely many diffeomorphisms and a compact support diffeomorphism g such that $g^i(U)$ $(i \in \boldsymbol{Z})$ are disjoint. Hence we have the following corollary.

Corollary 2.2. *Let M^n be an n-dimensional manifold diffeomorphic to $N^p \times \boldsymbol{R}^q$ $(q \geqq 1,\ p + q = n)$ for a compact manifold N^p, then any element of* $\mathrm{Diff}^r_c(M^n)_0$ $(1 \leqq r \leqq \infty,\ r \neq n + 1)$ *can be written as a product of two commutators.*

§3. Review of the Morse theory and handle decompositions

In our theorems, the assumptions are given in terms of handle decompositions. We review in this section several facts on the Morse theory for manifolds and handle decompositions ([10], [11]).

A function $f : M^n \longrightarrow \boldsymbol{R}$ on a compact n-dimensional manifold M^n without boundary is called a Morse function if the critical points are nondegenerate, that is, the Hessian matrices of f at the critical points are nondegenerate. For such a function f, the set of critical points is a finite set. The index of the Hessian matrix of f at a critical point is called the index of the critical point.

Any compact connected n-dimensional manifold M^n without boundary admits a Morse function $f : M^n \longrightarrow \boldsymbol{R}$ such that $f(M^n) = [0, n]$, the set of critical points of index k is contained in $f^{-1}(k)$ $(k = 0, \ldots, n)$ and $f^{-1}(0)$ and $f^{-1}(n)$ are one point sets ([11]).

Put $W_k = f^{-1}([0, k+1/2])$, and then this W_k is a compact manifold with boundary $\partial W_k = f^{-1}(k + 1/2)$. Let c_k be the number of critical points of index k. Then the manifold W_k is diffeomorphic to the manifold obtained from W_{k-1} by attaching c_k handles of index k $(k = 0, \ldots, n)$. This means the following.

Let $D^k \times D^{n-k}$ be the product of the k-dimensional disk D^k and the $(n-k)$-dimensional disk D^{n-k}. Let $\varphi_i : (\partial D^k) \times D^{n-k} \longrightarrow \partial W_{k-1}$ $(i = 1, \ldots, c_k)$ be diffeomorphisms with disjoint images. Let

$$W'_k = W_{k-1} \cup_{\bigsqcup_{i=1}^{c_k} \varphi_i} \bigsqcup_{i=1}^{c_k} (D^k \times D^{n-k})_i$$

be the space obtained from the disjoint union $W_{k-1} \sqcup \bigsqcup_{i=1}^{c_k} (D^k \times D^{n-k})_i$ by identifying $x \in (\partial D^k) \times D^{n-k} \subset (D^k \times D^{n-k})_i$ with $\varphi_i(x) \in \partial W_{k-1} \subset W_{k-1}$.

For given triangulations of W_{k-1} and of $(D^k \times D^{n-k})_i$, we have subdivisions of them such that φ_i after isotoped are piecewise linear isomorphisms to the images. Thus W'_k has a triangulation as a piecewise linear manifold.

On the other hand, by smoothing along the corner which is the image $\bigsqcup_{i=1}^{c_k} \varphi_i((\partial D^k) \times (\partial D^{n-k}))$, W'_k has a differentiable structure. This manifold W'_k is the manifold obtained from W_{k-1} by attaching c_k handles of index k $(k = 0, \ldots, n)$ which we stated. The image of $D^k \times D^{n-k}$ is called a handle of index k. We will simply write the handle of index k as $(D^k \times D^{n-k})_i$.

Then the manifold W_k is diffeomorphic to the manifold W'_k with boundary. It is better to say that the manifold W_k is obtained from the manifold W'_k by adding the collar of the boundary $\partial W'_k$.

By using the sequence of submanifolds $D^n \cong W_0 \subset W_1 \subset \cdots \subset W_n = M^n$ and the diffeomorphisms $W_k \cong W'_k$, M^n is decomposed into the union of the handles $(D^k \times D^{n-k})_i$ $(i = 1, \ldots, c_k; k = 0, \ldots, n)$. This decomposition into handles is called a handle decomposition of M. We write the handle decomposition as $D^n \cong W_0 \subset W_1 \subset \cdots \subset W_n = M^n$. This handle decomposition represents a piecewise linear structure as well as a differentiable structure. We call the image of $D^k \times \{0\}$ the core disk of the handle $(D^k \times D^{n-k})_i$ of index k, and the image of $\{0\} \times D^{n-k}$ its co-core disk.

For the above Morse function $f : M^n \longrightarrow \boldsymbol{R}$ and the constant function n, the function $n - f$ is a Morse function, and the critical points of the Morse function f of index k are nothing but the critical points of the Morse function $n - f$ of index $n - k$. Hence this gives rise to a handle decomposition of M^n called the dual handle decomposition. A handle decomposition and its dual handle decomposition can be considered identical as a decomposition of M^n into subsets. The handles of index k of the original handle decomposition corresponds to the handles of index $n - k$ of the dual handle decomposition. This duality switches the roles of core disks and co-core disks.

By choosing a Riemannian metric on the manifold M^n, the Morse function f defines the gradient vector field and the gradient flow Ψ_t. The singular points of the gradient vector field are precisely the critical points of f. The core disk and the co-core disk of a handle of a handle decomposition of M^n correspond to the local stable manifold and the local unstable manifold of the corresponding singular point p of the gradient flow Ψ_t, respectively ([10], [11]). Let e_i^k and e'^{n-k}_i denote the global stable manifold and the global unstable manifold, respectively, for the singular point p which is a critical point of index k of f. Then e_i^k and e'^{n-k}_i are diffeomorphic to $\boldsymbol{R}^k$ and $\boldsymbol{R}^{n-k}$, respectively. Then we know that the global stable manifolds and the global unstable manifolds of critical points give the cell decomposition $\bigcup_{k=0}^n \bigcup_{i=1}^{c_k} e_i^k$ and the dual cell decomposition $\bigcup_{k=0}^n \bigcup_{i=1}^{c_k} e'^{n-k}_i$ of M^n, respectively ([10]). The dual cell decomposition is the cell decomposition for the Morse function $n-f$. Consider the k-skeleton $X^{(k)}$ of the cell decomposition and the $(n-k-1)$-skeleton $X'^{(n-k-1)}$ of the dual cell decomposition:

$$X^{(k)} = \bigcup_{j \leqq k} \bigcup_{i=1}^{c_j} e_i^j \quad \text{and} \quad X'^{(n-k-1)} = \bigcup_{j \geqq k+1} \bigcup_{i=1}^{c_j} e'^{n-j}_i.$$

The boundary ∂W_k of W_k is transverse to the gradient flow Ψ_t, and hence $M \setminus (X^{(k)} \cup X'^{(n-k-1)})$ is diffeomorphic to $\partial W_k \times \boldsymbol{R}$ by the map

$$\partial W_k \times \boldsymbol{R} \ni (x, t) \longmapsto \Psi_t(x) \in M \setminus (X^{(k)} \cup X'^{(n-k-1)}).$$

Moreover $\Psi_t(\partial W_k)$ converges to $X^{(k)}$ as $t \longrightarrow -\infty$ and to $X'^{(n-k-1)}$ as $t \longrightarrow \infty$. Hence, $M \setminus X'^{(n-k-1)}$ is diffeomorphic to the interior $\mathrm{int}(W_k)$

of W_k and $X^{(k)}$ is a deformation retract of both W_k and $M \setminus X'^{(n-k-1)}$:

$$X^{(k)} \subset \mathrm{int}(W_k) \subset W_k \subset M \setminus X'^{(n-k-1)}.$$

Using the flow Ψ_t, for any neighborhood V of $X^{(k)}$ and for any compact subset A in $\mathrm{int}(W_k)$, we can construct an isotopy $\{G_t : \mathrm{int}(W_k) \longrightarrow \mathrm{int}(W_k)\}_{t\in[0,1]}$ with compact support such that $G_0 = \mathrm{id}_{\mathrm{int}(W_k)}$, $G_t|X^{(k)} = \mathrm{id}_{X^{(k)}}$ $(t \in [0,1])$ and $G_1(A) \subset V$. A similar statement is true for $X^{(k)} \subset M \setminus X'^{(n-k-1)}$.

