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A sharp bilinear restriction estimate for the sphere and its application to the wave-Schrödinger system

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Abstract.

We consider $2 \times 2 \to q$ type bilinear restriction estimates for transverse subsets of the sphere, for all $q > \frac{n+2}{n}$. Moreover, we give its application to the wave-Schrödinger system.

§1. Introduction and main results

In this paper, we consider bilinear restriction estimates for the sphere and its application to the wave-Schrödinger system in the three dimensions:

$$(1) \qquad \begin{cases} i\partial_t u + \Delta u &= vu, \\ \partial_t^2 v - \Delta v &= -|u|^2, \end{cases}$$

where u and v are complex and real-valued functions on $\mathbb{R}^3 \times \mathbb{R}$, respectively.

The Fourier restriction estimate has been studied by many mathematicians, since it is related to many other problems such as the Bochner-Riesz conjecture, the local smoothing conjecture for the wave equation and the Kakeya conjecture (see [5]). Also it has many applications to PDE. In particular, the author showed that the bilinear restriction estimate for the sphere plays an important role to improve the local and global well-posedness results of the Cauchy problem for (1) (see [1]). Thus, it is important to consider the bilinear restriction estimate for the sphere.

We denote the $n-1$ dimensional sphere by S^{n-1} and its induced Lebesgue measure by σ. Let S_1 and S_2 be any two subsets of S^{n-1} with boundary. Then we say that the "bilinear adjoint restriction estimate" (or bilinear extension estimate) $R^*_{S_1,S_2}(p \times p \to q)$ holds, if we have

$$\|(f_1 d\sigma)^\vee (f_2 d\sigma)^\vee\|_{L^q(\mathbb{R}^n)} \le C_{p,q,S_1,S_2} \|f_1\|_{L^p(S_1;d\sigma)} \|f_2\|_{L^p(S_2;d\sigma)}$$

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for all smooth functions f_1 and f_2 supported on S_1 and S_2 respectively, where $(f_j d\sigma)^\vee$ is the inverse Fourier transform of the measure $f_j d\sigma$.

Now let ε be a sufficiently small positive parameter and ρ_ε be a smooth cut-off function vanishing on $C\varepsilon$-neighborhood of $2S^{n-1} := \{\xi \in \mathbb{R}^n : |\xi| = 2\}$ and $C\sqrt{\varepsilon}$-neighborhood of the origin, where C is a universal large constant. Then our main result is the following:

Theorem 1.1. *Let $n \geq 3$, $\infty \geq q > \frac{n+2}{n}$ and set*

$$A = \max\left\{ \frac{1}{4} + (n+1)\left[\frac{1}{q} - \frac{n-1}{n+1}\right]_+, \frac{1}{2} \right\},$$

where $[\cdot]_+$ denotes the nonnegative part. Then we have

$$\|(f_1 d\sigma * f_2 d\sigma)\rho_\varepsilon\|_{L^{q'}(\mathbb{R}^n)} \lesssim \varepsilon^{-A} \|f_1\|_{L^2(S^{n-1};d\sigma)} \|f_2\|_{L^2(S^{n-1};d\sigma)} \tag{2}$$

for all $f_1, f_2 \in L^2(S^{n-1}; d\sigma)$, where the implicit constant depends only on n and q, and in the case where $q = \frac{4(n+1)}{4n-3}$ we have to modify the factor ε^{-A} to $\varepsilon^{1/2}|\log \varepsilon|$.

This type of estimate was first given by Bourgain for the cone [3] and has applications to PDE (see [4]). To prove the theorem, we need the nearly sharp estimates $R^*_{S_1,S_2}(2 \times 2 \to q)$, $q > \frac{n+2}{n}$, where "nearly sharp" means that the estimate fails, if $q < \frac{n+2}{n}$. In [6], Tao proves the nearly sharp bilinear restriction estimate for the paraboloid. His proof is applicable to hypersurfaces which are small perturbations of the paraboloid, in particular, small subsets of the sphere. In this paper, we consider the sphere directly and give an explicit dependence on the transversality (see Theorem 2.2 below).

Theorem 1.1 plays an important role to analyze a transverse interaction in bilinear estimates related to (1). Indeed, combining Theorem 1.1 with the result of [1], we have the following well-posedness result for (1).

Corollary 1.2. *The Cauchy problem for the wave-Schrödinger system (1) is locally well-posed for initial data $u_0 \in H^{s_1}$ and $(v_0, v_1) \in H^{s_2} \times H^{s_2-1}$, if $s_1 > -\frac{1}{8}$ and $s_2 > -\frac{1}{38}$. Moreover, if $s_1, s_2 > -\frac{45}{88642}$, then the global well-posedness holds.*

Throughout this paper, we use $N_r(S)$ to denote the r-neighborhood of a set S. Also we use $A \lesssim B$ and $O(A)$ to denote the estimate $|A| \leq CB$ and CA respectively, where C's are constants depending only on n and q. $A \sim B$ denotes the relation $B \lesssim A \lesssim B$. Moreover, for a large parameter R, $A \lessapprox B$ denotes the estimate $A \lesssim (\log R)^\nu B$ for all $\nu > 0$.

This paper is organized as follows. Section 2 is assigned for the proof of Theorem 1.1. In Section 3, we give an important tool, wave packet decomposition. In Section 4, we prove the crucial bilinear restriction estimate Theorem 2.2 via Proposition 4.1.

§2. Proof of Theorem 1.1

Let Γ_ε be a finitely overlapping covering of S^{n-1} by caps of size $\sqrt{\varepsilon}$, where a cap $\boldsymbol{\kappa}$ of center $\omega_0 \in S^{n-1}$ and size $0 < r \leq \pi$ is defined by $\boldsymbol{\kappa} = \{\omega \in S^{n-1} : \angle(\omega, \omega_0) \leq \frac{r}{2}\}$. Then we easily see that $\#\Gamma_\varepsilon \sim \varepsilon^{-\frac{n-1}{2}}$. The following proposition shows why ρ_ε is needed in the statement of Theorem 1.1.

Proposition 2.1. *Let $\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2 \in \Gamma_\varepsilon$.*
(i) If $\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 \subset N_{O(\varepsilon)}(2S^{n-1})$, then $\angle(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) \lesssim \sqrt{\varepsilon}$.
(ii) If $\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 \subset B_{O(\sqrt{\varepsilon})}(0)$, then $\pi - C\sqrt{\varepsilon} \leq \angle(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) \leq \pi$ for some large constant $C \gg 1$.

Proof of Proposition 2.1.
It is sufficient to prove the case $n = 2$. The proof is easy and so we omit the details. □

From Proposition 2.1, we see that ρ_ε yields the transversality in (2).

Now we prove Theorem 1.1. Take any $f_1, f_2 \in L^2(S^{n-1}; d\sigma)$. We decompose

$$f_j = \sum_{\boldsymbol{\kappa}_j \in \Gamma_\varepsilon} f_{\boldsymbol{\kappa}_j}, \tag{3}$$

where $\mathrm{supp} f_{\boldsymbol{\kappa}_j} \subset \boldsymbol{\kappa}_j$ $(j = 1, 2)$. Then, by the Hausdorff-Young inequality,

$$\begin{aligned} &\|(f_1 d\sigma * f_2 d\sigma)\rho_\varepsilon\|_{L^{q'}(\mathbb{R}^n)} \\ &\lesssim \sum_{\substack{(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) \in \Gamma_\varepsilon \times \Gamma_\varepsilon \\ C\sqrt{\varepsilon} \leq \angle(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) \leq \pi - C\sqrt{\varepsilon}}} \|(f_{\boldsymbol{\kappa}_1} d\sigma)^\vee (f_{\boldsymbol{\kappa}_2} d\sigma)^\vee\|_{L^q(\mathbb{R}^n)}, \end{aligned} \tag{4}$$

where $C \gg 1$ is some constant, in particular we may take $C = 10$. To analyze the effect of angular separation, we estimate the R.H.S. of (4) by

$$\sum_{\substack{\theta \in \sqrt{\varepsilon}\mathbb{N} \\ C\sqrt{\varepsilon} \leq \theta \leq \pi - C\sqrt{\varepsilon}}} \sum_{\substack{(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) \in \Gamma_\varepsilon \times \Gamma_\varepsilon \\ \theta \leq \angle(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) < \theta + \sqrt{\varepsilon}}} \|(f_{\boldsymbol{\kappa}_1} d\sigma)^\vee (f_{\boldsymbol{\kappa}_2} d\sigma)^\vee\|_{L^q(\mathbb{R}^n)}. \tag{5}$$

To estimate (5), we use the following bilinear restriction estimate:

Theorem 2.2. *Let* $\frac{n+2}{n} < q \leq \infty$, $10\sqrt{\varepsilon} \leq \theta \leq \pi - 10\sqrt{\varepsilon}$ *and let* $\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2 \in \Gamma_\varepsilon$ *with* $\theta \leq \angle(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) < \theta + \sqrt{\varepsilon}$. *Then the bilinear adjoint restriction estimate* $R^*_{\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2}(2 \times 2 \to q)$ *holds with constant*

$$C_{q,n} \min\left\{\frac{1}{\sqrt{\varepsilon}(\sin\theta)^2}, \frac{1}{(\sin\theta)^4}\right\}^{(n+1)\left[\frac{1}{q} - \frac{n-1}{n+1}\right]_+},$$

where $C_{q,n}$ *depends only on* q *and* n.

We remark that Theorem 2.2 is nearly sharp in the sense that $R^*_{\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2}(2 \times 2 \to q)$ fails for $q < \frac{n+2}{n}$, the endpoint case $q = \frac{n+2}{n}$ is still open.

