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Duality of Euler data for affine varieties

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Abstract.

We compare the Euler-Poincaré characteristic to the global Euler obstruction, in case of singular affine varieties, and point out a certain duality among their expressions in terms of strata of a Whitney stratification.

The local Euler obstruction was defined by MacPherson [MP], as a key ingredient for introducing Chern classes for singular spaces. Results on the local Euler obstruction have been obtained during the time by, among others, A. Dubson, M.-H. Schwartz, J.-P. Brasselet, G. Gonzalez-Sprinberg, B. Teissier, Lê D.T, J. Schürmann, J. Seade. Some of them are surveyed in [Br] and [Sch2]. For more recent results and generalizations one can look up [BLS, BMPS, Sch1, STV1, STV2].

For a connected singular algebraic closed affine space $Y \subset \mathbb{C}^N$ we have defined in [STV1] a *global Euler obstruction* $\mathrm{Eu}(Y)$. The definition in the global setting can be traced back to Dubson's viewpoint [Du]. It immediately follows that, for a *non-singular* Y, $\mathrm{Eu}(Y)$ equals the Euler characteristic $\chi(Y)$. The natural question that we address here is how these two "Euler data" compare to each other whenever Y is *singular.*

Both objects, Eu and χ, can be viewed as constructible functions with respect to some Whitney (b)-regular algebraic stratification of Y. Let us fix such a stratification $\mathcal{A} = \{\mathcal{A}_i\}_{i\in\Lambda}$ on Y. We first show how $\mathrm{Eu}(Y)$ and $\chi(Y)$ can be expressed in terms of strata such that the formulas are, in a certain sense, dual:

$$\mathrm{Eu}(Y) = \sum_{i\in\Lambda} \chi(\mathcal{A}_i)\,\mathrm{Eu}_Y(\mathcal{A}_i), \tag{0.1}$$

$$\chi(Y) = \sum_{i\in\Lambda} \mathrm{Eu}(\mathcal{A}_i)\chi(\mathrm{NMD}(\mathcal{A}_i)). \tag{0.2}$$

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The duality consists in the observation that the formulas are obtained one from another by interchanging Eu with χ. To the Euler characteristic $\chi(\mathcal{A}_i)$ of some stratum $\mathcal{A}_i$ in formula (0.1) corresponds the global Euler obstruction $\mathrm{Eu}(\mathcal{A}_i)$ of the same stratum in formula (0.2). The latter has the following meaning: as it will be explained in §1, the Euler obstruction $\mathrm{Eu}(\bar{\mathcal{A}}_i)$ of the algebraic closure $\bar{\mathcal{A}}_i$ of $\mathcal{A}_i$ in $\mathbb{C}^N$ is well defined and depends only on the open part $\mathcal{A}_i$. We may therefore set $\mathrm{Eu}(\mathcal{A}_i) := \mathrm{Eu}(\bar{\mathcal{A}}_i)$. In case of a point-stratum $\{y\}$, we set $\mathrm{Eu}(\{y\}) = 1$.

Let us explain how the "normal Euler data" $\chi(\mathrm{NMD}(\mathcal{A}_i))$ and $\mathrm{Eu}_Y(\mathcal{A}_i)$ fit into this correspondence. Both data are attached to a general slice $\mathcal{N}_i$ of complementary dimension of the stratum $\mathcal{A}_i$ at some point $p_i \in \mathcal{A}_i$.

Firstly, $\mathrm{NMD}(\mathcal{A}_i)$ stands for the *normal Morse data* of the stratum $\mathcal{A}_i$ (after Goresky-MacPherson's [GM]), i.e. the Morse data of $(\mathcal{N}_i, p_i)$, see §2.

Secondly, $\mathrm{Eu}_Y(\mathcal{A}_i)$ denotes the *normal Euler obstruction* of the stratum $\mathcal{A}_i$, i.e. the local Euler obstruction of $\mathcal{N}_i$ at p_i.

It is known that both data are independent on the choices of $\mathcal{N}_i$ and of p_i. We refer to §2 for the definitions and more details.

We finally consider the case when Y is a locally complete intersection with arbitrary singularities. We show (Proposition 3.1) how the difference $\chi(Y) - \mathrm{Eu}(Y)$ can be expressed in terms of Betti numbers of complex links and the polar invariants α_Y defined in §1. If the singularities are isolated then the difference $\chi(Y) - \mathrm{Eu}(Y)$ measures the total "quantity of slice-singularities" of Y, see (3.3).

