Advanced Studies in Pure Mathematics 51, 2008
Surveys on Geometry and Integrable Systems
pp. 401–488

Geometries and symmetries of soliton equations and integrable elliptic equations

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§1. Introduction

In the classical literature, a differential equation is called "integrable" if it can be solved by quadratures. A Hamiltonian system in $2n$-dimensions is *completely integrable* if it has n independent commuting Hamiltonians. By the Arnold-Liouville Theorem, such systems have action-angle variables that linearize the flow, and these can be found by quadrature. This concept of integrability can be extended to PDEs, and one class consists of evolution equations on function spaces that have Hamiltonian structures and are completely integrable Hamiltonian systems in the sense of Liouville, i.e., there exist action angle variables. We call this class of equations soliton equations. The model examples are the Korteweg-de Vries equation, the non-linear Schrödinger equation (NLS equation), and the Sine-Gordon equation (SGE equation). For example, the action-angle variables are constructed for the KdV equation in [33], for the NLS equation in [34], and for flows in the $SL(n)$-hierarchy in [5]. Besides the Hamiltonian formulation and complete integrability, these soliton equations have many other remarkable properties including:

(1) infinite families of explicit solutions,

(2) a hierarchy of commuting flows described by partial differential equations,

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Received March 18, 2002.

Revised January 24, 2003.

2000 *Mathematics Subject Classification.* Primary 37K25; Secondary 53A, 53C.

Research supported in part by NSF Grant DMS 9972172 and National Center for Theoretical Sciences, Taiwan.

(3) a Lax pair,

(4) an algebraic-geometric description of certain solutions,

(5) a scattering theory,

(6) an inverse scattering transform to solve the Cauchy problem,

(7) a construction of solutions using loop group factorizations (dressing actions).

We will give a brief review of properties (1), (2), (3), and (7), and refer the reader to [32, 46, 47, 53, 73, 74] for (4), and to [3, 4] for (5) and (6).

The existence of a Lax pair is one of the key properties of soliton equations. This was first constructed for the case of the KdV equation by Lax [49], who observed that the KdV equation can be written as the condition for an isospectral deformation of the Schrödinger operator on the line. Later, this was shown to be equivalent to the zero curvature condition of a family of connections ([2, 76]). Roughly speaking, a PDE for $q : \mathbb{R}^n \to \mathbb{R}^m$ is said to have a *zero curvature formulation* if there is a family of connections θ_λ on $\mathbb{R}^n$, (defined by q and its derivatives, and a holomorphic parameter λ defined in some open subset of $\mathbb{C}$) so that the condition for θ_λ to be flat for all λ is that q solve the given PDE. The connection $\theta_\lambda = \sum_{i=1}^n A_i dx_i$ is flat if $d\theta_\lambda = -\theta_\lambda \wedge \theta_\lambda$ for all λ, or equivalently the n operators $\{\frac{\partial}{\partial x_i} + A_i \mid 1 \leq i \leq n\}$ commute, i.e.,

$$\left[\frac{\partial}{\partial x_i} + A_i,\ \frac{\partial}{\partial x_j} + A_j\right] = 0, \quad i \neq j.$$

We call θ_λ a *Lax pair* if $n = 2$, and a *Lax n-tuple* for general n. A Lax n-tuple naturally gives rise to a loop group factorization, which in turn provides a method for constructing explicit solutions and symmetries of the equations.

Another class of integrable PDEs are non-linear elliptic equations. Although these equations do not have Hamiltonian formulations, they do have zero curvature formulations that give rise to loop group factorizations, and hence the techniques developed for soliton equations can also be used to construct solutions and symmetries of these elliptic equations. In particular, we can find solutions of the equation by factorizations. One class of model examples are the equations for harmonic maps from $\mathbb{C}$ to a compact Lie group.

Some goals of this paper are to give a brief survey of the following:

—A systematic construction of integrable hierarchies associated to a complex semi-simple Lie algebra and finite order automorphisms.

—Some geometric integrable PDEs arising in differential geometry.

—Construction of solutions using loop group factorizations.

Another goal of this paper is to put some known results of evolution soliton equations and integrable elliptic systems together so that we can compare and see similarities and differences in these two theories.

The literature of integrable systems and their applications to differential geometry is huge. In this article, the author only covers a part of these research areas and the references are correspondingly restricted. We refer readers to the following survey books and articles [1, 25, 32, 34, 38, 46, 53, 54, 59, 58, 63], and articles in this volume for more complete references.

G-hierarchy

The ZS-AKNS construction of the $n \times n$-hierarchy of soliton flows works equally well when we replace $sl(n, \mathbb{C})$ by any complex, simple Lie algebra $\mathcal{G}$. In fact, let $a \in \mathcal{G}$, $\mathcal{G}_a = \{y \in \mathcal{G} \mid [a, y] = 0\}$ the centralizer of a, and $\mathcal{G}_a^\perp = \{\xi \in \mathcal{G} \mid (\xi, y) = 0 \text{ for all } y \in \mathcal{G}_a\}$. Here $(\ ,\ )$ is a non-degenerate ad-invariant bilinear form of $\mathcal{G}$. It can be shown that there exists a sequence of polynomial differential operators on the space $C(\mathbb{R}, \mathcal{G}_a^\perp)$ of smooth functions from $\mathbb{R}$ to $\mathcal{G}_a^\perp$,

$$\{Q_{b,j}(u) \mid b \in \mathcal{G}_a,\ \mathcal{G}_b = \mathcal{G}_a,\ j \geq 0 \text{ integer}\}.$$

These $Q_{b,j}(u)$ are determined uniquely from the following recursive formula

$$\begin{aligned} &(Q_{b,j}(u))_x + [u, Q_{b,j}(u)] = [Q_{b,j+1}(u), a], \\ &Q_{b,0} = b, \quad Q_{a,1}(u) = u. \end{aligned}$$

The (b, j)-flow is $u_t = (Q_{b,j}(u))_x + [u, Q_{b,j}(u)]$, which commutes with the (b', j')-flow. The hierarchy of these commuting flows is called the $\mathfrak{g}$AKNS-hierarchy in [75], and the G-hierarchy in [69].

It follows from the recursive formula that u is a solution of the (b, j)-th flow if and only if

$$\theta_\lambda = (a\lambda + u)dx + (b\lambda^j + Q_{b,1}(u)\lambda^{j-1} + \cdots + Q_{b,j}(u))dt$$

is flat for all $\lambda \in \mathbb{C}$. In other words, θ_λ is a Lax pair of the (b, j)-flow.

There are several ways to construct subhierarchies from the G-hierarchy by finding suitable invariant submanifolds (sometimes called *restrictions*). For example, there are many works concerning the KdV type equations (cf. [26, 30, 31, 62, 70]). In this paper, we explain several restrictions of the G-hierarchy using finite order automorphisms of $\mathcal{G}$ (cf. [41, 48, 63, 70]).

σ-twisted G-hierarchy

If σ is an order k automorphism of the complex Lie group G, then the $(b, nk+1)$-flow in the G-hierarchy leaves $C(\mathbb{R}, \mathcal{G}_a^\perp \cap \mathcal{G}_0)$ invariant, where $\mathcal{G}_0$ is the fixed point set of $d\sigma_e$ on $\mathcal{G}$. The hierarchy of the restriction of these flows to $C(\mathbb{R}, \mathcal{G}_a^\perp \cap \mathcal{G}_0)$ is called the *σ-twisted G-hierarchy*. The Kupershmidt-Wilson hierarchy is an example with $\mathcal{G} = sl(n, \mathbb{C})$ and $k = n$ ([48]).

U-hierarchy

Suppose τ is a conjugate linear Lie algebra involution of $\mathcal{G}$, and $\mathcal{U}$ is the fixed point set of τ, i.e., $\mathcal{U}$ is a real form of $\mathcal{G}$. Then the (b, j)-flow leaves $C(\mathbb{R}, \mathcal{G}_a^\perp \cap \mathcal{U})$ invariant. The hierarchy restricted to $C(\mathbb{R}, \mathcal{G}_a^\perp \cap \mathcal{U})$ is called the U-hierarchy. For example, the NLS equation occurs as the second flow in the $SU(2)$-hierarchy, and the 3-wave equation as the first flow in the $SU(3)$-hierarchy.

U/U_0-hierarchy

Suppose τ is a conjugate-linear involution, and σ is an order k complex linear, Lie algebra automorphism of $\mathcal{G}$ such that $\sigma\tau = \tau^{-1}\sigma^{-1}$. Then the $(b, nk+1)$-flow leaves $C(\mathbb{R}, \mathcal{G}_a^\perp \cap \mathcal{U}_0)$ invariant, where $\mathcal{U}_0$ is the Lie subalgebra of $\mathcal{G}$ that is fixed by both σ and τ. The hierarchy restricted to $C(\mathbb{R}, \mathcal{G}_a^\perp \cap \mathcal{U}_0)$ is called the U/U_0-hierarchy. For example, the 3rd flow in the $SU(2)/SO(2)$-hierarchy is the modified KdV equation with $k = 2$.

U/U_0-system

Let U/U_0 be the rank n symmetric space given by involutions τ, σ of G, $\mathcal{U} = \mathcal{U}_0 + \mathcal{U}_1$ the Cartan decomposition, and $\mathcal{A}$ a maximal abelian subspace of $\mathcal{U}_1$. Let $\{a_1, \cdots, a_n\}$ be a basis of $\mathcal{A}$. By putting the $(a_1, 1), \cdots, (a_n, 1)$-flows in the U/U_0-hierarchy together, we get the U/U_0 *-system* for maps $v : \mathbb{R}^n \to \mathcal{U}_{\mathcal{A}}^\perp \cap \mathcal{U}_1$:

$$[a_i, v_{x_j}] - [a_j, v_{x_i}] = [[a_i, v], [a_j, v]], \quad i \neq j,$$

where $\mathcal{U}_{\mathcal{A}} = \{y \in \mathcal{U} \mid [y, \xi] = 0 \ \forall \ \xi \in \mathcal{A}\}$. Note that v is a solution of the U/U_0-system if and only if

$$\theta_\lambda = \sum_{j=1}^{n} (a_i \lambda + [a_i, v]) \ dx_i$$

is flat for all $\lambda \in \mathbb{C}$, i.e., θ_λ is a Lax n-tuple of the U/U_0-system.

The -1-flow associated to U

There is also a sequence of negative flows associated to G (cf. [26, 31, 69]). We review the first one in this sequence (the -1-flow) below.

Let $a, b \in \mathcal{U}$ such that $[a, b] = 0$. The -1-flow associated to U is the following system for $g : \mathbb{R}^2 \to U$:

$$(g^{-1} g_x)_t = [a, g^{-1} b g],$$

with constraint $g^{-1} g_x \in \mathcal{U}_a^\perp$. The -1-flow has a Lax pair

$$(a\lambda + g^{-1} g_x) dx + \lambda^{-1} g^{-1} b g \ dt.$$

Elliptic (G, τ)-systems

The m-th elliptic (G, τ)-system is the equation for $(u_0, \cdots, u_m)$: $\mathbb{C} \to \oplus_{i=0}^m \mathcal{G}$ so that

$$\theta_\lambda = \sum_{j=0}^{m} \lambda^{-j} u_j dz + \lambda^j \tau(u_j) d\bar{z}$$

is flat for all $\lambda \in S^1$. The first (G, τ)-system is the equation for harmonic maps from $\mathbb{R}^2$ to U, where U is the fixed point set of τ.

Elliptic (G, τ, σ)-systems

Suppose σ is an order k automorphism of G such that $\sigma\tau = \tau\sigma$. Let $\mathcal{G}_j$ denote the eigenspace of σ_* on $\mathcal{G}$ with eigenvalue $e^{\frac{2\pi i j}{k}}$. We call the m-th elliptic (G, τ)-system with constraints $u_i \in \mathcal{G}_{-i}$ the m-th elliptic (G, τ, σ)-system. Solutions of the first (G, τ, σ)-system is the equation for primitive maps studied by Burstall and Pedit [15].

Dressing actions

To explain the symmetries and the construction of solutions of integrable systems, we need the dressing action of Zakharov and Shabat

[76]. Suppose G_+, G_- are subgroups of G and the multiplication map from $G_+ \times G_-$ to G is a bijection. Then every $g \in G$ can be factored uniquely as $g = g_+g_-$ with $g_+ \in G_+$ and $g_- \in G_-$. Moreover, the space of right cosets G/G_- can be identified with G_+, so the canonical action of G_- on G/G_- by left multiplication, $g_- \cdot (gG_-) = g_-gG_-$, induces an action $*$ of G_- on G_+. The action $*$ is called the *dressing action*. The dressing action can be computed by factorization. In fact, $g_- * g_+ = \tilde{g}_+$, where $g_-g_+ = \tilde{g}_+\tilde{g}_-$ with $\tilde{g}_+ \in G_+$ and $\tilde{g}_- \in G_-$. If the multiplication map from $G_+ \times G_-$ to G is injective but only onto an open, dense subset of G, then the dressing action $*$ is a local action, but the corresponding Lie algebra action is well-defined.

Iwasawa and Gauss factorizations

There are two well-known factorizations associated to a complex simple Lie group G. The *Iwasawa factorization* is $G = KAN$, where K is a maximal compact subgroup of G, A is abelian, and N is unipotent. We also refer to $G = KB$ as the Iwasawa factorization of G, where $B = AN$ is a Borel subgroup. Let $\mathcal{A}$ be a Cartan subalgebra of $\mathcal{G}$, $\mathcal{N}_+, \mathcal{N}_-$ the spaces spanned by all positive and negative roots respectively, and A, N_+, N_- the corresponding Lie subgroups of G. Then the multiplication map from $N_- \times A \times N_-$ to G is injective and onto an open dense subset of G. The set N_-AN_+ is called a *big cell* of G. The so-called *Gauss factorization* associated to G refers to the fact that any g in the big cell can be factorized uniquely as n_-an_+ with $n_\pm \in N_\pm$ and $a \in A$. For example, for $G = SL(n, \mathbb{C})$, let $K = SU(n)$, B_n the subgroup of upper triangular matrices with real diagonal, A_n the subgroup of diagonal matrices, and $N_+(n), N_-(n)$ the subgroups of strictly upper and lower triangular matrices. The Iwasawa factorization of $SL(n, \mathbb{C})$ is KB_n, which can be done using the Gram-Schmidt process. The Gauss factorization for the big cell is $N_-(n)A_nN_+(n)$, which can be carried out using the Gaussian elimination.

Loop group factorizations

We review three types of loop group factorizations that are needed for the study of symmetries of soliton equations and elliptic integrable systems. Let $L(G)$ denote the group of smooth maps $f : S^1 \to G$, $L_+(G)$ the subgroup of $f \in L(G)$ that are the boundary values of holomorphic maps defined on $|\lambda| < 1$, and $L_-(G)$ the subgroup of $f \in L(G)$ that can be extended holomorphically to $|\lambda| > 1$ in S^2 and $f(\infty) = e$. Let U be a maximal compact subgroup of G, and $L_e(U)$ the subgroup of $f \in L(G)$ such that the image of f lies in U and $f(1) = e$ (the identity of G).

—The *Gauss loop group factorization* (or the *Birkhoff factorization*) states that there is an open dense subset L' of $L(G)$ such that any $g \in L'$ can be factored uniquely as $g_+ g_-$ with $g_\pm \in L_\pm(G)$.

—The *Iwasawa loop group factorization*, proved in [57], states that the multiplication map from $L_e(U) \times L_+(G)$ to $L(G)$ is a bijection.

—Let $\epsilon > 0$, $\mathcal{O}_\epsilon = \{\lambda \in \mathbb{C} \mid |\lambda| < \epsilon\}$, and $\mathcal{O}_{\frac{1}{\epsilon}} = \{\lambda \in S^2 = \mathbb{C} \cup \{\infty\} \mid |\lambda| > 1/\epsilon\}$. Let $\mathbb{C}^* = \{\lambda \in \mathbb{C} \mid \lambda \neq 0\}$, and $\Omega^\tau(G)$ the group of holomorphic maps $f : (\mathcal{O}_\epsilon \cup \mathcal{O}_{1/\epsilon}) \cap \mathbb{C}^* \to G$ that satisfy the (G,τ)-*reality condition* $\tau(f(1/\bar{\lambda})) = f(\lambda)$, $\Omega^\tau_+(G)$ the subgroup of $f \in \Omega^\tau(G)$ that extend holomorphically to $\mathbb{C}^*$, and $\Omega^\tau_-(G)$ the subgroup of $f \in \Omega^\tau(G)$ that extend holomorphically to $\mathcal{O}_\epsilon \cup \mathcal{O}_{1/\epsilon}$ and $f(\infty) = e$. McIntosh proved ([51]) that the multiplication map from $\Omega^\tau_-(G) \times \Omega^\tau_+(G)$ to $\Omega^\tau(G)$ is a bijection.

These loop group factorizations play central roles in the study of integrable PDEs.

Solutions of soliton flows via loop group factorizations

Let $\mathcal{O}_{1/\epsilon} = \{\lambda \in \mathbb{C} \mid |\lambda| > 1/\epsilon\}$, and $\Lambda^\tau(G)$ the group of holomorphic maps $f : \mathcal{O}_{1/\epsilon} \to G$ that satisfies the U-reality condition

$$\tau(f(\bar{\lambda})) = f(\lambda),$$

where τ is the involution on $\mathcal{G}$ that defines the real form $\mathcal{U}$. Note that $f(r) \in U$ for real r. Let $\Lambda^\tau_+(G)$ denote the subgroup of $f \in \Lambda^\tau(G)$ that extend holomorphically to $\mathbb{C}$, and $\Lambda^\tau_-(G)$ the subgroup of $f \in \Lambda^\tau(G)$ that extend holomorphically to $1/\epsilon < |\lambda| \leq \infty$ and $f(\infty) = e$. The Gauss loop group factorization implies that the multiplication map from $\Lambda^\tau_+(G) \times \Lambda^\tau_-(G)$ to $\Lambda^\tau(G)$ is injective and its image is open and dense.

The Lax pair θ_λ of a soliton flow in the U-hierarchy is a flat $\mathcal{G}$-valued connection 1-form that satisfies the U-reality condition $\tau(\theta_{\bar{\lambda}}) = \theta_\lambda$. So $\theta_\lambda(x,t)$ can be viewed as a map from $(x,t) \in \mathbb{R}^2$ to the Lie algebra $\Lambda^\tau_+(\mathcal{G})$ of $\Lambda^\tau_+(G)$, or equivalently a $\Lambda^\tau_+(\mathcal{G})$-valued connection 1-form on $\mathbb{R}^2$. Therefore the trivialization $E_\lambda(x,t)$ of $\theta_\lambda(x,t)$ can be viewed as a map from $\mathbb{R}^2$ to $\Lambda^\tau_+(G)$. Given $g_- \in \Lambda^\tau_-(G)$, let $\tilde{E}(x,t)$ denote the dressing action of g_- on $E(x,t)$, i.e., $\tilde{E}(x,t)$ is obtained using the Gauss loop group factorization to factor $g_- E(x,t) = \tilde{E}(x,t)\tilde{g}(x,t)$ with $\tilde{E}(x,t) \in \Lambda^\tau_+(G)$ and $\tilde{g}(x,t) \in \Lambda^\tau_-(G)$ for each (x,t). It can be shown that $\tilde{E}(x,t)$ is again a trivialization of some solution of the soliton flow. This defines an action of $\Lambda^\tau_-(G)$ on the space of solutions, which we denote by $*$. Moreover, 0 is a solution. If $g_- \in \Lambda_-^\tau(G)$ is rational, then $g_- * 0$ can be computed explicitly and is a rational function of exponentials. These are the pure soliton solutions. For general $g_- \in \Lambda_-^\tau(G)$, $g_- * 0$ is a local analytic solution of the soliton flow. Algebraic geometric solutions are included in the orbit $\Lambda_-^\tau(G) * 0$. To construct general rapidly decaying solutions for the flows in the U-hierarchy, we need a new type of loop group factorization. Namely, factor fg as $\tilde{g}\tilde{f}$, where $f, \tilde{f} \in L_+(G)$ so that $f_b, \tilde{f}_b$ are the identity $e \in G$ at $\lambda = -1$ up to infinite order and $g, \tilde{g}$ are loops in U that have an essential singularity at $\lambda = -1$. Here $f_b(\lambda)$ and $\tilde{f}_b(\lambda)$ denote the B-component of $f(\lambda)$ and $\tilde{f}(\lambda)$ in the Iwasawa factorization $G = UB$ for each λ. For more details see section 5.3 and [69].

Solutions of elliptic systems via loop group factorizations

The Lax pair θ_λ of the m-th (G, τ)-system satisfies the (G, τ)-reality condition

$$\tau(g(1/\bar{\lambda})) = g(\lambda).$$

The trivialization E of θ_λ is a map from $\mathbb{C}$ to $\Omega_+^\tau(G)$. It follows from the McIntosh loop group factorization that the dressing action of $\Omega_-^\tau(G)$ induces an action on the space of solutions of the (G, τ)-systems. Since there are constant solutions for the (G, τ)-system, the $\Omega_-^\tau(G)$-orbits through these constant solutions give rise to a class of solutions. But these are not all the solutions. The (G, τ)-reality condition implies that the restriction of the trivialization E of a solution to the unit circle $\mid \lambda \mid = 1$ lies in U, i.e., E can be viewed as a map from $\mathbb{C}$ to $L(U)$. Dorfmeister, Pedit and Wu ([27]) use meromorphic maps and the Iwasawa loop group factorization $L(G) = L_e(U)L_+(G)$ to give a method of constructing all local solutions of the (G, τ)-systems. This is the so-called the *Weierstrass representation* or the *DPW method.*

Although methods of constructing solutions for both the U-hierarchy and the elliptic (G, τ)-systems are similar in spirit, initial data and techniques used are somewhat different. Moreover, while there is a canonical choice of initial data used in the factorization method to solve soliton flows, there is no clear canonical choice of meromorphic data for the (G, τ)-hierarchy. Since the (G, τ)-hierarchy contains the equation for harmonic maps from a domain of $\mathbb{R}^2$ to U, the main interest has been to understand the relation between the initial meromorphic data of the factorization method and the global geometry; for example, to find properties of meromorphic data which corresponds to a harmonic map from a complete surface M to U. This has been done when M is S^2 and more generally for harmonic maps of finite uniton numbers ([72, 14, 39]), and also when M is T^2 ([45, 56, 13]). For a detailed survey of results concerning harmonic maps, loop groups, and integrable systems, we refer the reader to [38].

When we study a geometric problem concerning maps f from a manifold M to a homogeneous space U/U_0, it is often useful to find a good lifting $\tilde{f} : M \to U$ and write down the geometric condition imposed on the map f in terms of the flat $\mathcal{U}$-valued 1-form $\tilde{f}^{-1}d\tilde{f}$. If there is a natural holomorphic deformation $F_\lambda : M \to G$ of such maps so that $F_0 = \tilde{f}$ and the flatness of $F_\lambda^{-1}dF_\lambda$ for all λ is equivalent to the flatness of $\tilde{f}^{-1}d\tilde{f}$ in some natural coordinate system on M, then the corresponding geometric PDE is often an integrable system with a zero curvature formulation.

Integrable systems in differential geometry

One of the main interests in classical differential geometry is to find natural geometric conditions for surfaces in $\mathbb{R}^3$ so that there are many explicit solutions and deformations. It is now known that the Gauss-Codazzi equations for surfaces with constant mean curvature, constant Gaussian curvature, and isothermic surfaces in $\mathbb{R}^3$ studied by classical differential geometers are integrable systems and Bäcklund and Ribaucour transformations can be constructed naturally using loop group factorizations (cf. [6, 10, 12, 17, 18, 37, 44, 56, 58, 67, 70, 71]).

In this paper, we give a brief review of the following subset of the known integrable geometric problems:

(i) The Gauss-Codazzi equations of n-submanifolds with constant sectional curvature in $\mathbb{R}^m$, S^m and hyperbolic space $\mathbb{H}^m$ are the U/U_0-system associated to certain real Grassmannian manifolds U/U_0 (cf. [10, 19, 36, 66, 68]).

(ii) The Gauss-Codazzi equations of flat Lagrangian submanifolds of $\mathbb{C}P^n$ is the $SU(n+1)/SO(n+1)$-system.

(iii) Indefinite affine spheres in $\mathbb{R}^3$ are given by solutions of the -1-flow in the $SL(3,\mathbb{R})/\mathbb{R}^+$-hierarchy ([7]).

(iv) Solutions of the -1-flow in the U/U_0-hierarchy give rise to harmonic maps from the Lorentz space $\mathbb{R}^{1,1}$ to U/U_0. (These are called sigma-models by physicists.)

(v) The first elliptic (G, τ)-system is the equation for harmonic maps from $\mathbb{R}^2$ to U ([72]). The first elliptic (G, τ, σ)-system is the equation for harmonic maps from $\mathbb{R}^2$ to the symmetric space U/U_0 if the order of σ is two ([13]), where U_0 is the fixed point set of σ in U.

(vi) The equation for minimal surfaces in $\mathbb{C}P^2$ is the first $(SL(3, \mathbb{C}), \tau, \sigma)$-system, where τ, σ gives the 3-symmetric space $SU(3)/T^2$ ([11, 9]).

(vii) Equations for minimal Lagrangian surfaces in $\mathbb{C}P^2$, minimal Legendrian surfaces in S^5, and minimal Lagrangian cones in $\mathbb{R}^6 = \mathbb{C}^3$ are given by the first $(SL(3, \mathbb{C}), \tau, \sigma)$-system, where τ, σ give the 6-symmetric space $SU(3)/SO(2)$ ([52]).

(viii) The equation for Hamiltonian stationary surfaces in $\mathbb{C}P^2$ is the second elliptic system associated to the 4-symmetric space $SU(3)/SU(2)$ ([43]).

Note that there may be several geometric problems associated to one integrable system. For example:

—The SGE equation is the equation for surfaces in $\mathbb{R}^3$ with Gaussian curvature $K = -1$, and is also the equation for harmonic maps from $\mathbb{R}^{1,1}$ to S^2. The reason here is that if M is a surface in $\mathbb{R}^3$ with $K = -1$, then the second fundamental form II of M is conformally equivalent to the flat Lorentzian metric and the Gauss map $\nu : M \to S^2$ is harmonic when M is equipped with metric II.

—The $U(n)/O(n)$-system is the equation for flat Lagrangian submanifolds in $\mathbb{R}^{2n}$ that lie in S^{2n-1}, the equation for flat Lagrangian submanifolds in $\mathbb{C}P^{n-1}$, and also the equation for flat Egoroff metrics. These three geometries are related as follows: the preimage of a flat Lagrangian submanifold in $\mathbb{C}P^{n-1}$ via the Hopf fibration $\pi : S^{2n-1} \to \mathbb{C}P^{n-1}$ is a flat Lagrangian submanifold in $\mathbb{R}^{2n}$ that lies in S^{2n-1}, and the induced metrics on these flat Lagrangian submanifolds are flat Egoroff metrics.

We now know that there is a very large collection of integrable geometric PDEs. In this paper, we only discuss a small subset of these examples. For a more extensive review of integrable systems in differential geometry, we refer the readers to the books [37, 58, 67], articles in this volume, and references therein.

Most of the integrable geometric PDEs mentioned in this paper are either the U/U_0-system, the -1-flow, or the (G, τ, σ)-systems. We would like to end this introduction by proposing a program: Find geometric problems whose equations are given by the U/U_0-system, the -1-flow, or the m-th (G, τ, σ)-system. Most examples given in this paper have $U = O(m)$ or $SU(m)$. We believe the success of this program for general compact Lie groups U should provide new natural classes of submanifolds in symmetric spaces and in homogeneous Riemannian manifolds with exceptional holonomy. This program also makes sense for any semi-simple, non-compact, Lie group U. We believe that interesting classes of submanifolds in pseudo-Riemannian symmetric spaces, in $SL(n, \mathbb{R})$-geometry (affine geometry), and in U-geometry will arise naturally from this program.

The author would like to thank Martin Guest for many helpful comments and suggestions.

§2. Soliton equations

We review the method of constructing a hierarchy of $n \times n$ soliton flows developed by Zakharov-Shabat [76] and Ablowitz-Kaup-Newell-Segur [2]. Their method works equally well if we replace the algebra of $n \times n$ matrices by a general semi-simple, complex Lie algebra $\mathcal{G}$ (cf. [41, 60, 69, 75]). We also review the construction of new hierarchies of flows by restricting the G-hierarchy to submanifolds naturally associated to finite order automorphisms of $\mathcal{G}$. Many interesting equations in differential geometry and mathematical physics are flows in these restricted hierarchies.

2.1. The G-hierarchy

Let $\langle\ ,\ \rangle$ be a non-degenerate, ad-invariant bilinear form on $\mathcal{G}$, $a \in \mathcal{G}$ regular, i.e., the centralizer $\mathcal{G}_a$ of a in $\mathcal{G}$ is abelian, and $\mathcal{G}_a^\perp = \{\xi \in \mathcal{G} \mid \langle \xi, \mathcal{G}_a \rangle = 0\}$. Let $\mathcal{S}(\mathbb{R}, \mathcal{G}_a^\perp)$ denote the space of rapidly decaying maps from $\mathbb{R}$ to $\mathcal{G}_a^\perp$.

There is a unique family of $\mathcal{G}$-valued maps $Q_{b,j}(u)$ parametrized by $\{b \in \mathcal{G} \mid \mathcal{G}_b = \mathcal{G}_a\}$ and positive integer j that satisfies the following conditions:

$$(Q_{b,j}(u))_x + [u, Q_{b,j}(u)] = [Q_{b,j+1}(u), a], \quad Q_{b,0}(u) = b, \tag{2.1.1}$$

$$\sum_{j=0}^{\infty} Q_{b,j}(u)\lambda^{-j} \text{ is conjugate to } b \text{ as an asymptotic expansion.} \tag{2.1.2}$$

These conditions imply that $Q_{b,j}(u)$ is a polynomial in $u, \partial_x u, \ldots, \partial_x^{j-1}u$ (cf. [60, 69]). The *G-hierarchy* is a family of evolution equations on $\mathcal{S}(\mathbb{R}, \mathcal{G}_a^\perp)$ parametrized by (b,j), where $b \in \mathcal{G}_a$ such that $\mathcal{G}_b = \mathcal{G}_a$ and j is a positive integer. The (b,j)-flow is

$$u_t = (Q_{b,j}(u))_x + [u, Q_{b,j}(u)] = [Q_{b,j+1}(u), a]. \tag{2.1.3}$$

Recall that a $\mathcal{G}$-valued connection 1-form $\theta = \sum_{i=1}^n A_i(x)dx_i$ is *flat* if

$$d\theta = -\theta \wedge \theta,$$

i.e.,

$$-(A_i)_{x_j} + (A_j)_{x_i} + [A_i, A_j] = 0, \quad 1 \le i < j \le n.$$

The flatness of θ is equivalent to the solvability of the following linear system:

$$E_{x_i} = EA_i, \quad 1 \le i \le n. \tag{2.1.4}$$

Note that (2.1.4) can also be written as $E^{-1}dE = \theta$.

Definition 2.1.1. Let θ be a flat $\mathcal{G}$-valued connection 1-form on $\mathbb{R}^n$. A map $E : \mathbb{R}^n \to G$ is called *a trivialization of* θ if $E^{-1}dE = \theta$. A trivialization E of θ is called a *frame* of θ if E satisfies the initial condition $E(0) = e$, where e is the identity element of G.