Applying Theorem 2.2, we estimate (5) by

$$\begin{aligned} &\sum_{\substack{\theta \in \sqrt{\varepsilon}\mathbb{N} \\ C\sqrt{\varepsilon} \leq \theta \leq \pi - C\sqrt{\varepsilon}}} \min\left\{\frac{1}{\sqrt{\varepsilon}(\sin\theta)^2}, \frac{1}{(\sin\theta)^4}\right\}^{(n+1)\left[\frac{1}{q} - \frac{n-1}{n+1}\right]_+} \\ &\times \sum_{\substack{(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) \in \Gamma_\varepsilon \times \Gamma_\varepsilon \\ \theta \leq \angle(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) < \theta + \sqrt{\varepsilon}}} \|f_{\boldsymbol{\kappa}_1}\|_{L^2(S^{n-1}; d\sigma)} \|f_{\boldsymbol{\kappa}_2}\|_{L^2(S^{n-1}; d\sigma)}. \end{aligned} \tag{6}$$

Since for given $\boldsymbol{\kappa}_1 \in \Gamma_\varepsilon$ there are at most $O(1)$ caps $\boldsymbol{\kappa}_2 \in \Gamma_\varepsilon$ such that $\theta \leq \angle(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) < \theta + \sqrt{\varepsilon}$, by the Schwarz inequality and the finitely overlapping property of Γ_ε, (6) is estimated by

$$\begin{aligned} &\sum_{\substack{\theta \in \sqrt{\varepsilon}\mathbb{N} \\ C\sqrt{\varepsilon} \leq \theta \leq \pi - C\sqrt{\varepsilon}}} \min\left\{\frac{1}{\sqrt{\varepsilon}(\sin\theta)^2}, \frac{1}{(\sin\theta)^4}\right\}^{(n+1)\left[\frac{1}{q} - \frac{n-1}{n+1}\right]_+} \\ &\times \|f_1\|_{L^2(S^{n-1}; d\sigma)} \|f_2\|_{L^2(S^{n-1}; d\sigma)}. \end{aligned} \tag{7}$$

Set $q_n = (n+1)\left[\frac{1}{q} - \frac{n-1}{n+1}\right]_+$. When $n \leq 5$, we see that (7) is estimated by

$$\begin{cases} \varepsilon^{-q_n - 1/4}\|f_1\|_{L^2}\|f_2\|_{L^2} & \text{if } \frac{n+2}{n} < q < \frac{4(n+1)}{4n-3}, \\ \varepsilon^{-1/2}|\log\varepsilon| \|f_1\|_{L^2}\|f_2\|_{L^2} & \text{if } q = \frac{4(n+1)}{4n-3}, \\ \varepsilon^{-1/2}\|f_1\|_{L^2}\|f_2\|_{L^2} & \text{if } q > \frac{4(n+1)}{4n-3}. \end{cases}$$

On the other hand, when $n \geq 6$, (7) is estimated by $\varepsilon^{-1/2}\|f_1\|_{L^2}\|f_2\|_{L^2}$ for all $q > \frac{n+2}{n}$. Hence we have completed the proof of Theorem 1.1. □

Thus it remains to prove Theorem 2.2. We give the proof in the next sections.

§3. Wave packet decomposition

Let R be a positive number with $\varepsilon \gg 1/R$ and let $\boldsymbol{\kappa} \in \Gamma_\varepsilon$. In this section, we expand the Fourier transform of a smooth function $F_{\boldsymbol{\kappa}}$ supported on $N_{\frac{1}{R}}(\boldsymbol{\kappa})$ by wave packets adapted to tiles of width $\sqrt{R}$ and length R. Let $\{e^{(l)}\}_{l=1}^n$ be the standard basis of $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, we set $x' = \pi_{e^{(n)}}^{\perp}(x)$ and $x'' = \pi_{e^{(n)}}(x)$, where $\pi_{e^{(n)}}^{\perp}$ and $\pi_{e^{(n)}}$ are the projections onto the plane perpendicular to $e^{(n)}$ and the axis $e^{(n)}$, respectively. Thus we have $x = (x', x'') \in \mathbb{R}^{n-1} \times \mathbb{R}$.

Let $\Gamma_R(\boldsymbol{\kappa})$ be a finitely overlapping covering of $\boldsymbol{\kappa}$ by caps κ of size $1/\sqrt{R}$, center $\omega(\kappa)$ and $\angle(\omega(\kappa_1), \omega(\kappa_2)) \sim 1/\sqrt{R}$ for all $\kappa_1, \kappa_2 \in \Gamma_R(\boldsymbol{\kappa})$. Thus we have

$$N_{\frac{1}{R}}(\boldsymbol{\kappa}) \subset \bigcup_{\kappa \in \Gamma_R(\boldsymbol{\kappa})} N_{\frac{1}{R}}(\kappa).$$

3.1. ω-coordinate and ω-tiles

Let $\omega \in S^{n-1}$ and let L_ω be a rotation such that $L_\omega \omega = e^{(n)} := (0, \cdots, 1)$. Then we define the unit vectors $\{e_\omega^{(l)}\}_{l=1}^n$ by $e_\omega^{(l)} := L_\omega^{-1} e^{(l)}$ $(l = 1, \cdots, n)$. In particular, $e_\omega^{(n)} = \omega$. We call $\{e_\omega^{(l)}\}$ ω-coordinate system. Set $x_\omega = L_\omega x$ for $x \in \mathbb{R}^n$.

Next we introduce ω-tiles. For $\omega \in S^{n-1}$, we define the fundamental tile $T_\omega(0)$ as the $\overbrace{\sqrt{R} \times \ldots \times \sqrt{R}}^{n-1 \text{ times}} \times R$-rectangle centered at the origin, the long side is in the direction ω. Set $X = \sqrt{R}\mathbb{Z}^{n-1}$ and $Y = R\mathbb{Z}$. Then for $\tilde{a} \in L_\omega^{-1}(X \times Y)$, we set $T_\omega(\tilde{a}) = T_\omega(0) + \tilde{a}$ and call ω-tile of center $\tilde{a}$. We denote the set of ω-tiles by $\mathbb{T}(\omega)$, namely $\mathbb{T}(\omega) := \{T_\omega(0) + \tilde{a} : \tilde{a} \in L_\omega^{-1}(X \times Y)\}$.

For a $\overbrace{\sqrt{R} \times \ldots \times \sqrt{R}}^{n-1 \text{ times}} \times R$-rectangle T, we denote the long side direction by $\omega(T)$, and call the direction of T. Also we denote the center of T by $\tilde{a}(T)$. We will often use $a(T)$ to denote $L_{\omega(T)}\tilde{a}(T)$.

3.2. Wave packet decomposition for the sphere

Following Tao's idea [6], we decompose functions into wave packets. Let $\boldsymbol{\kappa} \in \Gamma_\varepsilon$ and let F be a smooth function supported on $N_{\frac{1}{R}}(\boldsymbol{\kappa})$. We decompose

$$F = \sum_{\kappa \in \Gamma_R(\boldsymbol{\kappa})} F_\kappa, \tag{8}$$

where F_κ is smooth and $\operatorname{supp} F_\kappa \subset N_{\frac{1}{R}}(\kappa)$. Now we decompose $F_\kappa^\vee$ into wave packets adapted to $\omega(\kappa)$-tiles. We note that

$$F_\kappa^\vee(x) = (F \circ L_{\omega(\kappa)}^{-1})^\vee(x_{\omega(\kappa)}).$$

We take a Schwartz function η' on $\mathbb{R}^{n-1}$ such that $\operatorname{supp}\eta' \subset B_{1/(10\pi)}(0)$ and $\sum_{k\in\mathbb{Z}^{n-1}} \eta'(x'-k) = 1$ for all $x' \in \mathbb{R}^{n-1}$ (cf. [6]). Moreover, we take a Schwartz function η'' on $\mathbb{R}$ such that $\operatorname{supp}\eta'' \subset [-1/10\pi, 1/10\pi]$ and $\sum_{k\in\mathbb{Z}^{n-1}} \eta'(x''-k) = 1$ for all $x'' \in \mathbb{R}^{n-1}$. Then we decompose

$$\begin{aligned} F_\kappa^\vee(x) &= (F_\kappa \circ L_{\omega(\kappa)}^{-1})^\vee(x_{\omega(\kappa)}) \\ &= \sum_{a=(a',a'')\in X\times Y} \eta''\Big(\frac{x''_{\omega(\kappa)} - a''}{R}\Big)\eta'\Big(\frac{x'_{\omega(\kappa)} - a'}{\sqrt{R}}\Big) \\ &\qquad\qquad \times (F_\kappa \circ L_{\omega(\kappa)}^{-1})^\vee(x_{\omega(\kappa)}), \end{aligned}$$

where $x_{\omega(\kappa)} = (x'_{\omega(\kappa)}, x''_{\omega(\kappa)}) \in \mathbb{R}^{n-1}\times\mathbb{R}$. For $\omega \in S^{n-1}$, we can associate $a = (a', a'') \in X \times Y$ with the ω-tile with center $\tilde{a} := L_{\omega(\kappa)}a$ and thus we set

$$c_{T_{\omega(\kappa)}(\tilde{a})} = R^{\frac{n+1}{4}} M' \otimes M''[(F_\kappa \circ L_{\omega(\kappa)}^{-1})^\vee](a', a''),$$

where $M' \otimes M''$ is the tensor product of the Hardy-Littlewood maximal operators on $\mathbb{R}^{n-1}$ and $\mathbb{R}$, i.e.,

$$\begin{aligned} &M' \otimes M''[u](x', x'') \\ &:= \sup_{r''>0} \frac{1}{|D''_{r''}(x'')|} \int_{D''_{r''}(x'')} \left(\sup_{r>0} \frac{1}{|D'_{r'}(x')|} \int_{D'_{r'}(x')} |u(y', y'')|\, dy' \right) dy''. \end{aligned}$$