For another comparison of the Euler characteristic, namely to the total curvature, in case of an affine hypersurface, we send the reader to [ST].

§1. Global Euler obstruction

Since $Y \subset \mathbb{C}^N$ is affine, one has a well defined *link at infinity* of Y, denoted by $K_\infty(Y) := Y \cap S_R$. It follows from Milnor's finiteness argument [Mi, Cor. 2.8] and from standard isotopy arguments that $K_\infty(Y)$ does not depend on the radius R, provided that R is large enough.

Let $\widetilde{Y} = \mathrm{closure}\{(x, T_xY_{\mathrm{reg}}) \mid x \in Y_{\mathrm{reg}}\} \subset Y \times G(d, N)$ be the Nash blow-up of Y, where $G(d, N)$ is the Grassmannian of complex d-planes in $\mathbb{C}^N$. Let $\nu\colon \widetilde{Y} \to Y$ denote the natural projection and let $\widetilde{T}$ denote the restriction over $\widetilde{Y}$ of the bundle $\mathbb{C}^N \times U(d, N) \to \mathbb{C}^N \times G(d, N)$, where $U(d, N)$ is the tautological bundle over $G(d, N)$. This is the "Nash bundle" over $\widetilde{Y}$. We next consider a continuous, stratified vector field $\mathbf{v}$ on a subset $V \subset Y$. The restriction of $\mathbf{v}$ to V has a well-defined canonical lifting $\tilde{\mathbf{v}}$ to $\nu^{-1}(V)$ as a section of the Nash bundle $\widetilde{T} \to \widetilde{Y}$ (see e.g. [BS], Prop. 9.1).

We refer to [STV1] for other details concerning the following definition (which can be traced back to Dubson's approach), and in particular for the discussion on the independence on the choices:

Definition 1.1. Let $\tilde{\mathbf{v}}$ be the lifting to a section of the Nash bundle $\widetilde{T}$ of a stratified vector field $\mathbf{v}$ over $K_\infty(Y) = Y \cap S_R$, which is radial with respect to the sphere S_R. The obstruction to extend $\tilde{\mathbf{v}}$ as a nowhere zero section of $\widetilde{T}$ within $\nu^{-1}(Y \cap B_R)$ is a relative cohomology class $o(\tilde{\mathbf{v}}) \in H^{2d}(\nu^{-1}(Y \cap B_R), \nu^{-1}(Y \cap S_R)) \simeq H^{2d}(\tilde{Y})$.

One calls *global Euler obstruction of* Y, and denotes it by $\mathrm{Eu}(Y)$, the evaluation of $o(\tilde{\mathbf{v}})$ on the fundamental class of the pair $(\nu^{-1}(Y \cap B_R), \nu^{-1}(Y \cap S_R))$.

By obstruction theory, $\mathrm{Eu}(Y)$ is an integer and does not depend on the radius of the sphere defining the link at infinity $K_\infty(Y)$. We have shown in [STV1, Theorem 3.4] that $\mathrm{Eu}(Y)$ can be expressed in terms of polar multiplicities as follows, denoting $d = \dim Y$:

$$\mathrm{Eu}(Y) = \sum_{j=1}^{d+1} (-1)^{d-j+1} \alpha_Y^{(j)}, \tag{1.1}$$

where:

$$\alpha_Y^{(1)} := \text{the number of Morse points of a global generic linear function on} Y_{\mathrm{reg}}. \tag{1.2}$$

After taking a general hyperplane slice $H \cap Y$, the second number is $\alpha_Y^{(2)} := \alpha_{H\cap Y}^{(1)}$. This continues by induction and yields a sequence of non-negative integers:

$$\alpha_Y^{(1)}, \alpha_Y^{(2)}, \dots, \alpha_Y^{(d)},$$

which we complete by $\alpha_Y^{(d+1)} :=$ the number of points of the intersection of Y_{reg} with a global generic codimension d plane in $\mathbb{C}^N$.

