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Characteristic classes of singular varieties

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Abstract. This is a short and concise survey on recent results on the Milnor classes of global complete intersections. By definition the Milnor class of X equals the difference between the Chern-Schwartz-MacPherson and the Fulton-Johnson classes of X and we describe the results that express it in terms of the local and global invariants of the singular locus of X. In this survey we underline the characteristic cycle approach and its realtion to the vanishing Euler characteristic, as for instance to the Euler characteristic of the Milnor fibre in the hypersurface case.

We present some recent developments in the theory of characteristic classes of singular algebraic and analytic varieties. We would like, in particular, underline the characteristic cycle approach and the geometric insight given by this construction. For different approaches the reader may consult the excellent surveys [7] and [45].

Several different characteristic classes can be defined for a singular variety X: the Chern-Schwartz-MacPherson class $c_*(X)$, the Chern-Mather class $c_M(X)$, the Fulton class $c^F(X)$ and the Fulton-Johnson class $c^{FJ}(X)$. For nonsingular X they are all equal to the Poincaré dual of the Chern class $c(TX)$ of the tangent bundle. We present in this survey some results that answer the following question: how does the difference $c^F(X)-c_*(X)$ (or $c^{FJ}(X)-c_*(X)$) depend on the singularities of X?

The characteristic classes of singular varieties may be defined in different set-ups. For complex algebraic varieties they take values in the

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Chow group $A_*(\cdot)$. (They can be even defined for algebraic varieties over an arbitrary algebraically closed field of characteristic zero, cf. [21, 22].) For complex analytic spaces, assumed compact or such that they can be compactified, the characteristic classes take values in the Borel-Moore homology $H_*^{BM}(\cdot;\mathbb{Z})$. For complex algebraic varieties both approaches are linked by the cycle map $cl : A_*(\cdot) \to H_*^{BM}(\cdot;\mathbb{Z})$. For simplicity of exposition by variety we mean either a complex analytic space or an algebraic variety, and then by homology we mean $H_*^{BM}(\cdot;\mathbb{Z})$ or $A_*(\cdot)$ respectively.

We shall also include a brief review of Stiefel-Whitney classes that can be defined for real algebraic varieties or, in general, for Euler mod 2 triangulated spaces and that take values in $H_*^{BM}(\cdot;\mathbb{Z}_2)$.

If X is singular then the tangent bundle to X is not well-defined and therefore one cannot consider simply the characteristic classes of this bundle. Suppose X is a subvariety of a non-singular variety M. We recall briefely the definitions of the Fulton and Fulton-Johnson classes, see [16] for details. The idea is to find an object which plays the role of the normal bundle to X in M. Let C_XM be the *normal cone* to X in M and let $\mathcal{J}$ be the ideal sheaf of X in C_XM. Denote by $\mathcal{N}_XM = \mathcal{J}/\mathcal{J}^2$ the *conormal sheaf* of X in M. Let p be the blow-up of X in M. The exceptional divisor of p can be identified with the projectivization of C_XM. The *Segre class* of X in M is given by:

$$s(X,M) = s(C_XM) = p_*\big(\sum c_1(\mathcal{O}(1))^i \cap [\mathbb{P}C_XM]\big),$$

where $\mathcal{O}(1)$ denotes the canonical line bundle on $\mathbb{P}C_XM$.

The *Fulton class* of X is defined by

$$c^F(X) = c(TM|_X) \cap s(X,M).$$

The *Fulton-Johnson class* of X equals

$$c^{FJ}(X) = c(TM|_X) \cap s(\mathcal{N}_XM).$$

Both the Fulton and the Fulton-Johson classes are independent of the embedding of X into non-singular variety, cf. [16], Example 4.2.6. If X is regularly embedded in M (i. e. $\mathcal{J}$ is locally generated by a regular sequence) then the both classes coincide and

$$c^{FJ}(X) = c^F(X) = c(\tau_X) \cap [X],$$

where $\tau_X = TM|_X - N_XM$ is the *virtual tangent bundle to* X. In this case $N_XM = dual(\mathcal{J})$ is a vector bundle and is canonically isomorphic to C_XM, cf. [16], Appendix B7.

Another possibility is to recover the tangent bundle to singular X by the means of the Nash-blowing-up, see [26], [16], Example 4.2.9. Let $\nu : \tilde{X} \to X$ be the Nash blowing-up of X and let $\tilde{T}$ denote the vector bundle on $\tilde{X}$ that extends $\nu^* TX$. Then the *Chern-Mather class* $c_M(X)$ of X equals by definition

$$c_M(X) := \nu_*(c(\tilde{T}) \cap [\tilde{X}]).$$

In section 1 below we recall the definition of the Chern-Schwartz-MacPherson class $c_*(X)$ of X. Thus we have at least four different notions of characteristic classes that coincide for X nonsingular.