Here $D'_{r'}(x')$ (resp. $D''_{r''}(x'')$) is the $n-1$ dimensional ball (resp. interval in $\mathbb{R}$) of center x' (resp. x'') and radius r' (resp. length $2r''$), i.e., $D'_{r'}(x') := \{y' \in \mathbb{R}^{n-1} : |y' - x'| \le r'\}$ (resp. $D''_{r''}(x'') := [x'' - r'', x'' + r'']$). We set

$$\begin{aligned} &\varphi_{T_{\omega(\kappa)}(\tilde{a})}(x_{\omega(\kappa)}) \\ &= \frac{1}{c_{T_{\omega(\kappa)}(\tilde{a})}} \eta''\Big(\frac{x''_{\omega(\kappa)} - a''}{R}\Big)\eta'\Big(\frac{x'_{\omega(\kappa)} - a'}{\sqrt{R}}\Big)(F_\kappa \circ L_{\omega(\kappa)}^{-1})^\vee(x_{\omega(\kappa)}). \end{aligned}$$

Moreover we set $\phi_{T_{\omega(\kappa)}(\tilde{a})} = \varphi_{T_{\omega(\kappa)}(\tilde{a})} \circ L_{\omega(\kappa)}$ and thus $\phi_{T_{\omega(\kappa)}(\tilde{a})}(x) = \varphi_{T_{\omega(\kappa)}(\tilde{a})}(x_{\omega(\kappa)})$ for all $x \in \mathbb{R}^n$. Then we represent

$$F_\kappa^\vee(x) = \sum_{T\in\mathbb{T}(\omega(\kappa))} c_T \phi_T(x). \tag{9}$$

Combining (8) and (9), we obtain a decomposition of $F^\vee$:

$$F^\vee = \sum_{T\in\mathbb{T}_{\boldsymbol{\kappa}}} c_T\phi_T(x), \tag{10}$$

where $\mathbb{T}_{\boldsymbol{\kappa}} := \cup_{\kappa\in\Gamma_R(\boldsymbol{\kappa})}\mathbb{T}(\omega(\kappa))$. We call $\mathbb{T}_{\boldsymbol{\kappa}}$ the tiles associated to $\boldsymbol{\kappa}$.

We give properties of the decomposition (10). In particular, we find that ϕ_T is a wave packet concentrated on a tile T.

Proposition 3.1. *(i) The coefficients $\{c_T\}$ in (10) obey the bound*

$$\left(\sum_{T\in\mathbb{T}_{\boldsymbol{\kappa}}}|c_T|^2\right)^{\frac{1}{2}} \lesssim \|F\|_{L^2(\mathbb{R}^n)}.$$

(ii) For $T\in\mathbb{T}_{\boldsymbol{\kappa}}$, let $\kappa(T)$ be the cap in $\Gamma_R(\boldsymbol{\kappa})$ with the center $\omega(T)$. Then we have

$$supp\,\widehat{\phi_T} \subset L^{-1}_{\omega(T)}(L_{\omega(T)}N_{1/R}(\kappa(T)) + D'_{1/(10\pi\sqrt{R})}(0)\times D''_{1/(10\pi R)}(0)).$$

In particular, taking R sufficiently large, we find that $supp\,\widehat{\phi_T}$ is contained in the $2/R$-neighborhood of $\kappa(T)$.

(iii) For any $T\in\mathbb{T}_{\boldsymbol{\kappa}}$ and any $N, M\geq 0$, we have

$$|\phi_T(x)| \leq C(M,N)R^{-\frac{n+1}{4}}\left\langle\frac{d^\perp_{\omega(T)}(x,T)}{\sqrt{R}}\right\rangle^{-N}\left\langle\frac{d_{\omega(T)}(x,T)}{R}\right\rangle^{-M},$$

where $d^\perp_\omega(x,y) := dist(\pi^\perp_\omega(x),\pi^\perp_\omega(y))$ and $d_\omega(x,y) := dist(\pi_\omega(x),\pi_\omega(y))$.

(iv) Let $\mathbb{T}'_{\boldsymbol{\kappa}}$ be any subset of $\mathbb{T}_{\boldsymbol{\kappa}}$ with $\#\mathbb{T}'_{\boldsymbol{\kappa}} < \infty$. Then we have

$$\Big\|\sum_{T\in\mathbb{T}'_{\boldsymbol{\kappa}}}\phi_T\Big\|_{L^2(\mathbb{R}^n)} \lesssim (\#\mathbb{T}'_{\boldsymbol{\kappa}})^{\frac{1}{2}}.$$

Proof of Proposition 3.1.
The proof is similar to that of Lemma 4.1 in [6] and therefore we omit the proof. □

§4. Proof of Theorem 2.2

Let $\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2 \in \Gamma_\varepsilon$ with $\theta \leq \angle(\boldsymbol{\kappa}_1,\boldsymbol{\kappa}_2) < \theta + \sqrt{\varepsilon}$. Take q bigger than and arbitrarily close to $\frac{n+2}{n}$. For Theorem 2.2, by Lemma 2.4 of [8], Proposition 4.3 of [9] and the interpolation with the Tomas-Stein restriction estimate, it suffices to prove the following localized and $1/R$-spread bilinear adjoint restriction estimate:

$$\begin{aligned} &\|F^\vee_{\kappa_1} F^\vee_{\kappa_2}\|_{L^q(B_R(0))} \\ &\lesssim \min\left\{\frac{1}{\sqrt{\varepsilon}(\sin\theta)^2}, \frac{1}{(\sin\theta)^4}\right\}^{(1-\nu)(1-\frac{1}{q})} R^{\alpha-1} \end{aligned} \tag{11}$$

for all $R \geq 1$, $\alpha > 0$, $0 < \nu \ll 1$ and all smooth L^2-normalized functions F_{κ_1} and F_{κ_2} supported on $N_{1/R}(\kappa_1)$ and $N_{1/R}(\kappa_2)$, respectively, where the implicit constant depends only on n, q and α.

We use $R^*_{\kappa_1,\kappa_2}(2 \times 2 \to q, \alpha)$ to denotes the statement that the estimate (11) holds for all $R \geq 1$, $1 \gg \nu > 0$ and some $\alpha > 0$.

Our proof of (11) heavily depends on the idea of Tao [6] and Wolff [10]. The salient point in our proof is the geometric observation for the sphere, see Sections 4.6.3 and 4.6.5.

In view of Proposition 3.2 of [6], for (11) it suffices to prove the following inductive statement.

Proposition 4.1. *Suppose $\alpha > 0$ such that $R^*_{\kappa_1,\kappa_2}(2 \times 2 \to q, \alpha)$ holds. Then we have*

$$\begin{aligned} &\|F^\vee_{\kappa_1} F^\vee_{\kappa_2}\|_{L^q(B_R(0))} \\ &\leq C_{q,\alpha} \min\left\{\frac{1}{\sqrt{\varepsilon}(\sin\theta)^2}, \frac{1}{(\sin\theta)^4}\right\}^{(1-\nu)(1-\frac{1}{q})} \\ &\quad \times R^{\max\{(1-\delta)\alpha, C\delta\}+C\nu} R^{-1} \end{aligned} \tag{12}$$

for all $0 < \delta, \nu \ll 1$, where C's are constants independent of α, δ and ν, and $C_{q,\alpha}$ is some constant depending only on q, n and α.

Now we prove Proposition 4.1. By the wave packet decomposition at the scale R, the L.H.S. of (12) is rewritten as follows:

$$\Big\|\sum_{T_1 \in \mathbb{T}_{\kappa_1}} \sum_{T_2 \in \mathbb{T}_{\kappa_2}} c_{T_1}\phi_{T_1} c_{T_2}\phi_{T_2}\Big\|_{L^q(B_R(0))}, \tag{13}$$

where $\mathbb{T}_{\kappa_j}$ $(j = 1, 2)$ denote the tiles associated to κ_j. Let Ω_j be the set of directions of $\mathbb{T}_{\kappa_j}$, i.e. the set of centers of κ_j in $\Gamma_R(\kappa_j)$ $(j = 1, 2)$. Then we have $\#\Omega_j \lesssim (\varepsilon R)^{\frac{n-1}{2}}$ $(j = 1, 2)$. Now for each $\omega_j \in \Omega_j$ and $a' \in X := \sqrt{R}\mathbb{Z}^{n-1}$, we define the infinitely long stick of width $\sqrt{R}$ by

$$S_{\omega_j}(\tilde{a}') = \{T \in \mathbb{T}_{\kappa_j}(\omega_j) \,:\, a(T)' = a'\},$$

where $\mathbb{T}_{\boldsymbol{\kappa}_j}(\omega_j)$ is the set of ω_j-tiles (see the end of Section 3.1 for the notation $a(T)$). We set $\mathbb{S}(\omega_j) = \{S_{\omega_j}(\widetilde{a}')\}_{a' \in X}$ and $\mathbb{S}_{\boldsymbol{\kappa}_j} = \bigcup_{\omega_j \in \Omega_j} \mathbb{S}(\omega_j)$.