By careful choices of the Morse function and the Riemannian metric on M, the cell complexes $X^{(k)}$ and $X'^{(n-k-1)}$ become differentiably embedded simplicial complexes. Since we use this fact, we give here a sketch of the proof.

Proposition 3.1. *Let L be an ℓ-dimensional simplicial complex differentiably embedded in ∂W_k ($\ell \geqq k$). Then there is an $(\ell+1)$-dimensional simplicial complex $\widehat{L}$ differentiably embedded in W_k such that $\partial W_k \cap \widehat{L} = L$ and $\widehat{L}$ is a deformation retract of W_k.*

Sketch of the proof. The proof is roughly as follows: It is shown by the induction on k. For $k = 0$, we take the cone of L as $\widehat{L}$. We assume that the assertion is true for $k-1$ and we construct $\widehat{L}$ for W_k. First, for the handles $(D^k \times D^{n-k})_i$ of index k ($i = 1, \ldots, c_k$), we can deform the co-core disks $(\{0\}\times D^{n-k})_i$ so that the belt spheres $S_i^{n-k-1} = (\{0\}\times(\partial D^{n-k}))_i$ are in general position to L. In a neighborhood of a belt sphere S_i^{n-k-1}, L is isomorphic to the product of a small k-dimensional disk B^k and $S_i^{n-k-1} \cap L$. We can subdivide $L'_i = S_i^{n-k-1} \cap L$ so that L'_i becomes an $(\ell-k)$-dimensional simplicial complex. Then we subdivide L and the triangulation of $(D^k \times D^{n-k})_i$ $(\subset W_k)$ so that $B^k \times (\{0\} * L'_i)$ $(\subset (B^k \times D^{n-k})_i)$ becomes an $(\ell+1)$-dimensional subcomplex of the subcomplex $(B^k \times D^{n-k})_i$ of $(D^k \times D^{n-k})_i \subset W_k$ ($i = 1, \ldots, c_k$) after isotoping the triangulation of the handle $(D^k \times D^{n-k})_i$. Here, $\{0\}$ is the center of the co-core disk $(\{0\} \times D^{n-k})_i$ and we regard B^k as a small disk embedded in D^k. If remove $(\mathrm{Int}(B^k) \times D^{n-k})_i$ ($i = 1, \ldots, c_k$) from W_k, we obtain a piecewise linear manifold W''_{k-1} isomorphic to W_{k-1} and an ℓ-dimensional simplicial complex

$$L_1 = (L \setminus \bigcup_{i=1}^{c_k} B^k \times L'_i) \cup \bigcup_{i=1}^{c_k} (\partial B^k) \times (\{0\} * L'_i)$$

on $\partial W''_{k-1}$. By the induction hypothesis, we have an $(\ell+1)$-dimensional simplicial complex $\widehat{L}_1$ in W''_{k-1} such that $\partial W''_{k-1} \cap \widehat{L}_1 = L_1$ and $\widehat{L}_1$ is a deformation retract of W''_{k-1}. Since $W''_{k-1} \cup \bigcup_{i=1}^{c_k} B^k \times (\{0\} * L'_i)$ is a deformation retract of W_k, $\widehat{L} = \widehat{L}_1 \cup \bigcup_{i=1}^{c_k} B^k \times (\{0\} * L'_i)$ is the desired $(\ell+1)$-dimensional simplicial complex. Q.E.D.

As for this proposition, the case where L is the empty set corresponds to the construction of a k-dimensional simplicial complex K^k in W_k which is a deformation retract of W_k. In this case, the complex K^k is constructed from the core disks $(D^k \times \{0\})_i$ $(\subset (D^k \times D^{n-k})_i)$.

Corollary 3.2. *There is a k-dimensional simplicial complex K^k differentiably embedded in W_k which is a deformation retract of W_k.*

We note here that by careful choices of the Morse function and the Riemannian metric on M^n, we can make K^k be differentiably embedded and be the union of the stable manifolds of the gradient flow. Hence we have the following proposition.

Proposition 3.3. *Let $D^n \cong W_0 \subset W_1 \subset \cdots \subset W_n = M^n$ be a handle decomposition.*

(1). *There is a k-dimensional simplicial complex K^k differentiably embedded in W_k such that, for any neighborhood V of K^k and for any compact subset A in $\mathrm{int}(W_k)$, there is an isotopy $\{G_t : \mathrm{int}(W_k) \longrightarrow \mathrm{int}(W_k)\}_{t\in[0,1]}$ with compact support such that $G_0 = \mathrm{id}_{\mathrm{int}(W_k)}$, $G_t | K^k = \mathrm{id}_{K^k}$ $(t \in [0,1])$ and $G_1(A) \subset V$.*

(2). *There is an $(n-k-1)$-dimensional simplicial complex K'^{n-k-1} differentiably embedded in $M \setminus W_k$ such that, for any neighborhood V of K^k and for any compact subset A in $M \setminus K'^{n-k-1}$, there is an isotopy $\{G_t : M \setminus K'^{n-k-1} \longrightarrow M \setminus K'^{n-k-1}\}_{t\in[0,1]}$ with compact support such that $G_0 = \mathrm{id}_{M\setminus K'^{n-k-1}}$, $G_t | K^k = \mathrm{id}_{K^k}$ $(t \in [0,1])$ and $G_1(A) \subset V$.*

Remark 3.4. For a compact connected n-dimensional manifold M^n with boundary ∂M^n, we have a handle decomposition of the form $D^n \cong W_0 \subset W_1 \subset \cdots \subset W_k = M^n$ for some $k < n$. Then Proposition 3.3 (1) holds.

§4. Diffeomorphisms of manifolds with small spines

We study the group of diffeomorphisms of open manifolds to which the idea of the proof of Theorem 2.1 applies.

Theorem 4.1. *Let M^n be the interior of a compact n-dimensional manifold with handle decomposition with handles of indices not greater than $(n-1)/2$, then any element of $\mathrm{Diff}_c^r(M^n)_0$ $(1 \leqq r \leqq \infty,\ r \neq n+1)$ can be written as a product of two commutators.*

Proof. This theorem is a corollary to the following Proposition 4.2. For, by Proposition 3.3 (Remark 3.4), we can construct a k-dimensional simplicial complex K^k ($k \leqq (n-1)/2$) differentiably embedded in M^n such that, for any compact set A in M^n and for any neighborhood V of K^k, there is an isotopy $\{G_t\}_{t\in[0,1]}$ required in Proposition 4.2. Q.E.D.

Proposition 4.2. *Let M^n be an n-dimensional manifold. Assume that $2k+1 \leqq n$ and there is a finite k-dimensional simplicial complex K^k differentiably embedded in M^n such that for any compact set A in M^n and any neighborhood V of K^k, there is an isotopy $\{G_t : M^n \longrightarrow M^n\}_{t\in[0,1]}$ such that $G_0 = \mathrm{id}_{M^n}$, $G_t|K^k = \mathrm{id}_{K^k}$ ($t \in [0,1]$) and $G_1(A) \subset U$. Then any element of $\mathrm{Diff}^r_c(M^n)_0$ ($1 \leqq r \leqq \infty$, $r \neq n+1$) can be written as a product of two commutators.*

For the proofs of this proposition and the theorems for diffeomorphisms of compact manifolds we need the following lemmas. These lemmas should be well-known but we include their proofs for the completeness.