The recursive formula (2.1.1) implies that u is a solution of the (b,j)-flow (2.1.3) if and only if

$$\theta_\lambda = (a\lambda + u)\, dx + (b\lambda^j + Q_{b,1}(u)\lambda^{j-1} + \cdots + Q_{b,j}(u))\, dt \tag{2.1.5}$$

is a flat $\mathcal{G}$-valued connection 1-form on the (x,t) plane for all $\lambda \in \mathbb{C}$. In other words, the (b,j)-flow has a Lax pair.

The Cauchy problem with rapidly decaying initial data for the (b,j)-flow (2.1.3) in the G-hierarchy is solved by the inverse scattering method (cf. [3]).

Theorem 2.1.2. *([3]). Suppose $a \in \mathcal{G}$ such that $\mathcal{G}_a$ is a maximal abelian subalgebra $\mathcal{A}$ of $\mathcal{G}$. Then there is an open dense subset $\mathcal{S}_0$ of $\mathcal{S}(\mathbb{R}, \mathcal{A}^\perp)$ such that if $u_0 \in \mathcal{S}_0$, then the Cauchy problem for the (b,j)-flow in the G-hierarchy,*

$$\begin{cases} u_t = (Q_{b,j}(u))_x + [u, Q_{b,j}(u)], \\ u(x,0) = u_0(x), \end{cases}$$

has a unique solution u. Moreover, $u(x,t)$ is defined for all $(x,t) \in \mathbb{R}^2$ and $u(\cdot,t) \in \mathcal{S}(\mathbb{R}, \mathcal{A}^\perp)$.

The following is well-known, and the proof can be found in many places (cf. [1, 69]).

Theorem 2.1.3. *Let $X_{b,j}$ denote the vector field on $\mathcal{S}(\mathbb{R}, \mathcal{A}^\perp)$ defined by*

$$X_{b,j}(u) = (Q_{b,j}(u))_x + [u, Q_{b,j}(u)]. \tag{2.1.6}$$

Then $[X_{b,j}, X_{b',j'}] = 0$ for all $b, b' \in \mathcal{A}$ and positive integers j, j'. In other words, the (b,j)-flow commutes with the (b',j')-flow.

Example 2.1.4. *The $SL(2,\mathbb{C})$-hierarchy* (cf. [1, 54]).

Let $G = SL(2,\mathbb{C})$, $a = \text{diag}(i, -i)$. Then $\mathcal{G}_a = \mathcal{A} = \mathbb{C}a$, and

$$\mathcal{G} \cap \mathcal{A}^\perp = \left\{ \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \,\middle|\, q, r \in \mathbb{C} \right\}.$$

Let $u = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$. Use (2.1.1) and (2.1.2) and a direct computation to get the first three terms of $Q_{a,j}(u)$:

$$Q_{a,1}(u) = u, \quad Q_{a,2}(u) = \begin{pmatrix} \frac{iqr}{2} & \frac{iq_x}{2} \\ -\frac{ir_x}{2} & -\frac{iqr}{2} \end{pmatrix},$$

$$Q_{a,3}(u) = \begin{pmatrix} \frac{qr_x - q_x r}{4} & -\frac{q_{xx}}{4} + \frac{q^2 r}{2} \\ -\frac{r_{xx}}{4} + \frac{qr^2}{2} & -\frac{qr_x - q_x r}{4} \end{pmatrix}.$$

Then the (a,j)-flow, $j = 1, 2, 3$, in the $SL(2,\mathbb{C})$-hierarchy is the following evolution for q and r:

$$\begin{aligned}
q_t &= q_x, \quad r_t = r_x, \\
q_t &= \frac{i}{2}(q_{xx} - 2q^2 r), \quad r_t = -\frac{i}{2}(r_{xx} - 2qr^2), \\
q_t &= -\frac{q_{xxx}}{4} + \frac{3}{2} qrq_x, \quad r_t = -\frac{r_{xxx}}{4} + \frac{3}{2} qrr_x.
\end{aligned}$$

2.2. The U-hierarchy

Let τ be an involution of G such that its differential at the identity e (still denoted by τ) is a conjugate linear involution on the complex Lie algebra $\mathcal{G}$, and U the fixed point set of τ. The Lie algebra $\mathcal{U}$ of U is a real form of $\mathcal{G}$. Let $\mathcal{U}_a$ denote the centralizer of a in $\mathcal{U}$, and $\mathcal{U}_a^\perp$ the orthogonal complement of $\mathcal{U}_a$ in $\mathcal{U}$. Note that $\mathcal{U}_a^\perp = \mathcal{G}_a^\perp \cap \mathcal{U}$. It is known that the (b,j)-flow in the G-hierarchy leaves $\mathcal{S}(\mathbb{R},\mathcal{U}_a^\perp)$-invariant (for more details see [70]). The restriction of the flow (2.1.3) to $\mathcal{S}(\mathbb{R},\mathcal{U}_a^\perp)$ is the (b,j)-flow in the *U-hierarchy*. The Lax pair θ_λ defined by (2.1.5) is a $\mathcal{G}$-valued 1-form, and θ_λ satisfies the *U-reality condition*:

$$\tau(\theta_{\bar\lambda}) = \theta_\lambda. \tag{2.2.1}$$

Note that $\xi = \sum_j \xi_j \lambda^j$ satisfies the U-reality condition if and only if $\xi_j \in \mathcal{U}$ for all j.

Example 2.2.1. *The $SU(2)$-hierarchy.*

Let τ be the involution of $sl(2,\mathbb{C})$ defined by $\tau(\xi) = -\bar\xi^t$. Then the fixed point set of τ is the real form $\mathcal{U} = su(2)$. Let $a = \mathrm{diag}(i,-i) \in \mathcal{U}$. Then $\mathcal{U}_a = \mathcal{A} = \mathbb{R}a$, and

$$\mathcal{A}^\perp \cap \mathcal{U} = \left\{ \begin{pmatrix} 0 & q \\ -\bar q & 0 \end{pmatrix} \,\middle|\, q \in \mathbb{R} \right\}.$$

So $C(\mathbb{R},\mathcal{A}^\perp \cap \mathcal{U})$ can be identified as $C(\mathbb{R},\mathbb{C})$, and the $SU(2)$-hierarchy is the restriction of the $SL(2,\mathbb{C})$-hierarchy to the subspace $r = -\bar q$. The first three flows in the $SU(2)$-hierarchy are

$$\begin{aligned} q_t &= q_x, \\ q_t &= \frac{i}{2}(q_{xx} + 2|q|^2 q), \\ q_t &= -\frac{1}{4}(q_{xxx} + 6|q|^2 q). \end{aligned}$$

Note that the $(a,2)$-flow in the $SU(2)$-hierarchy is the NLS equation.

2.3. The σ-twisted G-hierarchy

Let σ be an order k group automorphism of G such that its differential at the identity e (still denoted by σ) is a complex linear Lie algebra homomorphism of $\mathcal{G}$. Let

$$\mathcal{G} = \mathcal{G}_0 + \cdots + \mathcal{G}_{k-1},$$

where $\mathcal{G}_j$ is the eigenspace of σ with eigenvalue $e^{\frac{2j\pi i}{k}}$. Note that $\mathcal{G}_i = \mathcal{G}_j$ if $i \equiv j \mod k$, and

$$[\mathcal{G}_j, \mathcal{G}_r] \subset \mathcal{G}_{j+r}.$$

Let $\mathcal{A}$ be a maximal abelian subspace in $\mathcal{G}_1$, and $a \in \mathcal{A}$ *regular in* $\mathcal{G}_1$, i.e.,

$$\{x \in \mathcal{G}_1 \mid [x,a] = 0\} = \mathcal{A}.$$

It is known (cf. [70]) that if the image of u lies in $\mathcal{G}_0 \cap \mathcal{G}_a^\perp$, then

$$Q_{b,j}(u) \in \mathcal{G}_{1-j}. \tag{2.3.1}$$

Since $a \in \mathcal{G}_1$, the right hand side of the $(b, mk+1)$-flow satisfies

$$(Q_{b,mk+1}(u))_x + [u, Q_{b,mk+1}(u)] = [Q_{b,mk+2}, a] \in \mathcal{G}_{-mk} = \mathcal{G}_0.$$

In other words, the $(b, mk+1)$-flow leaves $\mathcal{S}(\mathbb{R}, \mathcal{G}_a^\perp \cap \mathcal{G}_0)$ invariant. *The σ-twisted G-hierarchy* is the restriction of the $(b, mk+1)$-flow in the G-hierarchy to $\mathcal{S}(\mathbb{R}, \mathcal{G}_a^\perp \cap \mathcal{G}_0)$ for $m = 1, 2, \cdots$.

It follows from (2.3.1) that the Lax pair of the $(b, mj+1)$-flow in the σ-twisted G-hierarchy satisfies the (G, σ)*-reality condition*:

$$\sigma(\theta_{e^{-\frac{2\pi i}{k}}\lambda}) = \theta_\lambda. \tag{2.3.2}$$

Note that $\xi = \sum_j \xi_j \lambda^j$ satisfies the (G, σ)-reality condition if and only if $\xi_j \in \mathcal{G}_j$ for all j.

Example 2.3.1. *Kupershmidt-Wilson hierarchy* ([48]).

Let $G = SL(n, \mathbb{C})$, and σ the order n automorphism of $SL(n, \mathbb{C})$ defined by $\sigma(g) = C^{-1} g C$, where $C = e_{21} + e_{32} + \cdots + e_{n,n-1} + e_{1n}$ is the permutation matrix $(12 \cdots n)$. Here e_{ij} denote the (i,j)-th elementary matrix in $gl(n)$. The eigenspace $\mathcal{G}_k$ of σ on $sl(n, \mathbb{C})$ with eigenvalue $\alpha = e^{\frac{2\pi i k}{n}}$ is the space of all $y = (y_{ij})$ such that $y_{i+1,j+1} = \alpha^k y_{ij}$ for all $1 \le i, j \le n$. Let $a = \mathrm{diag}(1, \alpha, \cdots, \alpha^{n-1}) \in \mathcal{G}_1$, and $\mathcal{A} = \mathbb{C}a$. Then $\mathcal{A}$ is a maximal abelian subalgebra of $\mathcal{G}_1$. The $(SL(n, \mathbb{C}), \sigma)$-hierarchy is the restriction of the $(jn+1)$-th flow in the $sl(n, \mathbb{C})$-hierarchy to $\mathcal{S}(R, \mathcal{G}_0 \cap \mathcal{G}_a^\perp)$. For example, for $n = 2$,

$$\mathcal{G}_0 \cap \mathcal{G}_a^\perp = \left\{ \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix} \;\middle|\; q \in \mathbb{C} \right\}.$$

The first flow is the translation $q_t = q_x$, and the third flow is the complex modified KdV equation

$$q_t = \frac{1}{4}(q_{xxx} - 6q^2 q_x).$$

2.4. The U/U_0-hierarchy

Let τ be a conjugate linear involution of $\mathcal{G}$, $\mathcal{U}$ its fixed point set, and σ a complex linear, order k automorphism of $\mathcal{G}$ such that

$$\tau\sigma = \sigma^{-1}\tau^{-1} = \sigma^{-1}\tau.$$

Let $\mathcal{G}_j$ denote the eigenspace of σ with eigenvalue $e^{\frac{2\pi i j}{k}}$. We claim that $\tau(\mathcal{G}_j) \subset \mathcal{G}_j$. To see this, let $\xi_j \in \mathcal{G}_j$. Then

$$\sigma(\tau(\xi_j)) = \tau(\sigma^{-1}(\xi_j)) = \tau(\alpha^{-j}\xi_j) = \overline{\alpha^{-j}}\tau(\xi_j) = \alpha^j \tau(\xi_j),$$

where $\alpha = e^{\frac{2\pi i}{k}}$, proving the claim. Let $\mathcal{U}_j = \mathcal{G}_j \cap \mathcal{U}$. Then we have

$$\mathcal{U} = \mathcal{U}_0 + \cdots + \mathcal{U}_{k-1}.$$

Let $\mathcal{A} \subset \mathcal{U}_1$ be a maximal abelian subspace in $\mathcal{U}_1$. An element $a \in \mathcal{A}$ is *regular in* $\mathcal{U}_1$ if

$$\{\xi \in \mathcal{U}_1 \mid [\xi, a] = 0\} = \mathcal{A}.$$

Let $b \in \mathcal{A}$, and $u \in \mathcal{G}_a^\perp \cap \mathcal{U}_0$. Then $Q_{b,j}(u) \in \mathcal{U}_{1-j}$ for all $j \geq 0$ ([70]). So the $(b, mk+1)$-flow in the σ-twisted G-hierarchy leaves $\mathcal{S}(\mathbb{R}, \mathcal{G}_a^\perp \cap \mathcal{U}_0)$ invariant. The restriction of these flows to $\mathcal{S}(\mathbb{R}, \mathcal{G}_a^\perp \cap \mathcal{U}_o)$ is called the *U/U_o-hierarchy*.

The Lax pair θ_λ of the $(b, mk+1)$-flow in the U/U_0-hierarchy satisfies the *U/U_o-reality condition*:

$$\tau(\theta_{\bar{\lambda}}) = \theta_\lambda, \quad \sigma(\theta_\lambda) = \theta_{e^{\frac{2\pi i}{k}}\lambda}. \tag{2.4.1}$$

Note that $\xi = \sum_j \xi_j \lambda^j$ satisfies the U/U_0-reality condition if and only if $\xi_j \in \mathcal{U}_j$ for all j.

When the order of σ is 2, the condition $\tau\sigma = \sigma^{-1}\tau^{-1}$ implies that τ and σ commute, U/U_0 is a symmetric space, and $\mathcal{U} = \mathcal{U}_0 + \mathcal{U}_1$ is a Cartan decomposition for the symmetric space U/U_0.

Example 2.4.1. *The $SU(2)/SO(2)$-hierarchy.*

Let $\tau(\xi) = -\bar{\xi}^t$ and $\sigma(\xi) = -\xi^t$ be involutions of $sl(2, \mathbb{C})$ that give the symmetric space $SU(2)/SO(2)$. Then

$$\mathcal{U}_0 = so(2), \quad \mathcal{U}_1 = \{i\xi \mid \xi \in sl(2, \mathbb{R}), \text{ symmetric}\}$$

with $SU(2)/SO(2)$ the corresponding symmetric space. Let $a = \mathrm{diag}(i, -i)$. Then $\mathcal{G}_a^\perp \cap \mathcal{U}_0 = so(2)$. So $\mathcal{S}(\mathbb{R}, \mathcal{G}_a^\perp \cap \mathcal{U}_0)$ can be identified as $\mathcal{S}(\mathbb{R}, \mathbb{R})$. The $(a,1)$- and $(a,3)$-flow in the $SU(2)/SO(2)$-hierarchy are

$$q_t = q_x,$$
$$q_t = -\frac{1}{4}(q_{xxx} + 6q^2 q).$$

Note that the $(a,3)$-flow is *modified KdV equation* (mKdV equation).

Example 2.4.2. *The $SU(n)/SO(n)$-hierarchy.*

Let τ and ξ be involutions of $sl(n,\mathbb{C})$ defined by

$$\tau(\xi) = -\bar{\xi}^t, \quad \sigma(\xi) = -\xi^t.$$

Then $\tau\sigma = \sigma\tau$, $\mathcal{U} = su(n)$, $\mathcal{U}_0 = so(n)$, and $\mathcal{U}_1$ is the space of $iY \in su(n)$, where Y is real and symmetric with trace zero. The corresponding symmetric space is $\frac{U}{U_0} = \frac{SU(n)}{SO(n)}$. Let $\mathcal{A}$ denote the space of diagonal matrices in $su(n)$. Then $\mathcal{A}$ is a maximal abelian linear subspace in $\mathcal{U}_1$, and $\mathcal{A}^\perp \cap \mathcal{U}_0 = so(n)$. An element $a = i\text{diag}(a_1, \cdots, a_n)$ is *regular in* $\mathcal{U}_1$ if $a_1, \cdots, a_n$ are distinct. Let $a \in \mathcal{A}$ be a regular element, and $b = i\text{diag}(b_1, \cdots, b_n) \in \mathcal{A}$. The $(b,1)$-flow in the $\frac{SU(n)}{SO(n)}$-hierarchy on $\mathcal{S}(\mathbb{R}, so(n))$ is the *reduced n-wave equation*

$$(u_{ij})_t = \frac{b_i - b_j}{a_i - a_j}\,(u_{ij})_x + \sum_k \left(\frac{b_k - b_j}{a_k - a_j} - \frac{b_i - b_k}{a_i - a_k}\right) u_{ik}u_{kj}, \quad i \neq j.$$

Example 2.4.3. Let U/U_0 be a symmetric space, $\mathcal{U} = \mathcal{U}_0 + \mathcal{U}_1$ a Cartan decomposition, $\mathcal{A}$ a maximal abelian subspace in $\mathcal{U}_1$, $a \in \mathcal{A}$ regular, and $b \in \mathcal{A}$. Note $\text{ad}(a)^{-1}$ maps $\mathcal{U}_a^\perp \cap \mathcal{U}_0$ and $\mathcal{U}_a^\perp \cap \mathcal{U}_1$ isomorphically onto $\mathcal{U}_a^\perp \cap \mathcal{U}_1$ and $\mathcal{U}_a^\perp \cap \mathcal{U}_0$ respectively. So $\text{ad}(b)\text{ad}(a)^{-1}(\mathcal{U}_a^\perp \cap \mathcal{U}_0) \subset \mathcal{U}_a^\perp \cap \mathcal{U}_0$. The recursive formula (2.1.1) implies that

$$Q_{b,1}(u) = \text{ad}(b)\text{ad}(a)^{-1}(u). \tag{2.4.2}$$

So the $(b,1)$-flow in the U/U_0-hierarchy is the equation for maps $u : \mathbb{R}^2 \to \mathcal{U}_a^\perp \cap \mathcal{U}_0$:

$$u_t = \text{ad}(b)\text{ad}(a)^{-1}(u_x) + [u, \text{ad}(b)\text{ad}(a)^{-1}(u)]. \tag{2.4.3}$$

This is the *reduced n-wave equation associated to U/U_0*, which has a Lax pair

$$\theta_\lambda = (a\lambda + u)dx + (b\lambda + \text{ad}(b)\text{ad}(a)^{-1}(u))dt. \tag{2.4.4}$$

2.5. The U/U_0-system

Let τ be a conjugate linear involution of $\mathcal{G}$, σ a complex linear involution of $\mathcal{G}$ such that $\tau\sigma = \sigma\tau$, $\mathcal{U}$ the fixed point set of τ, and U_0 the subgroup of U fixed by σ. Let $\mathcal{U} = \mathcal{U}_0 + \mathcal{U}_1$ denote the Cartan decomposition of the symmetric space U/U_0. Let $\mathcal{A}$ be a maximal abelian linear subspace of $\mathcal{U}_1$, and $a_1, \cdots, a_n$ a basis of $\mathcal{A}$. *The U/U_0-system* is the following system for $v : \mathbb{R}^n \to \mathcal{U}_{\mathcal{A}}^{\perp} \cap \mathcal{U}_1$:

$$(2.5.1) \qquad [a_i, v_{x_j}] - [a_j, v_{x_i}] = [[a_i, v], [a_j, v]], \quad i \neq j.$$

It has a Lax n-tuple,

$$(2.5.2) \qquad \theta_\lambda = \sum_{i=1}^{n} (a_i \lambda + [a_i, v]) dx_i,$$

which satisfies the U/U_0-reality condition (2.4.1). Moreover, the following statements are equivalent for smooth map $v : \mathbb{R}^n \to \mathcal{U}_{\mathcal{A}}^{\perp} \cap \mathcal{U}_1$:

(i) v is a solution of the U/U_0-system (2.5.1),
(ii) θ_λ defined by (2.5.2) is a flat $\mathcal{G}$-valued connection 1-form on $\mathbb{R}^n$ for all $\lambda \in C$,
(iii) θ_r is flat for some $r \in \mathbb{R}$.

We claim that the U/U_0-system is independent of the choice of basis of $\mathcal{A}$. If $b_1, \cdots, b_n$ is a basis of $\mathcal{A}$, then there exists a constant matrix (c_{ij}) such that $a_i = \sum_{j=1}^n c_{ij} b_j$. The U/U_0-system defined by the new base $b_1, \cdots, b_n$ is

$$[b_i, v_{y_j}] - [b_j, v_{y_i}] = [[b_i, v], [b_j, v]].$$

This is the same system as (2.5.1) if we make the coordinate transformation $y_i = \sum_{j=1}^n c_{ji} x_j$.

The U/U_0-system is given by the first commuting n-flows in the U/U_0-hierarchy, i.e.,

Proposition 2.5.1. *([68]). With the same notation as above, let $a_1, \cdots, a_n$ be a basis of $\mathcal{A}$ such that $a_1, \cdots, a_n$ are regular. Let $a = a_1$. Then $v : \mathbb{R}^n \to \mathcal{U}_{\mathcal{A}}^{\perp} \cap \mathcal{U}_1$ is a solution of the U/U_0-system* (2.5.1) *if and only if $u(x) = [a, v(x)]$ satisfies the $(a_j, 1)$-flow in the U/U_0-hierarchy,*

$$u_{x_j} = \mathrm{ad}(a_j)\mathrm{ad}(a)^{-1}(u_{x_1}) + [u, \mathrm{ad}(a_j)\mathrm{ad}(a)^{-1}(u)],$$

for all $1 \leq j \leq n$.

As a consequence of Theorem 2.1.2 and Proposition 2.5.1 we have

Corollary 2.5.2. *([68]). Suppose $a = a_1 \in \mathcal{A}$ is regular in $\mathcal{U}_1$. Then there exists an open dense subset $\mathcal{S}_0$ of $\mathcal{S}(\mathbb{R}, \mathcal{U}_a^{\perp} \cap \mathcal{U}_1)$ such that given any $v_0 \in \mathcal{S}_0$ there exists a unique solution v of* (2.5.1) *defined for all $x \in \mathbb{R}^n$ such that $v(x_1, 0, \cdots, 0) = v_0(x_1)$ and $v(\cdot, x_2, \cdots, x_n) \in \mathcal{S}(\mathbb{R}, \mathcal{U}_a^{\perp} \cap \mathcal{U}_1)$.*

Next we give some examples.

Example 2.5.3. The U-system.

Let τ be a conjugate linear involution of $\mathcal{G}$, and $\mathcal{U}$ the fixed point set of τ. Let $\tau_2(x,y) = (\tau(x), \tau(y))$ and $\sigma(x,y) = (y,x)$ be involutions of $\mathcal{G} \times \mathcal{G}$. Then $\tau_2\sigma = \sigma\tau_2$, and the corresponding symmetric space is $(U \times U)/\triangle(U) \simeq U$, where $\triangle(U)$ is the diagonal group $\{(g,g) \mid g \in U\}$. The $(U \times U)/\triangle(U)$-system is the U-system (2.5.1) for maps $v : \mathbb{R}^n \to \mathcal{A}^\perp \cap \mathcal{U}$, where $\mathcal{A}$ is a maximal abelian subalgebra of $\mathcal{U}$ and $\{a_1, \cdots, a_n\}$ is a basis of $\mathcal{A}$.

Example 2.5.4. The $\frac{O(2n)}{O(n)\times O(n)}$-system.

Here U/U_0 is the symmetric space $\frac{O(2n)}{O(n)\times O(n)}$, $G = O(2n, \mathbb{C})$, $\tau(g) = \bar{g}$, $\sigma(g) = \mathrm{I}_{n,n}\ g\ \mathrm{I}_{n,n}^{-1}$, where $I_{n,n}$ is the diagonal matrix with $a_{ii} = 1$ for $1 \le i \le n$ and $a_{ii} = -1$ for $n+1 \le i \le 2n$. So $\mathcal{U} = so(2n)$, $\mathcal{U}_0 = so(n) + so(n)$, and

$$\mathcal{U}_1 = \left\{ \begin{pmatrix} 0 & \xi \\ -\xi^t & 0 \end{pmatrix} \middle| \ \xi \in gl(n, \mathbb{R}) \right\}.$$

The linear subspace $\mathcal{A}$ spanned by

$$\{a_i = -e_{i,n+i} + e_{n+i,i} \mid 1 \le i \le n\}$$

is a maximal abelian subspace of $\mathcal{U}_1$, and

$$\mathcal{U}_1 \cap \mathcal{A}^\perp = \left\{ \begin{pmatrix} 0 & F \\ -F^t & 0 \end{pmatrix} \middle| F = (f_{ij}) \in gl(n, \mathbb{R}), f_{ii} = 0 \text{ for } 1 \le i \le n. \right\}.$$

The corresponding U/U_0-system (2.5.1) written in terms of $F = (f_{ij})$ is

$$\begin{cases} (f_{ij})_{x_i} + (f_{ji})_{x_j} + \sum_k f_{ki} f_{kj} = 0, & \text{if } i \ne j, \\ (f_{ij})_{x_j} + (f_{ji})_{x_i} + \sum_k f_{ik} f_{jk} = 0, & \text{if } i \ne j, \\ (f_{ij})_{x_k} = f_{ik} f_{kj}, & \text{if } i, j, k \text{ are distinct.} \end{cases} \tag{2.5.3}$$

The Lax n-tuple θ_λ (2.5.2), written in matrix form is

$$\theta_\lambda = \begin{pmatrix} \delta F^t - F\delta & -\lambda\delta \\ \lambda\delta & -F^t\delta + \delta F \end{pmatrix}, \quad \text{where } \delta = \mathrm{diag}(dx_1, \cdots, dx_n). \tag{2.5.4}$$

Note that the first and the third equations of (2.5.3) imply that $\delta F^t - F\delta$ is flat, and the second and third equations of (2.5.3) imply that $-F^t\delta + \delta F$ is flat.

Example 2.5.5. The $\frac{U(n)}{O(n)}$-system.

Here $\mathcal{G} = gl(n, \mathbb{C})$, $\tau(\xi) = -\bar{\xi}^t$, and $\sigma(\xi) = -\xi^t$. Then $\mathcal{U} = u(n)$, $\mathcal{U}_0 = o(n)$, and

$$\mathcal{U}_1 = \{iF \mid F = (f_{ij}) \in gl(n, \mathbb{R}), f_{ij} = f_{ji}\}.$$

The linear subspace $\mathcal{A}$ spanned by

$$\{a_j = ie_{jj} \mid 1 \le j \le n\}$$

is a maximal abelian subspace of $\mathcal{U}_1$, and

$$\mathcal{U}_1 \cap \mathcal{A}^\perp = \{iF \mid F = (f_{ij}) \in gl(n, \mathbb{R}),\ f_{ij} = f_{ji},\ f_{ii} = 0 \text{ for } 1 \le i, j \le n\}.$$

The corresponding U/U_0-system written in terms of F is the restriction of system (2.5.3) to the linear subspace of $F = (f_{ij})$ such that $f_{ij} = f_{ji}$. So the $\frac{U(n)}{O(n)}$-system is the system for symmetric $F = (f_{ij})$:

$$\text{(2.5.5)} \qquad \begin{cases} (f_{ij})_{x_i} + (f_{ij})_{x_j} + \sum_k f_{ik} f_{jk} = 0, & \text{if } i \ne j, \\ (f_{ij})_{x_k} = f_{ik} f_{kj}, & \text{if } i, j, k \text{ are distinct.} \end{cases}$$

Or, equivalently, $[\delta, F] = \delta F - F\delta$ is flat.

Example 2.5.6. The $\frac{SU(n)}{SO(n)}$-system.

Here $\mathcal{G} = sl(n, \mathbb{C})$, $\tau(\xi) = -\bar{\xi}^t$, and $\sigma(\xi) = -\xi^t$ for $\xi \in \mathcal{G}$. Then $\mathcal{U} = su(n)$, $\mathcal{U}_0 = so(n)$, and

$$\mathcal{U}_1 = \{iF \mid F = (f_{ij}) \in gl(n, \mathbb{R}), f_{ij} = f_{ji}, \sum_{i=1}^n f_{ii} = 0\}.$$

The linear subspace $\mathcal{A}$ spanned by

$$\{b_j = i(e_{jj} - e_{11}) \mid 2 \le j \le n\}$$

is a maximal abelian linear subspace of $\mathcal{U}_0$, and

$$\mathcal{A}^\perp \cap \mathcal{U}_1 = \{iF \mid F = (f_{ij}) \in gl(n, \mathbb{R}), f_{ij} = f_{ji}, f_{ii} = 0 \text{ for } 1 \le i, j \le n\}.$$

The $\frac{SU(n)}{SO(n)}$-system is

$$\text{(2.5.6)} \qquad [b_i, F_{t_j}] - [b_j, F_{t_i}] = [b_i, F], [b_j, F]]. \quad 2 \le i \ne j \le n.$$

Example 2.5.7. The $U/U_0 = \frac{GL(2,\mathbb{H})}{(\mathbb{R}^+ \times SU(2))^2}$-system ([8]).

Here $\mathcal{G} = gl(4,\mathbb{C})$. For $X \in gl(4,\mathbb{C})$, write

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}, \quad X_i \in gl(2,\mathbb{C}).$$

Let τ be the involution of $\mathcal{G}$ defined by

$$\tau(X) = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \overline{X} \begin{pmatrix} J^{-1} & 0 \\ 0 & J^{-1} \end{pmatrix}, \quad \text{where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The fixed point set $\mathcal{U}$ of τ is the subalgebra of $X \in \mathcal{G}$ such that $J\bar{X}_i J^{-1} = X_i$ for all $1 \le i \le 4$, i.e., X_i lies in the fixed point set of the involution of $gl(2,\mathbb{C})$ defined by $\tau_0(Y) = J\bar{Y}J^{-1}$. A direct computation implies that the fixed point set of τ_0 is

$$\left\{ \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \,\middle|\, z_1, z_2 \in \mathbb{C} \right\} = \mathbb{R} \times su(2).$$

Note that $\mathbb{R} \times su(2)$ is isomorphic to the quaternions $\mathbb{H}$ as an associative algebra via the isomorphism

$$\mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(It is easy to check that $\mathbf{ij} = \mathbf{k}$, $\mathbf{jk} = \mathbf{i}$, $\mathbf{ki} = \mathbf{j}$.) So we can view $\mathcal{U} = gl(2,\mathbb{H})$, i.e., the algebra of 2×2 matrices with entries in the quaternions $\mathbb{H}$.