Of course $\alpha_Y^{(k)}$ depends on the embedding of Y into $\mathbb{C}^N$. Nevertheless, these invariants (and therefore, by the equality (1.1), $\mathrm{Eu}(Y)$ too) depend only on some Zariski open part of Y. Now, for a stratum $\mathcal{A}_i$ from the stratification $\mathcal{A} = \{\mathcal{A}_i\}_{i\in\Lambda}$ of Y, the global Euler obstruction $\mathrm{Eu}(\bar{\mathcal{A}}_i)$ of its Zariski closure $\bar{\mathcal{A}}_i$ is well-defined. However, since we have seen that this depends only on the open part $\mathcal{A}_i$, we can use the notation

$\mathrm{Eu}(\mathcal{A}_i)$ for $\mathrm{Eu}(\bar{\mathcal{A}}_i)$. This convention explains the occurrence of $\mathrm{Eu}(\mathcal{A}_i)$ instead of $\mathrm{Eu}(\bar{\mathcal{A}}_i)$ in formula (0.2).

If the highest dimensional stratum is denoted by $\mathcal{A}_0$, then we have $\bar{\mathcal{A}}_0 = Y$ and therefore $\mathrm{Eu}(Y) = \mathrm{Eu}(\mathcal{A}_0)$.

§2. The dual formula

The equality (0.1) was explained in [STV1]. It follows by Dubson's [Du, Theorem 1] applied to our setting. In case of germs of spaces a similar formula was proved in [BLS, Theorem 3.1] by using the Lefschetz slicing method. A different proof may be derived from [BS, Theorem 4.1]. For a more general proof, in terms of constructible functions, we send to [Sch2, (5.65)].

We now give a proof of the equality (0.2). This can be viewed as a global index theorem, similar to Kashiwara's local index theorem (see for this [Sch2, (5.38), (5.38)]). Our proof will only use the equality (1.1).

Definition 2.1 (cf. [GM]). The *complex link* of a space germ (X, x) is the general fibre in the local Milnor-Lê fibration defined by a general (linear) function germ at x. Up to homotopy type, this does not depend on the stratification or the choices of the representatives of the space or of the general function.

Let $\mathrm{CL}_Y(\mathcal{A}_i)$ denote the *complex link of the stratum* $\mathcal{A}_i$ of Y. This is by definition the complex link of the germ $(\mathcal{N}_i, p_i)$, where $\mathcal{N}_i$ is a generic slice of Y at some $p_i \in \mathcal{A}_i$, of codimension equal to the dimension of $\mathcal{A}_i$. Let us remark that the complex link of a point-stratum $\{y\}$ is precisely the complex link of the germ (Y, y).

Let $\mathrm{Cone}(\mathrm{CL}_Y(\mathcal{A}_i))$ denote the cone over this complex link. We denote by $\mathrm{NMD}(\mathcal{A}_i)$ the *normal Morse data* at some point of $\mathcal{A}_i$, that is the pair of spaces $(\mathrm{Cone}(\mathrm{CL}_Y(\mathcal{A}_i)), \mathrm{CL}_Y(\mathcal{A}_i))$. After Goresky and MacPherson [GM], the local normal Morse data are local invariants up to homotopy and do not depend on the various choices in cause. The complex link of the highest dimensional stratum $\mathcal{A}_0$ is empty, and we set by definition $\chi(\mathrm{NMD}(\mathcal{A}_0)) = 1$. In the same case, for the normal Euler obstruction we have $\mathrm{Eu}_Y(\mathcal{A}_0) = 1$ by definition.

Theorem 2.2. *Let $Y \subset \mathbb{C}^N$ be an algebraic closed affine space and let $\mathcal{A} = \{\mathcal{A}_i\}_{i\in\Lambda}$ be some Whitney stratification of Y. Then:*

$$\chi(Y) = \sum_{i\in\Lambda} \mathrm{Eu}(\mathcal{A}_i)\chi(\mathrm{NMD}(\mathcal{A}_i)). \tag{2.1}$$

Proof. Take an affine Lefschetz pencil of hyperplanes in $\mathbb{C}^N$ defined by a linear function $l_H\colon \mathbb{C}^N \to \mathbb{C}$. By the genericity of the pencil, there are only finitely many stratified Morse singularities of the pencil, each one contained in a different slice. By the definition (1.2), the number of stratified Morse points on a stratum $\mathcal{A}_i$ of dimension > 0 is precisely $\alpha_{\bar{\mathcal{A}}_i}^{(\dim \mathcal{A}_i)}$.