Lemma 4.3. *Let M^n be a compact n-dimensional manifold. Let K^k and L^ℓ be k-dimensional and ℓ-dimensional finite simplicial complexes, respectively. Let $f : K^k \longrightarrow M^n$ and $g : L^\ell \longrightarrow M^n$ be differentiable maps and assume that f is an embedding. If $k+\ell+1 \leqq n$ then there is an isotopy $\{F_t : M^n \longrightarrow M^n\}_{t\in[0,1]}$ ($F_0 = \mathrm{id}$) such that $F_1(f(K^k)) \cap g(L^\ell) = \emptyset$.*

Proof. We construct the isotopy F_t, skeleton by skeleton. We consider that $K^k \subset M^n$ and let $K^{(m)}$ be the m skeleton of K^k ($m = 0, \dots, k$). Assume that there is an isotopy $\{F^m_t\}_{t\in[0,1]}$ such that $F^m_1(K^{(m)}) \cap g(L^\ell) = \emptyset$. Assume also that the number of $(m+1)$-dimensional simplices of K^k is N_{m+1} and for $0 \leqq u < N_{m+1}$, we obtained an isotopy $\{F^{(m+1),u}_t\}_{t\in[0,1]}$ such that

$$F^{(m+1),u}_1(K^{(m)} \cup (\sigma^{m+1}_1 \cup \cdots \cup \sigma^{m+1}_u)) \cap g(L^\ell) = \emptyset.$$

For the $(m+1)$-dimensional simplex σ^{m+1}_{u+1} of K^k, $F^{(m+1),u}_1(\sigma^{m+1}_{u+1})$ is differentiably embedded in M^n. We take the normal bundle ν of $F^{(m+1),u}_1(\sigma^{m+1}_{u+1})$, and take the image U under the exponential map of a small disk bundle in ν. Let $\pi : U \longrightarrow F^{(m+1),u}_1(\sigma^{m+1}_{u+1})$ be the projection. We may assume that for neighborhoods V_0 and V_1 of $\partial\sigma^{m+1}_{u+1}$ in σ^{m+1}_{u+1} such that $\partial\sigma^{m+1}_{u+1} \subset V_0 \subset \overline{V_0} \subset V_1$, $\pi^{-1}(V_1) \cap L^\ell = \emptyset$ and $\pi^{-1}(\sigma^{m+1}_{u+1} \setminus V_0)$ does not intersect other $(m+1)$-dimensional simplices of $F^{(m+1),u}_1(K^k)$. Since this normal bundle is trivial, we have a projection $p : U \longrightarrow \boldsymbol{R}^{n-m-1}$ (of rank $n-m-1$) such that $p^{-1}(0) = F^{(m+1),u}_1(\sigma^{m+1}_{u+1})$. Note that $p(g(L^\ell) \cap U)$ is a finite union of the images under differentiable maps of simplices of dimension not greater than ℓ. Since $\ell \leqq n-k-1 \leqq n-(m+1)-1$, $p(g(L^\ell) \cap U)$ is a nowhere dense subset of $\boldsymbol{R}^{n-m-1}$. Take a point q close to 0 in $p(U) - p(g(L^\ell)) \subset p(U) \subset \boldsymbol{R}^{n-m-1}$. Let $\{F_t^{(m+1),u+1}\}_{t\in[0,1]}$ be an isotopy with support in U such that $\pi(F_t^{(m+1),u+1}(x)) = x$, $p(F_t^{(m+1),u+1}(x)) = t\mu(x)q$ for $x \in \sigma_{u+1}^{m+1}$, where $\mu : \sigma_{u+1}^{m+1} \longrightarrow [0,1]$ is a C^∞ function such that $\mu(x) = 1$ for $x \in \sigma_{u+1}^{m+1} \setminus V_1$ and $\mu(x) = 0$ for $x \in V_0$. Then $F_1^{(m+1),u+1}(\sigma_{u+1}^{m+1}) \cap g(L) = \emptyset$.

Thus we obtain an isotopy $\{F_t^{m+1}\}_{t\in[0,1]}$ such that $F_1^{m+1}(K^{(m+1)}) \cap g(L^\ell) = \emptyset$ as the composition of $\{F_t^{(m+1),N_k-1}\}_{t\in[0,1]}, \ldots, \{F_t^{(m+1),0} = F_t^m\}_{t\in[0,1]}$.

Note that the support of the isotopy $\{F_t\}_{t\in[0,1]}$ can be made to be an arbitrarily small compact neighborhood of K^k. Q.E.D.

Remark 4.4. Under the notation of Lemma 4.3, if $k + \ell = n$, then we obtain F_t such that $F_1(f(K^{(k-1)})) \cap g(L^\ell) = \emptyset$, $F_1(f(K^k)) \cap g(L^{(\ell-1)}) = \emptyset$ and the intersection $F_1(f(\sigma^k)) \cap g(\tau^\ell)$ is transverse for each k-dimensional simplex σ^k of K^k and each ℓ-dimensional simplex τ^ℓ of L^ℓ, where $L^{(\ell-1)}$ denotes the $(\ell-1)$ skeleton of L^ℓ. For, we can proceed as in the proof of Lemma 4.3, and for the modification with respect to a k-dimensional simplex σ_{u+1}^k of K^k, we can use a regular value of the projection $p : U \longrightarrow \boldsymbol{R}^\ell$ to the fiber of the normal bundle of $F_1^{(k),u}(\sigma_{u+1}^k)$.

Lemma 4.5. *Let M^n be an n-dimensional manifold. Let K^k be a k-dimensional finite simplicial complex differentiably embedded in M^n. If $2k+1 \leqq n$, then there are an isotopy $\{F_t : M^n \longrightarrow M^n\}_{t\in[0,1]}$ with compact support ($F_0 = \mathrm{id}$) and an open neighborhood U of K^k such that $(F_1)^\ell(U)$ ($\ell \in \boldsymbol{Z}$) are disjoint.*

Proof. There is a neighborhood V of K^k such that for any neighborhood W of K^k ($K^k \subset W \subset \overline{W} \subset V$) and a compact subset $A \subset V$, there is an isotopy $\{G_t : M^n \longrightarrow M^n\}_{t\in[0,1]}$ with support in V such that $G_0 = \mathrm{id}$ and $G_1(A) \subset W$. The neighborhood U is defined by using the structure of normal bundles of each simplex of K^k used in the proof of Lemma 4.3. Then the isotopy is defined skeleton by skeleton by using the normal bundle projections.

By Lemma 4.3 applied to $g(L^\ell) = K^k$, there is an isotopy $\{h_t\}_{t\in[0,1]}$ such that $h_0 = \mathrm{id}$ and $h_1(K^k) \cap K^k = \emptyset$. We may assume that the support of the isotopy $\{h_t\}_{t\in[0,1]}$ is contained in V.

Take a neighborhood W_0 of K^k and W_1 of $h_1(K^k)$ such that $W_0 \cap W_1 = \emptyset$. Then $W = W_0 \cap (h_1)^{-1}(W_1)$ is a neighborhood of K^k such that

$W \cap h_1(W) = \emptyset$. Here we can take W_0 and W_1 such that their closures $\overline{W_0}$ and $\overline{W_1}$ are compact, and then $\overline{W}$ is compact.

For W and $h_1(\overline{W})$, we have an isotopy $\{G_t : M^n \longrightarrow M^n\}_{t\in[0,1]}$ with support in V such that $G_0 = \mathrm{id}$ and $G_1(h_1(\overline{W})) \subset W$.