Let σ be the involution on $gl(4,\mathbb{C})$ defined by

$$\sigma(Y) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} Y \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}^{-1}.$$

Then $\sigma\tau = \tau\sigma$, and

$$\mathcal{U}_0 = \left\{ \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \,\middle|\, P, Q \in \mathbb{R} \times su(2) \right\},$$

$$\mathcal{U}_1 = \left\{ \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix} \,\middle|\, P, Q \in \mathbb{R} \times su(2) \right\}.$$

Let

$$a_1 = \begin{pmatrix} 0 & \mathbf{k} \\ \mathbf{k} & 0 \end{pmatrix} \quad a_2 = \begin{pmatrix} 0 & \mathbf{j} \\ -\mathbf{j} & 0 \end{pmatrix}. \tag{2.5.7}$$

The space $\mathcal{A}$ spanned by a_1 and a_2 is a maximal abelian linear subspace of $\mathcal{U}_1$, and $\mathcal{A}^\perp \cap \mathcal{U}_1$ is the space of all matrices of the form

$$v = \begin{pmatrix} 0 & \begin{pmatrix} p_1 & p_2 \\ -\bar{p}_2 & \bar{p}_1 \end{pmatrix} \\ \begin{pmatrix} q_1 & \bar{p}_2 \\ -p_2 & \bar{q}_1 \end{pmatrix} & 0 \end{pmatrix}. \tag{2.5.8}$$

The $\frac{GL(2,\mathbb{H})}{(\mathbb{R}\times SU(2))^2}$-system is

$$\begin{cases} (p_2)_y + i(p_2)_x = -|p_1|^2 + |q_1|^2, \\ (\bar{q}_1 + p_1)_x = 2i\ (\bar{p}_2 - p_2)(\bar{q}_1 - p_1), \\ (\bar{q}_1 - p_1)_y = -2(p_2 + \bar{p}_2)(\bar{q}_1 + p_1). \end{cases} \tag{2.5.9}$$

Its Lax pair (2.5.2) is $\theta_\lambda =$

$$\begin{pmatrix} \begin{pmatrix} Z & W \\ -\bar{W} & \bar{Z} \end{pmatrix} & \mathbf{k}\lambda \\ \mathbf{k}\lambda & \begin{pmatrix} -\bar{Z} & \bar{W} \\ -W & -Z \end{pmatrix} \end{pmatrix} dx + \begin{pmatrix} \begin{pmatrix} -iZ & Y \\ -\bar{Y} & i\bar{Z} \end{pmatrix} & \mathbf{j}\lambda \\ -\mathbf{j}\lambda & \begin{pmatrix} -i\bar{Z} & -\bar{Y} \\ Y & iZ \end{pmatrix} \end{pmatrix} dy,$$

where

$$Z = -2ip_2, \quad W = i(\bar{q}_1 - p_1), \quad Y = (\bar{q}_1 + p_1). \tag{2.5.10}$$

Let $p_2 = \beta_1 + i\beta_2, \quad \beta_1, \beta_2 \in \mathbb{R}$. Equate the imaginary part of the first equation in system (2.5.9) to get $(\beta_2)_y + (\beta_1)_x = 0$. So there exists u such that $\beta_1 = -\frac{u_y}{8}$ and $\beta_2 = \frac{u_x}{8}$, and hence $p_2 = \frac{-u_y + iu_x}{8}$. Substitute this into (2.5.9) to get

$$\begin{cases} u_{xx} + u_{yy} = 8(|p_1|^2 - |q_1|^2), \\ (\bar{q}_1 + p_1)_x = \frac{u_x}{2}(\bar{q}_1 - p_1), \\ (\bar{q}_1 - p_1)_y = \frac{u_y}{2}(\bar{q}_1 + p_1). \end{cases} \tag{2.5.11}$$

System (2.5.11) is gauge equivalent to system (2.5.9). To see this, we recall that v is a solution of the U/U_0-system (2.5.1) if and only if θ_0 is flat. So θ_0 is a $(\mathbb{R} \times su(2)) \times (\mathbb{R} \times su(2))$-valued flat connection 1-form. The $\mathbb{R} \times \mathbb{R}$-component of θ_0 is

$$\theta_0^0 = \begin{pmatrix} 2\beta_2\mathbf{1} & 0 \\ 0 & -2\beta_2\mathbf{1} \end{pmatrix} dx + \begin{pmatrix} -2\beta_1\mathbf{1} & 0 \\ 0 & 2\beta_1\mathbf{1} \end{pmatrix} dy = \begin{pmatrix} \frac{du}{4}\,\mathbf{1} & 0 \\ 0 & -\frac{du}{4}\,\mathbf{1} \end{pmatrix}.$$

Let

$$g = \begin{pmatrix} e^{u/4}\mathbf{1} & 0 \\ 0 & e^{-u/4}\mathbf{1} \end{pmatrix}.$$

The gauge transformation of θ_0 by g is

$$g * \theta_0 = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \text{where}$$

$$\tau_1 = \begin{pmatrix} \frac{i(u_y dx - u_x dy)}{4} & i(\bar{q}_1 - p_1)dx + (\bar{q}_1 + p_1)dy \\ i(q_1 - \bar{p}_1)dx - (q_1 + \bar{p}_1)dy & -\frac{i(u_y dx - u_x dy)}{4} \end{pmatrix},$$

$$\tau_2 = \begin{pmatrix} \frac{i(u_y dx - u_x dy)}{4} & -i(q_1 - \bar{p}_1)dx - (q_1 + \bar{p}_1)dy \\ -i(\bar{q}_1 - p_1)dx + (\bar{q}_1 + p_1)dy & -\frac{i(u_y dx - u_x dy)}{4} \end{pmatrix}.$$

The connection $g * \theta_0$ is flat if and only if (u, p_1, q_1) is a solution of (2.5.11). So $g * \theta_\lambda$ is a Lax pair of (2.5.11). In other words, system (2.5.11) is gauge equivalent to the $\frac{GL(2,\mathbb{H})}{(\mathbb{R}^+ \times SU(2))^2}$-system.

Suppose (u, p_1, q_1) is a solution of (2.5.11) and q_1 is real. Let $p_1 = B_1 + iB_2$. Equate the imaginary part of the second and third equations of (2.5.11) to get

$$(B_2)_x = -u_x B_2/2, \quad (B_2)_y = -u_y B_2/2.$$

So $B_2 = ce^{-u/2}$ for some constant c, and (2.5.11) becomes the following system for real functions u, q_1, B_1.

$$\text{(2.5.12)} \qquad \begin{cases} u_{xx} + u_{yy} = 8(c^2 e^{-u} + B_1^2 - q_1^2), \\ (q_1 + B_1)_x = \frac{u_x}{2}(q_1 - B_1), \\ (q_1 - B_1)_y = \frac{u_y}{2}(q_1 + B_1). \end{cases}$$

If p_1 is also real, i.e., $c = 0$ in (2.5.12), then system (2.5.12) becomes the following system for real functions u, q_1, p_1:

$$\text{(2.5.13)} \qquad \begin{cases} u_{xx} + u_{yy} = 8(p_1^2 - q_1^2), \\ (q_1 + p_1)_x = \frac{u_x}{2}(q_1 - p_1), \\ (q_1 - p_1)_y = \frac{u_y}{2}(q_1 + p_1). \end{cases}$$

2.6. The -1-flow

Let $\mathcal{U}$ be the real form defined by the involution τ on $\mathcal{G}$, $a, b \in \mathcal{U}$ such that $[a, b] = 0$. The -1-*flow associated to* U *defined by* a, b is the following system for $u : \mathbb{R}^2 \to \mathcal{U} \cap \mathcal{U}_a^\perp$:

$$u_t = [a, g^{-1}bg], \quad \text{where } g^{-1}g_x = u. \tag{2.6.1}$$

This system has a Lax pair

$$\theta_\lambda = (a\lambda + u)dx + \lambda^{-1}g^{-1}bg\, dt, \quad \text{where } g : \mathbb{R}^2 \to U, \quad g^{-1}g_x = u.$$

Note that θ_λ satisfies the U-reality condition (2.2.1).

Theorem 2.6.1. *([68]). The -1-flow* (2.6.1) *commutes with all the flows in the U-hierarchy.*

Let σ be an order k automorphism of $\mathcal{G}$ such that $\tau\sigma = \sigma^{-1}\tau^{-1}$ and $\mathcal{U} = \mathcal{U}_0 + \cdots + \mathcal{U}_{k-1}$ as in section 2.4 Let $a \in \mathcal{U}_1$ a regular element, and $b \in \mathcal{U}_{k-1}$ such that $[a, b] = 0$, then the right hand side of (2.6.1) is a vector field on $C(\mathbb{R}, \mathcal{U}_0 \cap \mathcal{U}_a^\perp)$. The flow (2.6.1) restricted to the space $C(\mathbb{R}, \mathcal{U}_0 \cap \mathcal{U}_a^\perp)$ of smooth maps from $\mathbb{R}$ to $\mathcal{U}_0 \cap \mathcal{U}_a^\perp$ is *the -1-flow associated to U/U_0*, and θ_λ satisfies the U/U_0-reality condition (2.4.1).

We can also write the -1-flow (2.6.1) associated to U (or U/U_0) as an equation for $g : \mathbb{R}^2 \to U$ ($g : \mathbb{R}^2 \to U_0$ respectively):

$$(g^{-1}g_x)_t = [a, g^{-1}bg], \tag{2.6.2}$$

where $g^{-1}g_x \in \mathcal{U}_a^\perp$ ($\in \mathcal{U}_0 \cap \mathcal{U}_a^\perp$ respectively). Its Lax pair is

$$\theta_\lambda = (a\lambda + g^{-1}g_x)dx + \lambda^{-1}g^{-1}bgdt. \tag{2.6.3}$$

Example 2.6.2. The -1-flow associated to $SU(2)/SO(2)$.

Let $a = \mathrm{diag}(i, -i)$, and $b = -\frac{a}{4}$. Let $g = \begin{pmatrix} \cos\frac{q}{2} & \sin\frac{q}{2} \\ -\sin\frac{q}{2} & \cos\frac{q}{2} \end{pmatrix}$. Then $u = g^{-1}g_x = \frac{1}{2}\begin{pmatrix} 0 & q_x \\ -q_x & 0 \end{pmatrix}$, and the -1-flow (2.6.2) associated to $SU(2)/SO(2)$ is the sine-Gordon equation (SGE equation):

$$q_{xt} = \sin q,$$

and its Lax pair is

$$\theta_\lambda = \begin{pmatrix} i\lambda & \frac{q_x}{2} \\ -\frac{q_x}{2} & -i\lambda \end{pmatrix} dx - \frac{i\lambda}{4}\begin{pmatrix} \cos q & \sin q \\ \sin q & \cos q \end{pmatrix} dt.$$

Example 2.6.3. The -1-flow associated to $U/U_0 = SL(3, \mathbb{C})/\mathbb{R}^+$.

Here $\mathbb{R}^+$ is the subgroup

$$\mathbb{R}^+ = \{\operatorname{diag}(r, r^{-1}, 1) \mid r > 0\}$$

of $SL(3,\mathbb{R})$, $G = SL(3,\mathbb{C})$, $\tau(g) = \bar{g}$, and σ is the order 6 automorphism of $SL(3,\mathbb{C})$ defined by

$$\sigma(g) = C\ (g^t)^{-1}C^{-1}, \quad \text{where } C = \begin{pmatrix} 0 & \alpha^2 & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and } \alpha = e^{\frac{2\pi i}{3}}.$$

The induced automorphism σ on $sl(3,\mathbb{C})$ is

$$\sigma(A) = -CA^tC^{-1}.$$

Note that the order of σ is 6, σ is complex linear on $sl(3,\mathbb{C})$, and $\sigma\tau = \tau^{-1}\sigma^{-1}$. Let $\beta = e^{\frac{2\pi i}{6}} = e^{\frac{\pi i}{3}}$. A direct computation implies that Y_j lies in the eigenspaces $\mathcal{G}_j$ of β^j if

$$Y_0 = \begin{pmatrix} s & 0 & 0 \\ 0 & -s & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 & 0 & s_1 \\ s_2 & 0 & 0 \\ 0 & s_1 & 0 \end{pmatrix},$$

$$Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s \\ -s & 0 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & -2s \end{pmatrix},$$

$$Y_4 = \begin{pmatrix} 0 & 0 & s \\ 0 & 0 & 0 \\ 0 & -s & 0 \end{pmatrix}, \quad Y_5 = \begin{pmatrix} 0 & s_1 & 0 \\ 0 & 0 & s_2 \\ s_2 & 0 & 0 \end{pmatrix}.$$

The fixed point set of τ is $\mathcal{U} = sl(3,\mathbb{R})$, and $\mathcal{U}_j = sl(3,\mathbb{R}) \cap \mathcal{G}_j$.

Let

$$a = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in \mathcal{U}_1, \quad b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in \mathcal{U}_{-1} = \mathcal{U}_5.$$

Note that $[a,b] = 0$. The fixed point set U_0 of σ on U is the abelian group

$$U_0 = \{\operatorname{diag}(r, r^{-1}, 1) \mid r > 0\}.$$

A smooth map $g : \mathbb{R}^2 \to U_0$ is of the form

$$g = \begin{pmatrix} e^w & 0 & 0 \\ 0 & e^{-w} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some smooth function w. So $g^{-1}g_x = w_x \mathrm{diag}(1,-1,0)$, and

$$g^{-1}bg = \begin{pmatrix} 0 & e^{-2w} & 0 \\ 0 & 0 & e^w \\ e^w & 0 & 0 \end{pmatrix}.$$

Hence the -1-flow (2.6.2) associated to $SL(3,\mathbb{R})/U_0$ is

$$w_{xt}\mathrm{diag}(1,-1,0) = (e^w - e^{-2w})\mathrm{diag}(1,-1,0),$$

i.e., the *Tzitzeica equation*:

$$w_{xt} = e^w - e^{-2w}. \tag{2.6.4}$$

The corresponding Lax pair θ_λ (2.6.3) is

$$\theta_\lambda = \begin{pmatrix} w_x & 0 & \lambda \\ \lambda & -w_x & 0 \\ 0 & \lambda & 0 \end{pmatrix} dx + \lambda^{-1} \begin{pmatrix} 0 & e^{-2w} & 0 \\ 0 & 0 & e^w \\ e^w & 0 & 0 \end{pmatrix} dt. \tag{2.6.5}$$

Note that θ_λ satisfies the $\frac{SL(3,\mathbb{R})}{\mathbb{R}^+}$-reality condition:

$$\overline{\theta_{\bar\lambda}} = \theta_\lambda, \quad -C\theta_\lambda^t C^{-1} = \theta_{\beta\lambda}, \quad \text{where } \beta = e^{\frac{2\pi i}{6}} = e^{\frac{\pi i}{3}}. \tag{2.6.6}$$

Example 2.6.4. The -1-flow associated to $SL(n,\mathbb{R})/(\mathbb{R}^+)^{n-1}$.

Let $\mathcal{G} = sl(n,\mathbb{C})$, $\tau(\xi) = \bar\xi$, and $\sigma(\xi) = C\xi C^{-1}$, where

$$C = \mathrm{diag}(1, \alpha, \cdots, \alpha^{n-1})$$

and $\alpha = e^{\frac{2\pi i}{n}}$. Then the order of σ is n and $\tau\sigma = \sigma^{-1}\tau^{-1}$. The fixed point set $\mathcal{U}$ of τ is $sl(n,\mathbb{R})$. The eigenspace $\mathcal{G}_j$ of σ with eigenvalue $e^{\frac{2\pi j i}{n}}$ is spanned by $\{e_{i+j,i} \mid i = 1, \cdots, n\}$, where e_{ij} is the elementary matrix and $e_{ij} = e_{i'j'}$ if $i \equiv i'$ and $j \equiv j'$ mod n. Let $\mathcal{U}_j = \mathcal{G}_j \cap \mathcal{U}$, $a = e_{21} + e_{32} + \cdots + e_{n,n-1} + e_{1n}$, and $b = e_{12} + e_{23} + \cdots + e_{n-1,n} + e_{n1}$. Then $a \in \mathcal{U}_1$, $b \in \mathcal{U}_{-1}$, and $[a,b] = 0$. Let $g = \mathrm{diag}(e^{u_1}, \cdots, e^{u_n})$ with $\sum_i u_i = 0$. So $g^{-1}g_x = \mathrm{diag}((u_1)_x, \cdots, (u_n)_x)$. The -1-flow (2.6.2) associated to G, τ, σ is:

$$(u_i)_{xt} = e^{u_i - u_{i-1}} - e^{u_{i+1} - u_i}, \quad 1 \le i \le n,$$

where $u_{n+1} = u_1$ and $u_0 = u_n$.

2.7. The hyperbolic systems

Let $\mathcal{U}$ be the real form defined by the involution τ on $\mathcal{G}$. *The hyperbolic U-system* is the following system for $(u_0, u_1, v_0, v_1) : \mathbb{R}^2 \to \prod_{i=1}^4 \mathcal{U}$:

$$(2.7.1) \qquad \begin{cases} (u_1)_t = [u_1, v_0], \\ (u_0)_t = (v_0)_x + [u_1, v_1] + [u_0, v_0], \\ (v_1)_x = -[u_0, v_1]. \end{cases}$$

It has a Lax pair

$$(2.7.2) \qquad \Omega_\lambda = (u_1\lambda + u_0)dx + (\lambda^{-1}v_1 + v_0)dt,$$

which satisfies the U-reality condition (2.2.1).

Let σ be an order k automorphism of $\mathcal{G}$ such that $\tau\sigma = \sigma^{-1}\tau$, and $\mathcal{U} = \mathcal{U}_0 + \cdots + \mathcal{U}_{k-1}$, where $\mathcal{U}_j$ is the intersection of $\mathcal{U}$ and the $e^{\frac{2\pi ij}{k}}$-eigenspace of σ. *The hyperbolic U/U_0-system* is the restriction of the hyperbolic U-system (2.7.1) to $(u_0, u_1, v_0, v_1) \in \mathcal{U}_0 \times \mathcal{U}_1 \times \mathcal{U}_0 \times \mathcal{U}_{-1}$. The corresponding Lax pair (2.7.2) satisfies the U/U_0-reality condition (2.4.1).

§3. Geometries associated to soliton equations

We give geometric interpretations of certain soliton equations. For example, solutions of the $O(2n)/O(n) \times O(n)$-system give rise to orthogonal coordinates of $\mathbb{R}^n$ and flat submanifolds in $\mathbb{R}^{2n}$, solutions of the $U(n)/O(n)$-system give Egoroff flat metrics and flat Lagrangian submanifolds of $\mathbb{R}^{2n} = \mathbb{C}^n$, a subclass of solutions of the $GL(2,\mathbb{H})/(SU(2) \times \mathbb{R}^+)^2$-system give rise to Bonnet pairs in $\mathbb{R}^3$, and the -1-flow associated to $SL(3,\mathbb{R})/\mathbb{R}^+$ (the Tzitzeica equation) is the Gauss-Codazzi equation for affine spheres in the affine 3-space. If U/U_0 is a rank n symmetric space, then we can associate to each solution of the U/U_0-system a flat n-submanifold in U/U_0 and a flat n-submanifold in the tangent space of U/U_0. We also give a brief review of the relation between harmonic maps from $\mathbb{R}^{1,1}$ to U and solutions of the hyperbolic U-system.

3.1. The method of moving frames

Let (N, g) be an $(n + k)$-dimensional Riemannian manifold, ∇ the Levi-Civita connection of g, and $X : M^n \to N$ an immersion. We set up some notation next. The first fundamental form I is the induced metric.

Let ξ be a normal vector field on M, v a tangent vector field, $(\nabla_v \xi)^t$ and $(\nabla_v \xi)^\nu$ the tangential and normal components of $\nabla_v \xi$ respectively. The induced normal connection on the normal bundle $\nu(M)$ is defined by

$$\nabla_v^\nu \xi = (\nabla_v \xi)^\nu.$$

The second fundamental form II is a smooth section of $S^2(T^*M) \otimes \nu(M)$ defined by

$$\mathrm{II}_\xi(v_1, v_2) = -g(\nabla_{v_1} \xi, v_2).$$

Next we express $\mathrm{I}, \mathrm{II}, \nabla^\nu$ using a moving frame. Let $(e_1, \cdots, e_{n+k})$ be a local orthonormal frame on M such that $e_1, \cdots, e_n$ are tangent to M. We use the following index convention:

$$1 \leq A, B, C \leq n+k, \quad 1 \leq i, j, k \leq n, \quad n+1 \leq \alpha, \beta, \gamma \leq n+k.$$

Let w_A denote the dual coframe of e_A, and write

$$\nabla e_A = \sum_B w_{BA} e_B, \quad w_{AB} + w_{BA} = 0.$$

Then we have

$$dX = \sum_i w_i e_i,$$

$$dw_A = -\sum_B w_{AB} \wedge w_B, \quad w_{AB} + w_{BA} = 0,$$

$$dw_{AB} = -\sum_C w_{AC} \wedge w_{CB} + \sum_{CD} \tilde{R}_{ABCD}\, w_C \wedge w_D,$$

where $\tilde{R}_{ABCD}$ are the coefficients of the Riemann tensor of g. The first fundamental form of M is $\mathrm{I} = \sum_i w_i^2$. Let $w_{i\alpha} = \sum_j h_{ij}^\alpha w_j$. Since $w_\alpha = 0$, $dw_\alpha = -\sum_i w_{\alpha i} \wedge w_i = 0$. This implies that $h_{ij}^\alpha = h_{ji}^\alpha$. The second fundamental form and the normal connection are

$$\mathrm{II} = \sum_{i,\alpha} w_i w_{i\alpha} e_\alpha = \sum_{i,j,\alpha} h_{ij}^\alpha w_i w_j e_\alpha,$$

$$\nabla^\nu(e_\alpha) = \sum_i w_{\beta\alpha} e_\beta$$

respectively. The normal curvature is the curvature of ∇^ν, i.e.,

$$\Omega_{\alpha\beta}^\nu = dw_{\alpha\beta} + \sum_\gamma w_{\alpha\gamma} \wedge w_{\gamma\beta}.$$

The normal bundle is *flat* if the normal connection is flat, i.e., $\Omega^{\nu}_{\alpha\beta} = 0$ for all α, β. The Levi-Civita connection of I is (w_{ij}), and the curvature is

$$\sum_{kl} R_{ijkl} w_k \wedge w_l = dw_{ij} + \sum_k w_{ik} \wedge w_{kj} = -\sum_\alpha w_{i\alpha} \wedge w_{\alpha j} + \tilde{R}_{ijkl}.$$

Note that given $\mathrm{I}, \mathrm{II}, \nabla^{\perp}$ is the same as given $w_i, w_{i\alpha}, w_{\alpha\beta}$. Moreover, the Levi-Civita connection of I can be obtained by solving

$$dw_i = -\sum_j w_{ij} \wedge w_j, \quad w_{ij} + w_{ji} = 0.$$

The Gauss-Codazzi equation is
(3.1.1)

$$\begin{cases} dw_{ij} + \sum_k w_{ik} \wedge w_{kj} = -\sum_\alpha w_{i\alpha} \wedge w_{\alpha j} + \sum_{kl} \tilde{R}_{ijkl} w_k \wedge w_l, \\ dw_{i\alpha} = -\sum_A w_{iA} \wedge w_{A\alpha} + \sum_{kl} \tilde{R}_{i\alpha kl} w_k \wedge w_l, \\ dw_{\alpha\beta} + \sum_\gamma w_{\alpha\gamma} \wedge w_{\gamma\beta} = -\sum_i w_{\alpha i} \wedge w_{i\beta} + \sum_{ij} \tilde{R}_{\alpha\beta ij} w_i \wedge w_j. \end{cases}$$

The Fundamental Theorem for Submanifolds states that I, II and ∇^ν together with the Gauss-Codazzi equation (3.1.1) determine the submanifold M up to isometries of N.

The mean curvature vector field is defined as the trace of II with respect to I, i.e.,

$$H = \mathrm{tr}_{\mathrm{I}}\mathrm{II} = \sum_{i\alpha} h^\alpha_{ii} e_\alpha.$$

The normal bundle $\nu(M)$ is said to be *non-degenerate* if the dimension of the space of all shape operators of M, $\{A_v \mid v \in \nu(M)_p\}$ is equal to $\mathrm{codim}(M)$ for all $p \in M$.

If $X : M \to N$ is a submanifold of a space form N^{n+k}, then the frame $F = (X, e_1, \cdots, e_{n+k})$ given above is a lift of X to $\mathrm{Iso}(N)$ and the Gauss-Codazzi equation for M is exactly the flatness of $F^{-1}dF$. When M satisfies certain geometric conditions, we often can find special coordinates and frames F on M so that $F^{-1}dF$ takes a special simple form. If moreover, such submanifolds admit a natural holomorphic deformation, then the Gauss-Codazzi equation for M is likely to be an integrable system. On the other hand, if the Lax n-tuple of the U/U_0-system can be interpreted as the connection 1-form of a submanifold, then we can read its geometry from the Lax n-tuple. This gives a natural method to find interesting submanifolds whose equations are integrable. We have had some success when U is an orthogonal group or a unitary group, but very little is known for other simple Lie groups U.

3.2. Orthogonal coordinates and the $\frac{O(2n)}{O(n)\times O(n)}$-system

A local coordinate system $(x_1, \cdots, x_n)$ of $\mathbb{R}^n$ is called an *orthogonal coordinate system* if the flat metric written in this coordinate system is diagonal, i.e., of the form $\sum_{i=1}^n g_{ii}(x)dx_i^2$. The theory of orthogonal coordinate systems of $\mathbb{R}^n$ was studied extensively by classical differential geometers (cf. Darboux [24]).

An elementary computation gives:

Proposition 3.2.1. *The Levi-Civita connection 1-form (w_{ij}) of the metric $ds^2 = \sum_{i=1}^n b_i^2 dx_i^2$ is*

$$w_{ij} = \frac{(b_i)_{x_j}}{b_j} dx_i - \frac{(b_j)_{x_i}}{b_i} dx_j.$$

So the Levi-Civita connection 1-form w of $ds^2 = \sum_{i=1}^n b_i^2 dx_i^2$ written in matrix form is

$$w = (w_{ij}) = \delta F - F^t \delta, \quad \text{where} \quad f_{ij} = \begin{cases} \frac{(b_i)_{x_j}}{b_j}, & \text{if } i \neq j \\ f_{ii} = 0, & \text{if } 1 \leq i \leq n. \end{cases}$$

Let $gl_*(n)$ denote the space of $\xi = (\xi_{ij}) \in gl(n, \mathbb{R})$ such that $\xi_{ii} = 0$ for $1 \leq i \leq n$. Recall that $F = (f_{ij}) : \mathbb{R}^n \to gl_*(n)$ is a solution of the $\frac{O(2n)}{O(n)\times O(n)}$-system (2.5.3) if and only if both $\delta F - F^t \delta$ and $\delta F^t - F\delta$ are flat connection 1-forms. Note that both connections have the same form as the Levi-Civita connections of orthogonal metrics. In the rest of the section, we try to answer the following question: Are there orthogonal coordinate systems of $\mathbb{R}^n$ whose Levi-Civita connections are $\delta F - F^t \delta$ and $\delta F^t - F\delta$?

Given $F : \mathbb{R}^n \to gl_*(n)$, there is a diagonal metric whose Levi-Civita connection 1-form is

$$w = \delta F - F^t \delta$$

if and only if there exist positive functions $b_1, \cdots, b_n$ so that

$$(b_i)_{x_j} = f_{ij} b_j, \quad i \neq j. \tag{3.2.1}$$

However, if system (3.2.1) is solvable for $b_1, \cdots, b_n$, then the mixed derivatives must be equal. This implies that

$$(f_{ij})_{x_k} = f_{ik} f_{kj}, \quad i, j, k \text{ distinct}. \tag{3.2.2}$$

It is a classical result that this condition is also sufficient for (3.2.1) to be solvable:

Theorem 3.2.2. *Given a smooth function $F = (f_{ij}) : \mathbb{R}^n \to gl_*(n)$, system* (3.2.1) *is solvable for* $(b_1, \cdots, b_n)$ *if and only if F satisfies* (3.2.2). *Moreover, given n smooth one variable functions $b_1^0, \cdots, b_n^0$, there exists a unique local solution $(b_1, \cdots, b_n)$ of* (3.2.1) *such that $b_i(0, \cdots, x_i, 0, \cdots, 0) = b_i^0(x_i)$.*

Corollary 3.2.3. *The space of local n-dimensional orthogonal metrics that have the same Levi-Civita connection 1-form is parametrized by n smooth positive functions of one variable.*

If F is a solution of the $\frac{O(2n)}{O(n)\times O(n)}$-system (2.5.3), then F is a solution of (3.2.2). So by Theorem 3.2.2 we can construct orthogonal coordinates of $\mathbb{R}^n$, whose Levi-Civita connections are $\delta F - F^t\delta$ and $\delta F^t - F\delta$. Therefore we have

Proposition 3.2.4. *Let $F = (f_{ij})$ be a solution of the $\frac{O(2n)}{O(n)\times O(n)}$-system* (2.5.3), *$\tau_1 = \delta F^t - F\delta$, $\tau_2 = \delta F - F^t\delta$, and $a_1^0, \cdots, a_n^0, b_1^0, \cdots, b_n^0$ smooth positive functions of one variable. Then there exist unique flat local orthogonal metrics $g_1 = \sum_{i=1}^n a_i^2(x)dx_i^2$ and $g_2 = \sum_{i=1}^n b_i(x)^2 dx_i^2$ such that*

(i) $a_i(0, \cdots, x_i, 0, \cdots) = a_i^0(x_i)$ and $b_i(0, \cdots, x_i, 0, \cdots) = b_i^0(x_i)$,

(ii) the Levi-Civita connection 1-form for g_1 and g_2 are τ_1 and τ_2 respectively,

(iii) there exist $O(n)$-valued maps $A = (\xi_1, \cdots, \xi_n)$ and $B = (\eta_1, \cdots, \eta_n)$ such that $A^{-1}dA = \tau_1$ and $B^{-1}dB = \tau_2$,

(iv) there exist ϕ and ψ defined on a neighborhood of the origin in $\mathbb{R}^n$ such that

$$d\phi = \sum_i a_i \eta_i dx_i, \quad d\psi = \sum_i b_i \xi_i dx_i,$$

(v) ϕ and ψ are local orthogonal coordinates on $\mathbb{R}^n$ with Levi-Civita connection τ_1 and τ_2 respectively.

The next theorem states that a subclass of orthogonal coordinate systems of $\mathbb{R}^n$ can be obtained using trivializations of τ_1 and τ_2.

Theorem 3.2.5. *Let $F = (f_{ij})$ be a solution of* (2.5.3), *and $A = (a_{ij})$, $B = (b_{ij})$ smooth $O(n)$-valued maps defined on an simply connected domain $\mathcal{O}$ of $\mathbb{R}^n$ satisfying*

$$A^{-1}dA = \delta F^t - F\delta, \quad B^{-1}dB = \delta F - F^t\delta. \tag{3.2.3}$$

If a_{mj}, b_{mj} never vanish on $\mathcal{O}$ for all $1 \leq j, m \leq n$, then:

(i) $ds_m^2 = a_{m1}^2 dx_1^2 + \cdots + a_{mn}^2 dx_n^2$ is a flat metric with $\delta F - F^t \delta$ as its Levi-Civita connection,

(ii) $d\tilde{s}_m^2 = b_{m1}^2 dx_1^2 + \cdots + b_{mn}^2 dx_n^2$ is a flat metric with $\delta F^t - F\delta$ as its Levi-Civita connection,

(iii) there exists a smooth map $X : \mathcal{O} \to gl(n, \mathbb{R})$ such that $dX = B\delta A^t$,

(iv) the m-th column X_m and the m-th row Y_m of X are local orthogonal coordinates for $\mathbb{R}^n$ such that the standard metric on $\mathbb{R}^n$ written in these coordinates are ds_m^2 and $d\tilde{s}_m^2$ respectively.

Proof. Let ξ_i denote the i-th column of A. W claim that

$$(\xi_j)_{x_k} = f_{jk}\xi_k, \quad j \neq k. \tag{3.2.4}$$

Note that (3.2.3) gives

$$\xi_i \cdot d\xi_j = f_{ji} dx_i - f_{ij} dx_j, \quad i \neq j. \tag{3.2.5}$$

This implies

$$(\xi_j)_{x_k} \cdot \xi_i = 0, \quad \text{if } i, j, k \text{ distinct.}$$

Since $\xi_j \cdot \xi_j = 1$, $(\xi_j)_{x_k} \cdot \xi_j = 0$. By (3.2.5), $\xi_k \cdot (\xi_j)_{x_k} = f_{jk}$. This proves (3.2.4). Equate each coordinate of (3.2.4) to get

$$(a_{mj})_{x_k} = f_{jk} a_{mk}, \quad 1 \leq m \leq n, \ j \neq k.$$

By Proposition 3.2.1, the Levi-Civita connection of ds_m^2 is

$$\frac{(a_{mj})_{x_k}}{a_{mk}} dx_j - \frac{(a_{mk})_{x_j}}{a_{mj}} dx_k = f_{jk} dx_j - f_{kj} dx_k,$$

i.e., the Levi-Civita connection 1-form for ds_m^2 is $\delta F - F^t \delta$. This proves (i). A similar argument gives (ii).