We denote an element of $\mathbb{S}_{\boldsymbol{\kappa}_j}$ by S_j and rewrite (13) in terms of sticks:

$$(14) \qquad \| \sum_{S_1 \in \mathbb{S}_{\boldsymbol{\kappa}_1}} \sum_{S_2 \in \mathbb{S}_{\boldsymbol{\kappa}_2}} \sum_{T_1 \in S_1} \sum_{T_2 \in S_2} c_{T_1} \phi_{T_1} c_{T_2} \phi_{T_2} \|_{L^q(B_R(0))}.$$

In cases of L^2-norm, we exploit almost orthogonality of wave packets. We have the following lemma, which plays an important role to estimate (14) (cf. [6]):

Lemma 4.2. *Let $S_1 \in \mathbb{S}_{\boldsymbol{\kappa}_1}$ and $S_2 \in \mathbb{S}_{\boldsymbol{\kappa}_2}$. Then for any subsets $S_{1,sub} \subset S_1$ and $S_{2,sub} \subset S_2$, we have*

$$\| \sum_{T_1 \in S_{1,sub}} \sum_{T_2 \in S_{2,sub}} \phi_{T_1} \phi_{T_2} \|_{L^2(\mathbb{R}^n)} \lesssim (\sin\theta)^{-\frac{1}{2}} R^{-\frac{n+2}{4}},$$

where the implicit constant depends only on n.

We omit the proof.

In the next subsections, we estimate (14).

4.1. Separating minor and major contributive portions

We first remove some minor portions from the sum in (14). By the triangle inequality, (14) is estimated by the sum of the followings:

$$(15) \qquad \| \sum_{\textit{minor portion}} c_{T_1} \phi_{T_1} c_{T_2} \phi_{T_2} \|_{L^q(B_R(0))},$$

$$(16) \qquad \| \sum_{\textit{major portion}} c_{T_1} \phi_{T_1} c_{T_2} \phi_{T_2} \|_{L^q(B_R(0))},$$

where the minor portion is the case: $|c_{T_1}| \le R^{-100n}$ or $|c_{T_2}| \le R^{-100n}$, and the major portion is the remainder case.

For the minor portion (15) we easily obtain the desired bound . In fact, we see that the minor portion (15) is estimated by $R^{-\frac{n}{2}}$.

We consider the major portion in the next sections.

4.2. Major portion 1; coarse-scale decomposition

By (i) of Proposition 3.1 and the L^2-normalization of $F_{\boldsymbol{\kappa}_j}$, we have $\sum_{T \in \mathbb{T}_{\boldsymbol{\kappa}_j}} |c_T|^2 \lesssim 1$. Thus the major portion is

$$\bigcup_{S_j \in \mathbb{S}_{\boldsymbol{\kappa}_j}} \{T_j \in S_j \,:\, R^{-100n} \le |c_{T_j}| \lesssim 1\} \quad (j = 1, 2).$$

Dyadically pigeonholing on the size of $|c_{T_j}|$, we estimate (16) by

$$\sum_{\substack{R^{-100n}\le\gamma_1\lesssim 1;\\ dyadic}}\ \sum_{\substack{R^{-100n}\le\gamma_2\lesssim 1;\\ dyadic}} \Big\| \sum_{S_1\in\mathbb{S}_{\kappa_1}} \sum_{T_1\in S_1(\gamma_1)} c_{T_1}\phi_{T_1} \times \sum_{S_2\in\mathbb{S}_{\kappa_2}} \sum_{T_2\in S_2(\gamma_2)} c_{T_2}\phi_{T_2}\Big\|_{L^q(B_R(0))}, \tag{17}$$

where $S_j(\gamma_j) := \{T_j \in S_j : \gamma_j \le |c_{T_j}| < 2\gamma_j\}$. We easily see that

$$\#\{\gamma_j; dyadic : R^{-100n} \le \gamma_j \lesssim 1\} \lesssim \log R \quad (j = 1, 2).$$

Moreover, we easily see that

$$\sum_{S_j\in\mathbb{S}_{\kappa_j}} \#S_j(\gamma_j) \lesssim \gamma_j^{-2} \quad (j = 1, 2). \tag{18}$$

Then we crudely estimate (17) by

$$(\log R)^2 \sup_{R^{-100n}\le\gamma_1,\gamma_2\lesssim 1} \Big\| \sum_{S_1\in\mathbb{S}_{\kappa_1}} \sum_{T_1\in S_1(\gamma_1)} c_{T_1}\phi_{T_1} \times \sum_{S_2\in\mathbb{S}_{\kappa_2}} \sum_{T_2\in S_2(\gamma_2)} c_{T_2}\phi_{T_2}\Big\|_{L^q(B_R(0))}. \tag{19}$$

In (19), since $|c_{T_j}/\gamma_j| \lesssim 1$, we can absorb the factor c_{T_j}/γ_j harmlessly into ϕ_{T_j} and thus it suffices to consider

$$(\log R)^2 \sup_{R^{-100n}\le\gamma_1,\gamma_2\lesssim 1} \gamma_1\gamma_2 \Big\| \sum_{S_1\in\mathbb{S}_{\kappa_1}} \sum_{T_1\in S_1(\gamma_1)} \phi_{T_1} \times \sum_{S_2\in\mathbb{S}_{\kappa_2}} \sum_{T_2\in S_2(\gamma_2)} \phi_{T_2}\Big\|_{L^q(B_R(0))}. \tag{20}$$

In what follows, in (20), we concentrate on the factor

$$\Big\| \sum_{S_1\in\mathbb{S}_{\kappa_1}} \sum_{S_2\in\mathbb{S}_{\kappa_2}} \sum_{T_1\in S_1(\gamma_1)} \sum_{T_2\in S_2(\gamma_2)} \phi_{T_1}\phi_{T_2}\Big\|_{L^q(B_R(0))}, \tag{21}$$

for all $R^{-100n} \le \gamma_1, \gamma_2 \lesssim 1$, and thus, for the desired estimate (12), it suffices to estimate (21) by

$$\min\left\{\frac{1}{\sqrt{\varepsilon}(\sin\theta)^2}, \frac{1}{(\sin\theta)^4}\right\}^{(1-\nu)(1-\frac{1}{q})} R^{\max\{(1-\delta)\alpha, C\delta\}+C\nu} R^{-1}\gamma_1^{-1}\gamma_2^{-1}.$$

In (21), we may assume that $\#S_1(\gamma_1) \geq 1$ and $\#S_2(\gamma_2) \geq 1$. We set $\mathbb{S}_{\boldsymbol{\kappa}_j}(\gamma_j) = \{S_j \in \mathbb{S}_{\boldsymbol{\kappa}_j}(\gamma_j) \,:\, \#S_j(\gamma_j) \geq 1\}$. We easily see that

$$\#\mathbb{S}_{\boldsymbol{\kappa}_j}(\gamma_j) \lesssim \gamma_j^{-2}. \tag{22}$$

To employ the inductive argument, for any $1 \gg \delta > 0$, we make a coarse-scale decomposition. Let $\mathbb{B}$ be a finitely overlapping covering of $B_R(0)$ by balls B of radius $R^{1-\delta}$. Then we estimate (21) by

$$\sum_{B\in\mathbb{B}} \| \sum_{S_1\in\mathbb{S}_{\boldsymbol{\kappa}_1}(\gamma_1)} \sum_{S_2\in\mathbb{S}_{\boldsymbol{\kappa}_2}(\gamma_2)} \sum_{T_1\in S_1(\gamma_1)} \sum_{T_2\in S_2(\gamma_2)} \phi_{T_1}\phi_{T_2} \|_{L^q(B)}. \tag{23}$$

We will easily see that

$$\#\mathbb{B} \lesssim R^{n\delta}. \tag{24}$$

4.3. Major portion 2; local and global portions

For $\gamma_1, \gamma_2 > 0$, we introduce some relation "$\sim_{\gamma_1,\gamma_2}$" between sticks in $\mathbb{S}_{\boldsymbol{\kappa}_1}(\gamma_1) \cup \mathbb{S}_{\boldsymbol{\kappa}_2}(\gamma_2)$ and balls $B \in \mathbb{B}$, which is the same as Tao's one [6] except for the dependence on γ_1 and γ_2.

Let $\mathbb{Q}$ be a finitely overlapping covering of $B_R(0)$ by balls Q of radius $\sqrt{R}$. Then we set

$$\mathbb{S}_{\boldsymbol{\kappa}_j}(Q;\gamma_j) = \left\{S_j \in \mathbb{S}_{\boldsymbol{\kappa}_j}(\gamma_j) \,:\, S_j \cap R^\delta Q \neq \emptyset\right\} \quad (j = 1, 2),$$

where $R^\delta Q$ denotes the ball with the same center as Q and radius $R^\delta\sqrt{R}$.

Let μ_1, μ_2 be dyadic numbers ≥ 1 or 0. We set

$$\begin{aligned} &\mathbb{Q}(\mu_1,\mu_2;\gamma_1,\gamma_2) \\ &\quad = \left\{Q \in \mathbb{Q} \,:\, \mu_j \leq \#\mathbb{S}_{\boldsymbol{\kappa}_j}(Q;\gamma_j) < 2\mu_j \ (j = 1, 2)\right\}. \end{aligned} \tag{25}$$

Then we see that μ_j ranges at most $0 \leq \mu_j \lesssim \varepsilon^{\frac{n-1}{2}} R^{\frac{n-1}{2}+(n-1)\delta}$ $(j = 1, 2)$. Since μ_j dyadically varies, we see that the possible number of μ_j $\lesssim \log R$ for $j = 1, 2$, and therefore the possible number of pairs (μ_1, μ_2) $\lesssim (\log R)^2$.