Let F_t be the composition of G_t and h_t. Then $F_1(W) \subset W$. Since $F_1(W) \cap K^k = \emptyset$, we can take a neighborhood U of K^k such that $U \subset W$ and $U \cap F_1(W) = \emptyset$. Then $U \subset W \setminus F_1(W)$ and for $\ell > 0$, $(F_1)^\ell(U) \subset (F_1)^\ell(W) \setminus (F_1)^{\ell+1}(W)$. Hence $(F_1)^\ell(U)$ ($\ell \in \boldsymbol{Z}$, $\ell \geqq 0$) are disjoint. Then $(F_1)^\ell(U)$ ($\ell \in \boldsymbol{Z}$) are disjoint. Q.E.D.

Proof of Proposition 4.2. By Lemma 4.5, there are a neighborhood U of K^k and an element g of $\mathrm{Diff}^r_c(M^n)_0$ such that $g^i(U)$ ($i \in \boldsymbol{Z}$) are disjoint. For any element $f \in \mathrm{Diff}^r_c(M^n)_0$, by the assumption of the proposition, there is an isotopy $\{G_t : M^n \longrightarrow M^n\}_{t\in[0,1]}$ such that $G_0 = \mathrm{id}$, $G_t|K^k = \mathrm{id}_{K^k}$ and $G_1(\mathrm{supp}(\{f_t\}_{t\in[0,1]})) \subset U$. Then by the argument of Theorem 2.1, $G_1 \circ f \circ {G_1}^{-1}$ can be written as a product of two commutators in $\mathrm{Diff}^r_c(M^n)_0$. Hence f can be written as product of two commutators in $\mathrm{Diff}^r_c(M^n)_0$. Q.E.D.

§5. Diffeomorphisms of compact manifolds

If a compact manifold M has a decomposition into nice pieces, we can show that any element of $\mathrm{Diff}^r(M)_0$ can be written as a composition of diffeomorphisms to which we can apply Theorem 4.1, and that any element of $\mathrm{Diff}^r(M)_0$ can be written as a product of a bounded number of commutators.

Theorem 5.1. *Let M^n be a compact n-dimensional manifold. Let P^p and Q^q be p-dimensional and q-dimensional finite simplicial complexes differentiably embedded in M^n, respectively. Assume that $p+q+2 \leqq n$ and that $P^p \cap Q^q = \emptyset$. Then any element $f \in \mathrm{Diff}^r(M^n)_0$ ($1 \leqq r \leqq \infty$) can be written as a product $f = g \circ h$ such that $g \in \mathrm{Diff}^r_c(M^n \setminus k(Q^q))_0$ and $h \in \mathrm{Diff}^r_c(M^n \setminus P^p)_0$, where $k \in \mathrm{Diff}^r(M^n)_0$, $k(Q^q) \cap P^p = \emptyset$, and $\mathrm{Diff}^r_c(M^n \setminus k(Q^q))_0$ and $\mathrm{Diff}^r_c(M^n \setminus P^p)_0$ are considered as subgroups of $\mathrm{Diff}^r(M^n)_0$, respectively.*

By using Theorems 5.1 and 4.1, we obtain the following theorem.

Theorem 5.2. *Let M^n be a compact n-dimensional manifold. If M^n has a handle decomposition without handles of middle indices, that is, if there is a handle decomposition with p-handles, where $2p+2 \leqq n$ or $2p-2 \geqq n$, then any element of $\mathrm{Diff}^r(M^n)_0$ ($1 \leqq r \leqq \infty$, $r \neq n+1$) can be written as a product of four commutators. In particular, if M^{2m} is a $(2m)$-dimensional compact manifold with a handle decomposition* *without handles of index m, any element of* $\mathrm{Diff}^r(M^{2m})_0$ $(1 \leqq r \leqq \infty$, $r \neq 2m+1)$ *can be written as a product of four commutators.*

Proof. We look at the handle decomposition and the dual handle decomposition of M^n. Then by using the core disks of the handles of indices not greater than $q = (n-2)/2$ of the handle decomposition and the dual handle decomposition, we obtain q-dimensional simplicial complexes P^q and Q^q such that $P^q \cap Q^q = \emptyset$. Since $q+q+2 \leqq n$, any element $f \in \mathrm{Diff}^r(M^n)_0$ can be written as a product $f = g \circ h$ such that $g \in \mathrm{Diff}^r_c(M^n \setminus k(Q^q))_0$ and $h \in \mathrm{Diff}^r_c(M^n \setminus P^q)_0$ by Theorem 5.1, where $k \in \mathrm{Diff}^r(M^n)_0$. By Proposition 3.3, $M^n \setminus P^q$ and $M^n \setminus Q^q$ as well as $M^n \setminus k(Q^q)$ satisfy the assumption of Theorem 4.1. Hence g and h can be written as product of two commutators in $\mathrm{Diff}^r(M^n)_0$. Thus Theorem 5.2 is proved. Q.E.D.

Proof of Theorem 5.1. Let $\{f_t\}_{t\in[0,1]}$ be the isotopy such that $f_0 = \mathrm{id}$ and $f_1 = f$. Let $F : [0,1] \times M^n \longrightarrow M^n$ be the trace of the isotopy: $F(t,x) = f_t(x)$.

We look at the image $F([0,1] \times P^p) \subset M^n$. Since $p+1+q \leqq n-1$, by Lemma 4.3, there is an isotopy $\{k_s\}_{s\in[0,1]}$ $(k_0 = \mathrm{id}, k_1 = k)$ such that $F([0,1] \times P^p) \cap k(Q^q) = \emptyset$.

Let U be a neighborhood of $F([0,1] \times P^p)$ and V be a neighborhood of $k(Q^q)$ such that $U \cap V = \emptyset$.

Let ξ be the vector field on $[0,1] \times M^n$ given by

$$\frac{\partial}{\partial t} + \Big(\frac{\mathrm{d} f_{t+s}(x)}{\mathrm{d}s}\Big)_{s=0}$$

at $(t, f_t(x))$. This ξ generates the isotopy f_t. Let η be a vector field on $[0,1] \times M^n$ with support in $[0,1] \times U$ such that $\eta = \xi$ on a neighborhood of $\{(t, f_t(x_0)) \mid x_0 \in P^p,\ t \in [0,1]\}$. Then $\eta = \partial/\partial t$ on $[0,1] \times V$ which is a neighborhood of $[0,1] \times k(Q^q)$. Then η generates an isotopy $\{g_t\}_{t\in[0,1]}$ such that g_t is the identity on the neighborhood V of $k(Q^q)$ and $g_t(x) = f_t(x)$ for x in a neighborhood of P^p. Put $h = {g_1}^{-1} f_1$, then h is the identity in a neighborhood of P^p, and it is isotopic to the identity as an element of $\mathrm{Diff}^r(M^n)$. Put $h_t = {g_t}^{-1} \circ f_t$. Then h_t is the identity on a neighborhood of P^p.

Thus $f = g \circ h$ and $g \in \mathrm{Diff}^r_c(M^n \setminus k(Q^q))_0$ and $h \in \mathrm{Diff}^r_c(M^n \setminus P^p)_0$. Q.E.D.

Remark 5.3. In the proof of Theorem 5.1, the decomposition of a diffeomorphism uses only the fact that $F([0,1] \times P^p) \cap k(Q^q) = \emptyset$.