Since F is a solution of (2.5.3),

$$\theta_\lambda = \begin{pmatrix} \delta F^t - F\delta & -\delta\lambda \\ \delta\lambda & \delta F - F^t \delta \end{pmatrix}$$

is flat. Let $h = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Then $h^{-1} dh = \theta_0 = \begin{pmatrix} \delta F^t - F\delta & 0 \\ 0 & \delta F - F^t \delta \end{pmatrix}$. The gauge transformation of θ_λ by h is

$$\Theta_\lambda = h\theta_\lambda h^{-1} - dh h^{-1} = \begin{pmatrix} 0 & -\lambda A \delta B^t \\ \lambda B \delta A^t & 0 \end{pmatrix}.$$

Since θ_λ is flat for all $\lambda \in \mathbb{C}$, $\Theta_\lambda = h * \theta_\lambda$ is flat for all λ, i.e., $d\Theta_\lambda = -\Theta_\lambda \wedge \Theta_\lambda$. This gives

$$\lambda \begin{pmatrix} 0 & -d\zeta^t \\ d\zeta & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} -\zeta^t \wedge \zeta & 0 \\ 0 & -\zeta \wedge \zeta^t \end{pmatrix} = 0,$$

where $\zeta = B\delta A^t$. Compare coefficients of λ to get $d\zeta = 0$. Since $\mathcal{O}$ is simply connected, there exists X such that

$$dX = B\delta A^t. \tag{3.2.6}$$

This proves (iii). Equate the m-th column and row of (3.2.6) to get

$$dX_m = B(a_{m1}dx_1, \cdots, a_{mn}dx_n)^t. \tag{3.2.7}$$

Recall that $A = (\xi_1, \cdots, \xi_n)$. Write equation (3.2.7) using columns of A and B to get $dX_m = \sum_{i=1}^n a_{mi}dx_i\eta_i$. Let η_i denote the i-th row of X, and $Y = X^t$. Then $dY = A\delta B^t$ and $dY_m = \sum_{i=1}^n b_{mi}dx_i\xi_i$. This proves (iv). Q.E.D.

3.3. Flat submanifolds and the $\frac{O(2n)}{O(n)\times O(n)}$-system

The $\frac{O(2n)}{O(n)\times O(n)}$-system can also be viewed as the Gauss-Codazzi equations for flat n-dimensional submanifolds in $\mathbb{R}^{2n}$ with flat and non-degenerate normal bundles. In fact, there is an isomorphism from the space of local n-dimensional flat submanifolds in $\mathbb{R}^{2n}$ with flat and non-degenerate normal bundle modulo rigid motions to the space of $(F, c_1, \cdots, c_n)$, where F is a local solution of the $\frac{O(2n)}{O(n)\times O(n)}$-system (2.5.3) and $c_1, \cdots, c_n$ are positive functions of one variable. We state this more precisely in the following two known theorems (cf. [68]):

Theorem 3.3.1. *Let M^n be an n-dimensional flat submanifold of $\mathbb{R}^{2n}$ with flat and non-degenerate normal bundle. Then there exist local coordinates $x_1, \cdots, x_n$, a parallel normal frame $e_{n+1}, \cdots, e_{2n}$, an $O(n)$-valued map $A = (a_{ij})$, and a map $b = (b_1, \cdots, b_n)$ such that the fundamental forms of M are*

$$\begin{cases} \mathrm{I} = \sum_{i=1}^n b_i^2 dx_i^2, \\ \mathrm{II} = \sum_{i,j=1}^n b_i a_{ji} dx_i e_{n+j}. \end{cases} \tag{3.3.1}$$

Moreover, let $f_{ij} = (b_i)_{x_j}/b_j$ for $1 \leq i \neq j \leq n$, $f_{ii} = 0$ for $1 \leq i \leq n$, and $F = (f_{ij})$. Then F is a solution of the $\frac{O(2n)}{O(n)\times O(n)}$-system (2.5.3).

Theorem 3.3.2. *Let F be a solution of the $\frac{O(2n)}{O(n)\times O(n)}$-system* (2.5.3), *and $b_1^0, \cdots, b_n^0$ be n smooth positive functions of one variable. Then there exist an open subset $\mathcal{O}$ of the origin in $\mathbb{R}^n$, smooth maps $A : \mathcal{O} \to O(n)$ and*

$$\phi = \begin{pmatrix} g & X \\ 0 & 1 \end{pmatrix} : \mathcal{O} \to GL(2n+1, \mathbb{R}) \tag{3.3.2}$$

with $g : \mathcal{O} \to O(2n)$, $X : \mathcal{O} \to \mathbb{R}^{2n}$, and $b_1, \cdots, b_n : \mathcal{O} \to \mathbb{R}$ such that

$$A^{-1}dA = \delta F^t - F\delta,$$

$$\phi^{-1}d\phi = \begin{pmatrix} 0 & -A\delta & 0 \\ \delta A^t & \delta F - F^t\delta & \varpi \\ 0 & 0 & 0 \end{pmatrix},$$

$$b_i(0, \cdots, x_i, 0, \cdots) = b_i^0(x_i), \quad 1 \le i \le n,$$

where $\varpi = (b_1 dx_1, \cdots, b_n dx_n)^t$. Moreover,

(i) X is an immersion of a flat n-dimensional submanifolds of $\mathbb{R}^{2n}$ with flat and non-degenerate normal bundle,

(ii) $g = (e_{n+1}, \cdots, e_{2n}, e_1, \cdots, e_n)$ is a local orthonormal frame for X such that $e_{n+1}, \cdots, e_{2n}$ are parallel normal field,

(iii) $b_i(0, \cdots, x_i, 0, \cdots, 0) = b_i^0(x_i)$ for $1 \le i \le n$,

(iv) the fundamental forms of the immersion X are given as in (3.3.1),

(v) the Levi-Civita connection for the induced metric is $\delta F - F^t\delta$.

É Cartan proved that a flat n-dimensional submanifold cannot be locally isometrically immersed in S^{n+k} if $k < n-1$, but can be locally isometrically immersed into S^{2n-1}. Moreover, the normal bundle of a flat n-dimensional submanifold of S^{2n-1} is flat, and is non-degenerate viewed as a submanifold of $\mathbb{R}^{2n}$. By Theorem 3.3.1, flat n-dimensional submanifolds in S^{2n-1} give rise to solutions of the $\frac{O(2n)}{O(n)\times O(n)}$-system (2.5.3). This gives the following theorem of Tenenblat ([66]):

Theorem 3.3.3. ([66]). *Let $X : M^n \to S^{2n-1}$ be an immersion of a flat submanifold. Then there exist local coordinates $x_1, \cdots, x_n$, a parallel normal frame $e_{n+2}, \cdots, e_{2n}$, and a smooth $O(n)$-valued map $A = (a_{ij})$ such that*

$$\mathrm{I} = \sum_{i=1}^{n} a_{1i}^2 dx_i^2, \quad \mathrm{II} = \sum_{i=1, j=2}^{n} a_{1i}a_{ji}dx_i e_{n+j}.$$

Set $f_{ij} = (a_{1i})_{x_j}/a_{1j}$ for $i \neq j$, $f_{ii} = 0$, and $F = (f_{ij})$. Let $e_i = X_{x_i}/a_{1i}$ for $1 \leq i \leq n$, and $g = (X, e_{n+2}, \cdots, e_{2n}, e_1, \cdots, e_n)$. Then F is a solution of the $\frac{O(2n)}{O(n)\times O(n)}$-system (2.5.3) *and*

$$g^{-1}dg = \begin{pmatrix} 0 & -A\delta \\ \delta A^t & -F^t\delta + \delta F \end{pmatrix}. \tag{3.3.3}$$

Conversely, let $F = (f_{ij})$ be a solution of (2.5.3), *$A = (a_{ij})$ an $O(n)$-valued map such that $A^{-1}dA = \delta F^t - F\delta$ and $g \in O(2n)$ a solution of* (3.3.3). *If $a_{ij} > 0$ for all $1 \leq j \leq n$ on an open subset $\mathcal{O}$ of $\mathbb{R}^n$, then the i-th column of g is an immersion of a flat submanifold in S^{2n-1} and the corresponding solution of* (2.5.3) *is F.*

Corollary 3.3.4. *Let $M^n \subset S^{2n-1}$ be a flat submanifold, ξ a parallel normal field such that the shape operator A_ξ is non-degenerate. Then ξ is an immersion of a flat n-submanifold in S^{2n-1}. Moreover, the solution of* (2.5.3) *corresponding to ξ is the same as the one corresponding to M.*

3.4. Egoroff metrics and the $\frac{U(n)}{O(n)}$-system

The $\frac{U(n)}{O(n)}$-system is the restriction of the $\frac{O(2n)}{O(n)\times O(n)}$-system to the subspace of symmetric real $n \times n$ matrices F. We have seen that each solution F of the $\frac{O(2n)}{O(n)\times O(n)}$-system gives rises to an $o(n)$-connection of some flat diagonal metric. In this section, we show that such a diagonal metric takes a special form:

Proposition 3.4.1. *Let $ds^2 = \sum_{i=1}^n b_i^2 dx_i^2$ be a metric, $f_{ij} = (b_i)_{x_j}/b_j$ for $1 \leq i \neq j \leq n$, and $F = (f_{ij})$. Then $F = F^t$ if and only if there exists a function ϕ such that $b_i^2 = \phi_{x_i}$ for all $1 \leq i \leq n$.*

Proof. Since $f_{ij} = \frac{(b_i)_{x_j}}{b_j}$, $F^t = F$ if and only if $\frac{(b_i)_{x_j}}{b_j} = \frac{(b_j)_{x_i}}{b_i}$, $i \neq j$. This is equivalent to $(b_i^2)_{x_j} = (b_j^2)_{x_i}$ for all $i \neq j$. Q.E.D.

Definition 3.4.2. An *Egoroff* metric is a flat metric of the form $\sum_{i=1}^n \phi_{x_i} dx_i^2$ for some smooth function ϕ.

It follows from Proposition 3.2.4, Theorem 3.2.5, and Proposition 3.4.1 that:

Theorem 3.4.3. *Let F be a solution of the $\frac{U(n)}{O(n)}$-system* (2.5.5), *and $a_1, \cdots, a_n$ smooth positive functions of one variable. Then there* *exists a smooth function ϕ defined on a simply connected open subset $\mathcal{O}$ of $\mathbb{R}^n$ such that*

$$\phi_{x_i}(0,\cdots,0,x_i,0,\cdots,0)=a_i^2(x_i)$$

for $1\le i\le n$ and the Levi-Civita connection for $\sum_{i=1}^n \phi_{x_i}dx_i^2$ is $[\delta,F]$. Moreover, let $A=(a_{ij})$ be an $O(n)$-valued map such that $A^{-1}dA=[\delta,F]$. Then:

(i) $ds_m^2=\sum_{i=1}^n a_{mi}^2dx_i^2$ is an Egoroff metric with $[\delta,F]$ as its Levi-Civita connection,

(ii) there exists a smooth map X from $\mathcal{O}$ to the space of symmetric matrices such that $dX=A\delta A^t$,

(iii) the m-th column X_m of X is a local orthogonal coordinate system for $\mathbb{R}^n$ and the flat metric of $\mathbb{R}^n$ written in this coordinate system is ds_m^2 as in (i).

3.5. Flat Lagrangian submanifolds and the $\frac{U(n)}{O(n)}$-system

In this section, we explain the relation between solutions of the $\frac{U(n)}{O(n)}$-system and the Gauss-Codazzi equations for flat, Lagrangian submanifolds of $\mathbb{R}^{2n}$. If these submanifolds also lie in S^{2n-1}, then they are invariant under the S^1-action of the Hopf fibration. Hence the projection of these submanifolds are flat Lagrangian submanifolds of $\mathbb{C}P^{n-1}$. For more details of the geometry of flat Lagrangian submanifolds of $\mathbb{C}P^{n-1}$ see [20].

Let $\langle\ ,\ \rangle$ and w be the standard inner product and symplectic form on $\mathbb{C}^n=\mathbb{R}^{2n}$ respectively, i.e.,

$$\langle X,Y\rangle=\mathrm{Re}(\bar{X}^tY),\quad w(X,Y)=\mathrm{Im}(\bar{X}^tY),\quad X,Y\in\mathbb{C}^n.$$

Write $Z\in\mathbb{C}^n$ as $Z=X+iY\in\mathbb{R}^n+i\mathbb{R}^n$, and $A\in gl(n,\mathbb{C})$ as $A=B+iC$ with $B,C\in gl(n,\mathbb{R})$. Then $A\in gl(n,\mathbb{C})$ is identified as $\begin{pmatrix}B & -C\\ C & B\end{pmatrix}$ in $gl(2n,\mathbb{R})$. This identifies $u(n)$ as the following subalgebra of $o(2n)$:

$$u(n)=\left\{\begin{pmatrix}B & -C\\ C & B\end{pmatrix}\in o(2n)\ \middle|\ B\in o(n), C\in gl(n,\mathbb{R})\text{ symmetric}\right\}.$$

The standard complex structure on $\mathbb{R}^{2n}$ is

$$J\begin{pmatrix}X\\ Y\end{pmatrix}=\begin{pmatrix}-Y\\ X\end{pmatrix}.$$

Definition 3.5.1. An n-dimensional submanifold M of $\mathbb{C}^n = \mathbb{R}^{2n}$ is *Lagrangian* if $w(v_1, v_2) = 0$ for all $v_1, v_2 \in TM$, or equivalently, $J(TM) = \nu(M)$.

The proposition below follows from the definition of Lagrangian submanifold:

Proposition 3.5.2. *Let $X : M^n \to \mathbb{R}^{2n}$ be a Lagrangian submanifold, and $(e_1, \cdots, e_n)$ a local orthonormal tangent frame. Then $(Je_1, \cdots, Je_n)$ is an orthonormal normal frame. Moreover, if*

$$g = (Je_1, \cdots, Je_n, e_1, \cdots, e_n)$$

then $g^{-1}dg$ is a $u(n)$-valued 1-form, i.e., it is of the form $\begin{pmatrix} \xi & -\eta \\ \eta & \xi \end{pmatrix}$, where ξ is an $o(n)$-valued 1-form and η is a 1-form with values in the space of symmetric matrices. Conversely, if M^n has a local orthonormal frame $g = (e_{n+1}, \cdots, e_{2n}, e_1, \cdots, e_n)$ such that $e_1, \cdots, e_n$ are tangent to M and $g^{-1}dg$ is $u(n)$-valued 1-form, then M is Lagrangian.

Proposition 3.5.3. *Let $F = (f_{ij})$ be the solution of the $\frac{O(2n)}{O(n)\times O(n)}$-system (2.5.3) corresponding to the flat n-submanifold M of $\mathbb{R}^{2n}$ with flat and non-degenerate normal bundle as in Theorem 3.3.1. Then the following statements are equivalent:*

(i) F is a solution of the $\frac{U(n)}{O(n)}$-system (2.5.5),

(ii) $F = F^t$,

(iii) M is Lagrangian.

Proof. It is obvious that (i) and (ii) are equivalent. Let $x_1, \cdots, x_n$, $b_1, \cdots, b_n$, $e_{n+1}, \cdots, e_{2n}$, and $A = (a_{ij})$ be as in Theorem 3.3.1. Let $e_i = \frac{X_{x_i}}{b_i}$, and $g = (e_{n+1}, \cdots, e_{2n}, e_1, \cdots, e_n)$. Then

$$g^{-1}dg = \begin{pmatrix} 0 & -A\delta \\ \delta A^t & [\delta, F] \end{pmatrix}.$$

To prove (ii) implies (iii), let

$$\phi = (\tilde{e}_{n+1}, \cdots, \tilde{e}_{2n}, e_1, \cdots, e_n) = g \begin{pmatrix} A & 0 \\ 0 & \mathrm{I} \end{pmatrix}.$$

Then $\tilde{e}_{n+1}, \cdots, \tilde{e}_{2n}$ are normal to X, and

$$\phi^{-1}d\phi = \begin{pmatrix} [\delta, F] & -\delta \\ \delta & [\delta, F] \end{pmatrix}. \tag{3.5.1}$$

Proposition 3.5.2 implies that M is Lagrangian.

(iii) implies (ii): Since M is Lagrangian and $(e_{n+1}, \cdots, e_{2n})$ is an orthonormal normal frame, $(Je_{n+1}, \cdots Je_{2n})$ is an orthonormal tangent frame for M. So there exists an $O(n)$-valued map h such that

$$\tilde{g} := (e_{n+1}, \cdots, e_{2n}, Je_{n+1}, \cdots, Je_{2n}) = g \begin{pmatrix} I & 0 \\ 0 & h^{-1} \end{pmatrix}.$$

Then

$$\tilde{g}^{-1} d\tilde{g} = \begin{pmatrix} 0 & -A\delta h^{-1} \\ h\delta A^t & h(\delta F - F^t\delta)h^{-1} - dhh^{-1} \end{pmatrix}.$$

But $\tilde{g}^{-1}d\tilde{g}$ is $u(n)$-valued. Hence

$$(3.5.2) \qquad \begin{cases} h(\delta F - F^t\delta)h^{-1} - dhh^{-1} = 0, \\ A\delta h^t = h\delta A^t \end{cases}$$

The second equation of (3.5.2) gives $a_{ik}h_{jk} = h_{ik}a_{jk}$ for all i, j, k. This implies that the i-th rows (or i-th columns) of A and h are proportional. Since both A and h are in $O(n)$, we have $h = A$. So $h^{-1}dh = A^{-1}dA = \delta F - F^t\delta$. But $A^{-1}dA = \delta F^t - F\delta$. Hence

$$\delta F^t - F\delta = \delta F - F^t\delta.$$

Equate the ij-th entry of the above equation to get $f_{ji}dx_i - f_{ij}dx_j = f_{ij}dx_i - f_{ji}dx_j$. So F is symmetric. Q.E.D.

As a consequence of Proposition 3.5.3, Theorem 3.3.3, and Corollary 3.3.4, we have

Corollary 3.5.4. *Let F be a solution of the $\frac{U(n)}{O(n)}$-system (2.5.5), $A = (a_{ij})$ an $O(n)$-valued map satisfying $A^{-1}dA = [\delta, F]$, and $\tilde{g}$ an $U(n)$-valued map satisfying*

$$\tilde{g}^{-1}d\tilde{g} = \begin{pmatrix} 0 & -A\delta A^t \\ A\delta A^t & 0 \end{pmatrix}.$$

(Here $U(n)$ is embedded as a subgroup of $O(2n)$). Let e_{m+n} denote the m-th column of $\tilde{g}$ for $1 \leq m \leq n$. If $a_{m1}, \cdots, a_{mn}$ never vanishes in an open subset $\mathcal{O}$ of $\mathbb{R}^n$, then $e_{n+m} : \mathcal{O} \to S^{2n-1}$ is an n-dimensional immersed flat submanifold of S^{2n-1} that is Lagrangian in $\mathbb{R}^{2n}$. Conversely, if M^n is a flat submanifold of S^{2n-1} that is Lagrangian in $\mathbb{R}^{2n}$, then F defined in Theorem 3.3.3 is a solution of the $\frac{U(n)}{O(n)}$-system (2.5.5).

Proposition 3.5.5. *Let F be a solution of the $\frac{U(n)}{O(n)}$-system* (2.5.5), *M^n a flat submanifold of S^{2n-1} corresponding to F as in Corollary 3.5.4, and $\pi : S^{2n-1} \to \mathbb{C}P^{n-1}$ the Hopf fibration. Then $M = \pi^{-1}(\pi(M))$ and $\pi(M)$ is a flat Lagrangian submanifold of $\mathbb{C}P^{n-1}$.*

Proof. Let S^1 act on $\mathbb{R}^{2n} = \mathbb{C}^n$ by

$$e^{is} \cdot (z_1, \cdots, z_n) = (e^{is}z_1, e^{is}z_2, \cdots, e^{is}z_n).$$

This action leaves S^{2n-1} invariant, the orbit space S^{2n-1}/S^1 is $\mathbb{C}P^{n-1}$, and the projection $\pi : S^{2n-1} \to \mathbb{C}P^{n-1}$ is the Hopf fibration.

It suffices to show that M is invariant under the S^1-action on S^{2n-1}. Let X be the immersion, $(x_1, \cdots, x_n)$, $g = (X, e_{n+2}, \cdots, e_{2n}, e_1, \cdots, e_n)$, and $A = (a_{ij})$ as in Theorem 3.3.3. First we change coordinates from $x_1, \cdots, x_n$ to $t_1, \cdots, t_n$ such that

$$\begin{cases} x_1 = t_1 - t_2 - \cdots - t_n \\ x_j = t_j + t_1, \ 2 \leq j \leq n. \end{cases}$$

Then $\frac{\partial}{\partial t_1} = \frac{\partial}{\partial x_1} + \cdots \frac{\partial}{\partial x_n}$. Since

$$A^{-1}\frac{\partial A}{\partial t_1} = [\delta, F]\left(\frac{\partial}{\partial t_1}\right) = [\mathrm{I}_n, F] = 0,$$

we have $\frac{\partial A}{\partial t_1} = 0$. Here I_n is the identity $n \times n$ matrix. Let $\tilde{g} = g\begin{pmatrix} \mathrm{I}_n & 0 \\ 0 & A^t \end{pmatrix}$. Since $A^{-1}dA = [\delta, F]$,

$$\tilde{g}^{-1}d\tilde{g} = \begin{pmatrix} 0 & -A\delta A^t \\ A\delta A^t & 0 \end{pmatrix}.$$

So we have

$$\tilde{g}^{-1}\frac{\partial \tilde{g}}{\partial t_1} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \tag{3.5.3}$$

This implies that

$$\tilde{g}(t_1, \cdots, t_n) = e^{it_1}\tilde{g}(1, t_2, \cdots, t_n).$$

But the first column of g and of $\tilde{g}$ are the immersion X. So X is invariant under the S^1-action on S^{2n-1}. Q.E.D.

It follows from elementary submanifold theory that $\tilde{M}$ is a flat Lagrangian submanifold of $\mathbb{C}P^{n-1}$ if and only if $\pi^{-1}(\tilde{M})$ is a flat submanifold of S^{2n-1} that is Lagrangian in $\mathbb{R}^{2n}$. Hence the Gauss-Codazzi equations for flat, Lagrangian submanifolds of $\mathbb{C}P^{n-1}$ is the $\frac{U(n)}{O(n)}$-system (2.5.5), or equivalently the $\frac{SU(n)}{SO(n)}$-system.

3.6. Bonnet pairs in $\mathbb{R}^3$ and the $\frac{GL(2,\mathbb{H})}{(SU(2)\times\mathbb{R}^+)^2}$-system

Let $X : M \to \mathbb{R}^3$ be an immersion. Locally, there exists a conformal coordinate system (x, y), i.e., the induced metric is of the form $\mathrm{I} = e^u(dx^2 + dy^2)$ for some smooth function u. Let H denote the mean curvature function of M. Since $\mathrm{II} - H\mathrm{I}$ is traceless, there is a smooth complex valued function $h = h_1 + ih_2$ such that

$$\mathrm{II} - H\ \mathrm{I} = h_1(dx^2 - dy^2) - 2h_2 dxdy = \mathrm{Re}(hdz^2),$$

where $z = x + iy$. The two fundamental forms of M are

$$(3.6.1)\qquad \begin{cases} \mathrm{I} = e^u(dx^2 + dy^2), \\ \mathrm{II} = H\ \mathrm{I} + \mathrm{Re}(h(dx + idy)^2) \\ \quad = (He^u + h_1)dx^2 - 2h_2 dxdy + (He^u - h_1)dy^2. \end{cases}$$

Let $e_1 = Xe^{-\frac{u}{2}}$, $e_2 = Xe^{-\frac{u}{2}}$, and $e_3 = e_1 \times e_2$, where $\times$ is the cross-product. Let w_1, w_2, w_3 be the dual coframe: $w_1 = e^{\frac{u}{2}}dx$, $w_2 = e^{\frac{u}{2}}dy$, $w_3 = 0$. Let $g = (e_1, e_2, e_3)$, and $(w_{ij}) = g^{-1}dg$, i.e., $de_i = \sum_{j=1}^3 w_{ji}e_j$, $1 \le i \le 3$. Then

$$(3.6.2)\qquad \begin{cases} w_{12} = \frac{1}{2}(u_y dx - u_x dy), \\ w_{13} = (He^{\frac{u}{2}} + h_1 e^{-\frac{u}{2}})dx - h_2 e^{-\frac{u}{2}}dy, \\ w_{23} = -h_2 e^{-\frac{u}{2}}dx + (He^{\frac{u}{2}} - h_1 e^{-\frac{u}{2}})dy, \end{cases}$$

The Gauss-Codazzi equations for M express the flatness of (w_{ij}), i.e.,

$$dw_{ij} = -\sum_{k=1}^3 w_{ik} \wedge w_{kj}, \quad i \neq j.$$

Write this equation in terms of $u, H, h = h_1 + ih_2$ to get
(3.6.3)

$$\begin{cases} u_{xx} + u_{yy} = -2\left(H^2 e^u - (h_1^2 + h_2^2)e^{-u}\right), \\ (He^{\frac{u}{2}} + h_1 e^{-\frac{u}{2}})_y + (h_2 e^{-\frac{u}{2}})_x = \frac{1}{2}\left(u_y(He^{\frac{u}{2}} - h_1 e^{-\frac{u}{2}}) - u_x h_2 e^{-\frac{u}{2}}\right), \\ (He^{\frac{u}{2}} - h_1 e^{-\frac{u}{2}})_x + (h_2 e^{-\frac{u}{2}})_y = \frac{1}{2}\left(u_x(He^{\frac{u}{2}} + h_1 e^{-\frac{u}{2}}) - u_y h_2 e^{-\frac{u}{2}}\right). \end{cases}$$

A surface M in $\mathbb{R}^3$ is called *isothermic* if there exists a conformal line of curvature coordinate system, i.e., there is a coordinate system (x, y) such that both I and II are diagonalized, or equivalently, $h_2 = 0$ in (3.6.1). In this case, the Gauss-Codazzi equations (3.6.3) become

$$\begin{cases} u_{xx} + u_{yy} = -2\left(H^2 e^u - h_1^2 e^{-u}\right), \\ (He^{\frac{u}{2}} + h_1 e^{-\frac{u}{2}})_y = \frac{1}{2} u_y (He^{\frac{u}{2}} - h_1 e^{-\frac{u}{2}}), \\ (He^{\frac{u}{2}} - h_1 e^{-\frac{u}{2}})_x = \frac{1}{2} u_x (He^{\frac{u}{2}} + h_1 e^{-\frac{u}{2}}). \end{cases}$$

This implies that (u, p_1, q_1) is a solution of (2.5.13), where $p_1 = \frac{ih_1}{2}\, e^{-u/2}$ and $q_1 = \frac{H}{2}\, e^{u/2}$.

A pair of surfaces $(M, \tilde{M})$ in $\mathbb{R}^3$ is called a *Bonnet pair* if there is an isometry $f : M \to \tilde{M}$ so that $\tilde{H} = H \circ f$, where $\tilde{H}$ and H are the mean curvature functions of M and $\tilde{M}$ respectively and H is not a constant function. The following is a consequence of the Gauss-Codazzi equation (cf. [8]):

Proposition 3.6.1. *([8]). Let $(M, \tilde{M})$ be a Bonnet pair in $\mathbb{R}^3$. Then away from umbilic points there exist a conformal coordinate system (x, y), and smooth real functions u, h_1 and h_2 such that the two fundamental forms for $M, \tilde{M}$ are as follows:*

$$\text{(3.6.4a)} \qquad \begin{cases} \mathrm{I} = e^u (dx^2 + dy^2), \\ \mathrm{II} = H\ \mathrm{I} + \mathrm{Re}(h(dx + i dy)^2) \\ \quad = (He^u + h_1) dx^2 - 2h_2 dx dy + (He^u - h_1) dy^2, \end{cases}$$

$$\text{(3.6.4b)} \qquad \begin{cases} \tilde{\mathrm{I}} = e^u (dx^2 + dy^2), \\ \tilde{\mathrm{II}} = H\ \tilde{\mathrm{I}} + \mathrm{Re}(\bar{h}(dx + i dy)^2) \\ \quad = (He^u + h_1) dx^2 + 2h_2 dx dy + (He^u - h_1) dy^2. \end{cases}$$

Since both (u, H, h_1, h_2) and $(u, H, h_1, -h_2)$ are solutions of (3.6.3), we get

$$\text{(3.6.5)} \qquad \begin{cases} u_{xx} + u_{yy} = 2(-H^2 e^u + (h_1^2 + h_2^2) e^{-u}), \\ (he^{u/2} + h_1 e^{-u/2})_y = \frac{u_y}{2}(He^{u/2} - h_1 e^{-u/2}, \\ (h_2 e^{-u/2})_x = -\frac{u_x}{2} h_2 e^{-u/2}, \\ (He^{u/2} - h_1 e^{-u/2})_x = \frac{u_x}{2}(He^{u/2} + h_1 e^{-u/2}, \\ (h_2 e^{-u/2})_y = -\frac{u_y}{2}(h_2 e^{-u/2}. \end{cases}$$

Note that the third and the fifth equations of (3.6.5) imply $(h_2)_x = (h_2)_y = 0$. So h_2 is a constant, and we have

Theorem 3.6.2. *([8]). Let $(M, \tilde{M})$ be a Bonnet pair in $\mathbb{R}^3$, and (u, H, h_1, h_2) the corresponding solution of (3.6.5). Then h_2 is a constant. Moreover, set $p_1 = \frac{i}{2}(h_1 - ih_2)e^{-\frac{u}{2}}$, and $q_1 = \frac{1}{2}He^{\frac{u}{2}}$. Then (u, p_1, q_1) is a solution of the $\frac{GL(2,\mathbb{H})}{(\mathbb{R}^+ \times SU(2))^2}$-system (2.5.11). Conversely, if (u, p_1, q_1) is a solution of system (2.5.11) and q_1 is real, then there is a Bonnet pair with fundamental forms given by (3.6.4), where $H = 2q_1e^{-u/2}$ and $h_1 - ih_2 = -2i\, p_1 e^{u/2}$.*

3.7. Curved flats in symmetric spaces

Let U/U_0 be a rank n Riemannian symmetric space, σ the corresponding involution on U, $\mathcal{U} = \mathcal{U}_0 + \mathcal{U}_1$ the eigenspace decomposition of $d\sigma_e$ on $\mathcal{U}$ corresponding to eigenvalues 1 and -1, $\mathcal{A}$ a maximal abelian linear subspace of $\mathcal{U}_1$, and $a_1, \cdots, a_n$ an orthonormal basis of $\mathcal{A}$. In this section, we associate to each solution of the U/U_0-system (2.5.1) a flat submanifold in $\mathcal{U}_1$. We also review the construction of curved flats in U/U_0 given by Ferus and Pedit [35].

Theorem 3.7.1. *Let $v : \mathbb{R}^n \to \mathcal{U}_1 \cap \mathcal{A}^\perp$ be a solution of the U/U_0-system (2.5.1), and $E(x, \lambda)$ the frame of the corresponding Lax n-tuple θ_λ (2.5.2), i.e.,*

$$E^{-1}dE = \theta_\lambda = \sum_{i=1}^{n}(a_i\lambda + [a_i, v])dx_i, \quad E(x, 0) = e,$$

Set $Y = \frac{\partial E}{\partial \lambda}E^{-1}\big|_{\lambda=0}$. Then Y is an immersed flat submanifold in $\mathcal{U}_1$ such that the tangent plane of Y is a maximal abelian subalgebra of $\mathcal{U}_1$ at every point. Conversely, locally all such flat submanifolds in $\mathcal{U}_1^0$ can be constructed this way, where $\mathcal{U}_1^0$ is the subset of regular points in $\mathcal{U}_1$.