According to the Lefschetz slicing method applied to singular spaces (see e.g. [GM]), the space Y is obtained from a generic hyperplane slice $Y \cap \mathcal{H}$ of the pencil, to which are attached cones over the complex links of each singularity of the pencil. Goresky and MacPherson have proved that the Milnor data of a stratified Morse function germ is the $(\dim \mathcal{A}_i)$-times suspension of $\mathrm{NMD}(\mathcal{A}_i)$. At the level of Euler characteristic, we then have:

$$\chi(Y) = \chi(Y \cap \mathcal{H}) + \sum_{i \in \Lambda} (-1)^{\dim \mathcal{A}_i} \alpha_{\bar{\mathcal{A}}_i}^{(1)} \chi(\mathrm{NMD}(\mathcal{A}_i)), \tag{2.2}$$

The sign $(-1)^{\dim \mathcal{A}_i}$ is due to the repeated suspension of the normal Morse data. By convention, for 0 dimensional strata $\mathcal{A}_i$ we put $\alpha_{\bar{\mathcal{A}}_i}^{(1)} := 1$, and therefore $\mathrm{Eu}(\bar{\mathcal{A}}_i) = 1$. We apply formula (2.2) to $Y \cap \mathcal{H}$ and to the successive generic slicings in decreasing dimensions. In the resulting equality, we get the sum of all the coeffients of $\chi(\mathrm{NMD}(\mathcal{A}_i))$, for each $i \in \Lambda$. We may then identify this sum to $\mathrm{Eu}(\bar{\mathcal{A}}_i)$ via the formula (1.1). This ends our proof. Q.E.D.

§3. Case of locally complete intersections

We consider here the case of a locally complete intersection $Y \subset \mathbb{C}^N$ of dimension d, with arbitrary singularities. Being a locally complete intersection implies however that the complex link of any stratum $\mathcal{A}_i$ is homotopy equivalent to a bouquet of spheres of dimension equal to $\mathrm{codim}_Y \mathcal{A}_i - 1$, by Lê's result [Lê]. Let $b_{d-\dim \mathcal{A}_i - 1}(\mathrm{CL}_Y(\mathcal{A}_i))$ denote the Betti number of this complex link. One can then write the formula (2.2) in the following form:

$$\chi(Y) = \chi(Y \cap \mathcal{H}) + (-1)^d (\alpha_Y^{(1)} + \beta_Y^{(1)}) \tag{3.1}$$

where $\beta_Y^{(1)}$ collects the contributions from all the lower dimensional strata in the sum (2.2), more precisely, under our assumption we have:

$$\beta_Y^{(1)} := \sum_{i \in \Lambda \setminus \{0\}} \alpha_{\bar{\mathcal{A}}_i}^{(1)} b_{d-\dim \mathcal{A}_i - 1}(\mathrm{CL}_Y(\mathcal{A}_i)).$$

According to their definitions, $\alpha_Y^{(1)}$ and $\beta_Y^{(1)}$ are both non-negative integers. Their sum represents the number of d-cells which have to be attached to $Y \cap \mathcal{H}$ in order to obtain Y.

Let us define $\beta_Y^{(k)}$ for $k \geq 2$, by:

$$\beta_Y^{(2)} := \beta_{Y\cap\mathcal{H}}^{(1)}$$

and so on by induction, for successive slices of Y, as in case of the $\alpha_Y^{(k)}$-series defined before. [1]

After repeatedly applying (3.1), and then using (1.1), we get the following expression of the difference among the two Euler data:

Proposition 3.1.

$$\chi(Y) - \mathrm{Eu}(Y) = \sum_{k=1}^{d}(-1)^{d-k+1}\beta_Y^{(k)}. \tag{3.2}$$

Remark 3.2. Let us see what becomes this difference in case Y is a hypersurface, or a locally complete intersection, with *isolated singularities.* For an isolated singular point $q \in Y$, let $\mu_q^{\langle d-1\rangle}(Y)$ denote the Milnor number of the local complete intersection $(Y \cap \mathcal{H}, q)$ which is the result of slicing Y by a generic hyperplane $\mathcal{H}$. In case Y is a hypersurface, this is the second highest Milnor-Teissier number in the sequence $\mu_q^*(Y)$. We get:

$$\chi(Y) - \mathrm{Eu}(Y) = (-1)^d \sum_{q\in \mathrm{Sing}\, Y} \mu_q^{\langle d-1\rangle}(Y). \tag{3.3}$$

Since by convention $\alpha_{\{q\}}^{(1)} = 1$, and since $b_{d-1}(\mathrm{CL}_Y(\{q\})) = \mu_q^{\langle d-1\rangle}(Y)$, formula (3.3) is indeed a particular case of formula (3.2). This can be also proved by using the local Euler obstruction formula [BLS, Theorem 3.1].

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