Remark 5.4. For a compact manifold M we have a handle decomposition. For a compact odd-dimensional manifold M^{2m+1}, M^{2m+1} is covered by two open sets U_1 and U_2 which are neighborhoods of the union of handles of indices not greater than m and the union of dual handles of indices not greater than m. Then by the fragmentation lemma ([1]), there is a neighborhood $\mathcal{N}$ of the identity in $\mathrm{Diff}^r(M^{2m+1})_0$ such that every element f of $\mathcal{N}$ can be written as a product $f = g \circ h$, where $g \in \mathrm{Diff}^r_c(U_1)_0$ and $h \in \mathrm{Diff}^r_c(U_2)_0$. Hence by Theorem 4.1, every element f of $\mathcal{N}$ can be written as a product of four commutators of elements of $\mathrm{Diff}^r(M^{2m+1})_0$ ($1 \leqq r \leqq \infty$, $r \neq 2m+2$). For a compact even-dimensional manifold M^{2m}, M^{2m} is covered by three open sets U_1, U_2 and U_3. Here, U_1 and U_2 are neighborhoods of the union of handles of indices not greater than $m-1$ and the union of dual handles of indices not greater than $m-1$, and U_3 is a disjoint union of open balls which is a neighborhood of the union of m handles. Then by the fragmentation lemma, there is a neighborhood $\mathcal{N}$ of the identity in $\mathrm{Diff}^r(M^{2m})_0$ such that every element f of $\mathcal{N}$ can be written as a product $f = a \circ g \circ h$, where $g \in \mathrm{Diff}^r_c(U_1)_0$, $h \in \mathrm{Diff}^r_c(U_2)_0$ and $a \in \mathrm{Diff}^r_c(U_3)_0$. Hence by Theorem 4.1, every element f of $\mathcal{N}$ can be written as a product of six commutators of elements of $\mathrm{Diff}^r(M^{2m})_0$ ($1 \leqq r \leqq \infty$, $r \neq 2m+1$).

§6. Diffeomorphisms of odd-dimensional compact manifolds

In [2], Burago, Ivanov and Polterovich proved that for a closed 3-dimensional manifold M^3, any element of $\mathrm{Diff}^r(M^3)_0$ can be written as a product of ten commutators. Their method together with the general position argument in the previous sections gives the following theorem.

Theorem 6.1. *Let M^{2m+1} be a compact $(2m+1)$-dimensional manifold. Then any element of* $\mathrm{Diff}^r(M^{2m+1})_0$ ($1 \leqq r \leqq \infty$, $r \neq 2m+2$) *can be written as a product of six commutators.*

Proof. We look at the handle decomposition and the dual handle decomposition of M^{2m+1}. Then by using the core disks of the handles of indices not greater than m of the handle decomposition and the dual handle decomposition, we obtain m-dimensional simplicial complexes P^m and Q^m such that $P^m \cap Q^m = \emptyset$. By Proposition 3.3, $M^{2m+1} \setminus P^m$ and $M^{2m+1} \setminus Q^m$ satisfy the assumption of Theorem 4.1. Then the theorem follows from the following theorem and Theorems 2.1 and 4.1. Q.E.D.

Theorem 6.2. *Let M^{2m+1} be a compact $(2m+1)$-dimensional manifold. Let P^m and Q^m be m-dimensional finite simplicial complexes differentiably embedded in M^{2m+1}, respectively. Assume that $P^m \cap Q^m = \emptyset$. Then any element $f \in \mathrm{Diff}^r(M^{2m+1})_0$ ($1 \leqq r \leqq \infty$) can be written as a product $f = a \circ g \circ h$ such that $a \in \mathrm{Diff}^r_c(\bigsqcup_i U_i)_0$, $g \in \mathrm{Diff}^r_c(M^{2m+1} \setminus$*

$k(Q^m))_0$ and $h \in \mathrm{Diff}^r_c(M^{2m+1} \setminus k'(P^m))_0$, where $\bigsqcup_i U_i$ is a disjoint union of $(2m+1)$-dimensional open balls U_i embedded in M^{2m+1}, k, $k' \in \mathrm{Diff}^r(M^{2m+1})_0$, and $\mathrm{Diff}^r_c(\bigsqcup_i U_i)_0$, $\mathrm{Diff}^r_c(M^{2m+1} \setminus k(Q^m))_0$ and $\mathrm{Diff}^r_c(M^{2m+1} \setminus k'(P^m))_0$ are considered as subgroups of $\mathrm{Diff}^r(M^{2m+1})_0$, respectively.

For the proof of Theorem 6.2, we need several lemmas.

Lemma 6.3. *Let $P^{(m-1)}$ and $Q^{(m-1)}$ be the $m-1$ skeletons of P^m and Q^m, respectively. Then any element $f \in \mathrm{Diff}^r(M^{2m+1})_0$ can be written as a product $f = g \circ h$ such that $g \in \mathrm{Diff}^r_c(M^{2m+1} \setminus k(Q^m))_0$ and $h \in \mathrm{Diff}^r_c(M^{2m+1} \setminus P^{(m-1)})_0$, where $k \in \mathrm{Diff}^r(M^{2m+1})_0$ and $k(Q^m) \cap P^m = \emptyset$. Moreover there is an isotopy $\{h_t\}_{t\in[0,1]}$ such that $h_0 = \mathrm{id}$, $h_1 = h$, h_t is the identity in a neighborhood of $P^{(m-1)}$, and for $H(t,x) = h_t(x)$, $H([0,1] \times P^m) \cap k(Q^{(m-1)}) = \emptyset$ and, for m-dimensional simplices τ^m of P^m and σ^m of Q^m, the intersection $H([0,1] \times \tau^m) \cap k(\sigma^m)$ is transverse. Thus $H([0,1] \times P^m) \cap k(Q^m)$ is a finite set.*

Proof. Let $\{f_t\}_{t\in[0,1]}$ be the isotopy such that $f_0 = \mathrm{id}$ and $f_1 = f$. Let $F : [0,1] \times M^{2m+1} \longrightarrow M^{2m+1}$ be the trace of the isotopy: $F(t,x) = f_t(x)$. As in the proof of Theorem 5.1, we look at the image $F([0,1] \times P^m) \subset M^{2m+1}$.

Since the dimension of the manifold is $2m+1$, by Lemma 4.3 and Remark 4.4, there is an isotopy $\{k_s\}_{s\in[0,1]}$ ($k_0 = \mathrm{id}$, $k_1 = k$) such that $F([0,1] \times P^m) \cap k(Q^{(m-1)}) = \emptyset$, $F([0,1] \times P^{(m-1)}) \cap k(Q^m) = \emptyset$ and $k(\sigma^m)$ is transverse to $F([0,1] \times \tau^m)$ for each pair of m-dimensional simplices σ^m of Q^m and τ^m of P^m. Hence, $F([0,1] \times P^m) \cap k(Q^m)$ is a finite set:

$$F([0,1] \times P^m) \cap k(Q^m) = \{F(t_i,u_i) \mid i = 1, \ldots, r\} \subset M^{2m+1}.$$

We proceed as in the proof of Theorem 5.1. We can take an isotopy g_t fixing a neighborhood of $k(Q^m)$ and $g_t = f_t$ in a small neighborhood of $P^{(m-1)}$. Then for $H(t,x) = h_t(x) = {g_t}^{-1} \circ f_t(x)$, h_t is the identity on a neighborhood of $P^{(m-1)}$. Thus the intersection $H([0,1] \times P^m) \cap k(Q^m)$ is transverse. Since $H(t_i,u_i) = F(t_i,u_i)$,

$$\begin{aligned} H([0,1] \times P^m) \cap k(Q^m) &= \{F(t_i,u_i) \mid i = 1, \ldots, r\} \\ &= \{H(t_i,u_i) \mid i = 1, \ldots, r\} \subset M^{2m+1}. \end{aligned}$$

Q.E.D.