Proof. Write $E_\lambda(x) = E(x,\lambda)$. Since $E^{-1}dE = \sum_i (a_i\lambda + [a_i, v])dx_i$, a direct computation gives

$$\begin{aligned} dY &= \left(\frac{\partial}{\partial\lambda}(dE)\right) E^{-1} - \frac{\partial E}{\partial\lambda} E^{-1} dE E^{-1}\Big|_{\lambda=0} \\ &= \frac{\partial}{\partial\lambda}\left(E\left(\sum_i a_i\lambda + [a_i, v]\right)\ dx_i\right) E^{-1}\Big|_{\lambda=0} \\ &\quad - YE_0\left(\sum_i [a_i, v]\ dx_i\right) E_0^{-1} \\ &= \sum_i (YE_0[a_i, v]E_0^{-1} + E_0 a_i E_0^{-1} - YE_0[a_i, v]E_0^{-1})\ dx_i \\ &= \sum_{i=1}^n E_0 a_i E_0^{-1} dx_i. \end{aligned}$$

Because θ_λ satisfies the U/U_0-reality condition, $\tau(E_\lambda) = E_{\bar\lambda}$ and $\sigma(E_\lambda) = E_{-\lambda}$. So $E_0 \in U_0$. Let $e_i = E_0\ a_i\ E_0^{-1}$. Since $a_1, \cdots, a_n$ are orthonormal and $E_0(x) \in U_0$, $\{e_i \mid 1 \le i \le n\}$ is an orthonormal tangent frame of Y. Hence Y is an immersion, and the induced metric is $\sum_{i=1}^n dx_i^2$.

Since $\mathrm{ad}(a_1)^2, \cdots, \mathrm{ad}(a_n)^2$ are commuting symmetric operators, there exist a set Λ of linear functionals of $\mathcal{A}$, an orthonormal common basis $\{p_\alpha \mid \alpha \in \Lambda\}$ for $\mathcal{A}^\perp \cap \mathcal{U}_1$, and an orthonormal basis $\{k_\alpha \mid \alpha \in \Lambda\}$ for $\mathcal{K} \cap \mathcal{K}_a^\perp$ such that $\mathrm{ad}(a)(p_\alpha) = \alpha(a) = k_\alpha$, $\mathrm{ad}(a)(k_\alpha) = -\alpha(a)p_\alpha$, for all $1 \le i \le n$ and $\alpha \in \Lambda$. Then $e_\alpha = E_0 p_\alpha E_0^{-1}$ is an orthonormal normal frame for the immersion Y in $\mathcal{U}_1$. Write the solution v of the U/U_0-system as $v(x) = \sum_\alpha v_\alpha(x)p_\alpha$ with respect to the decomposition $\mathcal{A}^\perp \cap \mathcal{U}_1 = \sum_\alpha \mathbb{R}\, p_\alpha$. Since $dE_0 = E_0 \sum_i [a_i, v]dx_i$, a direct computation gives

$$\begin{aligned} de_i &= E_0[E_0^{-1}dE_0, a_i]E_0^{-1} = \sum_{i,j} E_0[[a_j, v], a_i]E_0^{-1}dx_j, \\ &= -\sum_{j,\alpha} v_\alpha \alpha(a_i)\alpha(a_j)dx_j e_\alpha. \end{aligned}$$

Hence $w_{i\alpha} = v_\alpha \alpha(a_i) \sum_j \alpha(a_j)dx_j$. So the normal curvature $\sum_\alpha w_{i\alpha} \wedge w_{j\alpha}$ is zero.

To prove the converse, let M be a flat submanifold of $\mathcal{U}_1$ such that TM_p is a maximal abelian subalgebra of $\mathcal{U}_1$. Let x be a local flat, orthonormal coordinate of M, and $e_i = \frac{\partial}{\partial x_i}$ the orthonormal frame. Let $\mathcal{A}$ be a maximal abelian subspace of $\mathcal{U}_1$. Then every maximal abelian subspace of $\mathcal{U}_1$ is of the form $k\mathcal{A}k^{-1}$ for some $k \in U_0$. Since $M \subset \mathcal{U}_1^0$, we may assume that there exist $\mathcal{A}$-valued maps ξ_i and a U_0-valued map g such that $e_i = g\xi_i g^{-1}$. It follows from $(de_i, e_j) = 0$ and the fact that $(\ , \ )$ is ad-invariant that $d\xi_i = 0$ for all i. So $\xi_i = a_i$ is constant. In other words, $e_i = ga_ig^{-1}$. Note that if h is an A-valued map, then $e_i = ga_ig^{-1} = gha_ih^{-1}g^{-1}$. Choose an A-valued map h so that $(dh)h^{-1} = -\pi(g^{-1}dg)$, where π is the projection onto $\mathcal{U}_{\mathcal{A}}^{\perp} \cap \mathcal{U}_0$. Let $\tilde{g} = gh$. Then $e_i = \tilde{g}a_i\tilde{g}^{-1}$ and $\tilde{g}^{-1}d\tilde{g} \in \mathcal{U}_{\mathcal{A}}^{\perp} \cap \mathcal{U}_0$. Let X be the immersion of M into $\mathcal{U}_1$. Then $dX = \sum_{i=1}^n e_i dx_i = \sum_{i=1}^n \tilde{g}a_i\tilde{g}^{-1}dx_i$. Hence $(\tilde{g}a_i\tilde{g}^{-1})_{x_j} = (\tilde{g}a_j\tilde{g}^{-1})_{x_i}$ for all i, j. This implies $[\tilde{g}^{-1}\tilde{g}_{x_j}, a_i] = [\tilde{g}^{-1}\tilde{g}_{x_i}, a_j]$. So there exists an $\mathcal{A}^{\perp} \cap \mathcal{U}_1$-valued map v such that $\tilde{g}^{-1}\tilde{g}_{x_i} = [a_i, v]$. But this means $\sum_{i=1}^n [a_i, v]dx_i$ is flat and v is a solution of (2.5.1). Q.E.D.

Given an involution σ of U, there is a natural U-action on U defined by $g * x = gx\sigma(g)^{-1}$. The orbit at e is

$$M = \{g\sigma(g)^{-1} \mid g \in U\}.$$

Since the isotropy subgroup at e is U_0, the orbit M is diffeomorphic to U/U_0. Next we claim that M is totally geodesic. To see this, note that the map $f(g) = (\sigma(g))^{-1}$ is an isometry of U. So the fixed point set F of f is a totally geodesic submanifold of U. Note that $df_e = -d\sigma_e$. So $TF_e = \mathcal{U}_1$, and the dimension of F is equal to $\dim(\mathcal{U}_1)$. But M is fixed by f and $TM_e = \{x - d\sigma_e(x) \mid x \in \mathcal{U}\} = \mathcal{U}_1$. So M is an open subset of F. This proves the claim. This is the classical Cartan embedding of the symmetric space U/U_0 in U as a totally geodesic submanifold.

Note that U_0 acts on U/U_0 $(g\cdot(hU_0) = ghU_0)$. An element $x \in U/U_0$ is *regular* if the U_0-orbit at x is a principal orbit.

Theorem 3.7.2. *([35]). With the same assumption as in Theorem 3.7.1, set $\psi(x) = E(x,1)E(x,-1)^{-1}$. Then ψ is an immersed flat submanifold of the symmetric space U/U_0 which is tangent to a flat of U/U_0 at every point. Conversely, locally all such flat submanifolds in N' can be constructed this way, where N' is the open dense subset of regular points in U/U_0.*

Proof. The reality condition implies that $E(x,1) \in U$ and

$$\psi(x) = E(x,1)E(x,-1)^{-1} = E(x,1)\sigma(E(x,1))^{-1}.$$

So the image of ψ lies in the symmetric space $U/U_0 = \{g\sigma(g)^{-1} \mid g \in U\}$. A direct computation gives

$$\psi^{-1}d\psi = 2\sum_{i=1}^{n} E_{-1}a_iE_{-1}^{-1}dx_i.$$

Thus ψ is a flat immersion into U/U_0 and $2(x_1, \cdots, x_n)$ is an orthonormal coordinate for the induced metric. The rest of the theorem can be proved in a similar manner as for Theorem 3.7.1. Q.E.D.

Ferus and Pedit called the flat submanifolds obtained in Theorem 3.7.2 *curved flats*.

3.8. Indefinite affine spheres in $\mathbb{R}^3$ and the -1-flow

Affine geometry (cf. [55]) studies the geometry of hypersurfaces in $\mathbb{R}^{n+1}$ invariant under the affine transformations $x \mapsto Ax + v$, where $A \in SL(n+1, \mathbb{R})$ and $v \in \mathbb{R}^{n+1}$. There are three local affine invariants, the affine metric, the Fubini cubic form, and the third fundamental form. These invariants satisfy certain integrability conditions, the Gauss-Codazzi equations. We first give a brief description of these invariants for affine surfaces in $\mathbb{R}^3$, then explain the relation between the Tzitzeica equation and indefinite affine spheres. Recall that the Tzitzeica equation is the -1-flow associated to $SL(3, \mathbb{R})/\mathbb{R}^+$ (see Example 2.6.3).

Let $X : M \to \mathbb{R}^3$ be a surface with non-degenerate second fundamental form, $g = (e_1, e_2, e_3)$ a local frame on M such that e_1, e_2 are tangent to M, e_3 is transversal to M, and $\det(e_1, e_2, e_3) = 1$. Let w^i denote the dual coframe of e_i, i.e., $dX = w^1e_1 + w^2e_2$. Let (w_j^i) denote the $sl(3, \mathbb{R})$-valued 1-form $g^{-1}dg$, i.e., $de_i = \sum_{j=1}^3 w_i^j e_j, \quad 1 \le i \le 3$. Then we have the structure equation:

$$\begin{cases} dw^i = -\sum_{j=1}^3 w_j^i \wedge w^j = \sum_{j=1}^3 w^j \wedge w_j^i, \; 1 \le i \le 2 \\ dw_i^j = -\sum_{k=1}^3 w_k^j \wedge w_i^k, \quad 1 \le i, j \le 3. \end{cases} \tag{3.8.1}$$

Since $w^3 = 0$ on M,

$$w_i^3 = \sum_{j=1}^{2} h_{ij}w^j, \quad h_{ij} = h_{ji}. \tag{3.8.2}$$

A direct computation shows that the quadratic form

$$ds^2 = |\det(h_{ij})|^{-\frac{1}{4}} \sum_{ij} h_{ij} w^i w^j \tag{3.8.3}$$

is invariant under change of affine frames, and it is called the *affine metric* of M. An affine surface is called *definite* or *indefinite* if the affine metric is definite or indefinite respectively.

We can choose a vector field e_3 transversal to M so that

$$w_3^3 + \frac{1}{4} d(\log|\det(h_{ij})|) = 0. \tag{3.8.4}$$

Then

$$\nu = |\det(h_{ij})|^{\frac{1}{4}} e_3$$

is an affine invariant. The vector field ν is called the *affine normal* of M.

Take the exterior derivative of (3.8.2) to get

$$\sum_{j=1}^{2} dh_{ij} + h_{ij} w_3^3 + \sum_{k=1}^{2} (h_{ik} w_j^k + h_{kj} w_i^k) \wedge w^j = 0 \tag{3.8.5}$$

for $1 \leq j \leq 2$. Define h_{ijk} by

$$\sum_{k=1}^{2} h_{ijk} w^k = dh_{ij} + h_{ij} w_3^3 + \sum_{k=1}^{2} h_{ik} w_j^k + h_{kj} w_i^k. \tag{3.8.6}$$

Then (3.8.5) implies that $h_{ijk} = h_{ikj}$. But $h_{ij} = h_{ji}$. So h_{ijk} is symmetric in i, j, k. *The Fubini-Pick cubic form,*

$$J = \sum_{i,j,k} h_{ijk} w^i w^j w^k,$$

is an affine invariant.

Exterior differentiation of (3.8.4) gives $\sum_i w_3^i \wedge w_i^3 = 0$. Write $w_3^i = \sum_j \ell^{ij} w_j^3$. The *third fundamental form,*

$$\mathrm{III} = h^{\frac{1}{4}} w_3^i w_i^3,$$

is also an affine invariant. The trace of III with respect to the affine metric ds^2 is the *affine mean curvature*

$$L = \frac{1}{2} |\det(h_{ij})|^{\frac{1}{4}} \sum_{ij} h_{ij} \ell^{ij}.$$

The three affine invariants ds^2, J and III are completely determined by w^i and w^A_B, which satisfy the Gauss-Codazzi equations for affine surfaces. Conversely, suppose ds^2, J and III are given and satisfy the Gauss-Codazzi equations. Then h_{ij}, h_{ijk}, w^i, w^3_i, w^i_3, and w^3_3 can be computed from these three invariants. Moreover, we can find w^j_i by solving the linear system consisting of (3.8.6) and the first equation of (3.8.1). Then the Gauss-Codazzi equations, written in terms of w^i, w^A_B, are (3.8.1), i.e., the connection

$$\Omega = \begin{pmatrix} w^A_B & \tau \\ 0 & 0 \end{pmatrix}$$

is flat, where $\tau = (w^1, w^2, 0)^t$. Hence there exists

$$\psi = \begin{pmatrix} g & X \\ 0 & 1 \end{pmatrix}$$

such that $\psi^{-1}d\psi = \Omega$, where $g = (e_1, e_2, e_3) \in SL(3, \mathbb{R})$ and $X \in \mathbb{R}^3$. It follows that X is an immersion, e_1, e_2 are tangent to X, e_3 is the affine normal, and ds^2, J and III are the affine metric, Fubini-Pick form, and the third fundamental form for X respectively. This is the fundamental theorem for affine surfaces in $\mathbb{R}^3$.

A surface is called a *proper affine sphere* if there exists $p_0 \in \mathbb{R}^3$ such that the affine normal line $p + t\nu(p)$ passes through p_0 for all $p \in M$. We explain below the well-known fact (cf. [7]) that the equation for proper affine spheres with indefinite affine metric is the Tzitzeica equation (2.6.4).

Let w be a solution of the Tzitzeica equation (2.6.4), and θ_λ the corresponding Lax pair defined by (2.6.5), and $E(x, t, \lambda)$ the solution of

$$E^{-1}dE = \theta_\lambda, \quad E(0, 0, \lambda) = e.$$

(Here e is the identity matrix in $SL(3, \mathbb{R})$.) Fix a non-zero $r \in \mathbb{R}$, let $e_i(x, t)$ denote the i-th column of $E(x, t, r)$. We claim that $X = -e_3$ is an immersed indefinite affine sphere. To see this, we first note that

$$\theta_r = \begin{pmatrix} w_x dx & r^{-1}e^{-2w}dy & rdx \\ rdx & -w_x dx & r^{-1}e^w dy \\ r^{-1}e^w dy & rdx & 0 \end{pmatrix} = E(r)^{-1}dE(r).$$

Since θ_r is an $sl(3, \mathbb{R})$-valued flat 1-form, $E(r)$ is a map from $\mathbb{R}^2$ to $SL(3, \mathbb{R})$. Fix r, and let e_i denote $e_i(r)$. Equate each column of $dE(r) =$

$E(r)\theta_r$ to get

$$\begin{cases} dX = -de_3 = -rdx\ e_1 - r^{-1}e^w dy\ e_2, \\ de_1 = w_x dx\ e_1 + rdx\ e_2 + r^{-1}e^w dy\ e_3, \\ de_2 = r^{-1}e^{-2w}dy\ e_1 - w_x dx\ e_2 + rdx\ e_3. \end{cases}$$

This implies that e_1, e_2 are tangent to X,

$$\begin{aligned} &w^1 = -rdx, \quad w^2 = -r^{-1}e^w dy, \\ &w_1^3 = r^{-1}e^w dy = -w^2, \quad w_2^3 = rdx = -w^1, \\ &w_3^1 = -w^1, \quad w_3^2 = -w^2, \quad w_3^3 = 0. \end{aligned}$$

So $h_{11} = h_{22} = 0$, $h_{12} = -1$ and the affine metric is $2e^w dxdy$. Since $\det(h_{ij}) = -1$ and $w_3^3 = 0$, (3.8.4) is satisfied. Hence the affine normal is

$$\nu = |\det(h_{ij})|^{\frac{1}{4}} e_3 = e_3.$$

But $X = -e_3$ implies that all affine normal lines pass through the origin. In other words, X is an indefinite proper affine sphere.

Conversely, suppose X is an indefinite proper affine sphere in $\mathbb{R}^3$. We want to show that there exist a special coordinate system and a special affine frame so that the Gauss-Codazzi equation for X as an affine sphere is the Tzitzeica equation. First note that there exist a local asymptotic coordinate system (x, y) and a smooth function w such that the affine metric is

$$ds^2 = 2e^{2w} dxdy.$$

Let $e_1 = X_x$, $e_2 = X_y$, and e_3 parallel to the affine normal such that

$$\det(e_1, e_2, e_3) = 1.$$

Then

$$w^1 = dx, \quad w^2 = dy, \quad w_1^3 = e^{2w}dy, \quad w_2^3 = e^{2w}dx.$$

So $\det(h_{ij}) = -e^{4w}$. We may assume that all affine normal lines pass through the origin. So $X = fe_3$ for some function f. Exterior differentiation of $X = fe_3$ gives

$$w^1 e_1 + w^2 e_2 = df e_3 + f w_3^3 e_3 + f(w_3^1 e_1 + w_3^2 e_2).$$

Equate the coefficients of e_3 to get $df + fw_3^3 = 0$. Since e_3 is parallel to the affine normal, w_3^3 satisfies (3.8.4). Therefore $f = c \mid \det(h_{ij}) \mid^{1/4} =$

ce^w for some constant c. By rescaling, we may assume $c = 1$. Equate coefficients of e_1 and e_2 to get

$$w_3^1 = e^{-w}dx, \quad w_3^2 = e^{-w}dy.$$

Therefore $\ell^{11} = \ell^{22} = 0$ and $\ell^{12} = \ell^{21} = e^{-3w}$. So the affine mean curvature is $L = 1$. Use (3.8.5) to get

$$w_1^2 = -\frac{h_{111}}{2}e^{-2w}dx, \quad w_2^1 = -\frac{h_{222}}{2}e^{-2w}dy.$$

Use $dw^i = -\sum_j w_j^i \wedge w^j$ to conclude that $w_1^1 = w_x dx, \quad w_2^2 = w_y dy$. Substitute w_A^B into $dw_j^i = -\sum_A w_A^i \wedge w_j^A$ for $i = 1, j = 2$ and $j = 1, i = 2$ to get

$$(h_{111}e^{-w})_y = 0, \quad (h_{222}e^{-w})_x = 0.$$

So $h_{111} = u_1(x)e^w$ and $h_{222} = u_2(y)e^w$ for some smooth function u_1, u_2 of one variable. By making a coordinate change to $(\tilde{x}, \tilde{y}) = (\tilde{x}(x), \tilde{y}(y))$, we may assume that

$$w_2^1 = e^{-w}dy, \quad w_1^2 = e^{-w}dx.$$

To summarize, we have shown that

$$g^{-1}dg = \begin{pmatrix} w_x dx & e^{-w}dy & e^{-w}dx \\ e^{-w}dx & w_y dy & e^{-w}dy \\ e^{2w}dy & e^{2w}dx & -dw \end{pmatrix},$$

where $g = (e_1, e_2, e_3)$. Change the frame g to $\tilde{g} = g\,\mathrm{diag}(1, e^{-w}, e^w)$. Then

$$\tilde{g}^{-1}d\tilde{g} = \begin{pmatrix} w_x dx & e^{-2w}dy & dx \\ dx & -w_x dx & e^w dy \\ e^w dy & dx & 0 \end{pmatrix}.$$

This is the Lax pair θ_λ (2.6.5) at $\lambda = 1$. So w is a solution of the Tzitzeica equation.

3.9. The -1 flow, hyperbolic system, and the sigma model

Let $\mathbb{R}^{1,1}$ denote the Lorentz space equipped with metric $2dxdt$. In this section, we discuss the relation between harmonic maps from $\mathbb{R}^{1,1}$ to a Lie group U and solutions of the -1-flow and the hyperbolic U-system.

First we recall a theorem of Uhlenbeck ([72]):

Theorem 3.9.1. *([72]). Let $s : \mathbb{R}^{1,1} \to U$ be a smooth map, $A = \frac{1}{2}s^{-1}s_x$, and $B = \frac{1}{2}s^{-1}s_y$. Then the following statements are equivalent:*

(i) *s is harmonic,*
(ii) $A_t = -B_x = [A, B]$,
(iii) $\Omega_\lambda = (1-\lambda)A\ dx + (1-\lambda^{-1})B\ dt$ *is flat for all* $\lambda \in \mathbb{C} \setminus 0$.

Corollary 3.9.2. *([72]). Suppose* $\theta_\lambda = (1-\lambda)Adx + (1-\lambda^{-1})Bdt$ *is flat for all* $\lambda \in \mathbb{C}\setminus 0$, *and* E_λ *satisfies* $E_\lambda^{-1}dE_\lambda = \theta_\lambda$. *Then* $s = E_{-1}$ *is a harmonic map from* $\mathbb{R}^{1,1}$ *into* U *such that* $s^{-1}ds = 2Adx + 2Bdt$.

The following Proposition is well-known:

Proposition 3.9.3. *Let* $i : N_0 \to N$ *be a totally geodesic submanifold of* N. *A smooth map* $s : M \to N_0$ *is a harmonic map if and only if* $i \circ s : M \to N$ *is a harmonic map.*

Proposition 3.9.4. *([68]). Let* $\mathcal{U}$ *be the real form of* $\mathcal{G}$ *defined by the involution* τ, $a, b \in \mathcal{U}$ *such that* $[a, b] = 0$, $g : \mathbb{R}^2 \to U$ *a solution of the* -1*-flow* (2.6.2) *associated to* U, *and* $E_\lambda(x,t) = E(x,t,\lambda)$ *the frame for the corresponding Lax pair* θ_λ (2.6.3), *i.e.,*

$$E^{-1}dE = (a\lambda + g^{-1}g_x)dx + \lambda^{-1}g^{-1}bg\ dt, \quad E(x,t,0) = e.$$

Then $s = E_{-1}E_1^{-1}$ *is a harmonic map from* $\mathbb{R}^{1,1}$ *to* U. *Moreover, if* σ *is an order* $k = 2m$ *automorphism such that* $\tau\sigma = \sigma^{-1}\tau$, $a \in \mathcal{U} \cap \mathcal{G}_1$, *and* $b \in \mathcal{U} \cap \mathcal{G}_{-1}$, *then* $s = E_{-1}E_1^{-1}$ *is a harmonic map from* $\mathbb{R}^{1,1}$ *to the symmetric space* U/H, *where* H *is the fixed point set of the involution* σ^m *and* $\mathcal{G}_j$ *is the eigenspace of* σ *on* $\mathcal{G}$.

Proof. Note that the gauge transformation of θ_λ by E_1 is $E_1 * \theta_\lambda = E_1\theta_\lambda E_1^{-1} - dE_1E_1^{-1} = (1-\lambda)E_1aE_1^{-1}\ dx + (1-\lambda^{-1})E_1g^{-1}bgE_1^{-1}\ dt$. Let $\psi_\lambda = E_\lambda E_1^{-1}$. Then $\psi_\lambda^{-1}d\psi_\lambda = E_1 * \theta_\lambda$. By Corollary 3.9.2, $s = \psi_{-1} = E_{-1}E_1^{-1}$ is a harmonic map from $\mathbb{R}^{1,1}$ to U and $s^{-1}s_x$, $s^{-1}s_t$ are conjugate to $2a$, $2b$ respectively.

If the order k of σ is 2, then $\tau\sigma = \sigma^{-1}\tau = \sigma\tau$ and σ leaves $\mathcal{U}$ invariant. Let K denote the fixed point set of σ in U, and $\mathcal{U} = \mathcal{K} + \mathcal{P}$ the decomposition of eigenspaces of σ on $\mathcal{U}$ with eigenvalues $1, -1$ respectively. Note that the reality condition is

$$\tau(\theta_{\bar\lambda}) = \theta_\lambda, \quad \sigma(\theta_\lambda) = \theta_{-\lambda}.$$

So E_λ satisfies the reality condition

$$\tau(E_{\bar\lambda}) = \theta_\lambda, \quad \sigma(E_\lambda) = E_{-\lambda}.$$

This implies that $s = E_{-1}E_1^{-1} = E_{-1}\sigma(E_{-1})^{-1}$. So the image of s lies in the totally geodesic submanifold $M = \{g\sigma(g)^{-1} \mid g \in U\}$ of U. But

M is the Cartan embedding of the symmetric space U/U_0 into U as a totally geodesic submanifold. It follows from Proposition 3.9.3 that s is a harmonic map to the symmetric space $M = U/K$.

If the order of σ is $k = 2m$, an even integer, then E_λ satisfies the (G, τ, σ)-reality condition

$$\tau(E_{\bar{\lambda}}) = E_\lambda, \quad \sigma(E_\lambda) = E_{e^{\frac{2\pi i}{2m}}\lambda}.$$

Hence we have $\sigma^m(E_\lambda) = E_{-\lambda}$. So $E_1 = \sigma^m(E_{-1})$, and $s = E_{-1}\sigma^m(E_{-1})^{-1}$. Therefore s is a harmonic map from $\mathbb{R}^{1,1}$ into the symmetric space U/H. Q.E.D.

Example 3.9.5. Let $G = SL(2,\mathbb{C})$, $\tau(\xi) = -\bar{\xi}^t$, and $\sigma(\xi) = -\xi^t$. Note that

$$SU(2) = \left\{ \begin{pmatrix} w & z \\ -\bar{z} & \bar{w} \end{pmatrix} \;\middle|\; z, w \in \mathbb{C}, |w|^2 + |z|^2 = 1 \right\} = S^3,$$

and the Cartan embedding of $SU(2)/SO(2)$ is the totally geodesic 2-sphere

$$\left\{ \begin{pmatrix} w & ir \\ ir & -\bar{w} \end{pmatrix} \;\middle|\; r \in \mathbb{R},\ z \in \mathbb{C},\ r^2 + |w|^2 = 1. \right\}$$

The -1-flow associated to $SU(2)/SO(2)$ is the SGE equation, so solutions of the SGE equation give rise to harmonic maps from $\mathbb{R}^{1,1}$ to S^2.

Example 3.9.6. Let $w : \mathbb{R}^2 \to \mathbb{R}$ be a solution of the Tzitzeica equation (2.6.4), the -1-flow associated to the homogeneous space $\frac{SL(3,\mathbb{R})}{\mathbb{R}^+}$ given in Example 2.6.3. For this example, the order of σ is 6. Let θ_λ be the corresponding Lax pair (2.6.5), and E_λ the frame of θ_λ. A direct computation shows that

$$\sigma^3(\xi) = -PA^tP, \quad \text{where } P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So the fixed point set of the involution σ^3 in $sl(3,\mathbb{R})$ is $so(2,1)$, where $SO(2,1)$ is the isometry group of the quadratic form $2x_1x_2 + x_3^2$ on $\mathbb{R}^3$. Hence $E_{-1}E_1^{-1}$ is a harmonic map from $\mathbb{R}^{1,1}$ to the symmetric space $\frac{SL(3,\mathbb{R})}{SO(2,1)}$.

The proof of Proposition 3.9.4 also implies

Proposition 3.9.7. *Let $(u_0, u_1, v_0, v_1) : \mathbb{R}^2 \to \prod_{i=1}^4 \mathcal{U}$ be a solution of the hyperbolic system* (2.7.1) *associated to U, and $E(x, t, \lambda)$ the frame of the corresponding Lax pair* (2.7.2). *Then $s = E(\cdot, \cdot, -1)E(\cdot, \cdot, 1)^{-1}$ is a harmonic map from $\mathbb{R}^{1,1}$ to U. Moreover, if σ is an involution on $\mathcal{U}$, $\mathcal{U} = \mathcal{U}_0 + \mathcal{U}_1$ is the eigenspace decomposition, and $u_0, v_0 \in \mathcal{U}_0$, $u_1, v_1 \in \mathcal{U}_1$, then the image of s lies in the symmetric space U/U_0 (embedded in U via the Cartan embedding) and is a harmonic map from $\mathbb{R}^{1,1}$ to U/U_0.*

§4. Dressing actions and factorizations

Suppose that G is a Lie group, and that G_+, G_- are subgroups of G such that the multiplication maps $G_+ \times G_- \to G$ and $G_- \times G_+ \to G$ defined by $(g_+, g_-) \mapsto g_+g_-$ and $(g_-, g_+) \mapsto g_-g_+$ respectively are bijections. Thus given any $g \in G$, there exist unique $g_+ \in G_+$ and $g_- \in G_-$ so that $g = g_+g_-$, and unique $h_+ \in G_+$ and $h_- \in G_-$ such that $g = h_-h_+$. The dressing action of G_+ on G_- is defined as follows: Factor g_+g_- as $\tilde{g}_-\tilde{g}_+$ with $\tilde{g}_\pm \in G_\pm$. Then the dressing action of G_+ on G_- is $g_+ * g_- = \tilde{g}_-$. The dressing action of G_- on G_+ is defined similarly.

If the multiplication maps are injective and the images are open dense subsets of G, then the dressing actions are defined on an open neighborhood of the identity e in $G_\pm$. Moreover, the corresponding Lie algebra actions are well-defined.

There are two well known factorizations for a semi-simple Lie group, the Iwasawa and the Gauss factorizations. The analogous loop group factorizations are those given by Pressley and Segal in [57] and the Birkhoff factorization respectively. The dressing actions of these loop group factorizations play important roles in finding solutions and explaining the hidden symmetries of integrable systems. We will review these loop group factorizations.

4.1. Iwasawa and Gauss factorizations

Let G be a complex, semi-simple Lie group, U a maximal compact subgroup, and B a Borel subgroup. *The Iwasawa factorization* of G is $G = UB$, i.e., every $g \in G$ can be factored uniquely as ub, where $u \in U$ and $b \in B$. Let A be a maximal abelian subgroup of G, and N_+ and N_- the unipotent subgroups generated by the set of positive roots and negative roots with respect to a fixed simple root system of $\mathcal{A}$ respectively. The multiplication map from $N_- \times A \times N_+$ to G is injective and the image is an open dense subset of G, called a *big cell* of G. *The Gauss factorization* is the factorization of a big cell of G, i.e., every element g in the big cell can be factored uniquely as n_-an_+ with $n_\pm \in N_\pm$ and $a \in A$. Let $B_+ = AN_+$. We call the factorization of g in the big cell as n_-b_+ with $n_- \in N_-$ and $b_+ \in B_+$ again the Gauss factorization.

Example 4.1.1. Let $G = SL(n, \mathbb{C})$, $\triangle_+(n)$ the subgroup of upper triangular $g \in SL(n, \mathbb{C})$, and $\triangle_-(n)$ the subgroup of lower triangular matrix $g \in SL(n, \mathbb{C})$ with 1's on the the diagonal. The multiplication maps $\triangle_+(n) \times \triangle_-(n) \to SL(n, \mathbb{C})$ and $\triangle_-(n) \times \triangle_+(n) \to SL(n, \mathbb{C})$ are injective and the images are open and dense. Moreover, the factorization of $g \in SL(n, \mathbb{C})$ can be carried out using Gaussian elimination on rows and columns of g. This is the Gauss factorization of $SL(n, \mathbb{C})$.

Example 4.1.2. Let $G = SL(n, \mathbb{C})$, $U = SU(n)$, and $B_+(n)$ the subgroup of upper triangular matrices with real diagonal entries. The multiplication maps $B_+(n) \times U(n) \to SL(n, \mathbb{C})$ and $U(n) \times B_+(n) \to SL(n, \mathbb{C})$ are bijective. Moreover, the factorization can be carried out by applying the Gram-Schmidt process on rows and columns of g. This is the Iwasawa factorization of $SL(n, \mathbb{C})$.