Example 0.1. (Hypersurface with an isolated singularity).
Let $L \to M$ a line bundle, M nonsingular, and let X be the zero scheme of a holomorphic section f of L. Suppose that, moreover, X has an isolated singularity $SingX = \{p_0\}$. Then

$$\begin{aligned} c^{FJ}(X) &= c^F(X) = c(TM - L) \cap [X] \\ c_*(X) &= c(TM - L) \cap [X] + (-1)^n \mu_n [p_0] \\ c_M(X) &= c(TM - L) \cap [X] + (-1)^n (\mu_n + \mu_{n-1}) [p_0]. \end{aligned}$$

where $TM - L$ is the virtual tangent bundle of X, μ_n is the Milnor number of f at p_0 and μ_{n-1} is the Milnor number of the generic hyperplane section of f at p_0.

We shall study the general hypersurface case in section 2 below.

§1. Chern-Schwartz-MacPherson classes and characteristic cycles

We recall some of the basic results on Chern-Schwartz-MacPherson classes and characteristic cycles. For the details the reader is refered to [7, 17, 21, 26, 32, 36].

1.1. Constructible functions

For a variety X we denote by $F(X)$ the group of integer-valued *constructible functions on* X i.e. finite sums

$$\alpha = \sum_i n_i \mathbb{1}_{V_i}$$

where V_i are subvarieties of X. There are many interesting operations on constructible functions: sum, product, pull-back, push-forward, specialization, duality, and Euler integral inherited from sheaf theory by taking the index of a constructible complex of sheaves. Recall that for a constructible complex of sheaves $\mathcal{F}_\bullet$ on X its index is the stalkwise Euler characteristic $p \to \chi(\mathcal{F}_\bullet)(p) = \sum(-1)^i \dim H^i(\mathcal{F}_\bullet)_p$. It is a constructible function. Note that this definition is purely local so the global properties of $\mathcal{F}_\bullet$ are lost. The operations on constructible functions can be defined independently by means of Euler integral, see [42], [33], [20].

If X is compact then the *Euler integral* of α is defined as the weighted Euler characteristic: $\int \alpha\, d\chi := \sum_i n_i \chi(V_i)$. For a proper map $f: X \to Y$ the *proper push-forward* $f_*: F(X) \to F(Y)$ is given by

$$(f_*\alpha)(y) := \int_{f^{-1}(y)} \alpha\, d\chi.$$

Let $f: X \to S$ be a morphism to a curve and let s_0 be a nonsingular point of S. Denote $X_0 = f^{-1}(s_0)$. The specialization homomorphism $\mathrm{sp}: F(X) \to F(X_0)$, or nearby Euler characteristic, is given by the Euler integral on the Milnor fibre of f. That is, at $p \in X_0$ and for α as above

$$\mathrm{sp}\,(\alpha)(p) = \int_{F_p} \alpha\, d\chi = \sum_i n_i \chi(F_p \cap V_i), \tag{1}$$

where F_p is the Milnor fibre of f at p. That is, $F_p = f^{-1}(s) \cap B(p, \varepsilon)$, where, in local systems of coordinates, $B(p,\varepsilon)$ denotes the ball centered at p of radius ε and s is chosen so that $0 < |s - s_0| \ll \varepsilon \ll 1$.

1.2. Chern-Schwartz-MacPherson classes

The *Chern-Schwartz-MacPherson class* (the CSM class for short) c_* is the unique transformation from constructible functions $F(\cdot)$ to homology $H_*(\cdot)$ and satisfying:

(1) $f_* c_*(\alpha) = c_* f_*(\alpha)$ for a proper morphism $f: X \to Y$.
(2) $c_*(\alpha + \beta) = c_*(\alpha) + c_*(\beta)$,
(3) $c_*(\mathbb{1}_X) = c(TX) \cap [X]$ for X nonsingular.

Its existence was conjectured by Deligne and Grothendieck and proven by MacPherson in [26]. They are, by the Alexander duality isomorphism, equal to the characteristic classes introduced by M.-H. Schwartz, cf. [38, 9]

By a theorem of Verdier [41] the CSM class commutes with specialization: $c_* \circ \mathrm{sp} = \mathrm{Sp} \circ c_*$, where $\mathrm{Sp}: H_*(X) \to H_*(X_0)$ is the specialization on homology, see [22] for the Chow group counterpart.

1.3. Characteristic cycles

Let M be a nonsingular variety of dimension n and let T^*M denote the cotangent bundle of M. We consider $\mathcal{L}(M)$: the free abelian group generated by the set of conical Lagrangian subvarieties of T^*M. Thus each element of $\mathcal{L}(M)$ is an integral combination of irreducible Lagrangian subvarieties that can be described as follows. Let V be a closed subvariety of M and let $Reg(V) = V \setminus Sing(V)$ denote the set of regular points of V. The *conormal space* to V in M

$$T_V^*M := Closure\left\{(x,\xi) \in T^*M \mid x \in Reg(V),\ \xi|_{T_x Reg(V)} \equiv 0\right\},$$

is a conical Lagrangian subvariety and each irreducible conical Lagrangian subvariety of T^*M is the conormal space of an irreducible subvariety of M. For a subvariety $X \subset M$ let $\mathcal{L}(X)$ denote the subgroup of $\mathcal{L}(M)$ given by the conical Lagrangian subvarieties of T^*M over X. We call an element of $\mathcal{L}(X)$ a *conical Lagrangian cycle over* X.