We would like to decompose an element $\overline{h}$ close to h as a composition of an element $a \in \mathrm{Diff}^r_c(\bigsqcup_i U_i)_0$, where $\bigsqcup_i U_i$ is a disjoint union of

$(2m+1)$-dimensional open balls U_i embedded in M^{2m+1}, an element $\overline{g} \in \mathrm{Diff}^r_c(M^{2m+1} \setminus k(Q^m))_0$ and an element $\overline{h}' \in \mathrm{Diff}^r_c(M^{2m+1} \setminus k'(P^m))_0$:

$$\overline{h} = a \circ \overline{g} \circ \overline{h}'.$$

By the classical result of Whitney [18], we have the following lemma.

Lemma 6.4. *Let $\{h_t\}_{t\in[0,1]}$ be an isotopy which is the identity in a neighborhood of $P^{(m-1)}$ and put $H(t,x) = h_t(x)$. Let $V \subset P$ be the complement of a neighborhood of $P^{(m-1)}$ where $h_t = \mathrm{id}$. Then there is an isotopy $\{\overline{h}_t\}_{t\in[0,1]}$ fixing a neighborhood of $P^{(m-1)}$ such that its trace $\overline{H} : [0,1]\times M^{2m+1} \longrightarrow M^{2m+1}$ is close to $H : [0,1]\times M^{2m+1} \longrightarrow M^{2m+1}$ and $\overline{H}|[0,1] \times V$ is an immersion outside of a finite subset. Moreover the image $\overline{H}([0,1] \times V) \subset M^{2m+1} \setminus (P^{(m-1)} \cup k(Q^{(m-1)}))$ has finitely many double point curves which is in general position with respect to the curves $\overline{H}([0,1] \times \{v\})$ $(v \in V)$. If $m \geqq 2$ these double point curves are disjoint, and if $m = 1$, there are at most finitely many triple points and cusps.*

Lemma 6.5. *For generic $\overline{h} = \overline{h}_1 \in \mathrm{Diff}^r_c(M^{2m+1} \setminus P^{(m-1)})_0$ given by Lemma 6.4, $\overline{h}$ can be decomposed as $\overline{h} = a \circ \overline{g} \circ \overline{h}'$, where $a \in \mathrm{Diff}^r_c(\bigsqcup_i U_i)_0$, $\bigsqcup_i U_i$ is a disjoint union of $(2m+1)$-dimensional open balls U_i embedded in M^{2m+1}, $\overline{g} \in \mathrm{Diff}^r_c(M^{2m+1} \setminus k(Q^m))_0$ and $\overline{h}' \in \mathrm{Diff}^r_c(M^{2m+1} \setminus P^m)_0$.*

For the proof of Lemma 6.5, we need to find the open balls U_i. These balls are neighborhoods of embedded arcs or embedded trees in $M^{2m+1} \setminus (P^m \cup k(Q^{(m-1)}))$. This is a construction essentially due to Burago, Ivanov and Polterovich ([2])

Let

$$\overline{H}([0,1] \times P^m) \cap k(Q^m) = \{\overline{H}(s_i, v_i) \mid i = 1, \ldots, r\} \subset M^{2m+1} \setminus (P^m \cup k(Q^{(m-1)})).$$

We look at $\overline{H}([s_i,1] \times \{v_i\})$. For generic $\overline{H}$, $\overline{H}([s_i,1] \times \{v_i\})$ does not intersect $P^m \cup k(Q^m)$ other than $\overline{H}(s_i, v_i) \in k(Q^m)$

If $m \geqq 2$, then for generic $\overline{H}$, $\overline{H}([s_i,1] \times \{v_i\})$ does not intersect the double point curves.

If $m = 1$, then $\overline{H}([s_i,1] \times \{v_i\})$ may intersect the double point curves. For generic $\overline{H}$, the intersection consists of finitely many points $\overline{H}(s_{i,i_1}, v_i) = \overline{H}(s'_{i,i_1}, v'_{i,i_1})$ $(i_1 = 1, \ldots, j_i)$, where we only take the double points such that $s'_{i,i_1} > s_{i,i_1}$. For the double points where $s'_{i,i_1} > s_{i,i_1}$, we look at the curve $\overline{H}([s'_{i,i_1}, 1] \times \{v'_{i,i_1}\})$. For generic $\overline{H}$, $\overline{H}([s'_{i,i_1}, 1] \times \{v'_{i,i_1}\})$ does not intersect $P^1 \cup k(Q^1)$ but may intersect the double point curves at finitely many points again. $\overline{H}(s'_{i,i_1,i_2}, v'_{i,i_1}) = \overline{H}(s''_{i,i_1,i_2}, v''_{i,i_1,i_2})$ $(i_2 = 1, \ldots, j_{i,i_1})$. Then for $s''_{i,i_1,i_2} > s'_{i,i_1,i_2}$, we look at the curve $\overline{H}([s''_{i,i_1,i_2}, 1] \times \{v''_{i,i_1,i_2}\})$.

We continue this process and obtain trees consisting of arcs of the form $\overline{H}([s,1] \times \{v\})$ starting at the points of the intersection $\overline{H}([0,1] \times P^1) \cap k(Q^1)$ bifurcating at the double points which are the intersections of the arcs and the forward image $\overline{h}_t(P^1)$ of P^1 under the isotopy. Note that the branches of the trees are finitely many. It is because outside of small neighborhoods of the tangencies of the double point curves and the curves $\overline{H}([0,1] \times \{v\})$ $(v \in V)$ and outside of small neighborhoods of triple points and cusps, there exists a positive real number δ such that two intersecting points $\overline{H}(s_0, v)$, $\overline{H}(s_1, v)$ of the double point curves and $\overline{H}([0,1] \times \{v\})$ satisfy $|s_0 - s_1| > \delta$. Thus we obtain final branches which look like $\overline{H}([s^{(k)}_{i,i_1,i_2,\ldots,i_k}, 1] \times \{v^{(k)}_{i,i_1,i_2,\ldots,i_k}\})$. Note also that the tree intersect $P^1 \cup k(Q^1)$ only at the starting point $\overline{H}(s_i, v_i) \in k(Q^1)$.

Proof of Lemma 6.5 *for* $m \geqq 2$. If $m \geqq 2$, using the curves $\overline{H}([s_i,1] \times \{v_i\})$, we can define an isotopy $\{a_t\}_{t\in[0,1]}$ $(a_0 = \mathrm{id})$ with support in neighborhoods of $\overline{H}([s_i,1] \times \{v_i\})$ such that $(a_1 \circ \overline{h})(P^m) \cap k(Q^m) = \emptyset$ and there is an isotopy $\{h'_t\}_{t\in[0,1]}$ such that $h'_0 = \mathrm{id}$, $h'_1 = a_1 \circ \overline{h}$ and $h'_t(P^m) \cap k(Q^m) = \emptyset$ $(t \in [0,1])$.

We take a small neighborhood U_i of $\overline{H}([s_i,1] \times \{v_i\})$ diffeomorphic to the $(2m+1)$-dimensional ball. We take these U_i to be disjoint and the intersection of U_i and $\overline{H}([0,1] \times P^m)$ or $k(Q^m)$ is described as follows.