4.2. Factorizations of loop groups

Let G be a complex, semi-simple Lie group, τ an involution of G that gives the compact real form U, and σ an order k automorphism of G. Let B be a Borel subgroup of G such that $G = UB$ is the Iwasawa factorization. Given an open subset $\mathcal{O}$ of S^2, let $\mathrm{Hol}(\mathcal{O}, G)$ denote the group all holomorphic maps $f : \mathcal{O} \to G$ with multiplication defined by $(fg)(\lambda) = f(\lambda)g(\lambda)$. Let $\epsilon > 0$,

$$\begin{aligned} &S^2 = \mathbb{C} \cup \{\infty\}, \quad \mathbb{C}^* = \{\lambda \in \mathbb{C} \mid \lambda \neq 0\}, \\ &\mathcal{O}_\epsilon = \{\lambda \in \mathbb{C} \mid |\lambda| < \epsilon\}, \quad \mathcal{O}_{1/\epsilon} = \{\lambda \in S^2 \mid |\lambda| > 1/\epsilon\}. \end{aligned}$$

To explain symmetries of soliton flows we need to consider the following groups:

$$\begin{aligned} \Lambda(G) &= \mathrm{Hol}(\mathbb{C} \cap \mathcal{O}_{1/\epsilon}, G), \\ \Lambda_+(G) &= \mathrm{Hol}(\mathbb{C}, G), \\ \Lambda_-(G) &= \{f \in \mathrm{Hol}(\mathcal{O}_{1/\epsilon}, G) \mid f(\infty) = e\}. \end{aligned}$$

The following is the Gauss loop group factorization (the Birkhoff factorization):

Theorem 4.2.1. The Gauss loop group factorization. *The multiplication maps from $\Lambda_+(G) \times \Lambda_-(G)$ and $\Lambda_-(G) \times \Lambda_+(G)$ to $\Lambda(G)$ are injective and the images are open and dense. In particular, there exists an open dense subset $\Lambda(G)_0$ of $\Lambda(G)$ such that given $g \in \Lambda(G)_0$, g can be factored uniquely as $g = g_+g_- = h_-h_+$ with $g_+, h_+ \in \Lambda_+(G)$ and $g_-, h_- \in \Lambda_-(G)$.*

Suppose $\tau\sigma = \sigma^{-1}\tau$. Let $\mathcal{G} = \mathcal{G}_0 + \cdots + \mathcal{G}_{k-1}$ be the eigenspace decomposition of σ, and $\mathcal{U}_j = \mathcal{U} \cap \mathcal{G}_j$. Let $\hat{\tau}$ and $\tilde{\sigma}$ be the automorphism of $\Lambda(G)$ defined by

$$\hat{\tau}(g)(\lambda) = \overline{\tau(g(\bar{\lambda}))}, \tag{4.2.1a}$$

$$\tilde{\sigma}(g)(\lambda) = \sigma(g(e^{-2\pi i/k}\ \lambda)). \tag{4.2.1b}$$

Let $\Lambda^\sigma(G)$ denote the fixed point set of $\tilde{\sigma}$ on $\Lambda(G)$. Since $\tau\sigma = \sigma^{-1}\tau$, a direct computation implies that $\tilde{\tau}$ leaves $\Lambda^\sigma(G)$ invariant. Let $\Lambda^{\tau,\sigma}(G)$ denote the subgroup of $g \in \Lambda(G)$ that is fixed by $\hat{\tau}$ and $\tilde{\sigma}$. Let $\Lambda^\tau(G)$ denote the subgroup of $\Lambda(G)$ fixed by $\hat{\tau}$. Then

$$\begin{aligned}
\Lambda^\tau(G) &= \{f \in \Lambda(G) \mid f \text{ satisfies } U\text{--reality condition (2.2.1)}\}, \\
\Lambda^{\tau,\sigma}(G) &= \{f \in \Lambda(G) \mid f \text{ satisfies } U/U_0\text{--reality condition (2.4.1)}\}, \\
\Lambda^\tau_\pm(G) &= \Lambda^\tau(G) \cap \Lambda_\pm(G), \\
\Lambda^{\tau,\sigma}_\pm(G) &= \Lambda^{\tau,\sigma}(G) \cap \Lambda_\pm(G).
\end{aligned}$$

Corollary 4.2.2. *Suppose $g \in \Lambda(G)$ is factored as $g = g_+g_-$ with $g_+ \in \Lambda_+(G)$ and $g_- \in \Lambda_-(G)$. If $\tau\sigma = \sigma^{-1}\tau$, then*

(i) *$g \in \Lambda^\tau(G)$ implies that $g_\pm \in \Lambda^\tau_\pm(G)$,*
(ii) *$g \in \Lambda^{\tau,\sigma}(G)$ implies that $g_\pm \in \Lambda^{\tau,\sigma}_\pm(G)$.*

To explain symmetries of the elliptic integrable systems, we need to consider the (G, τ)-reality condition

$$\tau(g(1/\bar{\lambda}) = g(\lambda), \tag{4.2.2}$$

and the following groups:

$$L(G) = C^\infty(S^1, G),$$
$$L_+(G) = \{f \in L(G) \mid f \text{ extends holomorphically to } \mid \lambda \mid \ < 1\},$$
$$L_e(U) = \{f \in C^\infty(S^1, U) \mid f(1) = e\},$$

$$\Omega^\tau(G) = \{f \in \mathrm{Hol}((\mathcal{O}_\epsilon \cup \mathcal{O}_{1/\epsilon}) \cap \mathbb{C}^*, G) \mid f \text{ satisfies the } (G,\tau)\text{−reality condition (4.2.2)}\},$$
$$\Omega^\tau_+(G) = \{f \in \Omega^\tau(G) \mid f \text{ extends holomorphically to } \mathbb{C}^*\},$$
$$\Omega^\tau_-(G) = \{f \in \Omega^\tau(G) \mid f \text{ extends holomorphically to } \mathcal{O}_\epsilon \cup \mathcal{O}_{1/\epsilon} \text{ and} f(\infty) = \mathrm{I}\}.$$

For the (G,τ,σ)-system, we also need the subgroup of $g \in \Omega^\tau(G)$ that satisfies the (G,τ,σ)-*reality condition*:

$$\tau(g(1/\bar{\lambda})) = g(\lambda), \quad \sigma(g(e^{-\frac{2\pi i}{k}}\lambda)) = g(\lambda). \tag{4.2.3}$$

Theorem 4.2.3. *([51]). The multiplication map from $\Omega^\tau_-(G) \times \Omega^\tau_+(G)$ to $\Omega^\tau(G)$ is a bijection.*

Corollary 4.2.4. *Given $g_+ \in \Omega^\tau_+(G)$ and $g_- \in \Omega^\tau_-(G)$, g_-g_+ can be factored uniquely as $\tilde{g}_+\tilde{g}_-$ with $\tilde{g}_+ \in \Omega^\tau_+(G)$ and $\tilde{g}_- \in \Omega^\tau_-(G)$. Moreover, if $\tau\sigma = \sigma\tau$ and g_+, g_- satisfy the (G,τ,σ)-reality condition* (4.2.3), *then $\tilde{g}_+ \in \Omega^{\tau,\sigma}_+(G)$ and $\tilde{g}_- \in \Omega^{\tau,\sigma}_-(G)$.*

Theorem 4.2.5. The Iwasawa loop group factorization *([57]). The multiplication maps $L_e(U) \times L_+(G) \to L(G)$ and $L_+(G) \times L_e(U) \to L(G)$ are bijections.*

§5. Symmetries of the U-hierarchy

Let τ be a conjugate linear involution of G, and U its fixed point set. In this Chapter, we assume U is compact. Let $\mathcal{A}$ be a maximal abelian subalgebra of $\mathcal{U}$. The U-hierarchy has three types of symmetries, i.e., three actions on the space of solutions of the (b,j)-flow in the U-hierarchy:

—An action of the infinite dimensional abelian algebra $\hat{\mathcal{A}}_+$ of polynomial maps from $\mathbb{C}$ to $\mathcal{A} \otimes \mathbb{C}$.

—An action of $\Lambda^\tau_-(G)$.

—An action of the subgroup of $f \in L_+^\tau(G)$ such that the infinite jet of $f_b - \mathrm{I}$ at $\lambda = -1$ is 0, where $f = f_u f_b$ with $f_u \in U$, $f_b \in B$, and $G = UB$ is the Iwasawa factorization.

The first two symmetries arise naturally from the dressing actions of the factorization theorems given in section 4.2. The third action comes from the dressing action of a new factorization.

5.1. The action of $\Lambda_-^\tau(G)$

It follows from the Gauss loop group factorization 4.2.1 that the group $\Lambda_-^\tau(G)$ acts on $\Lambda_+^\tau(G)$ by local dressing action. This induces an action of $\Lambda_-^\tau(G)$ on the space of germs of solutions of the (b, j)-flow (2.1.3) in the U-hierarchy at the origin as follows: Let $u : \mathbb{R}^2 \to \mathcal{A}^\perp \cap \mathcal{U}$ be a solution of the (b, j)-th flow (2.1.3), θ_λ the corresponding Lax pair (2.1.5), and $E(x, t, \lambda)$ the frame of u, i.e., E is the solution of

$$\begin{cases} E^{-1}E_x = a\lambda + u, \\ E^{-1}E_t = \sum_{i=0}^{j} Q_{b,j-i}(u)\lambda^i, \\ E(0, \lambda) = e. \end{cases}$$

Then $E(x, t)(\lambda) = E(x, t, \lambda)$ is holomorphic in $\lambda \in \mathbb{C}$, i.e., $E(x, t) \in \Lambda_+(G)$ for all (x, t). Since θ_λ satisfies the U-reality condition $\tau(\theta_{\bar\lambda}) = \theta_\lambda$, $E(x, t)$ satisfies $\tau(E(x, t)(\bar\lambda)) = E(x, t)(\lambda)$. In other words, $E(x, t) \in \Lambda_+^\tau(G)$. Given $g \in \Lambda_-^\tau(G)$, by the Gauss loop group factorization 4.2.1 and Corollary 4.2.2 there is an open subset $\mathcal{O}$ of the origin in $\mathbb{R}^2$ such that the dressing action of g at $E(x, t)$ is defined for all $(x, t) \in \mathcal{O}$. Let $g * E(x, t)$ denote the dressing action of g at $E(x, t)$. This is obtained as follows: Factor $gE(x, t)$ as

$$gE(x, t) = \tilde{E}(x, t)\tilde{g}(x, t)$$

with $\tilde{E}(x, t) \in \Lambda_+^\tau(G)$, and $\tilde{g}(x, t) \in \Lambda_-^\tau(G)$. Then

$$g * E(x, t) = \tilde{E}(x, t).$$

Expanding $\tilde{g}(x)(\lambda)$ at $\lambda = \infty$ we have:

$$\tilde{g}(x, t)(\lambda) = I + g_1(x, t)\lambda^{-1} + g_2(x, t)\lambda^{-2} + \cdots.$$

The following results are known (cf. [70]):

(i) $\tilde{u} = u + [a, g_1]$ is again a solution of the (b, j)-flow.

(ii) $g * u = \tilde{u}$ defines an action of $\Lambda_-^\tau(G)$ on the space of local solutions of the (b, j)-flow.

(iii) $\tilde{E}(x, \lambda)$ is the frame of $\tilde{u}$.

(iv) Suppose $g \in \Lambda^\tau(G)$ is a rational map with only simple poles, i.e., g satisfies the U-reality condition and is of the form $g(\lambda) = I + \sum_{j=1}^k \frac{\xi_j}{\lambda - \alpha_j}$ for some $\xi_j \in \mathcal{G}$ and $\alpha_j \in \mathbb{C} \setminus \{0\}$. Then $g * u$ can be computed explicitly in terms of g, u, E.

(v) If U is compact, $g \in \Lambda^\tau(G)$ is rational, and u is a smooth solution defined on all $(x, t) \in \mathbb{R}^2$ that is rapidly decaying in x, then $g * u$ is also defined on all $\mathbb{R}^2$ and is rapidly decaying in x.

We claim that $\Lambda_-^{\tau,\sigma}(G)$ acts on the space of solutions of flows in the U/U_0-hierarchy. To see this, let E be the frame of a solution u of the $(b, mk+1)$-flow in the U/U_0-hierarchy. Then $E(x, t, \cdot) \in \Lambda_+^{\tau,\sigma}(G)$. By Corollary 4.2.2, $g * E$ satisfies the (G, τ, σ)-reality condition. Hence $g * u$ is a solution of the $(b, mk+1)$-flow in the U/U_0-hierarchy. We give an explicit example next.

Example 5.1.1. ([10]). Let $\tau(y) = \bar{y}$, and $\sigma(y) = \mathrm{I}_{n,n}\, y\, \mathrm{I}_{n,n}^{-1}$ be the involutions of $O(2n, \mathbb{C})$ that give the symmetric space $\frac{O(2n)}{O(n)\times O(n)}$ as in Example 2.5.4. Let $v = \begin{pmatrix} 0 & F \\ -F^t & 0 \end{pmatrix}$ be a solution of the $\frac{O(2n)}{O(n)\times O(n)}$-system (2.5.3), θ_λ the corresponding Lax n-tuple (2.5.4), and E the frame of v. We give an explicit construction of the action of certain rational map with two poles in $\Lambda_-^{\tau,\sigma}(G)$ on v below. Let W, Z be two unit vectors of $\mathbb{R}^n$, π the projection of $\mathbb{C}^{2n}$ onto the complex linear subspace spanned by $\begin{pmatrix} W \\ iZ \end{pmatrix}$, $s \in \mathbb{R}$ non-zero, and

$$h_{is,\pi}(\lambda) = \left(\pi + \frac{\lambda - is}{\lambda + is}(\mathrm{I} - \pi)\right)\left(\bar{\pi} + \frac{\lambda + is}{\lambda - is}(\mathrm{I} - \bar{\pi})\right). \tag{5.1.1}$$

A direct computation shows that $h_{is,\pi} \in \Lambda_-^{\tau,\sigma}(G)$. Then we have:

(1) $h_{is,\pi}E(x) = \tilde{E}(x)h_{is,\tilde{\pi}(x)}$, where $\tilde{\pi}(x)$ is the Hermitian projection onto the complex linear subspace spanned by

$$\begin{pmatrix} \tilde{W}(x) \\ i\tilde{Z}(x) \end{pmatrix} = E(x, -is)^{-1}\begin{pmatrix} W \\ iZ \end{pmatrix}.$$

(2) $(\tilde{W}, i\tilde{Z})^t$ is a solution of the following first order system:

$$\begin{pmatrix} \tilde{W} \\ i\tilde{Z} \end{pmatrix}_{x_j} = -(-is\, a_j + [a_j, v]) \begin{pmatrix} \tilde{W} \\ i\tilde{Z} \end{pmatrix}. \tag{5.1.2}$$

(3) Given v, system (5.1.2) is solvable for $\begin{pmatrix} \tilde{W} \\ i\tilde{Z} \end{pmatrix}$ if and only if v is a solution of (2.5.3).

(4) Let $\xi = (\xi_{ij})$, $\phi(\xi) = \xi - \sum_i \xi_{ii} e_{ii}$, and $\tilde{F} = F - 2s\ \phi(\tilde{\pi})$. Then

$$h_{is,\pi} * \begin{pmatrix} 0 & F \\ -F^t & 0 \end{pmatrix} = \begin{pmatrix} 0 & \tilde{F} \\ -\tilde{F}^t & 0 \end{pmatrix}.$$

$\tilde{E} = h_{is,\pi} * E$ is the frame of $\tilde{F} = h_{is,\pi} * F$.

5.2. The orbit $\Lambda_-^\tau(G) * 0$

Note that $u = 0$ is a trivial solution of the (b,j)-flow in the U-hierarchy and the corresponding Lax pair is $\theta_\lambda = a\lambda dx + b\lambda^j dt$. So the frame E^0 of $u = 0$ is $E^0(x,t,\lambda) = \exp(ax\lambda + b\lambda^j t)$. Given $g \in \Lambda_-^\tau(G)$, to compute $g * 0$, the first step is to factor

$$gE^0(x,t) = \tilde{E}(x,t)\tilde{g}(x,t), \tag{5.2.1}$$

with $\tilde{E}(x,t) \in \Lambda_+^\tau(G)$, $\tilde{g}(x,t) \in \Lambda_-^\tau(G)$. The second step is to expand $\tilde{g}(x,t)$ as

$$\tilde{g}(x,t)(\lambda) = \mathrm{I} + \tilde{g}_1(x,t)\lambda^{-1} + \tilde{g}_2(x,t)\lambda^{-2} + \cdots.$$

Then $g * 0 = [a, \tilde{g}_1]$ is a solution of the (b,j)-flow.

The orbit $\Lambda_-^\tau(G) * 0$ contains several interesting classes of solutions (cf. [69]):

(1) If $g \in \Lambda_-^\tau(G)$, then $g * 0$ is a local analytic solution of the (b,j)-flow.

(2) If $g \in \Lambda_-^\tau(G)$ is a rational map, then the factorization 5.2.1 can be carried out using residue calculus and linear algebra. In fact, $\tilde{g}$ can be given explicitly in terms of a rational function of exponentials. Hence $g * 0$ can be written explicitly as a rational function of exponentials. Moreover, $(g * 0)(x,t)$ is defined for all $(x,t) \in \mathbb{R}^2$, is rapidly decaying as $|x| \to \infty$ for each $t \in \mathbb{R}$, and is a pure soliton solution.

(3) If $g \in \Lambda_-^{\tau}(G)$ such that $g^{-1}(\lambda)ag(\lambda)$ is a polynomial in λ^{-1}, then $g * 0$ is a finite type solution, and $g * 0$ can be obtained either by solving a system of compatible first order differential equations or by algebraic geometric methods.

5.3. Rapidly decaying solutions

A global solution $u(x,t)$ of the (b,j)-flow in the U-hierarchy is called a *Schwartz class solution* if $u(x,t)$ is rapidly decaying as $|x| \to \infty$ for each $t \in \mathbb{R}$. The orbit $\Lambda_-^{\tau}(G) * 0$ contains soliton solutions, which are Schwartz class solutions. But most Schwartz class solutions of the (b,j)-flow do not belong to this orbit. We need to use a different loop group factorization than the ones given in section 4.2 to construct general Schwartz class solutions.

Since we assume U is the compact real form of G, there is a Borel subgroup B such that $G = UB$ (the Iwasawa factorization). Let SD^{τ} denote the group of meromorphic maps $g : \mathbb{C} \setminus \mathbb{R} \to G$ that satisfying the following conditions:

(a) g has an asymptotic expansion $g(\lambda) \sim I + g_1\lambda^{-1} + g_2\lambda^{-2} + \cdots$ at $\lambda = \infty$,

(b) g satisfies the U-reality condition (2.2.1),

(c) $\lim_{s\to 0^{\pm}} g(r+is) = g_{\pm}(r)$ is smooth,

(d) $h_+ - I$ is in the Schwartz class, where $g_+ = v_+h_+$ with $v_+ \in U$ and $h_+ \in B$. The U-reality condition implies that $g_- = \tau(g_+)$.

Let D_-^{τ} denote the subgroup of $g \in SD^{\tau}$ that is holomorphic in $\mathbb{C} \setminus \mathbb{R}$. Then D_-^{τ} is isomorphic to the subgroup of $f \in L_+(G)$ such that $f - \mathrm{I}$ vanishes up to infinite order at $\lambda = -1$. To see this, we consider the following linear fractional transformation

$$\lambda = \phi(z) = i(1-z)/(1+z). \tag{5.3.1}$$

Note that ϕ has the following properties:

(i) ϕ maps the unit circle $|z| = 1$ to the real axis,
(ii) $\phi(-1) = \infty$,
(iii) ϕ maps the unit disk $|z| < 1$ to the upper half plane.

Theorem 5.3.1. *([69]). Let $g \in D_-^{\tau}$, ϕ the linear fractional transformation defined by* (5.3.1)*, and $\Phi(g)(z) = g(\phi(z))$ for $|z| \neq 1$. Then:*

(1) Φ is one to one,

(2) $\Phi(g_+)(e^{i\theta})$ *and* $\Phi(g_-)(e^{i\theta})$ *are the limit of* $\Phi(g)(z)$ *as* $z \to e^{i\theta}$ *with* $|z| < 1$ *and* $|z| > 1$ *respectively. Moreover,* $\Phi(g_+) \in L_+^\tau(G)$.

(3) $\Phi(g)$ *satisfies the* (G, τ)*-reality condition,* $\tau(\Phi(g)(1/\bar{\lambda})) = \Phi(g)(\lambda)$.

(4) Let $G = UB$ *be the Iwasawa factorization of* G. *Factor* $\Phi(g) = f_u f_b$ *with* $f_u \in U$ *and* $f_b \in B$. *Then the infinite jet of* $f_b - \mathrm{I}$ *at* $z = -1$ *is zero.*

The above Theorem identifies $g \in D_-^\tau$ with $\Phi(g) \in L_+^\tau(G)$, whose B-component is equal to the identity up to infinite order at $z = -1$. Recall that the frame of the trivial solution $u = 0$ of the (b, j)-flow in the U-hierarchy is $E^0(x, t)(\lambda) = e^{ax\lambda + b\lambda^j t}$. Note that $\Phi(E^0(x, t))(z)$ is smooth for all $z \in S^1$ except at $z = -1$, where it has an essential singularity. So we *cannot* use the dressing action from the Gauss factorization $L(G) = L_e(U)L_+(G)$ and the identification Φ to induce an action of D_-^τ at $u = 0$. However, we can still factor gE^0 as $E\tilde{g}$ with $E \in \Lambda_+^\tau(G)$ and $\tilde{g} \in D_-^\tau$. Intuitively speaking, the essential singularity at $z = -1$ is compensated by the infinite flatness of $\tau(\Phi(g_+))^{-1}\Phi(g_+)(z)$ at $z = -1$.

Theorem 5.3.2. *([69]). There is an open dense subset* $\mathcal{D}$ *of* SD^τ *such that if* $g \in \mathcal{D}$ *then for each* $(x, t) \in \mathbb{R}^2$, *We can factor*

$$g(\lambda)e^{ax\lambda + b\lambda^j t} = E(x, t, \lambda)g(x, t, \lambda)$$

uniquely such that $E(x, t, \cdot) \in \Lambda_+^\tau(G)$ *and* $g(x, t, \cdot) \in SD^\tau$. *Moreover,*

(i) $E^{-1}E_x$ *is of the form* $a\lambda + u(x, t)$,

(ii) $u(x, t)$ *is a Schwartz class solution of the* (b, j)*-flow,*

(iii) E *is the frame of* u.

Let $g\sharp 0$ denote the solution u constructed in Theorem 5.3.2. Then:

Theorem 5.3.3. *([69]).*

(1) $(D_-^\tau \sharp 0) \cap (\Lambda_-^\tau(G) * 0) = \{0\}$.
(2) $SD^\tau \sharp 0$ *is open and dense in the space of Schwartz class solutions of the* (b, j)*-flow in the* U*-hierarchy.*

5.4. The action of an infinite dimensional abelian group

Let $j > 0$ be an integer, $b \in \mathcal{A}$, $\xi_{b,j} \in \Lambda_+^\tau(\mathcal{G})$ defined by $\xi_{b,j}(\lambda) = b\lambda^j$, and $e_{b,j}(t)$ the one-parameter subgroup of $\Lambda_+^\tau(G)$ defined by $\xi_{b,j}$, i.e.,

$$e_{b,j}(t)(\lambda) = e^{b\lambda^j t}.$$

Let $\hat{A}_+$ be the subgroup of $\Lambda_+^\tau(G)$ generated by $\{e_{b,j}(t) \mid b \in \mathcal{A}, j \in \mathbb{N}, t \in \mathbb{R}\}$. The Lie algebra $\hat{\mathcal{A}}_+$ of $\hat{A}_+$ is the subalgebra of $\Lambda_+^\tau(\mathcal{G})$ generated by $\{\xi_{b,j} \mid b \in \mathcal{A}, j \in \mathbb{N}\}$. It follows from Theorem 5.3.2 that given $f \in \Lambda_-^\tau(G)$ we can factor $f^{-1}e_{a,1}(x) = E(x)m^{-1}(x)$ with $E(x) \in \Lambda_+^\tau(G)$ and $m(x) \in SD^\tau$. Expand $m(x)(\lambda)$ at $\lambda = \infty$ to get

$$m(x)(\lambda) = \mathrm{I} + m_1(x)\lambda^{-1} + m_2(x)\lambda^{-2} + \cdots .$$

Define $\mathcal{F}(f) = u^f := [a, m_1]$. Then $\mathcal{F}$ is a map from $SD^\tau(G)$ to $\mathcal{S}(\mathbb{R}, \mathcal{A}^\perp \cap \mathcal{U})$.

Given $b \in \mathcal{A}$, a positive integer j, and $f \in SD^\tau$, it follows from Theorem 5.3.2 that we can factor

$$f^{-1}e_{a,1}(x)e_{b,j}(t) = E(x,t)m(x,t)^{-1}$$

with $E(x,t) \in \Lambda_+^\tau(G)$ and $m(x,t) \in SD^\tau$. A straightforward direct computation implies that (cf. [69]):

(i) $E^{-1}E_x(x,t,\lambda)$ must be of the form $a\lambda + u(x,t)$ and $u = [a, m_1]$, where m_1 is the coefficient of λ^{-1} of the expansion of m as

$$m(x,t)(\lambda) = \mathrm{I} + m_1(x,t)\lambda^{-1} + m_2(x,t)\lambda^{-2} + \cdots .$$

(ii) u is a solution of the (b,j)-flow in the U-hierarchy.

By definition of the dressing action, $m(x,t) = (e_{a,1}(x)e_{b,j}(t)) * f$. Hence the (b,j)-flow arises naturally from the dressing action of $\hat{A}_+ \subset \Lambda_+^\tau(G)$ on SD^τ. Moreover,

$$\mathcal{F}(e_{b,j} * f) = u^{e_{b,j}(t)*f} = \phi_{b,j}(t)(u^f) = \phi_{b,j}(t)(\mathcal{F}(f)),$$

where $\phi_{b,j}(t)$ is the one-parameter subgroup generated by the vector field $X_{b,j}$ corresponding the (b,j)-flow in the U-hierarchy (i.e., $X_{b,j}$ defined by (2.1.6)). In other words, the infinitesimal dressing action of the abelian algebra $\hat{\mathcal{A}}_+$ on SD^τ gives the U-hierarchy of commuting flows. For more details of this discussion see [69].

5.5. Geometric transformations

Let $g \in \Lambda_-^{\tau,\sigma}(G)$, v a solution of some integrable system associated to U/U_0, and E the frame of v. Factor gE as $\tilde{E}\tilde{g}$ with $\tilde{E}(x,t) \in \Lambda_+^{\tau,\sigma}(G)$ and $\tilde{g} \in \Lambda_-^{\tau,\sigma}(G)$. Then $g * v = \tilde{v}$, and $g * E = \tilde{E}$ is the frame of $g * v$. We have seen in Chapter 3 that geometries associated to solutions of an integrable system can often be read from their frames at some special value $\lambda = \lambda_0$. So $E_{\lambda_0} \mapsto g * E_{\lambda_0}$ gives rise to a geometric transformation for the corresponding geometries. For example, it is known that solutions of the SGE equation correspond to surfaces in $\mathbb{R}^3$ with constant Gaussian curvature $K = -1$. The Lax pair of the SGE equation satisfies the $SU(2)/SO(2)$-reality condition, and the dressing action of $g_{is,\pi}(\lambda) = \pi + \frac{\lambda - is}{\lambda + is}(\mathrm{I} - \pi)$ on the space of solutions of the SGE equation corresponds to the classical Bäcklund transformation of surfaces in $\mathbb{R}^3$ with $K = -1$ (cf. [70]). In this section, we use another example to demonstrate this correspondence between the dressing action and geometric transformations. We describe the geometric transformation of flat submanifolds that corresponds to the action of the rational element $h_{is,\pi}$ (defined by (5.1.1)) on the solutions of the $\frac{O(2n)}{O(n)\times O(n)}$-system.

First, we need to recall the following definition given by Dajczer and Tojeiro in [21, 22].

Definition 5.5.1. Let M^n and $\tilde{M}^n$ be submanifolds of S^{2n-1} with flat normal bundle. A vector bundle isomorphism $P : \nu(M) \to \nu(\tilde{M})$, which covers a diffeomorphism $\ell : M \to \tilde{M}$, is called a *Ribaucour Transformation* if P satisfies the following properties:

(a) If ξ is a parallel normal vector field of M, then $P \circ \xi \circ \ell^{-1}$ is a parallel normal field of $\tilde{M}$.

(b) Let $\xi \in \nu_x(M)$, and $\gamma_{x,\xi}$ the normal geodesic with ξ as the tangent vector at $t = 0$. Then for each $\xi \in \nu(M)_x$, $\gamma_{x,\xi}$ and $\gamma_{\ell(x),P(\xi)}$ intersect at a point that is equidistant from x and $\ell(x)$ (the distance depends on x).

(c) If η is an eigenvector of the shape operator A_ξ of M, then $\ell_*(\eta)$ is an eigenvector of the shape operator $A_{P(\xi)}$ of $\tilde{M}$. Moreover, the geodesics $\gamma_{x,\eta}$ and $\gamma_{\ell(x),\ell_*(\eta)}$ intersect at a point equidistant to x and $\ell(x)$.