Let λ_1 be a dyadic number ≥ 1 or 0. Then we set

$$\begin{aligned} &\mathbb{S}_{\boldsymbol{\kappa}_1}(\lambda_1,\mu_1,\mu_2;\gamma_1,\gamma_2) = \{S_1 \in \mathbb{S}_{\boldsymbol{\kappa}_1}(\gamma_1) \,: \\ &\qquad \lambda_1 \leq \#\left\{Q \in \mathbb{Q}(\mu_1,\mu_2;\gamma_1,\gamma_2) \,:\, S_1 \cap R^\delta Q \neq \emptyset\right\} < 2\lambda_1\right\}. \end{aligned} \tag{26}$$

Thus each $S_1 \in \mathbb{S}_{\boldsymbol{\kappa}_1}(\lambda_1,\mu_1,\mu_2;\gamma_1,\gamma_2)$ intersects about λ_1 (slight enlargement of) balls $Q \in \mathbb{Q}(\mu_1,\mu_2;\gamma_1,\gamma_2)$. We easily see that λ_1 ranges $0 \leq \lambda_1 \lesssim R^{\frac{n}{2}}$ and therefore the possible values of $\lambda_1 \lesssim \log R$.

Let $\lambda_1, \mu_1, \mu_2 \geq 1$ and $1 \gtrsim \gamma_1, \gamma_2 \geq R^{-100n}$. For each stick $S_1 \in \mathbb{S}_{\boldsymbol{\kappa}_1}(\lambda_1, \mu_1, \mu_2; \gamma_1, \gamma_2)$, let $B(S_1, \lambda_1, \mu_1, \mu_2; \gamma_1, \gamma_2)$ be the ball in $\mathbb{B}$ which maximizes the quantity

$$\begin{aligned}\#\{Q \in &\mathbb{Q}(\mu_1, \mu_2; \gamma_1, \gamma_2) : \\ &S_1 \cap R^\delta Q \neq \emptyset ,\, Q \cap B(S_1, \lambda_1, \mu_1, \mu_2; \gamma_1, \gamma_2) \neq \emptyset\}.\end{aligned}$$

(Choose one, if such ball is not unique.)

We first define the relation $\sim_{\lambda_1, \mu_1, \mu_2; \gamma_1, \gamma_2}$ between sticks in $\mathbb{S}_{\boldsymbol{\kappa}_1}(\gamma_1)$ and balls in $\mathbb{B}$ by defining $S_1 \sim_{\lambda_1, \mu_1, \mu_2; \gamma_1, \gamma_2} B$ if $S_1 \in \mathbb{S}_{\boldsymbol{\kappa}_1}(\lambda_1, \mu_1, \mu_2; \gamma_1, \gamma_2)$ and $B \in \mathbb{B}$ with $B \subset 10B(S_1, \lambda_1, \mu_1, \mu_2; \gamma_1, \gamma_2)$. Then we define $S_1 \sim_{\gamma_1, \gamma_2} B$ if one has $S_1 \sim_{\lambda_1, \mu_1, \mu_2; \gamma_1, \gamma_2} B$ for some dyadic numbers $\lambda_1, \mu_1, \mu_2 \geq 1$. We also define $\sim_{\gamma_1, \gamma_2}$ between $\mathbb{S}_{\boldsymbol{\kappa}_2}(\gamma_2)$ and $\mathbb{B}$ by a completely symmetrical procedure. For this relation, we have the following lemma, which is the same as Proposition 5.1 of [6].

Lemma 4.3. *Let $1 \gtrsim \gamma_1, \gamma_2 \geq R^{-100n}$. Then we have*

$$\#\{B \in \mathbb{B} \,:\, S \sim_{\gamma_1, \gamma_2} B\} \lesssim (\log R)^3$$

for all $S \in \mathbb{S}_{\boldsymbol{\kappa}_1}(\gamma_1) \cup \mathbb{S}_{\boldsymbol{\kappa}_2}(\gamma_2)$.

Now we estimate (23) by sum of the following two terms:

$$\sum_{B \in \mathbb{B}} \Big\| \sum_{\substack{S_1 \in \mathbb{S}_{\boldsymbol{\kappa}_1}(\gamma_1) \\ S_1 \sim_{\gamma_1 \cdot \gamma_2} B}} \sum_{\substack{S_2 \in \mathbb{S}_{\boldsymbol{\kappa}_2}(\gamma_2) \\ S_2 \sim_{\gamma_1 \cdot \gamma_2} B}} \sum_{T_1 \in S_1(\gamma_1)} \sum_{T_2 \in S_2(\gamma_2)} \phi_{T_1} \phi_{T_2} \Big\|_{L^q(B)}, \tag{27}$$

$$\sum_{B \in \mathbb{B}} \Big\| \sum_{\substack{S_1 \in \mathbb{S}_{\boldsymbol{\kappa}_1}(\gamma_1),\, S_2 \in \mathbb{S}_{\boldsymbol{\kappa}_2}(\gamma_2) \\ S_1 \not\sim_{\gamma_1 \cdot \gamma_2} B \text{ or } S_2 \not\sim_{\gamma_1 \cdot \gamma_2} B}} \sum_{T_1 \in S_1(\gamma_1)} \sum_{T_2 \in S_2(\gamma_2)} \phi_{T_1} \phi_{T_2} \Big\|_{L^q(B)}. \tag{28}$$

We call (27) local portion and (28) global portion.

The local portion (27) is estimated by the inductive hypothesis and a way similar to Wolff [10]. Thus it remains to estimate the global portion (28).

4.4. Global portion 1; interpolation setting

Our aim is to bound the global portion (28) by

$$\min\left\{\frac{1}{\sqrt{\varepsilon}(\sin\theta)^2}\,,\ \frac{1}{(\sin\theta)^4}\right\}^{(1-\nu)\left(1-\frac{1}{q}\right)} R^{C\delta + C\nu} R^{-1} \gamma_1^{-1} \gamma_2^{-1} \tag{29}$$

for all $\nu > 0$ and some constants C depending only on n and p.

By (24) and the symmetry of the relation $\sim_{\gamma_1,\gamma_2}$ with respect to $\mathbb{S}_{\kappa_1}(\gamma_1)$ and $\mathbb{S}_{\kappa_2}(\gamma_2)$, it suffices for (29) to prove

$$\begin{aligned}(30)\quad &\Big\|\sum_{\substack{S_1\in\mathbb{S}_{\kappa_1}(\gamma_1)\\ S_1\not\sim_{\gamma_1,\gamma_2}B}}\ \sum_{S_2\in\mathbb{S}_{\kappa_2}(\gamma_2)}\ \sum_{T_1\in S_1(\gamma_1)}\ \sum_{T_2\in S_2(\gamma_2)}\phi_{T_1}\phi_{T_2}\Big\|_{L^q(B)}\\ &\lesssim \min\left\{\frac{1}{\sqrt{\varepsilon}(\sin\theta)^2},\ \frac{1}{(\sin\theta)^4}\right\}^{(1-\nu)(1-\frac{1}{q})} R^{C\delta+C\nu}R^{-1}\gamma_1^{-1}\gamma_2^{-1},\end{aligned}$$

for all $B\in\mathbb{B}$.

We prove (30) by interpolating between bilinear L^1 and L^2 estimates below:

$$(31)\quad \Big\|\sum_{\substack{S_1\in\mathbb{S}_{\kappa_1}(\gamma_1)\\ S_1\not\sim_{\gamma_1,\gamma_2}B}}\ \sum_{S_2\in\mathbb{S}_{\kappa_2}(\gamma_2)}\ \sum_{T_1\in S_1(\gamma_1)}\ \sum_{T_2\in S_2(\gamma_2)}\phi_{T_1}\phi_{T_2}\Big\|_{L^1(B)}\lesssim\gamma_1^{-1}\gamma_2^{-1}$$

and

$$\begin{aligned}(32)\quad &\Big\|\sum_{\substack{S_1\in\mathbb{S}_{\kappa_1}(\gamma_1)\\ S_1\not\sim_{\gamma_1,\gamma_2}B}}\ \sum_{S_2\in\mathbb{S}_{\kappa_2}(\gamma_2)}\ \sum_{T_1\in S_1(\gamma_1)}\ \sum_{T_2\in S_2(\gamma_2)}\phi_{T_1}\phi_{T_2}\Big\|_{L^2(B)}\\ &\lesssim R^{C\delta+C\nu}\min\left\{\frac{1}{\sqrt{\varepsilon}(\sin\theta)^2},\ \frac{1}{(\sin\theta)^4}\right\}^{\frac{1}{2}(1-\nu)} R^{-\frac{n+2}{4}}\gamma_1^{-1}\gamma_2^{-1}\end{aligned}$$

for some constant $C>0$ depending only on n.

4.5. Global portion 2; bilinear L^1-estimate

By the Schwarz inequality, the L.H.S. of (31) is estimated by

$$(33)\quad \Big\|\sum_{\substack{S_1\in\mathbb{S}_{\kappa_1}(\gamma_1)\\ S_1\not\sim_{\gamma_1,\gamma_2}B}}\ \sum_{T_1\in S_1(\gamma_1)}\phi_{T_1}\Big\|_{L^2(B)}\Big\|\sum_{S_2\in\mathbb{S}_{\kappa_2}(\gamma_2)}\ \sum_{T_2\in S_2(\gamma_2)}\phi_{T_2}\Big\|_{L^2(B)}.$$

By (iv) of Proposition 3.1 and (18), we estimate (33) by $\gamma_1^{-1}\gamma_2^{-1}$, which is the desired estimate (31).