We put a coordinate $(x_1, x_2, \ldots, x_{m+1}, x_{m+2}, \ldots, x_{2m+1}) \in (-2,2)^{2m+1}$ on U_i such that, for $\varepsilon_i > 0$,

$$\begin{aligned}
&k(Q^m) \cap U_i = \{0\} \times \{0\}^m \times (-2,2)^m,\\
&\overline{H}((s_i - 2\varepsilon_i(1-s_i), 1] \times \{v_i\}) \cap U_i = (-2,1] \times \{0\}^{2m},\\
&\overline{h}_{s_i+t(1-s_i)}(P^m) \cap U_i = \{t\} \times (-2,2)^m \times \{0\}^m \quad (t \in [-\varepsilon_i, 1]).
\end{aligned}$$

Take an isotopy $\{a_t\}_{t\in[0,1]}$ with support in $\bigsqcup_{i=1}^r U_i$ such that on each U_i, $a_0 = \mathrm{id}$ and, for $(x_1, x_2, \ldots, x_{2m+1}) \in [-\varepsilon_i, 1] \times [-1,1]^{2m} \subset (-2,2)^{2m+1}$,

$$a_t(x_1, x_2, \ldots, x_{2m+1}) = (x_1 - (1+\varepsilon_i)t, x_2, \ldots, x_{2m+1}).$$

Now $(a_1 \circ \overline{h}_1)(P^m) \cap k(Q^m) = \emptyset$. Moreover there is an isotopy $\{h'_t\}_{t\in[0,1]}$ from the identity to $a_1 \circ \overline{h}_1$ such that $h'_t(P^m) \cap k(Q^m) = \emptyset$ $(t \in [0,1])$.

The reason is that we can modify $\overline{h}_t$ on U_i by replacing by

$$a_{(u_i+\varepsilon_i)/(1+\varepsilon_i)} \circ \overline{h}_{s_i+u_i(1-s_i)}$$

for $t = s_i + u_i(1-s_i) \in [s_i - \varepsilon_i(1-s_i), 1]$, i.e., $u_i \in [-\varepsilon_i, 1]$. Then

$$\begin{aligned}
&(a_{(u_i+\varepsilon_i)/(1+\varepsilon_i)} \circ \overline{h}_{s_i+u_i(1-s_i)})(\{-\varepsilon_i\} \times [-1,1]^m \times \{0\}^m) \\
&= a_{(u_i+\varepsilon_i)/(1+\varepsilon_i)}(\{u_i\} \times [-1,1]^m \times \{0\}^m) \\
&= \{u_i - (u_i+\varepsilon_i)\} \times [-1,1]^m \times \{0\}^m \\
&= \{-\varepsilon_i\} \times [-1,1]^m \times \{0\}^m
\end{aligned}$$

Thus there is an isotopy $\{h'_t\}_{t\in[0,1]}$ such that $h'_0 = \mathrm{id}$, $h'_1 = a_1 \circ \overline{h}_1$ and $h'_t(P^m) \cap k(Q^m) = \emptyset$ $(t \in [0,1])$.

Then by the proof of Theorem 5.1 (Remark 5.3), $a_1 \circ \overline{h}$ can be written as a composition $a_1 \circ \overline{h} = \overline{g} \circ \overline{h}'$, where $\overline{g} \in \mathrm{Diff}^r_c(M^{2m+1} \setminus k(Q^m))_0$ and $\overline{h}' \in \mathrm{Diff}^r_c(M^{2m+1} \setminus P^m)_0$. Thus $\overline{h} = (a_1)^{-1} \circ \overline{g} \circ \overline{h}'$. Since $(a_1)^{-1} \in \mathrm{Diff}^r_c(\bigsqcup_{i=1}^r U_i)_0$, Lemma 6.5 for $m \geqq 2$ is proved. Q.E.D.

Proof of Lemma 6.5 *for* $m = 1$. If $m = 1$, then we will take U_i considering the intersection with the double point curves.

First take a small neighborhood U_i of $\overline{H}([s_i, 1] \times \{v_i\})$ as in the case where $m \geqq 2$. U_i has the coordinate $(-2,2)^3$ as before. We will modify U_i by using several isotopies.

We also take small neighborhoods $U_{i,i_1}, U_{i,i_1,i_2}, \ldots$ of the branches $\overline{H}([s'_{i,i_1}, 1] \times \{v'_{i,i_1}\})$ $(s'_{i,i_1} > s_i)$, $\overline{H}([s''_{i,i_1,i_2}, 1] \times \{v''_{i,i_1,i_2}\})$ $(s''_{i,i_1,i_2} > s'_{i,i_1})$, ... We put a coordinate $(x_1, x_2, x_3) \in (-2,3) \times (-2,2)^2$ on U_{i,i_1} such that

$$\begin{aligned}
&\overline{H}([s'_{i,i_1} - 2\varepsilon_{i,i_1}(1 - s'_{i,i_1}), 1] \times \{v'_{i,i_1}\}) \cap U_{i,i_1} = (-2,1] \times \{(0,0)\}, \\
&\overline{h}_{s'_{i,i_1}+t(1-s'_{i,i_1})}(P^1) \cap U_{i,i_1} = \{t\} \times (-2,2) \times \{0\} \quad (t \in [-\varepsilon_{i,i_1}, 1]),
\end{aligned}$$

and coordinates on $U_{i,i_1,i_2}, \ldots$ are taken in a similar way.

We take isotopies $\{a^{i,i_1}_t\}_{t\in[0,1]}$ with support in U_{i,i_1} such that $a^{i,i_1}_0 = \mathrm{id}$ and, for $(x_1,x_2,x_3) \in [-\varepsilon_{i,i_1}, 1] \times [-1,1]^2 \subset (-2,3) \times (-2,2)^2$,

$$a^{i,i_1}_t(x_1,x_2,x_3) = (x_1 + t(1+\varepsilon_{i,i_1}), x_2, x_3).$$

We also take isotopies $\{a^{i,i_1,i_2}_t\}_{t\in[0,1]}, \ldots$ with support in $U_{i,i_1,i_2}, \ldots$ in a similar way. Then we take U_i very thin so that

$$\Big(\big(\prod_{i_1,i_2,\ldots,i_k} a^{i,i_1,i_2,\ldots,i_k}_1\big) \circ \cdots \circ \big(\prod_{i_1,i_2} a^{i,i_1,i_2}_1\big) \circ \big(\prod_{i_1} a^{i,i_1}_1\big) \Big)(U_i)$$

does not intersect $\overline{H}([s_i,1] \times P^1)$ outside of a neighborhood of $\overline{H}([s_i,1] \times \{v_1\})$, where $\{a^{i,i_1,i_2,\ldots,i_k}_t\}_{t\in[0,1]}$ is the isotopy with support in a neighborhood $U_{i,i_1,i_2,\ldots,i_k}$ of the final branch $\overline{H}([s^{(k)}_{i,i_1,i_2,\ldots,i_k}, 1] \times \{v^{(k)}_{i,i_1,i_2,\ldots,i_k}\})$

defined in a similar way. Let

$$\overline{a} = \prod_{i=1}^{r} \Big(\prod_{i_1,i_2,\ldots,i_k} a_1^{i,i_1,i_2,\ldots,i_k} \Big) \circ \cdots \circ \Big(\prod_{i_1,i_2} a_1^{i,i_1,i_2} \Big) \circ \Big(\prod_{i_1} a_1^{i,i_1} \Big).$$

Then $\overline{a} \circ a_t \circ \overline{a}^{-1}$ is isotopic to the identity by the isotopy with support in the disjoint union of 3-dimensional open balls $\overline{a}(\bigsqcup_{i=1}^{r} U_i)$. By the construction, $((\overline{a} \circ a_1 \circ \overline{a}^{-1}) \circ \overline{h}_1)(P^1) \cap k(Q^1) = \emptyset$. We show that there is an isotopy $\{h'_t\}_{t \in [0,1]}$ from the identity to $(\overline{a} \circ a_1 \circ \overline{a}^{-1}) \circ \overline{h}_1$ such that $h'_t(P^1) \cap k(Q^1) = \emptyset$ $(t \in [0,1])$.