Dajczer and Tojeiro used geometric methods to prove the existence of Ribaucour transformations between flat n-submanifolds of S^{2n-1} in [21]. These Ribaucour transformations are exactly the ones obtained from dressing actions of $h_{is,\pi}$ given in Example 5.1.1, i.e.,

Theorem 5.5.2. *([10]). Let F be a solution of the $\frac{O(2n)}{O(n)\times O(n)}$-system (2.5.3), and E the frame of the corresponding Lax n-tuple (2.5.4). Let $h_{is,\pi} \in \Lambda_-^{\tau,\sigma}(G)$ defined by (5.1.1), $\tilde{F} = h_{is,\pi} * F$, and $\tilde{E} = h_{is,\pi} * E$ as in Example 5.1.1. Let M be a flat n-submanifold in S^{2n-1} associated to F as in Theorem 3.3.3. Then there exist a flat n-submanifold $\tilde{M}$ in S^{2n-1} and a Ribaucour transformation $P : \nu(M) \to \nu(\tilde{M})$ constructed from $\tilde{E} = h_{is,\pi} * E$ such that the solution of the $\frac{O(2n)}{O(n)\times O(n)}$-system for $\tilde{M}$ is $\tilde{F} = h_{z,\pi} * F$.*

5.6. The characteristic initial value problem for the -1-flow

Given $a, b \in \mathcal{U}$ such that $[a, b] = 0$, the -1-flow in the U-hierarchy defined by a, b is the following equation for $g : \mathbb{R}^2 \to U$

$$(g^{-1}g_x)_t = [a, g^{-1}bg] \tag{5.6.1}$$

with the constraint $g^{-1}g_x \in [a, \mathcal{U}]$. It has a Lax pair

$$\theta_\lambda = (a\lambda + g^{-1}g_x)dx + \lambda^{-1}g^{-1}bgdt.$$

Equation (5.6.1) is hyperbolic, and the x-, t-curves are the characteristics. The *characteristic initial value problem* (or the *degenerate Goursat problem*) is the initial value problem with initial data defined on two characteristic axes, i.e., given $h_1, h_2 : \mathbb{R} \to U$ satisfying $h_1^{-1}(h_1)_x \in [a, \mathcal{U}]$ and $h_1(0) = h_2(0)$, solve

$$\begin{cases} (g^{-1}g_x)_t = [a, g^{-1}bg], \\ g(x, 0) = h_1(x), \quad g(0, t) = h_2(t). \end{cases} \tag{5.6.2}$$

If we write $u = g^{-1}g_x$, $v = g^{-1}bg$, then the -1-flow equation (5.6.1) becomes the following system for (u, v),

$$u_t = [a, v], \quad v_x = -[u, v], \tag{5.6.3}$$

The Lax pair is

$$\theta_\lambda = (a\lambda + u)dx + \lambda^{-1}vdt. \tag{5.6.4}$$

Let M_b denote the adjoint U-orbit in $\mathcal{U}$ at b. Since $u(x, 0) = h_1^{-1}h_1'(x)$ and $v(0, t) = h_2(t)^{-1}bh_2(t) \in M_b$, the characteristic initial value problem

(5.6.2) becomes the following initial value problem for (5.6.3): given $\xi : \mathbb{R} \to [a, \mathcal{U}]$ and $\eta : \mathbb{R} \to M_b$, find $(u, v) : \mathbb{R}^2 \to [a, \mathcal{U}] \times M_b$ so that

$$\begin{cases} u_t = [a, v], & v_x = -[u, v], \\ u(x, 0) = \xi(x), & v(0, t) = \eta(t). \end{cases} \tag{5.6.5}$$

In [28], Dorfmeister and Eitner use the Gauss loop group factorization to construct all local solutions of the Tzitzeica equation (2.6.4). Their construction in fact solves the characteristic initial value problem (5.6.5) for the -1 flow in the U-hierarchy:

Theorem 5.6.1. *([28]). Let $\xi, \eta : \mathbb{R} \to [a, \mathcal{U}] \times M_b$ be smooth maps, and $L_+(x, \lambda)$ and $L_-(t, \lambda)$ solutions of*

$$\begin{cases} (L_+)^{-1}(L_+)_x = a\lambda + \xi(x), \\ L_+(0, \lambda) = I, \end{cases} \qquad \begin{cases} (L_-)^{-1}(L_-)_t = \lambda^{-1}\eta(t), \\ L_-(0, \lambda) = I, \end{cases}$$

respectively. Factor

$$L_-^{-1}(t, \lambda)L_+(x, \lambda) = V_+(x, t, \lambda)V_-^{-1}(x, t, \lambda) \tag{5.6.6}$$

with $V_\pm(x, t, \cdot) \in L_\pm(G)$ via the Gauss loop group factorization. Set

$$\phi(x, t, \lambda) = L_-(t, \lambda)V_+(x, t, \lambda) = L_+(x, \lambda)V_-(x, t, \lambda).$$

Then $\phi^{-1}\phi_x = a\lambda + u(x, t)$ and $\phi^{-1}\phi_t = \lambda^{-1}v(x, t)$ for some u, v, and (u, v) solves the initial value problem of the -1-flow (5.6.5) in the U-hierarchy.

Proof. Differentiate $\phi = L_-V_+ = L_+V_-$ to get

$$\phi^{-1}\phi_x = V_-^{-1}(a\lambda + \xi(x))V_- + V_-^{-1}(V_-)_x = V_+^{-1}(V_+)_x.$$

So $\phi^{-1}\phi_x \in \mathcal{L}_+(\mathcal{G})$ and

$$\phi^{-1}\phi_x = \pi_+(V_-^{-1}(a\lambda + \xi(x))V_-),$$

where $\pi_\pm$ is the projection of $\mathcal{L}(\mathcal{G})$ onto $\mathcal{L}_\pm(\mathcal{G})$ with respect to the decomposition $\mathcal{L}(\mathcal{G}) = \mathcal{L}_+(\mathcal{G}) + \mathcal{L}_-(\mathcal{G})$. Expand

$$V_-(x, t, \lambda) = I + m_1(x, t)\lambda^{-1} + \cdots .$$

A direct computation shows that

$$\pi_+(V_-^{-1}(a\lambda + \xi(x))V_-) = a\lambda + \xi(x) + [a, m_1(x, t)].$$

Hence $\phi^{-1}\phi_x = a\lambda + u(x,t)$, where $u = \xi + [a, m_1]$. A similar argument implies that

$$\phi^{-1}\phi_t = \pi_-(V_+^{-1}\lambda^{-1}\eta(t)V_+) = \lambda^{-1}g_0(x,t)\eta(t)g_0^{-1}(x,t),$$

where $g_0(x,t)$ is the constant term in the expansion

$$V_+(x,t,\lambda) = \sum_{j=0}^{\infty} g_j(x,t)\lambda^j.$$

This implies that (u,v) is a solution of (5.6.3), where

$$u(x,t) = \xi(x) + [a, m_1(x,t)], \quad v(x,t) = g_0(x,t)\eta(t)g_0(x,t)^{-1}.$$

It remains to prove (u,v) satisfies the initial conditions. This can be seen from the factorization 5.6.6. Note that

$$L_-(0,\lambda)^{-1}L_+(x,\lambda) = V_+(x,0,\lambda)V_-(x,0,\lambda)^{-1}.$$

Since $L_-(0,\lambda) = I$, the right hand side lies in $L_+(G)$. Hence $V_-(x,0,\lambda) = I$, which proves that $m_1(x,0) = 0$. Therefore $u(x,0) = \xi(x)$. A similarl argument implies that $v(0,t) = \eta(t)$. Q.E.D.

They also show that every local solution of (5.6.3) can be constructed using suitable $\xi(x)$ and $\eta(t)$. To see this, let (u,v) be a solution of (5.6.3), and $\phi(x,t,\lambda)$ the trivialization of the corresponding Lax pair:

$$\phi^{-1}d\phi = (a\lambda + u)dx + \lambda^{-1}v dt, \quad \phi(0,0,\lambda) = I.$$

Use the Gauss loop group factorization to factor

$$\phi(x,t,\cdot) = L_-(x,t,\cdot)V_+(x,t,\cdot) = L_+(x,t,\cdot)V_-(x,t,\cdot).$$

Differentiate the above equation to get

$$\begin{aligned}
L_-^{-1}(L_-)_x &= \pi_-(V_+(a\lambda + u)V_+^{-1}) = 0,\\
L_-^{-1}(L_-)_t &= \pi_-(V_+\lambda^{-1}vV_+^{-1}) = \lambda^{-1}g_0vg_0^{-1},\\
L_+^{-1}(L_+)_x &= \pi_+(V_-(a\lambda + u)V_-^{-1}) = a\lambda + u + [m_1, a],\\
L_+^{-1}(L_+)_t &= \pi_-(V_-\lambda^{-1}vV_-^{-1}) = 0,
\end{aligned}$$

where g_0 is the constant term in the power series expansion of $V_+(x,t,\lambda)$ in λ and m_1 is the coefficient of λ^{-1} of V_-. This implies that $(L_-)_x = 0$, $(L_+)_t = 0$, $L_+^{-1}(L_+)_x = a\lambda + u(x,0)$ and $L_-^{-1}(L_-)_t = \lambda^{-1}v(0,t)$.

Let τ be the involution of G with the real form U as its fixed point set, and σ an order k automorphism of G such that $\sigma\tau = \tau^{-1}\sigma^{-1}$. Let $\mathcal{G}_j$ denote the eigenspace of σ_* of $\mathcal{G}$ with eigenvalue $e^{\frac{2\pi ij}{k}}$, and $\mathcal{U}_j = \mathcal{U} \cap \mathcal{G}_j$. Let $a \in \mathcal{G}_1$, $b \in \mathcal{G}_{-1}$ such that $[a, b] = 0$. The -1-flow in the U/U_0-hierarchy is the restriction of the -1-flow in the U-hierarchy (5.6.3) to the space of maps $(u, v) : \mathbb{R}^2 \to [a, \mathcal{U}_{-1}] \times \mathrm{Ad}(U_0)b$. It is easy to see that the solution constructed for initial data $\xi : \mathbb{R} \to [a, \mathcal{U}_{-1}]$ and $\eta : \mathbb{R} \to \mathrm{Ad}(U_0)(b)$ in Theorem 5.6.1 is a solution of the -1-flow in the U/U_0-hierarchy. In other words, the characteristic initial value problem for the -1 flow in the U/U_0-hierarchy can be solved by the algorithm given in Theorem 5.6.1.

§6. Elliptic systems associated to G, τ, σ

Let G be a complex Lie group, and τ a conjugate linear involution of $\mathcal{G}$, and σ an order k complex linear automorphism of $\mathcal{G}$ such that $\tau\sigma = \sigma\tau$. Let $\mathcal{G}_j$ be the eigenspace of σ with eigenvalue $e^{\frac{2\pi ji}{k}}$. So we have $\mathcal{G}_j = \mathcal{G}_m$ if $j \equiv m \pmod k$, and

$$\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_1 + \cdots + \mathcal{G}_{k-1}, \quad [\mathcal{G}_j, \mathcal{G}_r] \subset \mathcal{G}_{j+r}.$$

We claim that $\tau(\mathcal{G}_j) \subset \mathcal{G}_{-j}$. To see this, let $\sigma(\xi_j) = \alpha^j \xi_j$, where $\alpha = e^{\frac{2\pi i}{k}}$. Then

$$\sigma(\tau(\xi_j)) = \tau(\sigma(\xi_j)) = \tau(\alpha^j \xi_j) = \bar{\alpha}^j \tau(\xi_j) = \alpha^{-j} \tau(\xi_j).$$

Let $\mathcal{U}$ be the fixed point set of τ, and U_σ denote the fixed point set of σ on U. Since $\sigma\tau = \tau\sigma$, $\sigma(U) \subset U$ and $\sigma \mid U$ is an order k automorphism of U. The quotient space U/U_σ is called a *k-symmetric space.*

We will construct the sequences of U- and U/U_σ- systems.

6.1. The m-th (G, τ)-system

The m-th (G, τ)-system (also called the *m-th elliptic U-system*) is the following system for $(u_0, \cdots, u_m) : \mathbb{C} \to \prod_{i=0}^m \mathcal{G}$:

$$(6.1.1) \qquad \begin{cases} (u_j)_{\bar{z}} = \sum_{i=0}^{m-j} [u_{i+j}, \tau(u_i)], & \text{if } 1 \le j \le m, \\ (u_0)_{\bar{z}} - (\tau(u_0))_z = \sum_{i=0}^{m} [u_i, \tau(u_i)]. \end{cases}$$

It has a Lax pair:

$$(6.1.2) \qquad \theta_\lambda = \sum_{j=0}^{m} u_j \lambda^{-j} dz + \tau(u_j) \lambda^j d\bar{z}.$$

Equation (6.1.1) is also referred to as the *m-th elliptic U-system*, where U is the fixed point set of τ.

The Lax pair (6.1.2) satisfies the (G,τ)-*reality condition* (4.2.2), i.e.,

$$\tau(\theta_{1/\bar{\lambda}}) = \theta_\lambda.$$

Note that $\xi = \sum_j \xi_j \lambda^j$ satisfies the (G,τ)-reality condition if and only if $\xi_{-j} = \tau(\xi_j)$ for all j.

6.2. The m-th (G,τ,σ)-system

The m-th (G,τ,σ)*-system* is the following equation for $(u_0,\cdots,u_m)$: $\mathbb{C} \to \oplus_{j=0}^m \mathcal{G}_{-j}$,

$$\begin{cases} (u_j)_{\bar{z}} = \sum_{i=0}^{m-j} [u_{i+j}, \tau(u_i)], \text{ if } 1 \le j \le m, \\ -(u_0)_{\bar{z}} + (\tau(u_0))_z + \sum_{j=0}^m [u_j, \tau(u_j)] = 0. \end{cases} \tag{6.2.1}$$

It has a Lax pair

$$\theta_\lambda = \sum_{i=0}^m u_i \lambda^{-i} dz + \tau(u_i)\lambda^i d\bar{z}. \tag{6.2.2}$$

Note that:

(i) The m-th (G,τ,σ)-system is the restriction of the m-th (G,τ)-system to the space of maps $(u_0,\cdots,u_m)$ with values in $\oplus_{j=0}^m \mathcal{G}_{-j}$.

(ii) The Lax pair of the m-th (G,τ,σ)-system satisfies the (G,τ,σ)-*reality condition* 4.2.3, i.e.,

$$\tau(\theta_{1/\bar{\lambda}}) = \theta_\lambda, \quad \sigma(\theta_\lambda) = \theta_{e^{\frac{2\pi i}{k}}\lambda}.$$

(iii) $\xi(\lambda) = \sum_j \xi_j \lambda^j$ satisfies the (G,τ,σ)-reality condition if and only if $\xi_j \in \mathcal{G}_j$ and $\xi_{-j} = \tau(\xi_j)$ for all j.

Let U be the fixed point set of τ, and U_σ the fixed point set of σ on U. System (6.2.1) will also referred to as the *m-th elliptic* U/U_σ*-system*.

A direct computation gives the following Proposition:

Proposition 6.2.1. *Let τ be an involution, and σ an order k automorphism of G such that $\sigma\tau = \tau\sigma$, and $1 \le m < \frac{k}{2}$. If $\psi : \mathbb{C} \to U$ is a map such that*

$$\psi^{-1}\psi_z = u_o + \cdots + u_m \in \mathcal{G}_0 + \mathcal{G}_{-1} + \cdots + \mathcal{G}_{-m}, \tag{6.2.3}$$

then $(u_0, \cdots, u_m)$ *is a solution of the* m*-th* (G, τ, σ)*-system. Conversely, if*

$(u_0, \cdots, u_m)$ *is a solution of the* m*-th* (G, τ, σ)*-system* (6.2.1)*, then there exists* $\psi : \mathbb{C} \to U$ *such that* $\psi^{-1} d\psi = \sum_{j=0}^{m} u_j dz + \tau(u_j) d\bar{z}$.

Definition 6.2.2. Let τ, σ, k, and U be as in Proposition 6.2.1, and $1 \leq m < \frac{k}{2}$. A map $\psi : \mathbb{C} \to U$ is called a (σ, m)-map if $\psi^{-1}\psi_z \in \oplus_{j=1}^{m} \mathcal{G}_{-j}$.

Definition 6.2.3. ([15]). Let U/U_σ denote the k-symmetric space (with $k \geq 3$) given by τ, σ, and $\pi : U \to U/U_\sigma$ the natural projection. A map $\phi : \mathbb{C} \to U/U_0$ is called *primitive* if there is a lift $\psi : \mathbb{C} \to U$ (i.e., $\pi \circ \psi = \phi$) so that $\psi^{-1}\psi_z \in \mathcal{G}_0 + \mathcal{G}_{-1}$. In other words, there is a lift ψ that is a $(\sigma, 1)$-map.

By Proposition 6.2.1, the equation for (σ, m)-maps is the m-th (G, τ, σ)-system, and the equation for primitive maps is the first (G, τ, σ)-system. We refer the reader to [15] for a more detailed study of primitive maps.

Example 6.2.4. Let $\mathcal{G} = sl(3, \mathbb{C})$, $\tau(\xi) = -\bar{\xi}^t$, and $D = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Let $\sigma(\xi) = -D\xi^t D^{-1}$. Note that $D^{-1} = D^t$, $\sigma^2(\xi) = \mathrm{diag}(-1, -1, 1)\xi\mathrm{diag}(-1, -1, 1)$, σ has order 4, and $\sigma\tau = \tau\sigma$. A direct computation shows that the eigenspaces $\mathcal{G}_j$ with eigenvalues $(\sqrt{-1}\,)^j$ are:

$$\mathcal{G}_0 = \left\{ \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix} \,\middle|\, \xi \in sl(2, \mathbb{C}) \right\},$$
$$\mathcal{G}_1 = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ ib & -ia & 0 \end{pmatrix} \,\middle|\, a, b \in \mathbb{C} \right\},$$
$$\mathcal{G}_2 = \mathbb{C}\ \mathrm{diag}(1, 1, -2),$$
$$\mathcal{G}_3 = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -ib & ia & 0 \end{pmatrix} \,\middle|\, a, b \in \mathbb{C} \right\}.$$

The 2nd (G, σ, τ)-system (or the second elliptic $SU(3)/SU(2)$-system) is the system for $(u_0, u_1, u_2) : \mathbb{C} \to \mathcal{G}_0 \times \mathcal{G}_{-1} \times \mathcal{G}_{-2}$:

$$(6.2.4) \qquad \begin{cases} (u_2)_{\bar{z}} = 0, \\ -(u_1)_{\bar{z}} + [u_1, \tau(u_0)] + [u_2, \tau(u_1)] = 0, \\ -(u_0)_{\bar{z}} + (\tau(u_0))_z + [u_0, \tau(u_0)] + [u_1, \tau(u_1)] = 0. \end{cases}$$

Example 6.2.5. ([38, 51]). Let $\mathcal{G} = sl(n, \mathbb{C})$, $\tau(\xi) = -\bar{\xi}^t$, and $\sigma(\xi) = C\xi C^{-1}$, where $C = \text{diag}(1, \alpha, \cdots, \alpha^{n-1})$ and $\alpha = e^{\frac{2\pi i}{n}}$. Then $\tau\sigma = \sigma\tau$, and the eigenspace $\mathcal{G}_j$ of σ is spanned by $\{e_{i,i+j} \mid 1 \le i \le n\}$. Here we use the notation that $e_{ij} = e_{i'j'}$ if $i \equiv i'$ and $j \equiv j'$ (mod n). The first (G, τ, σ)-system is the equation for $A_0 = \text{diag}(u_1, \cdots, u_n)$ and $A_1 = \sum_{i=1}^n v_i e_{i,i-1}$ so that

$$\theta_\lambda = (A_0 + A_1\lambda^{-1})\, dz - (\bar{A}_0^t + \lambda \bar{A}_1^t)\, d\bar{z}$$

is flat for all λ. If $v_i > 0$ for all $1 \le i \le n$ and $v_1 \cdots v_n = 1$, then flatness of θ_λ implies that we can write $u_i = (w_i)_z$ and $v_i = e^{w_i - w_{i-1}}$ for some $w_1, \cdots, w_n$. The first (G, τ, σ)-system written in terms of w_i's is the *2-dimensional elliptic periodic Toda lattice*:

$$2(w_i)_{z\bar{z}} = e^{2(w_{i+1} - w_i)} - e^{2(w_i - w_{i-1})}.$$

6.3. The normalized system

The normalized m-th (G, τ)-system or the *normalized m-th U-system* is the system for $v_1, \cdots, v_m : \mathbb{R}^2 \to \mathcal{G}$:

$$(6.3.1) \qquad (v_j)_{\bar{z}} = \sum_{i=1}^{m-j} [v_{i+j}, \tau(v_i)] - \sum_{i=1}^{m} [v_j, \tau(v_i)], \quad 1 \le j < m.$$

It has a Lax pair

$$(6.3.2) \qquad \Theta_\lambda = \sum_{j=1}^{m} (\lambda^{-j} - 1) v_j\, dz + (\lambda^j - 1)\tau(v_j)\, d\bar{z},$$

which satisfies the (G, τ)-reality condition (4.2.2).

We claim that the Lax pair (6.3.2) is gauge equivalent to the Lax pair (6.1.2) for the m-th (G, τ)-system (6.1.1). Hence system (6.3.1) and (6.1.1) are gauge equivalent. To see this, let $(u_0, \cdots, u_m)$ be a solution of (6.1.1),

$$\theta_\lambda = \sum_{j=0}^{m} u_j \lambda^{-j} dz + \tau(u_j)\lambda^j d\bar{z}$$

its Lax pair, and $E(z, \bar{z})(\lambda)$ the frame of θ_λ, i.e.,

$$E^{-1}E_z = \sum_{j=0}^{m} u_j \lambda^{-j}, \quad E^{-1}E_{\bar{z}} = \sum_{j=0}^{m} \tau(u_j)\lambda^j, \quad E(0,0)(\lambda) = e.$$

Let $g = E(\cdot,\cdot,1)$. Since θ_λ satisfies the (G,τ)-reality condition, E satisfies

$$\tau(E(\cdot,\cdot,1/\bar{\lambda})) = E(\cdot,\cdot,\lambda).$$

Hence $E(\cdot,\cdot,\lambda) \in U$ if $\mid \lambda \mid = 1$. In particular, $g \in U$. The gauge transformation of θ_λ by g is

$$\tilde{\theta}_\lambda = g\theta_\lambda g^{-1} - dgg^{-1} = \sum_{j=1}^{m}(\lambda^{-j} - 1)gu_j g^{-1}dz + (\lambda^j - 1)g\tau(u_j)g^{-1}d\bar{z}$$

$$= \sum_{j=1}^{m}(\lambda^{-j} - 1)gu_j g^{-1}dz + (\lambda^j - 1)\tau(gu_j g^{-1})d\bar{z}.$$

So $(v_1, \cdots, v_m)$ is a solution of (6.3.1), where $v_i = gu_j g^{-1}$, and

$$F(z,\bar{z})(\lambda) = E(z,\bar{z})(\lambda)(E(z,\bar{z})(1))^{-1}$$

is the frame of $\tilde{\theta}_\lambda$.

§7. Geometries associated to integrable elliptic systems

Let τ be the involution of G whose fixed point set is the maximal compact subgroup U of G, and σ an order k automorphism of G such that $\sigma\tau = \tau\sigma$. Let $\mathcal{G}_j$ denote the eigenspace of σ_* on $\mathcal{G}$ with eigenvalue $e^{\frac{2\pi j i}{k}}$. Since $\sigma\tau = \tau\sigma$, we have $\sigma(U) \subset U$, and $\sigma|_U$ is an order k automorphism of U. Let U_σ denote the fixed point set of σ in U. The quotient U/U_σ is a symmetric space if $k = 2$, is a *k-symmetric space* if $k > 2$.

It is known that the first (G,τ)-system is the equation for harmonic maps from $\mathbb{C}$ to U ([72]). The first (G,τ,σ)-system is the equation for harmonic maps from $\mathbb{C}$ to the symmetric space U/U_σ if the order of σ is 2, and is the equation for primitive maps if $k > 2$ ([15]). It is also known that a primitive map $\phi : \mathbb{C} \to U/U_\sigma$ is harmonic if U/U_σ is equipped with a U-invariant metric and $\mathcal{G}_1$ is isotropic ([15]).

The first (G,τ,σ)-system also arises naturally in the study of surfaces in symmetric spaces with certain geometric properties. For example, constant mean curvature surfaces of simply connected 3-dimensional space forms $N^3(c)$ ([56, 6]), minimal surfaces in $\mathbb{C}P^2$ ([11, 9]), minimal Lagrangian surfaces in $\mathbb{C}P^2$ ([50, 52]), minimal Legendrian surfaces in S^5 ([64, 42]), and special Lagrangian cone in $\mathbb{R}^6 = \mathbb{C}^3$ ([42, 52]).

The only known surface geometry associated to the m-th (G, τ, σ)-system for $m > 1$ was given by Hélein and Roman. They showed that the equations for Hamiltonian stationary surfaces in 4-dimension Hermitian symmetric spaces are the second elliptic system associated to certain 4-symmetric spaces (cf. [43]).

If the equation for surfaces with special geometric properties is the m-th (G, τ, σ)-system, then the techniques developed for integrable systems can be applied to study the corresponding surfaces. In particular, the finite type (or finite gap) solutions give rise to tori with given geometric properties. This has been done for constant mean curvature tori of $N^3(c)$ in [56] and [6], for minimal tori of $\mathbb{C}P^2$ in [11, 9], and for minimal Legendrian tori of S^5 in [64, 52].

We will give a very brief review of some of the results mentioned above. For more details, we refer the readers to [38, 39] for harmonic maps, to [56, 6] for constant mean curvature surfaces in 3-dimensional space forms, to [11, 9] for minimal surfaces in $\mathbb{C}P^2$, and to [43] for Hamiltonian stationary surfaces in four dimensional Hermitian symmetric spaces.

7.1. Harmonic maps from $\mathbb{R}^2$ to U and the first (G, τ)-system

First we state some results of Uhlenbeck ([72]) on harmonic maps from $\mathbb{C}$ or S^2 to $U(n)$.

Theorem 7.1.1. *([72]). Let G be a complex semi-simple Lie group, $\mathcal{U}$ the real form defined by the conjugate linear involution τ, $s : \mathbb{C} \to U$ a smooth map, and $A = -\frac{1}{2}s^{-1}s_z$. Then the following statements are equivalent:*

(i) s is harmonic,

(ii) $A_{\bar{z}} = -[A, \tau(A)]$,

(iii) $(\lambda^{-1} - 1)A\ dz + (\lambda - 1)\tau(A)\ d\bar{z}$ is flat for all $\lambda \in \mathbb{C} \setminus \{0\}$, i.e., A is a solution of the normalized $1st$ (G, τ)-system.

Corollary 7.1.2. *([72]). Suppose*

$$\theta_\lambda = (\lambda^{-1} - 1)A(z, \bar{z})dz + (\lambda - 1)\tau(A(z, \bar{z}))d\bar{z}$$

is a flat $\mathcal{G}$-valued 1-form for all $\lambda \in \mathbb{C} \setminus 0$, and E_λ the corresponding frame (i.e., $E_\lambda^{-1}dE_\lambda = \theta_\lambda$ and $E_\lambda(0) = e$). Then E_{-1} is a harmonic map from $\mathbb{C}$ to U.

Using the ellipticity of the harmonic map equation, Uhlenbeck proved that there are trivializations of the Lax pair of harmonic maps from S^2 to $U(n)$ that are polynomials in the spectral parameter:

Theorem 7.1.3. *([72]). Let $s : S^2 \to U(n)$ be a harmonic map, and E the frame of the corresponding Lax pair $\theta_\lambda = -\frac{\lambda^{-1}-1}{2}\, s^{-1}s_z\, dz - \frac{\lambda-1}{2}\, s^{-1}s_{\bar z}\, d\bar z$. Then there exist $\gamma \in L_e(U)$ and smooth maps $\pi_i : S^2 \to Gr(k_i, \mathbb{C}^n)$ such that $\gamma(\lambda)E(\cdot,\cdot,\lambda) = (\pi_1 + \lambda\pi_1^\perp)\cdots(\pi_r + \lambda\pi_r^\perp)$.*

A harmonic map from a domain of $\mathbb{C}$ to $U(n)$ is called a *finite uniton* if the corresponding Lax pair admits a trivialization that is polynomial in λ ([72]). The above theorem implies that all harmonic maps from S^2 to $U(n)$ are *finite unitons.*

A proof similar to that of Proposition 3.9.4 gives

Proposition 7.1.4. *Let τ be the involution of G that defines the real form U, $(u_0, u_1) : \mathbb{C} \to \mathcal{G} \times \mathcal{G}$ a solution of the first (G,τ)-system (6.1.1), and $E_\lambda(z,\bar z)$ the frame of the corresponding Lax pair (6.1.2). Then $s = E_{-1}E_1^{-1}$ is a harmonic map from $\mathbb{C}$ to U. Moreover, let σ be an involution of $\mathcal{G}$ that commutes with τ, and $\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_1$ the eigenspace decomposition of σ. If $(u_0, u_1) \in \mathcal{G}_0 \times \mathcal{G}_1$, then $s = E_{-1}E_1^{-1}$ is a harmonic map from $\mathbb{C}$ to the symmetric space U/U_0.*

Corollary 7.1.5. *Let τ be a conjugate liner involution, and σ an order $k = 2m$ automorphism of $\mathcal{G}$ such that $\sigma\tau = \tau\sigma$. Let (u_1, u_0) be a solution of the first (G,τ,σ)-system (6.2.4), and E the frame of the corresponding Lax pair θ_λ (defined by (6.2.2)). Then $s = E(\cdot,\cdot,-1)E(\cdot,\cdot,1)^{-1}$ is a harmonic map from $\mathbb{C}$ to the symmetric space U/H, where H is the fixed point set of the involution σ^m on U.*

The following Theorem is proved by Burstall and Pedit.

Theorem 7.1.6. *([15]). Let $(u_0, u_1) : \mathbb{C} \to \mathcal{G}_0 \times \mathcal{G}_{-1}$ be a solution of the first (G,τ,σ)-system, U/U_σ the k-symmetric space corresponding to (τ,σ), and $k > 2$. Let $\pi : U \to U/U_\sigma$ be the natural fibration. If $E(z,\bar z,\lambda)$ is a trivialization of the Lax pair of the first (G,τ,σ)-system, then $\phi = \pi \circ E(\cdot,\cdot,1)$ is primitive. Moreover, if U/U_0 is equipped with an invariant metric and $\mathcal{G}_{-1}$ is isotropic, then ϕ is harmonic.*

7.2. The first (G,τ,σ)-system and surface geometry

Let $N^n(c)$ denote the simply connected space form of constant sectional curvature c, i.e., $N^n(0) = \mathbb{R}^n$, $N^n(1) = S^n$ the unit sphere in

$\mathbb{R}^{n+1}$, and $N^n(-1) = \mathbb{H}^n = \{x \in \mathbb{R}^{n,1} \mid \langle x, x\rangle = -1\} \subset \mathbb{R}^{n,1}$. The Gauss map ϕ of a k-dimensional submanifold M in $N^n(c)$ is the map from M to the symmetric space $Y(k,c)$, where

$$Y(n,0) = \mathrm{Gr}(k, \mathbb{R}^n), \quad Y(n,1) = \mathrm{Gr}(k, \mathbb{R}^{n+1}), \quad Y(n,-1) = \mathrm{Gr}(k, \mathbb{R}^{n,1}).$$

A theorem of Ruh and Vilms states that the Gauss map of a k-submanifold with parallel mean curvature vector in $N^n(c)$ is harmonic. Moreover, the Gauss-Codazzi equation for constant mean curvature (CMC) surfaces in $N^3(c)$ is the first (G,τ,σ)-system, where τ,σ are the involutions that define the symmetric space $Y(3,c)$. Since the equation for harmonic maps from $\mathbb{C}$ to $Y(3,c)$ defined by τ,σ is the first (G,τ,σ)-system, techniques developed for the first (G,τ,σ)-system (or harmonic maps) can be used to study CMC surfaces in $N^3(c)$ (cf. [56, 6]).

There are natural definitions of Gauss maps for surfaces and Lagrangian surfaces in $\mathbb{C}P^2$, for Legendrian surfaces in S^5, and Lagrangian cones in $\mathbb{R}^6 = \mathbb{C}^3$. The target manifolds of these Gauss maps are now k-symmetric spaces. The minimality of surfaces is equivalent to their Gauss maps being primitive. Hence equations of these surfaces are the corresponding first (G,τ,σ)-system.

Example 7.2.1. Minimal surfaces in $\mathbb{C}P^2$

Let $f: M \to \mathbb{C}P^2$ be an immersed surface, $L \to \mathbb{C}P^2$ the tautological complex line bundle, z a local conformal coordinate on M, and f_0 a local cross section of $f^*(L)$. Choose f_1, f_2 so that $(f_0, f_1, f_2) \in SU(3)$ and

$$\mathbb{C}f_0 + \mathbb{C}f_1 = \mathbb{C}f_0 + \mathbb{C}\frac{\partial f_0}{\partial z}.$$

The Gauss map of M is the map ϕ from M to the flag manifold $Fl(\mathbb{C}^3)$ of $\mathbb{C}^3$ defined by $\phi(f) =$ the flag $(\mathbb{C}f_0, \mathbb{C}f_0 + \mathbb{C}f_1, \mathbb{C}^3)$. Note that $Fl(\mathbb{C}^3) = SU(3)/T^2$ is a 3-symmetric space given by $\tau(g) = (g^t)^{-1}$ and $\sigma(g) = CgC^{-1}$, where $C = \mathrm{diag}(1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}})$. It is proved in [11, 9] that M is minimal in $\mathbb{C}P^2$ if and only if the Gauss map $\phi: \mathbb{C} \to SU(3)/T^2$ is primitive.