4.6. Global portion 3; bilinear L^2-estimate

4.6.1. *Fine-scale decomposition.* Let $\mathbb{Q}$ be a finitely overlapping covering of $B_R(0)$ by balls Q with radius $\sqrt{R}$. Note that

$$(34)\quad \#\mathbb{Q}\lesssim R^{\frac{n}{2}}.$$

We make a fine-scale decomposition. Then the squared L.H.S. of (32) is estimated by

$$(35)\quad \sum_{\substack{Q\in\mathbb{Q}\\ Q\subset 2B}}\Big\|\sum_{\substack{S_1\in\mathbb{S}_{\kappa_1}(\gamma_1)\\ S_1\not\sim_{\gamma_1,\gamma_2}B}}\ \sum_{S_2\in\mathbb{S}_{\kappa_2}(\gamma_2)}\ \sum_{T_1\in S_1(\gamma_1)}\ \sum_{T_2\in S_2(\gamma_2)}\phi_{T_1}\phi_{T_2}\Big\|_{L^2(Q)}^2.$$

To obtain the desired estimate (32), we have to show that

$$(36)\qquad (35) \lesssim \min\left\{\frac{1}{\sqrt{\varepsilon}(\sin\theta)^2}, \frac{1}{(\sin\theta)^4}\right\}^{1-\nu} R^{C\delta+C\nu}R^{-\frac{n+2}{2}}\gamma_1^{-2}\gamma_2^{-2}.$$

We divide the sum with respect to S_j into two cases and thus we estimate (35) by the sum of the followings:

$$(37)\qquad \sum_{\substack{Q\in\mathbb{Q}\\ Q\subset 2B}} \Big\| \sum_{\substack{S_1\in\mathbb{S}_{\kappa_1}^{\not\sim_{\gamma_1,\gamma_2}B}(\gamma_1),\, S_2\in\mathbb{S}_{\kappa_2}(\gamma_2)\\ S_1\cap R^\delta Q=\emptyset \ or \ S_2\cap R^\delta Q=\emptyset}} \sum_{T_1\in S_1(\gamma_1)} \sum_{T_2\in S_2(\gamma_2)} \phi_{T_1}\phi_{T_2}\Big\|_{L^2(Q)}^2,$$

$$(38)\qquad \sum_{\substack{Q\in\mathbb{Q}\\ Q\subset 2B}} \Big\| \sum_{\substack{S_1\in\mathbb{S}_{\kappa_1}^{\not\sim_{\gamma_1,\gamma_2}B}(\gamma_1)\\ S_1\cap R^\delta Q\neq\emptyset}} \sum_{\substack{S_2\in\mathbb{S}_{\kappa_2}(\gamma_2)\\ S_2\cap R^\delta Q\neq\emptyset}} \sum_{T_1\in S_1(\gamma_1)} \sum_{T_2\in S_2(\gamma_2)} \phi_{T_1}\phi_{T_2}\Big\|_{L^2(Q)}^2,$$

where $\mathbb{S}_{\kappa_1}^{\not\sim_{\gamma_1,\gamma_2}B}(\gamma_1)$ denotes the set of $S_1\in\mathbb{S}_{\kappa_1}(\gamma_1)$ with $S_1\not\sim_{\gamma_1,\gamma_2} B$. We call (37) minor global portion and (38) major global portion.

In (37), by the condition $S_1\cap R^\delta Q=\emptyset$ or $S_2\cap R^\delta Q=\emptyset$, (iii) of Proposition 3.1 and (34), we obtain the desired bound. Thus it remains to consider (38).

4.6.2. *Major global portion 1; pigeonholing of* $\mathbb{Q}$ *and* $\mathbb{S}_{\kappa_1}$. We consider the major global portion (38). We first do the dyadic pigeonholing of $\mathbb{Q}$. Then we have

$$(39)\qquad \begin{aligned}(38) \lessapprox \sup_{\substack{\mu_1,\mu_2\geq 1:\\ dyadic}} \sum_{\substack{Q\in\mathbb{Q}(\mu_1,\mu_2;\gamma_1,\gamma_2)\\ Q\subset 2B}} \Big\| &\sum_{S_1\in\mathbb{S}_{\kappa_1}^{\not\sim_{\gamma_1,\gamma_2}B}(Q;\gamma_1)} \sum_{T_1\in S_1(\gamma_1)} \phi_{T_1}\\ &\times \sum_{S_2\in\mathbb{S}_{\kappa_2}(Q;\gamma_2)} \sum_{T_2\in S_2(\gamma_2)} \phi_{T_2}\Big\|_{L^2(Q)}^2,\end{aligned}$$

where $\mathbb{S}_{\kappa_1}^{\not\sim_{\gamma_1,\gamma_2}B}(Q;\gamma_1) := \left\{S_1\in\mathbb{S}_{\kappa_1}^{\not\sim_{\gamma_1,\gamma_2}B}(\gamma_1) \,:\, S_1\cap R^\delta Q\neq\emptyset\right\}$ and $\mathbb{S}_{\kappa_2}(Q;\gamma_2) := \{S_2\in\mathbb{S}_{\kappa_2}(\gamma_2) \,:\, S_2\cap R^\delta Q\neq\emptyset\}$. We do the dyadic pigeonholing of $\mathbb{S}_{\kappa_1}(\gamma_1)$. Note that $\mathbb{S}_{\kappa_1}^{\not\sim_{\gamma_1,\gamma_2}B}(Q;\gamma_1)\cap\mathbb{S}_{\kappa_1}(0,\mu_1,\mu_2;\gamma_1,\gamma_2) = \emptyset$ for all $Q\in\mathbb{Q}(\mu_1,\mu_2;\gamma_1,\gamma_2)$. Thus, the R.H.S. of (39) is estimated by

$$\begin{aligned}R^\nu \sup_{\mu_1,\mu_2,\lambda_1\geq 1} \sum_{\substack{Q\in\mathbb{Q}(\mu_1,\mu_2;\gamma_1,\gamma_2)\\ Q\subset 2B}} \Big\| &\sum_{S_1\in\mathbb{S}_1'(Q)} \sum_{T_1\in S_1(\gamma_1)} \phi_{T_1}\\ &\times \sum_{S_2\in\mathbb{S}_2'(Q)} \sum_{T_2\in S_2(\gamma_2)} \phi_{T_2}\Big\|_{L^2(Q)}^2,\end{aligned}$$

where $\mathbb{S}_1'(Q) := \mathbb{S}_{\boldsymbol{\kappa}_1}^{\not\sim_{\gamma_1,\gamma_2} B}(Q;\gamma_1) \cap \mathbb{S}_{\boldsymbol{\kappa}_1}(\lambda_1,\mu_1,\mu_2;\gamma_1,\gamma_2)$ and $\mathbb{S}_2'(Q) := \mathbb{S}_{\boldsymbol{\kappa}_2}(Q;\gamma_2)$. Thus, it suffices to show that

$$
\begin{aligned}
&\sum_{\substack{Q\in\mathbb{Q}(\mu_1,\mu_2;\gamma_1,\gamma_2)\\ Q\subset 2B}} \Big\| \sum_{S_1\in\mathbb{S}_1'(Q)} \sum_{S_2\in\mathbb{S}_2'(Q)} \sum_{T_1\in S_1(\gamma_1)} \sum_{T_2\in S_2(\gamma_2)} \phi_{T_1}\phi_{T_2} \Big\|_{L^2(Q)}^2 \\
&\lessapprox \min\left\{ \frac{1}{\sqrt{\varepsilon}(\sin\theta)^2}, \frac{1}{(\sin\theta)^4} \right\}^{1-\nu} R^{C\delta+C\nu} R^{-\frac{n+2}{2}} \gamma_1^{-2}\gamma_2^{-2}
\end{aligned}
\tag{40}
$$

for all $\lambda_1,\mu_1,\mu_2 \geq 1$.

4.6.3. *Major global portion 2; constraint from the supports of wave packets.* We first consider the summand of the L.H.S. of (40):

$$
\Big\| \sum_{S_1\in\mathbb{S}_1'(Q)} \sum_{S_2\in\mathbb{S}_2'(Q)} \sum_{T_1\in S_1(\gamma_1)} \sum_{T_2\in S_2(\gamma_2)} \phi_{T_1}\phi_{T_2} \Big\|_{L^2(Q)}^2 \tag{41}
$$

for all $Q \in \mathbb{Q}(\mu_1,\mu_2;\gamma_1,\gamma_2)$ with $Q \subset 2B$.

The factor (41) is estimated by the global L^2-norm and thus, by the Plancherel theorem, it suffices to consider

$$
\begin{aligned}
&\sum_{S_1\in\mathbb{S}_1'(Q),S_2\in\mathbb{S}_2'(Q)} \ \sum_{S_1'\in\mathbb{S}_1'(Q),S_2'\in\mathbb{S}_2'(Q)} \\
&\Big(\sum_{T_1\in S_1(\gamma_1)} \sum_{T_2\in S_2(\gamma_2)} \widehat{\phi_{T_1}\phi_{T_2}}, \sum_{T_1'\in S_1'(\gamma_1)} \sum_{T_2'\in S_2'(\gamma_2)} \widehat{\phi_{T_1'}\phi_{T_2'}} \Big)_{L^2}.
\end{aligned}
\tag{42}
$$

¿From the support properties of $\widehat{\phi_{T_1}\phi_{T_2}}$ and $\widehat{\phi_{T_1'}\phi_{T_2'}}$, (42) is further reduced to

$$
\begin{aligned}
&\sum_{S_1\in\mathbb{S}_1'(Q),S_2\in\mathbb{S}_2'(Q)} \ \sum_{\substack{S_1'\in\mathbb{S}_1'(Q),S_2'\in\mathbb{S}_2'(Q)\\ N_{2/R}(\kappa(S_1))+N_{2/R}(\kappa(S_2))\cap N_{2/R}(\kappa(S_1'))+N_{2/R}(\kappa(S_2'))\neq\emptyset}} \\
&\Big(\sum_{T_1\in S_1(\gamma_1)} \sum_{T_2\in S_2(\gamma_2)} \widehat{\phi_{T_1}\phi_{T_2}}, \sum_{T_1'\in S_1'(\gamma_1)} \sum_{T_2'\in S_2'(\gamma_2)} \widehat{\phi_{T_1'}\phi_{T_2'}} \Big)_{L^2(\mathbb{R}^n)},
\end{aligned}
\tag{43}
$$

where $\kappa(S_j)$ is the cap in $\Gamma_R(\boldsymbol{\kappa}_j)$ whose center corresponds to the direction of the stick S_j.