For the construction of h'_t, we define the local time $u_i \in [-\varepsilon_i, 1]$ on U_i $(1 \leqq i \leqq r)$ by $t = s_i + u_i(1 - s_i)$ as in the case where $m \geqq 2$. We can modify $\overline{h}_t$ on the union

$$U_i \cup \bigcup_{i_1} U_{i,i_1} \cup \bigcup_{i_1,i_2} U_{i,i_1,i_2} \cup \cdots \cup \bigcup_{i,i_1,i_2,\ldots,i_k} U_{i,i_1,i_2,\ldots,i_k}$$

for $t = s_i + u_i(1 - s_i) \in [s_i - \varepsilon_i(1 - s_i), 1]$ $(u_i \in [-\varepsilon_i, 1])$ and define h'_t there by

$$h'_t = (\overline{a} \circ a_{(u_i+\varepsilon_i)/(1+\varepsilon_i)} \circ \overline{a}^{-1}) \circ \overline{h}_{s_i+u_i(1-s_i)}.$$

Then this isotopy $\{h'_t\}_{t \in [0,1]}$ satisfies that $h'_0 = \mathrm{id}$, $h'_1 = (\overline{a} \circ a_1 \circ \overline{a}^{-1}) \circ \overline{h}_1$ and $h'_t(P^1) \cap k(Q^1) = \emptyset$ $(t \in [0,1])$.

Then by the proof of Theorem 5.1 (Remark 5.3), $(\overline{a} \circ a_1 \circ \overline{a}^{-1}) \circ \overline{h}_1$ can be written as a composition $(\overline{a} \circ a_1 \circ \overline{a}^{-1}) \circ \overline{h}_1 = \overline{g} \circ \overline{h}'$, where $\overline{g} \in \mathrm{Diff}^r_c(M^3 \setminus k(Q^1))_0$ and $\overline{h}' \in \mathrm{Diff}^r_c(M^3 \setminus P^1)_0$. Thus $\overline{h} = (\overline{a} \circ {a_1}^{-1} \circ \overline{a}^{-1}) \circ \overline{g} \circ \overline{h}'$. Since $\overline{a} \circ {a_1}^{-1} \circ \overline{a}^{-1} \in \mathrm{Diff}^r_c(\overline{a}(\bigsqcup_{i=1}^{r} U_i))_0$, Lemma 6.5 for $m = 1$ is proved. Q.E.D.

Proof of Lemma 6.5 *for* $m = 0$. This is an (easy) exceptional case. The only compact connected 1-dimensional manifold is the circle S^1. For $f \in \mathrm{Diff}^r(S^1)_0$ and $p \in S^1$, we take a point q distinct from p and $f(p)$, Let g be a C^r diffeomorphism of S^1 which coincides with f on a neighborhood of p and with the identity on a neighborhood of q. Then $h = g^{-1} \circ f$ is the identity on a neighborhood of p. Since g is isotopic to the identity as an element of $\mathrm{Diff}^r_c(S^1 \setminus \{q\})_0$, and h is isotopic to the identity as an element of $\mathrm{Diff}^r_c(S^1 \setminus \{p\})_0$, $f = g \circ h$ in a desired way. (Then any element of $\mathrm{Diff}^r(S^1)_0$ $(1 \leqq r \leqq \infty, r \neq 2)$ can be written as a product of four commutators as in Theorem 5.2.) Note that in this case the original isotopy for f is different from the composition of the isotopies for g and h. Q.E.D.

Proof of Theorem 6.2. For $h \in \mathrm{Diff}^r(M^{2m+1})$, let $\overline{h}$ be the diffeomorphism obtained by Lemma 6.4. By Lemma 6.5, $\overline{h}$ can be written as $\overline{h} = a \circ \overline{g} \circ \overline{h}'$, where $a \in \mathrm{Diff}^r_c(\bigsqcup_i U_i)_0$, $\overline{g} \in \mathrm{Diff}^r_c(M^{2m+1} \setminus k(Q^m))_0$ and $\overline{h}' \in \mathrm{Diff}^r_c(M^{2m+1} \setminus P^m)_0$. Then $h = a \circ \overline{g} \circ \overline{h}' \circ (\overline{h}^{-1}h)$. Since $\overline{h}^{-1}h$ is close to the identity, by Remark 5.4, $\overline{h}^{-1}h$ can be written as the product $\overline{h}^{-1}h = \widehat{h} \circ \widehat{g}$, where $\widehat{g} \in \mathrm{Diff}^r_c(M^{2m+1} \setminus k(Q^m))_0$ and $\widehat{h} \in \mathrm{Diff}^r_c(M^{2m+1} \setminus P^m)_0$. Then

$$h = a \circ \overline{g} \circ \overline{h}' \circ \widehat{h} \circ \widehat{g} = a \circ (\overline{g} \circ \widehat{g}) \circ \widehat{g}^{-1} \circ (\overline{h}' \circ \widehat{h}) \circ \widehat{g}.$$

Here $a \in \mathrm{Diff}^r_c(\bigsqcup_i U_i)_0$, $\overline{g} \circ \widehat{g} \in \mathrm{Diff}^r_c(M^{2m+1} \setminus k(Q^m))_0$ and $\widehat{g}^{-1} \circ (\overline{h}' \circ \widehat{h}) \circ \widehat{g} \in \mathrm{Diff}^r_c(M^{2m+1} \setminus \widehat{g}^{-1}(P^m))_0$. Thus Theorem 6.2 is shown. Q.E.D.

Remark 6.6. In Corollary 2.2 and Theorem 4.1, there is an open subset U of M^n and there is an element g of $\mathrm{Diff}^r_c(M^n)_0$ such that any element f of $\mathrm{Diff}^r_c(M^n)_0$ is conjugate to an element of $\mathrm{Diff}^r_c(U)_0$ and $g(U) \cap U = \emptyset$. Then any commutator $[a, b]$ in $\mathrm{Diff}^r_c(U)_0$ can be written as a product of 4 conjugates of g or g^{-1}. For, if a, $b \in \mathrm{Diff}^r_c(U)_0$, then by putting $c = g^{-1}ag$, $cb = bc$ and

$$\begin{aligned} aba^{-1}b^{-1} &= gcg^{-1}bgc^{-1}g^{-1}b^{-1} \\ &= gcg^{-1}c^{-1}cbgc^{-1}b^{-1}bg^{-1}b^{-1} \\ &= g(cg^{-1}c^{-1})(bcgc^{-1}b^{-1})(bg^{-1}b^{-1}). \end{aligned}$$

Thus for an n-dimensional manifold M^n satisfying the assumption of Corollary 2.2 or Theorem 4.1, any element f of $\mathrm{Diff}^r_c(M^n)_0$ can be written as a product of 8 conjugates of g or g^{-1} (($1 \leqq r \leqq \infty$, $r \neq n+1$). By this observation, Theorem 5.2 implies that for an even-dimensional compact manifold M^{2m} which has a handle decomposition without handles of the middle index m, there is an element g such that any element f of $\mathrm{Diff}^r(M^{2m})_0$ can be written as a product of 16 conjugates of g or g^{-1} ($1 \leqq r \leqq \infty$, $r \neq 2m+1$). Here g is taken so that g maps a neighborhood U of the union of the simplicial complexes P^k and Q^k in Theorem 5.2 to an open set $g(U)$ with $U \cap g(U) = \emptyset$. In a similar way, Theorem 6.1 implies that for an odd-dimensional compact manifold M^{2m+1}, there is an element g such that any element f of $\mathrm{Diff}^r(M^{2m+1})_0$ can be written as a product of 24 conjugates of g or g^{-1} ($1 \leqq r \leqq \infty$, $r \neq 2m+2$). This implies that these groups are meager in the terminology of the paper [2] as Polterovich pointed out to the author. Note that, for a perfect group, if there is an element g with the above property, then it is uniformly perfect.

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Graduate School of Mathematical Sciences
the University of Tokyo
Komaba Meguro, Tokyo 153-8914, Japan
E-mail address: `tsuboi@ms.u-tokyo.ac.jp`