Example 7.2.2. Minimal Legendrian surfaces in S^5

Let $v_1 = (1,0,0)^t$, and $\mathbb{R}^3$ the real part of $\mathbb{C}^3$. Let Fl_1 denote the $SU(3)$-orbit of $(v_1, \mathbb{R}^3)$, i.e.,

$$\begin{aligned} Fl_1 &= \{(gv_1, g(\mathbb{R}^3) \mid g \in SU(3)\} \\ &= \{(v, V) \mid v \in S^5, v \in V, V \text{ is Lagrangian linear subspace of } \mathbb{C}^3\} \\ &= \frac{SU(3)}{1 \times SO(2)} \end{aligned}$$

Note that Fl_1 is a 6-symmetric space corresponding to automorphisms τ, σ of $SL(3,\mathbb{C})$ defined by $\tau(g) = (\bar{g}^t)^{-1}$ and $\sigma(g) = R(g^t)^{-1}R^{-1}$, where R is the rotation that fixes the x-axis and rotate $\frac{\pi}{3}$ in the yz-plane.

Let α be the standard contact form on S^5. A surface M in S^5 is *Legendrian* if the restriction of α to M is zero. It is easy to see that M is Legendrian if and only if the cone

$$C(M) = \{tx \mid t > 0, x \in M\}$$

is Lagrangian in $\mathbb{C}^3$. If $M \subset S^5$ is Legendrian, then there is a natural map ϕ from M to Fl_1 defined by $\phi(x) = (x, V(x))$, where $V(x)$ is the real linear subspace $\mathbb{R}x + TM_x$. It is known that (cf. [42, 52, 50, 64]) that the following statements are equivalent:

(i) M is minimal Legendrian in S^5,

(ii) the cone $C(M)$ is minimal Lagrangian in $\mathbb{R}^6 = \mathbb{C}^5$,

(iii) the Gauss map $\phi : M \to Fl_1$ is primitive.

Let $\pi : S^5 \to \mathbb{C}P^2$ be the Hopf fibration, N a surface in $\mathbb{C}P^2$, and $\tilde{N}$ a horizontal lift of N in S^5 with respect to the connection α (the contact form). Then N is minimal Lagrangian in $\mathbb{C}P^2$ if and only $\tilde{N}$ is minimal Legendrian in S^5. Hence there are three surface geometries associated to the first (G, τ, σ)-system associated to the 6-symmetric space Fl_1: minimal Lagrangian surfaces in $\mathbb{C}P^2$, minimal Legendrian surfaces in S^5, and minimal Lagrangian cones in $\mathbb{R}^6$.

Example 7.2.3. Hamiltonian stationary surfaces in $\mathbb{C}P^2$

Let N be a Kähler manifold. Given a smooth function f on N, let X_f denote the Hamiltonian vector field associated to f. A Lagrangian submanifold M is called *Hamiltonian stationary* if it is a critical point of the area functional A with respect to any Hamiltonian deformation, i.e.,

$$\left.\frac{\partial}{\partial t}\right|_{t=0} A(\phi_t(M)) = 0$$

for all f, where ϕ_t is the one-parameter subgroup generated by X_f. This class of submanifolds was studied by Schoen and Wolfson in [61]. When N is a four dimensional Hermitian symmetric space U/H, Hélein and Romon proved that the Gauss-Codazzi equation for Hamiltonian stationary surfaces is the 2nd (G, τ, σ)-system, where τ is the involution that gives U and σ is an order four automorphism such that σ^2 gives rise to the natural complex structure of U/H. In particular, they proved that if M is a Hamiltonian stationary Lagrangian surface of $\mathbb{C}P^2$, then locally the Gauss-Codazzi equation for M is the 2nd (G, τ, σ)-system (6.2.4) given by Example 6.2.4. Conversely, if (u_0, u_1, u_2) is a solution of (6.2.4), then for each non-zero $r \in \mathbb{R}$, $E_r E_{-r}^{-1}$ is a Hamiltonian stationary Lagrangian surface of $\mathbb{C}P^2$, where E_λ is the frame of the Lax pair (6.2.2) corresponding to (u_0, u_1, u_2).

§8. Symmetries of the (G, τ)-systems

There have been extensive studies on harmonic maps from a Riemann surface to a compact Lie group U. For example, there are loop group actions, finite unitons, finite type solutions, and a method of constructing all local harmonic maps from meromorphic data. The equation for harmonic maps from $\mathbb{C}$ to U is the first (G, τ)-system. Most results for the first (G, τ)-system hold for the m-th (G, τ)-system as well. We will give a brief review here. For more details, see [27, 38, 39, 72].

8.1. The action of $\Omega_-^\tau(G)$

Let $u = (u_0, \cdots, u_m) : \mathbb{C} \to \prod_{i=0}^m \mathcal{G}$ be a solution of the m-th (G, τ)-system (6.1.1), θ_λ the corresponding Lax pair (6.1.2), and E the frame of θ_λ, i.e.,

$$\begin{cases} E^{-1}E_z = \sum_{i=0}^m u_i \lambda^{-i}, \\ E^{-1}E_{\bar{z}} = \sum_{i=0}^m \tau(u_i)\lambda^i, \\ E(0,0,\lambda) = \mathrm{I}. \end{cases}$$

Let $E(z, \bar{z})(\lambda) = E(z, \bar{z}, \lambda)$. Since θ_λ satisfies the (G, τ)-reality condition,

$$\tau(E(z, \bar{z})(1/\bar{\lambda})) = E(z, \bar{z})(\lambda),$$

i.e., $E(z, \bar{z}) \in \Omega_+^\tau(G)$. Given $g \in \Omega_-^\tau(G)$, we can use Theorem 4.2.3 to factor $gE(z, \bar{z}) = \tilde{E}(z, \bar{z})\tilde{g}(z, \bar{z})$ with $\tilde{E}(z, \bar{z}) \in \Omega_+^\tau(G)$ and $\tilde{g}(z, \bar{z}) \in \Omega_-^\tau(G)$ for z in an open subset of the origin, i.e., $\tilde{E}(z, \bar{z}) = g * E(z, \bar{z})$

the dressing action. A direct computation gives

$$\tilde{E}^{-1}\tilde{E}_z = -\tilde{g}_z\tilde{g}^{-1} + \tilde{g}\left(\sum_{j=0}^{m} u_j\lambda^{-j}\right)\tilde{g}^{-1}. \tag{8.1.1}$$

Since $\tilde{g}(z,\bar{z})(\lambda)$ is holomorphic at $\lambda = 0$, the right hand side of (8.1.1) has a pole of order at most m at $\lambda = 0$. Hence there exist some $\tilde{u}_0, \cdots, \tilde{u}_m$ such that

$$\tilde{E}^{-1}\tilde{E}_z = \sum_{j=0}^{m} \tilde{u}_j\lambda^{-j}.$$

Since $\tilde{E}$ satisfies the (G,τ)-reality condition $\tau(\tilde{E}(1/\bar{\lambda})) = \tilde{E}(\lambda)$, $\tilde{E}^{-1}d\tilde{E}$ satisfies the (G,τ)-reality condition (4.2.2). Hence

$$\tilde{E}^{-1}d\tilde{E} = \sum_{j=0}^{m} \lambda^{-j}\tilde{u}_j dz + \lambda^j\tau(\tilde{u}_j)\, d\bar{z}.$$

In other words, $\tilde{u} = (\tilde{u}_0, \cdots, \tilde{u}_m)$ is a solution of the m-th (G,τ)-system. Moreover, $g * u = \tilde{u}$ defines an action of $\Omega_-^\tau(G)$ on the space of solutions of the m-th (G,τ)-system. This gives the following theorem of Uhlenbeck [72] (see also [38, 40]).

Theorem 8.1.1. *([38, 72]). Let E be the frame of a solution u of the m-th (G,τ)-system (6.1.1), and $g \in \Omega_-^\tau(G)$. Then the dressing action $\tilde{E}(z,\bar{z}) = g * E(z,\bar{z})$ is the frame of another solution $\tilde{u} = g * u$. Moreover, $(g,u) \mapsto g * u$ defines an action of $\Omega_-^\tau(G)$ on the space of solutions of the m-th (G,τ)-system.*

If $g \in \Omega_-^\tau(G)$ is a rational map with only simple poles, then the factorization of $gE(z,\bar{z}) = \tilde{E}(z,\bar{z})\tilde{g}(z,\bar{z})$ with $\tilde{E}(z,\bar{z}) \in \Omega_+^\tau(G)$ and $\tilde{g}(z,\bar{z}) \in \Omega_-^\tau(G)$ can be computed by an explicit formula in terms of g and E. In fact, if g has only one simple pole at $\alpha \in \mathbb{C} \setminus S^1$, then the factorization can be done by one of the following methods:

(i) Equate the residues of both sides of

$$g(\lambda)E(z,\bar{z},\lambda) = \tilde{E}(z,\bar{z},\lambda)\tilde{g}(z,\bar{z},\lambda)$$

at the pole $\lambda = \alpha$ to get an algebraic formula for $g * u$ in terms of g and E.

(ii) Let $\tilde{\theta}_\lambda = \tilde{E}^{-1}d\tilde{E}$. Equate the coefficient of λ^j in $\tilde{\theta}_\lambda\tilde{g} = d\tilde{g} + \tilde{g}\theta_\lambda$ for each j to get a system of compatible ordinary differential equations. Then $g*u$ can be obtained from the solution of this system of compatible ODEs.

Example 8.1.2. ([72]). Let $G = GL(n, \mathbb{C})$, and $\tau(g) = (\bar{g}^t)^{-1}$. The fixed point set U of τ is $U(n)$. Let V be a complex linear subspace of $\mathbb{C}^n$, π the Hermitian projection of $\mathbb{C}^n$ onto V, $\pi^\perp = \mathrm{I} - \pi$, $\alpha \in \mathbb{C} \setminus S^1$, and

$$f_{\alpha,\pi}(\lambda) = \pi + \zeta_\alpha(\lambda)\pi^\perp,$$

where $\zeta_\alpha(\lambda) = \frac{(\lambda-\alpha)(\bar{\alpha}-1)}{(\bar{\alpha}\lambda-1)(1-\alpha)}$. Note that $f_{\alpha,\pi}$ satisfies the (G,τ)-reality condition:

$$\overline{f(1/\bar{\lambda}))}^t f(\lambda) = \mathrm{I}.$$

If E is the frame of a solution u of the m-th (G,τ)-system (6.1.1), then for each $(z, \bar{z})$ the factorization $f_{\alpha,\pi} E(z, \bar{z})$ must be of the form

$$f_{\alpha,\pi} E(z, \bar{z}) = \tilde{E}(z, \bar{z}) \tilde{f}_{\alpha,\tilde{\pi}(z,\bar{z})} \tag{8.1.2}$$

for some $\tilde{E}(z, \bar{z}) \in \Omega_+^\tau(G)$ and projection $\tilde{\pi}(z, \bar{z})$. Method (i) leads to the conclusion that the image $\tilde{V}(z, \bar{z})$ of $\tilde{\pi}(z, \bar{z})$ is

$$\tilde{V}(z, \bar{z}) = \overline{(E(z, \bar{z})(\alpha))}^t (V). \tag{8.1.3}$$

Moreover,

$$f_{\alpha,\pi} * E = f_{\alpha,\pi} E f^{-1}_{\alpha,\tilde{\pi}(z,\bar{z})} = (\pi + \zeta_\alpha(\lambda)\pi^\perp)\; E\; (\tilde{\pi} + \zeta_\alpha(\lambda)^{-1}\tilde{\pi}^\perp) \tag{8.1.4}$$

is the frame of $f_{\alpha,\pi} * u$. For example, if $a \in \mathcal{U}$ is a constant, then a is a constant solution of the 1st normalized (G,τ)-system (6.1.1) with Lax pair $\theta_\lambda = a(\lambda^{-1}dz + \lambda d\bar{z})$ and frame $E_\lambda(z) = \exp(a\lambda^{-1}z + a\lambda\bar{z})$. The corresponding harmonic map is

$$s = E_{-1}(z)E_1^{-1}(z) = \exp(-2a(z + \bar{z})) = \exp(-4ax), \quad z = x + iy,$$

which is a geodesic. Since E is given explicitly for the constant solution, $f_{\alpha,\pi} * a$ is given explicitly and so is the harmonic map $f_{\alpha,\pi} * s$.

8.2. The DPW method and harmonic maps with finite uniton number

It is well-known that minimal surfaces in $\mathbb{R}^3$ have Weierstrass representations, i.e., they can be constructed from meromorphic functions. Dorfmeister, Pedit, and Wu gave a construction (the DPW method) of harmonic maps using meromorphic maps and the Iwasawa loop group factorization (Theorem 4.2.5). They call this construction of harmonic maps the *Weierstrass representation of harmonic maps*. The equation for harmonic maps from $\mathbb{C}$ to U is the first normalized (G, τ)-system. The DPW method works for the m-th normalized (G, τ)-system (6.3.1) as well. To explain the DPW method, we need the Iwasawa loop group factorization $L(G) = L_e(U) \times L_+(G)$, i.e., every $g \in L(G)$ can be factored uniquely as $g = g_1 g_2$ with $g_1 \in L_e(U)$ and $g_2 \in L_+(G)$. Let U denote the fixed point set of τ. Recall that $L_e(U)$ is the subgroup of $g \in L(U)$ such that $g(1) = e$ and $L_+(G)$ the space of smooth loops $g : S^1 \to G$ that are boundary value of holomorphic maps defined in $| \lambda | < 1$. The following Theorem was proved in [27] for the first (G, τ)-system (the harmonic map equation), but their proof works for the m-th (G, τ)-system as well.

Theorem 8.2.1. *([27]). Let $\mathcal{O}$ be a simply connected, open subset of $\mathbb{C}$, and $\mu(z, \lambda) = \sum_{j \geq -m} h_j(z) \lambda^j$ holomorphic in $z \in \mathcal{O}$ and smooth in $\lambda \in S^1$. Let $H : \mathcal{O} \times S^1 \to G$ be a solution of*

$$\begin{cases} H^{-1} H_z = \sum_{j \geq -m} h_j(z) \lambda^j, \\ H^{-1} H_{\bar{z}} = 0. \end{cases}$$

Then:

(i) H can be factored as $H(z, \lambda) = F(z, \bar{z}, \lambda) \phi(z, \bar{z}, \lambda)$ such that $F(z, \bar{z}, \cdot) \in L_e(U)$ and $\phi(z, \bar{z}, \cdot) \in L_+(G)$.

(ii) $F^{-1} F_z$ is of the form $\sum_{i=1}^m (\lambda^{-i} - 1) f_i$ and $f_\mu = (f_1, \cdots, f_m)$ is a solution of the normalized m-th (G, τ)-system.

(iii) Every solution of the m-th (G, τ)-system can be constructed from some μ.

Proof. We give a sketch of the proof (for more details see [27]). Statement (i) follows from the Iwasawa loop group factorization 4.2.5. The Iwasawa loop group factorization $L(G) = L_e(U) L_+(G)$ implies that there is a Lie algebra factorization

$$\mathcal{L}(\mathcal{G}) = \mathcal{L}_e(\mathcal{U}) + \mathcal{L}_+(\mathcal{G}). \tag{8.2.1}$$

In fact, we can use Fourier series to write down the Lie algebra factorization easily: Given $\xi = \sum_{j \in \mathbb{Z}} \xi_j \lambda^j$, then $\xi = \eta + \zeta$, where

$$\eta = \sum_{j=1}^{\infty} (\xi_{-j} (\lambda^{-j} - 1) + \tau(\xi_{-j})(\lambda^j - 1)) \in \mathcal{L}_e(\mathcal{U}),$$

$$\zeta = b_0 + \sum_{j=1}^{\infty} (\xi_j - \tau(\xi_{-j})) \lambda^j \in \mathcal{L}_+(\mathcal{G}).$$

Since $F = H\phi^{-1}$, we have $F^{-1}dF = \phi H^{-1}dH\phi^{-1} - (d\phi)\phi^{-1}$. Let p_1, p_2 denote the projection of $\mathcal{L}(\mathcal{G})$ onto $\mathcal{L}_e(\mathcal{U})$ and $\mathcal{L}_+(\mathcal{G})$ with respect to (8.2.1). Since $(d\phi)\phi^{-1} \in \mathcal{L}_+(\mathcal{G})$ and $F^{-1}dF \in \mathcal{L}(\mathcal{U})$, we have $F^{-1}dF = p_1(\phi(H^{-1}dH)\phi^{-1})$. It follows from the fact that $\phi^{-1}d\phi \in \mathcal{L}_+(\mathcal{G})$ and $H^{-1}dH = \sum_{j\geq -m} h_j(z)\lambda^{-j}dz$ that we have $F^{-1}dF = \sum_{j=0}^m f_j(\lambda^{-j} - 1)dz + \tau(f_j)(\lambda^j - 1)d\bar{z}$ for some $f_0, \cdots, f_m$. This proves (ii).

Let $u = (u_1, \cdots, u_m)$ be a solution of the m-th (G, τ)-system, and E a trivialization of the corresponding Lax pair. To prove (iii), it suffices to find $h(z, \lambda)$ so that $g = Eh^{-1}$ is holomorphic in $z \in \mathcal{O}$. Since we want

$$g^{-1}dg = h\left(\sum_{j=1}^{m}(\lambda^{-j} - 1)u_j dz + (\lambda^j - 1)\tau(u_j)d\bar{z}\right)h^{-1} - dhh^{-1}$$

has no $d\bar{z}$ term, we must solve for h from $h^{-1}h_{\bar{z}} = \sum_{j=1}^m(\lambda^j - 1)\tau(u_j)$. Since the right hand side lies in $\mathcal{L}_+(\mathcal{G})$, $h(z, \cdot)$ lies in $L_+(G)$. Hence $g^{-1}dg = \sum_{j=1}^m(\lambda^{-j} - 1)hu_jh^{-1}dz$. Hence g is holomorphic in z and $g^{-1}g_z$ is of the form $\sum_{j\geq -m} h_j(z)\lambda^j$. Q.E.D.

It is proved in [27] that finite type solutions arise from constant normalized potentials. We give a brief explanation next. Let $\xi \in L(\mathcal{G})$, and $H = \exp(z\xi(\lambda))$. So $H^{-1}H_z = \xi(\lambda)$, $H^{-1}H_{\bar{z}} = 0$, and $H(0, \lambda) = e$. Factor

$$\exp(z\xi) = F(x, y)\phi(x, y), \tag{8.2.2}$$

with $F(x, y) \in L_e(U), \phi(x, y) \in L(G)$, where $z = x + iy$. Then

$$H\xi H^{-1} = \exp(z\xi)\xi\exp(-z\xi) = \xi = F\phi\xi\phi^{-1}F^{-1}.$$

This implies that

$$F^{-1}\xi F = \phi\xi\phi^{-1}. \tag{8.2.3}$$

Differentiate (8.2.2) to get $\xi dz = \phi^{-1}F^{-1}dF\phi + \phi^{-1}d\phi$. We obtain $\phi\xi\phi^{-1}dz = F^{-1}dF + d\phi\phi^{-1}$. Hence $F^{-1}dF = p_1(\phi\xi\phi^{-1}(dx + i\ dy))$, where p_1 is the projection of $\mathcal{L}(\mathcal{G})$ to $\mathcal{L}_e(\mathcal{U})$. By (8.2.3), we get $F^{-1}dF = p_1(F^{-1}\xi F(dx + i\ dy))$. But $d(F^{-1}\xi F) = [F^{-1}\xi F, F^{-1}dF]$. So we have

$$d(F^{-1}\xi F) = [F^{-1}\xi F, p_1(F^{-1}\xi F(dx + i\ dy))].$$

Equivalently,

$$\begin{cases} (F^{-1}\xi F)_x = [F^{-1}\xi F, p_1(F^{-1}\xi F)], \\ (F^{-1}\xi F)_y = [F^{-1}\xi F, p_1(\sqrt{-1}\ F^{-1}\xi F)]. \end{cases} \tag{8.2.4}$$

Let $\xi(\lambda) = \lambda^{d-m}V(\lambda)$. Then (8.2.4) becomes

$$\text{(8.2.5)} \qquad \begin{cases} (F^{-1}VF)_x = [F^{-1}VF, p_1(\lambda^{d-m}F^{-1}VF)], \\ (F^{-1}VF)_y = [F^{-1}VF, p_1(\sqrt{-1}\ \lambda^{d-m}F^{-1}VF)]. \end{cases}$$

Let $\eta = F^{-1}VF$. Then 8.2.5 becomes

$$\text{(8.2.6)} \qquad \begin{cases} \eta_x = [\eta, p_1(\lambda^{d-m}\eta)], \\ \eta_y = [\eta, p_1(i\ \lambda^{d-m}\eta)]. \end{cases}$$

Note that this equation leaves the following finite dimensional submanifold of $\mathcal{L}(\mathcal{G})$ invariant:

$$\mathcal{V}_d = \left\{ \eta \in \mathcal{L}(\mathcal{G}) \middle| \ \eta(\lambda) = \sum_{|\ j\ | \leq d} \eta_j \lambda^j \right\}.$$

Hence given V in $\mathcal{V}_d$, we can solve the ODE system ((8.2.6)) to get $\eta(x,y)$ such that $\eta(0,0) = V$. System (8.2.6) is solvable if $p_1(\lambda^{d-m}\eta dz)$ is flat. So there exists $F(x,y) \in L_e(U)$ such that

$$F^{-1}dF = p_1(\lambda^{d-m}\eta dz),$$

i.e., F is a trivialization of the Lax pair of a solution of the normalized m-th (G,τ)-system. This is the method of constructing finite type solutions developed by Pinkall and Sterling in [56] and Burstall, Ferus, Pedit and Pinkall in [13].

All local solutions can also be constructed from meromorphic data μ that are polynomial in λ^{-1}. To explain this, we need

Theorem 8.2.2. *([27]). With the same notation as in Theorem 8.2.1, then there exists a discrete set $S \subset \mathcal{O}$ such that for $z \in \mathcal{O} \setminus S$, H can be factored as*

$$H(z,\lambda) = g_-(z,\lambda)g_+(z,\lambda)$$

with $g_-(z,\cdot) \in L_-^\tau(G)$ and $g_+(z,\cdot) \in L_+^\tau(G)$ via the Gauss loop group factorization. Moreover,

(i) $g_-(z,\lambda)$ is holomorphic in $z \in \mathcal{O} \setminus S$ and has poles at $z \in S$,

(ii) $g_-^{-1}dg_- = \sum_{j=1}^m \lambda^{-j}\eta_j(z)dz$ for some $\mathcal{G}$-valued meromorphic map η_j on $\mathcal{O}$.

Note that if we factor g_- via the Iwasawa loop group factorization (Theorem 4.2.5), then the $L_e(U)$ factor of g_- is the same E constructed in Theorem 8.2.1. This follows from $g_- = Hg_+^{-1} = E\phi g_+^{-1} = E(\phi g_+^{-1})$. The converse is also true. In fact, we have

Corollary 8.2.3. *Let $\mu(\lambda, z) = \sum_{j=1}^m \lambda^{-j}\eta_j(z)$ such that η_j are meromorphic. If there exists $h(z,\lambda)$ satisfying $h^{-1}dh/dz = \mu$, then the $L_e(U)$-factor $E(z,\cdot)$ of $h(z,\cdot)$ is a trivialization of some solution $f_\mu = (f_1, \cdots, f_m)$ of the m-th (G,τ)-system, i.e.,*

$$E^{-1}dE = \sum_{j=1}^{m}(\lambda^{-j} - 1)f_j(z)dz + (\lambda^j - 1)\tau(f_j)d\bar{z}.$$

Moreover, every local solution of the m-th (G,τ)-system can be constructed this way.

The 1-form $\mu(z,\lambda) = \sum_{j=1}^m \eta_j(z)\lambda^{-j}$ is called the *meromorphic potential* or the *normalized potential*. However, for a general normalized potential μ the solution f_μ might have singularities. An important problem is to identify meromorphic potentials μ so that the corresponding solution f_μ of the m-th (G,τ)-system can be extended to a complete surface. Burstall and Guest [14] have identified μ's that give rise to harmonic maps with finite uniton number. We explain some of their results next.

Burstall and Guest noted that if $\mu = \lambda^{-1}h(z)$ is nilpotent and $h(z)$ has no simple poles, then the equation

$$\begin{cases} H^{-1}H_z = \lambda^{-1}h(z), \\ H^{-1}H_{\bar{z}} = 0 \end{cases} \tag{8.2.7}$$

can be solved by integrations. We use $G = SL(n,\mathbb{C})$ to explain this. Let $\mathcal{N}$ denote the strictly upper triangular matrices in $sl(n,\mathbb{C})$, and $h : \mathcal{O} \to \mathcal{N}$ meromorphic. To solve (8.2.7), we may assume

$$H(z,\lambda) = I + b_1(z)\lambda^{-1} + b_2(z)\lambda^{-2} + \cdots$$

with meromorphic b_j's. Equate coefficients of λ^j to get

$$(b_1)_z = h, \ (b_2)_z = b_1h, \ (b_3)_z = b_2h, \ \cdots .$$

Since $\mathcal{N}^n = 0$, if we assume that $h(z)$ has no simple poles and the initial data $b_j(0) = 0$ then $b_1, \cdots, b_{n-1}$ can be solved by integration, $b_j = 0$ for all $j \geq n$, and $H(z,\lambda)$ is a polynomial of degree $\leq n-1$ in λ^{-1}.

Motivated by this computation and Uhlenbeck's finite uniton solutions, they make the following definition:

Definition 8.2.4. A harmonic map s from a Riemann surface M to U is said to have *finite uniton number* if there is a meromorphic $h : M \to \mathcal{G}$ such that (8.2.7) has a solution $H(z, \lambda)$ satisfying the following conditions:

(i) $H(z, \lambda)$ is meromorphic in $z \in M$ and a polynomial in λ and λ^{-1},

(ii) $s = E(\cdot, -1)$, where $E(z, \cdot)$ is the $L_e(U)$-component of the Iwasawa factorization of $H(z, \cdot)$.

In other words, s is the harmonic map constructed from the normalized potential $\mu = \lambda^{-1} h(z)$.

Theorem 8.2.5. *([14, 39]). If M is a Riemann surface and $s : M \to U$ is harmonic map with finite uniton number, then there exists a complex extended solution H (associated to s) of the form*

$$H(z, \lambda) = \exp(\lambda^{-1} b_1(z) + \cdots + \lambda^{-r} b_r(z)),$$

where $b_1, \cdots, b_r$ are meromorphic maps from M to the nilpotent subalgebra $\mathcal{N}$ of the Iwasawa decomposition $\mathcal{G} = \mathcal{K} + \mathcal{A} + \mathcal{N}$. Moreover,

(i) integer r can be computed in terms of root system of G,

(ii) the maps $b_2, \cdots, b_r$ satisfies a meromorphic ordinary differential equation, which can be solved by quadrature for any choice of b_1.

In fact, the normalized potential μ corresponding to the harmonic map constructed by Theorem 8.2.5 is $\mu = \lambda^{-1}(b_1)_z$.

8.3. Some comparisons

Let G be a complex, semi-simple Lie group, and U the maximal compact subgroup of G, and τ the corresponding involution with fixed point U. We have discussed the constructions of solutions of soliton equations in the U-hierarchy in Chapter 5 and of equations in the (G, τ)-hierarchy in section 8.2. Loop group factorizations are used in both cases. In this section, we give a summary and some comparisons of these constructions of solutions for the two hierarchies. To make the exposition easier to follow, we will not give references in this section (for references see the previous sections).

Let $\mathcal{A}$ be a maximal abelian subalgebra of $\mathcal{U}$, and $a \in \mathcal{A}$ a regular element. For the U-hierarchy defined by a, the data we use to construct solutions for the (b,j)-flow in the $\mathcal{U}$-hierarchy of soliton equations is one of the following types of maps:

(i) f is a holomorphic map from a neighborhood of $\lambda = \infty$ in $S^2 = \mathbb{C} \cup \{\infty\}$ to G that satisfies the U-reality condition, $\tau(f(\bar{\lambda})) = f(\lambda)$, and $f(\infty) = \mathrm{I}$,

(ii) $f : \mathbb{R} \to G$ is smooth, has an asymptotic expansion at ∞, $f(\infty) = \mathrm{I}$, f is the boundary value of a holomorphic map on the upper half plane, and f_b is rapidly decaying at infinity, where $f = f_u f_b$ is the pointwise Iwasawa factorization of $G = UB$, i.e., $f_u \in U$ and $f_b \in B$,

(iii) $f = f_1 f_2$, where $f_1 : S^2 = \mathbb{C} \cup \{\infty\} \to G$ is a rational map of type (i) and f_2 is of type (ii),

(iv) $f = f_1 f_2$, where f_1 is of type (i) and f_2 is of type (ii).

To construct solutions, we start with an f of type (i), (ii), (iii), or (iv), then factor $f^{-1} e_{a,1}(x) e_{b,j}(t)$ as $E(x,t) m(x,t)^{-1}$ with $E(x,t) \in \Lambda_+^\tau(G)$ and $m(x,t)$ of type (i), (ii), (iii) or (iv) accordingly, where $e_{\xi,j}(t) = e^{\xi \lambda^j t}$. Then

$$u^f(x,t) = [a, m_1(x,t)]$$

is a solution of the (b,j)-flow in the U-hierarchy, where $m_1(x,t)$ is the coefficient of λ^{-1} in the expansion of $m(x,t)(\lambda)$ at $\lambda = \infty$:

$$m(x,t,\lambda) \sim \mathrm{I} + m_1(x,t)\lambda^{-1} + \cdots .$$

Moreover, we know:

(1) $u^f = u^g$ if and only if $f = hg$ for some A-valued map h.

(2) u^f is a local real analytic solution if f is of type (i).

(3) If f is of type (iii), then $u^f(x,t)$ is a solution defined for all $(x,t) \in \mathbb{R}^2$ and is rapidly decaying in x for each fixed t. The space of such solutions u^f is open and dense in the space of all rapidly decaying solutions.

(4) If u is a finite gap solution (an algebraic geometric solution described by theta functions), then there exists an f of type (i) such that $f^{-1} a f$ is a polynomial in λ^{-1} and $u = u^f$.

(5) If f is of type (i) and is a rational map from S^2 to G, then u^f is a pure soliton solution.

For the normalized m-th (G,τ)-system, we start with meromorphic potential $\mu(z,\lambda) = \sum_{j=1}^m \eta_j(z) \lambda^{-j}\, dz$. There are two steps to construct a solution:

Step 1. Find a solution $H(z,\lambda)$ of $H^{-1}dH = \mu$ that is smooth for all $\lambda \in S^1$ and meromorphic in $z \in \mathcal{O} \subset \mathbb{C}$.

Step 2. Factor H as $F\phi$ with $F \in L_e(U)$ and $\phi \in L_+(G)$. Then $F^{-1}F_z$ is of the form $\sum_{j=1}^{m}(\lambda^{-j}-1)v_i$ for some $v_1,\cdots,v_m$. Hence

$$v_\mu = (v_1,\cdots,v_m), \qquad s_\mu = F(\cdot,-1)$$

are a solution of the normalized m-th (G,τ)-system and a harmonic map from $\mathcal{O}$ to U respectively.

For the first normalized (G,τ)-system, to go beyond solutions with finite uniton numbers we note that:

—There is no simple condition on μ to guarantee that Step 1 can be done.

—Every local smooth solution can be constructed from some μ. However, in general, there is no canonical choice of μ.

—One of the main open problems is to identify the set of μ so that s_μ can be extended to a harmonic map on a closed surface.

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