Now let us consider the constraint

$$
N_{2/R}(\kappa(S_1)) + N_{2/R}(\kappa(S_2)) \cap N_{2/R}(\kappa(S_1')) + N_{2/R}(\kappa(S_2')) \neq \emptyset. \tag{44}
$$

We denote the center of $\kappa(S_j)$ (resp. $\kappa(S_j')$) by $\omega(S_j)$ (resp. $\omega(S_j')$).

We easily see that

$$
N_{2/R}(\kappa(S_1)) + N_{2/R}(\kappa(S_2)) \subset B_{C/\sqrt{R}}(\omega(S_1)+\omega(S_2))
$$

and

$$N_{2/R}(\kappa(S_1')) + N_{2/R}(\kappa(S_2')) \subset B_{C/\sqrt{R}}(\omega(S_1') + \omega(S_2'))$$

for some universal constant $C > 0$. Therefore, for $N_{2/R}(\kappa(S_1)) + N_{2/R}(\kappa(S_2)) \cap N_{2/R}(\kappa(S_1')) + N_{2/R}(\kappa(S_2')) \neq \emptyset$, it is required that

$$|\,\omega(S_1) + \omega(S_2) - \omega(S_1') - \omega(S_2')\,| \lesssim \frac{1}{\sqrt{R}}. \tag{45}$$

¿From (45), we find that

$$\begin{aligned} &\left|\left|\omega(S_1') - \frac{\omega(S_1)+\omega(S_2)}{2}\right| - \left|\frac{\omega(S_1)-\omega(S_2)}{2}\right|\right| \\ &\lesssim \frac{1}{\sqrt{R}|\,\omega(S_1)-\omega(S_2)\,|}, \end{aligned} \tag{46}$$

which is observed by Bourgain in [2]. Note that, by the transversality of $\kappa(S_1)$ and $\kappa(S_2)$, $|\,\omega(S_1) - \omega(S_2)\,| \gtrsim \sin\frac{\theta}{2}$ in the R.H.S. of (46).

Replacing the constraint (44) with (45) in (43), we have

$$\begin{aligned} &\sum_{S_1\in\mathbb{S}_1'(Q),S_2\in\mathbb{S}_2'(Q)} \quad \sum_{\substack{S_1'\in\mathbb{S}_1'(Q).S_2'\in\mathbb{S}_2'(Q) \\ |\,\omega(S_1)+\omega(S_2)-(\omega(S_1')+\omega(S_2'))\,|\lesssim 1/\sqrt{R}}} \\ &(\sum_{T_1\in S_1(\gamma_1)} \sum_{T_2\in S_2(\gamma_2)} \widehat{\phi_{T_1}\phi_{T_2}}\,, \sum_{T_1'\in S_1'(\gamma_1)} \sum_{T_2'\in S_2'(\gamma_2)} \widehat{\phi_{T_1'}\phi_{T_2'}}\,)_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{47}$$

Then, using the Schwarz inequality and the Plancherel theorem, we estimate (47) by

$$\begin{aligned} &\sum_{S_1\in\mathbb{S}_1'(Q),S_2\in\mathbb{S}_2'(Q)} \quad \sum_{\substack{S_1'\in\mathbb{S}_1'(Q).S_2'\in\mathbb{S}_2'(Q) \\ |\,\omega(S_1)+\omega(S_2)-(\omega(S_1')+\omega(S_2'))\,|\lesssim 1/\sqrt{R}}} \\ &\|\sum_{T_1\in S_1(\gamma_1)} \sum_{T_2\in S_2(\gamma_2)} \phi_{T_1}\phi_{T_2}\|_{L^2(\mathbb{R}^n)} \|\sum_{T_1'\in S_1'(\gamma_1)} \sum_{T_2'\in S_2'(\gamma_2)} \phi_{T_1'}\phi_{T_2'}\|_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{48}$$

By Lemma 4.2 and (46),

$$
\begin{aligned}
(48) \lessapprox\ & (\sin\theta)^{-1} R^{-\frac{n+2}{2}} \#\mathbb{S}_1'(Q)\#\mathbb{S}_2'(Q) \\
& \times \sup_{\substack{\omega_1\in\Omega_1\\ \omega_2\in\Omega_2}} \Bigg\{ (S_1', S_2') \in \mathbb{S}_1'(Q)\times\mathbb{S}_2'(Q) \,:\, \\
& |\,\omega_1+\omega_2-\omega(S_1)-\omega(S_2))\,| \lesssim \frac{1}{\sqrt{R}}, \\
& \left|\left|\omega(S_1')-\frac{\omega_1+\omega_2}{2}\right| - \left|\frac{\omega_1-\omega_2}{2}\right|\right| \lesssim \frac{1}{\sqrt{R}\sin\frac{\theta}{2}} \Bigg\},
\end{aligned} \tag{49}
$$

where recall that Ω_j $(j=1,2)$ is the set of centers of $\kappa_j \in \Gamma_R(\boldsymbol{\kappa}_j)$.

We estimate the R.H.S. of (49) by

$$
\begin{aligned}
& (\sin\theta)^{-1} R^{-\frac{n+2}{2}} \#\mathbb{S}_1'(Q)\#\mathbb{S}_2'(Q) \\
& \times \sup_{\substack{\omega_1\in\Omega_1\\ \omega_2\in\Omega_2}} \#\left\{ S_1\in\mathbb{S}_1'(Q) \,:\, \omega(S_1)\in\Omega_1\cap N_{\frac{C}{\sqrt{R}\sin\frac{\theta}{2}}}(\Upsilon(\omega_1,\omega_2)) \right\} \\
& \times \sup_{\substack{\omega_1,\omega'\in\Omega_1\\ \omega_2\in\Omega_2}} \#\left\{ S_2\in\mathbb{S}_2'(Q) \,:\, |\,\omega_1+\omega_2-\omega_1'-\omega(S_2)\,| \lesssim \frac{1}{\sqrt{R}} \right\},
\end{aligned} \tag{50}
$$

where

$$
\Upsilon(\omega_1,\omega_2) := \left\{ x\in\mathbb{R}^n \,:\, \left| x-\frac{\omega_1+\omega_2}{2} \right| = \left| \frac{\omega_1-\omega_2}{2} \right| \right\}.
$$

In (50), we easily see that

$$
\sup_{\substack{\omega_1,\omega'\in\Omega_1\\ \omega_2\in\Omega_2}} \#\left\{ S_2\in\mathbb{S}_2'(Q) \,:\, |\,\omega_1+\omega_2-\omega_1'-\omega(S_2)\,| \lesssim \frac{1}{\sqrt{R}} \right\} \lesssim 1.
$$

Moreover, we have $\#\mathbb{S}_2'(Q) \lesssim \mu_2$ for all $Q\in\mathbb{Q}(\mu_1,\mu_2;\gamma_1,\gamma_2)$. Hence we have

$$
\begin{aligned}
(50) \lessapprox\ & (\sin\theta)^{-1} R^{-\frac{n+2}{2}} \mu_2 \#\mathbb{S}_1'(Q) \\
& \times \sup_{\substack{\omega_1\in\Omega_1\\ \omega_2\in\Omega_2}} \#\left\{ S_1\in\mathbb{S}_1 \,:\, \omega(S_1)\in\Omega_1\cap N_{\frac{C}{\sqrt{R}\sin\frac{\theta}{2}}}(\Upsilon(\omega_1,\omega_2)) \right\}.
\end{aligned}
$$

Now we set

$$
\mathbb{S}_1''(Q) = \left\{ S_1\in\mathbb{S}_1'(Q) \,:\, \omega(S_1)\in\Omega_1\cap N_{\frac{C}{\sqrt{R}\sin\frac{\theta}{2}}}(\Upsilon(\omega_1,\omega_2)) \right\}.
$$

For the desired result (40), it remains to prove that

$$\begin{aligned}(51)\qquad &\sum_{\substack{Q\in\mathbb{Q}(\mu_1,\mu_2;\gamma_1,\gamma_2)\\ Q\subset 2B}} \#\mathbb{S}_1'(Q) \sup_{\substack{\omega_1\in\Omega_1\\ \omega_2\in\Omega_2}} \#\mathbb{S}_1''(Q)\\ &\lessapprox R^{C\delta+C\nu}\min\left\{\frac{1}{\sqrt{\varepsilon}\sin\theta},\frac{1}{(\sin\theta)^3}\right\}^{1-2\nu}\frac{\gamma_1^{-2}\gamma_2^{-2}}{\mu_2}.\end{aligned}$$

We prove this combinatorial estimate in the next subsections.

4.6.4. *Combinatorial estimate 1; preliminary estimates.* We prove the combinatorial estimate (51). The L.H.S. of (51) is estimated by

$$(52)\qquad \sup_{\substack{Q\in\mathbb{Q}(\mu_1,\mu_2;\gamma_1,\gamma_2)\\ Q\subset 2B}} \sup_{\substack{\omega_1\in\Omega_1\\ \omega_2\in\Omega_2}} \#\mathbb{S}_1''(Q) \sum_{\substack{Q\in\mathbb{Q}(\mu_1,\mu_2;\gamma_1,\gamma_2)\\ Q\subset 2B}} \#\mathbb{S}_1'(Q).$$

By an estimate similar to that in p.1378 of [6] and (22), we see that

$$\sum_{\substack{Q\in\mathbb{Q}(\mu_1,\mu_2;\gamma_1,\gamma_2)\\ Q\subset 2B}} \#\mathbb{S}_1'(Q) \lesssim \lambda_1\gamma_1^{-2}$$

and thus (52) is estimated by

$$\lambda_1\gamma_1^{-2} \sup_{\substack{Q\in\mathbb{Q}(\mu_1,\mu_2;\gamma_1,\gamma_2)\\ Q\subset 2B}} \sup_{\substack{\omega_1\in\Omega_1\\ \omega_2\in\Omega_2}} \#\mathbb{S}_1''(Q).$$

Hence, to prove the desired result (51), it suffices to show that

$$(53)\qquad \#\mathbb{S}_1''(Q) \lessapprox R^{C\delta+C\nu}\min\left\{\frac{1}{\sqrt{\varepsilon}\sin\theta},\frac{1}{(\sin\theta)^3}\right\}^{1-2\nu}\frac{\gamma_2^{-2}}{\lambda_1\mu_2}$$

for all $Q\in\mathbb{Q}(\mu_1,\mu_2;\gamma_1,\gamma_2)$ with $Q\subset 2B$ and all $\omega_1\in\Omega_1$, $\omega_2\in\Omega_2$.

4.6.5. *Combinatorial estimate 2; Crucial geometric observation and conclusion.* Our aim is to prove (53). Now we fix $Q_0\in\mathbb{Q}(\mu_1,\mu_2;\gamma_1,\gamma_2)$ with $Q_0\subset 2B$ and recall $B(S_1,\lambda_1,\mu_1,\mu_2;\gamma_1,\gamma_2)$ which is the ball in $\mathbb{B}$ maximizing the quantity

$$\#\{\, Q\in\mathbb{Q}(\mu_1,\mu_2;\gamma_1,\gamma_2)\ :\ S_1\cap R^\delta Q\,,\ Q\cap B(S_1,\lambda_1,\mu_1,\mu_2;\gamma_1,\gamma_2)\neq\emptyset\,\}$$

for each $S_1\in\mathbb{S}_{\kappa_1}(\lambda_1,\mu_1,\mu_2;\gamma_1,\gamma_2)$. Note that we have

$$\begin{aligned}(54)\qquad &R^{-(n+1)\delta}\lambda_1\\ &\lesssim \inf_{S_1\in\mathbb{S}_{\kappa_1}(\lambda_1,\mu_1,\mu_2;\gamma_1,\gamma_2)} \#\{\, Q\in\mathbb{Q}(\mu_1,\mu_2;\gamma_1,\gamma_2)\ :\\ &\qquad\qquad S_1\cap R^\delta Q\,,\ Q\cap B(S_1,\lambda_1,\mu_1,\mu_2;\gamma_1,\gamma_2)\neq\emptyset\,\}.\end{aligned}$$

Indeed, if not, then we have $\#\{ Q \in \mathbb{Q}(\mu_1, \mu_2; \gamma_1, \gamma_2) \,:\, S_1 \cap R^\delta Q \neq \emptyset \} \ll \lambda_1$, which contradicts that $S_1 \in \mathbb{S}_{\kappa_1}(\lambda_1, \mu_1, \mu_2; \gamma_1, \gamma_2)$, see (26).

Now let $S_1 \in \mathbb{S}_1''(Q_0)$. Then, since $S_1 \in \mathbb{S}_{\kappa_1}(\lambda_1, \mu_1, \mu_2; \gamma_1, \gamma_2)$ and $S_1 \not\sim_{\gamma_1,\gamma_2} B$, we have $B \not\subset 10B(S_1, \lambda_1, \mu_1, \mu_2; \gamma_1, \gamma_2)$. From this and $Q_0 \subset 2B$, we have (cf. p.1379 of [6])

$$dist(Q_0, 2B(S_1, \lambda_1, \mu_1, \mu_2; \gamma_1, \gamma_2)) \gtrsim R^{1-\delta}$$

and therefore, for any $Q \in \mathbb{Q}$ with $Q \cap B(S_1, \lambda_1, \mu_1, \mu_2; \gamma_1, \gamma_2) \neq \emptyset$, we have

$$R \gtrsim dist(Q_0, Q) \gtrsim R^{1-\delta},$$

where we have used the triangle inequality $dist(Q_0, Q) \leq dist(Q_0, 0) + dist(0, Q) \lesssim R$ for the upper bound. Thus, by (54), we have,

$$\begin{aligned} R^{-n\delta}\lambda_1 \lesssim \inf_{S_1 \in \mathbb{S}_{\kappa_1}(\lambda_1, \mu_1, \mu_2; \gamma_1, \gamma_2)} \#\{\, & Q \in \mathbb{Q}(\mu_1, \mu_2; \gamma_1, \gamma_2) \,: \\ & S_1 \cap R^\delta Q \,,\, R \gtrsim dist(Q_0, Q) \gtrsim R^{1-\delta} \,\}. \end{aligned}$$

On the other hand, by the definition of $\mathbb{Q}(\mu_1, \mu_2; \gamma_1, \gamma_2)$ (see (25)), for each $Q \in \mathbb{Q}(\mu_1, \mu_2; \gamma_1, \gamma_2)$ there are at least μ_2 sticks in $\mathbb{S}_{\kappa_2}(\gamma_2)$ which intersect $R^\delta Q$. Thus we have

$$\begin{aligned} & R^{-n\delta}\lambda_1\mu_2 \\ & \lesssim \inf_{S_1 \in \mathbb{S}_{\kappa_1}(\lambda_1, \mu_1, \mu_2; \gamma_1, \gamma_2)} \#\{\, (Q, S_2) \in \mathbb{Q}(\mu_1, \mu_2; \gamma_1, \gamma_2) \times \mathbb{S}_{\kappa_2}(\gamma_2) \,: \\ & \quad S_1 \cap R^\delta Q \neq \emptyset \,,\, S_2 \cap R^\delta Q \neq \emptyset \,,\, R \gtrsim dist(Q_0, Q) \gtrsim R^{1-\delta} \,\}. \end{aligned}$$

Summing over all S_1 in $\mathbb{S}_1''(Q_0)$, we obtain

$$\begin{aligned} & R^{-n\delta}\lambda_1\mu_2 \, \#\mathbb{S}_1''(Q_0) \\ & \lesssim \#\{\, (Q, S_1, S_2) \in \mathbb{Q}(\mu_1, \mu_2; \gamma_1, \gamma_2) \times \mathbb{S}_1''(Q_0) \times \mathbb{S}_{\kappa_2}(\gamma_2) \,: \\ & \quad S_1 \cap R^\delta Q \neq \emptyset \,,\, S_2 \cap R^\delta Q \neq \emptyset \,,\, R \gtrsim dist(Q_0, Q) \gtrsim R^{1-\delta} \,\}. \end{aligned} \tag{55}$$

Now we give the following crucial geometric observation.

Proposition 4.4. *For each $S_2 \in \mathbb{S}_{\kappa_2}$, we have*

$$\begin{aligned} & \#\{\, (Q, S_1) \in \mathbb{Q} \times \mathbb{S}_1''(Q_0) \,:\, S_1 \cap R^\delta Q \neq \emptyset \,, \\ & \qquad S_2 \cap R^\delta Q \neq \emptyset \,,\, R \gtrsim dist(Q_0, Q) \gtrsim R^{1-\delta} \,\} \\ & \lesssim R^{C\delta + C\nu} \min\left\{ \frac{1}{\sqrt{\varepsilon}\sin\theta} \,,\, \frac{1}{(\sin\theta)^3} \right\}^{1-2\nu}. \end{aligned}$$

We omit the proof. Combining this proposition with (55), we see that

$$R^{-n\delta}\lambda_1\mu_2\,\#\mathbb{S}_1''(Q_0) \lesssim R^{C\delta+C\nu}\min\left\{\frac{1}{\sqrt{\varepsilon}\sin\theta},\frac{1}{(\sin\theta)^3}\right\}^{1-2\nu}\#\mathbb{S}_{\kappa_2}(\gamma_2)$$

and thus we obtain the desired result (53), since $\#\mathbb{S}_{\kappa_2}(\gamma_2) \lesssim \gamma_2^{-2}$ by (22). Hence we have proved Theorem 2.2. □

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References

[1] T. Akahori, Global solutions of the wave-Schrödinger system below L^2, Hokkaido Math. J., **35** (2006), 779-813.

[2] J. Bourgain, On the restriction and multiplier problem in R^3, Lecture Notes in Math., 1469, Springer, Berlin, 1991, 179–191.

[3] J. Bourgain, Estimates for cone multipliers, In: Geometric Aspects of Functional Analysis, Israel, 1992-1994, Oper. Theory Adv. Appl., **77**, Birkhäuser, Basel, 1995, pp. 41–60.

[4] J. Bourgain and J. Colliander, On wellposedness of the Zakharov system, Internat. Math. Res. Notices, **11** (1996), 515–546.

[5] T. Tao, The Bochner-Riesz conjecture implies the restriction conjecture, Duke Math. J., **96** (1999), 363–375.

[6] T. Tao, A sharp bilinear restriction estimate for paraboloids, Geom. Funct. Anal., **13** (2003), 1359–1384.

[7] T. Tao, Some recent progress on the restriction conjecture, In: Fourier analysis and convexity, Appl. Numer. Harmon. Anal., 2004, Birkhäuser Boston, pp. 217–243.

[8] T. Tao and A. Vargas, A bilinear approach to cone multipliers. I. Restriction estimates, Geom. Funct. Anal., **10** (2000), 185–215.

[9] T. Tao, A. Vargas and L. Vega, A bilinear approach to the restriction and Kakeya conjectures, J. Amer. Math. Soc., **11** (1998), 967–1000.

[10] T. Wolff, A sharp bilinear cone restriction estimate, Ann. of Math. (2), **153** (2001), 661–698.

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