To a constructible function $\alpha \in F(X)$ we associate its *characteristic cycle* $\mathrm{Ch}(\alpha) \in \mathcal{L}(X)$ so that we get a group isomorphism $\mathrm{Ch} : F(X) \to \mathcal{L}(X)$. For instance, for a subvariety V, $\mathrm{Ch}(\mathbb{1}_V)$ can be defined by means of the characteristic cycle of a sheaf, cf. for instance [11], by

$$\mathrm{Ch}(\mathbb{1}_V) = \mathrm{Ch}(i_*\mathbb{C}_V),$$

where $i : V \hookrightarrow M$ is the inclusion. Then

$$T_V^*M = (-1)^{\dim V}\,\mathrm{Ch}(Eu_V),$$

where Eu_V denotes MacPherson's Euler obstruction [26]. (In literature there are two sign conventions in the definition of Ch that differ by $(-1)^{\dim M}$. We follow that of [21])

Let $f : (M,p) \to (\mathbb{C},0)$ be the germ of a holomorphic function and let $\alpha = \sum_i n_i \mathbb{1}_{V_i}$ be a constructible function on M. Let $\mathrm{sp}\,\alpha(p)$ be the specialization of α to the zero fibre of f as defined in (1). The difference $\mathrm{sp}\,\alpha(p) - \alpha(p)$ can be interpreted as the vanishing Euler characteristic. Suppose that the graph $Gr(df)$ of df, considered as a section of T^*M, intersects $\mathrm{Ch}(\alpha)$ only at $(p, df(p))$. Then by the index formula for the sheaf vanishing cycles due to Lê, Dubson, and Sabbah, cf. [13] and (4.5) and (4.6) of [32], the local intersection number of the cycles $\mathrm{Ch}(\alpha)$ and $Gr(df)$ equals

$$(2) \qquad (\mathrm{Ch}(\alpha).Gr(df))_{(p,df(p))} = -(\mathrm{sp}\,\alpha(p) - \alpha(p)).$$

Thus one may interpret $\mathrm{Ch}(\alpha)$ as the set of such covectors $(p,\xi) \in T^*M$ that the Euler integral of the fibers of functions $f : (M,p) \to$

$(C,0)$ with $df(p)=\xi$ changes at p. To be more precise, fix a Whitney stratification $\{S_j\}$ of M, such that each V_i is the union of strata. Then, by Thom-Mather theory, there is no change of topology of fibers of $f|_{V_i}$ if $(p,\xi)\notin \bigcup T^*_{S_j}M$. In particular, $\mathrm{Ch}(\alpha)=\sum_i n_i T^*_{S_i}M$ with integer coefficients n_i. In general these coefficients may be zero or negative. By (2) they are determined by the vanishing Euler characteristic of such f that $Gr(df)$ intersects $T^*_{S_i}M$ at a generic point.

Example 1.1. Let $p\in X\subset M$. The coefficient of T^*_pM in $\mathrm{Ch}(\mathbb{1}_X)$ equals

$$1-\chi(lk_{\mathbb{C}}(X,p))$$

where $lk_{\mathbb{C}}(X,p)$ is the complex link of X at p (in local coordinates the intersection of X with generic hyperplane near p).

There are operations of proper push-forward and specialization on conical Lagrangian cycles defined geometrically. Ch is a natural transformation in the sense that it commutes with these operations and the corresponding operations on constructible functions, cf. [17], [21], [32],

(1) $f_*\,\mathrm{Ch}(\alpha)=\mathrm{Ch}\,f_*(\alpha)$ for proper morphisms $f:X\to Y$
(2) $\mathrm{Ch}(\alpha+\beta)=\mathrm{Ch}(\alpha)+\mathrm{Ch}(\beta)$
(3) $\mathrm{Ch}(\mathrm{sp}(\alpha))=\mathrm{Sp}(\mathrm{Ch}(\alpha))$.

By a formula of Sabbah [32], (1.2.1), for $\alpha\in F(X)$

$$c_*(\alpha):=(-1)^{n-1}c\,(TM|_X)\cap\pi_*\left(c(\mathcal{O}(1))^{-1}\cap[\mathbb{P}\,\mathrm{Ch}\,\alpha]\right), \tag{3}$$

where $\mathcal{O}(1)$ is the canonical line bundle on $\mathbb{P}T^*M$ and $\pi:\mathbb{P}T^*M|X\to X$ denotes the projection. Using Sabbah's own words "cela montre que la théorie des classes de Chern de [26] se ramène à une théorie de Chow sur T^*M, qui ne fait intervenir que des classes fondamentales".

The Chern-Mather class of V, see [26], equals

$$c_M(V)=c_*(Eu_V)=(-1)^{n-1-\dim V}c\,(TM|_V)\cap\pi_*\left(c(\mathcal{O}(-1))\cap[\mathbb{P}T^*_VM]\right). \tag{4}$$

Remark 1.2. The CSM class and the Euler obstruction are closely related to the geometry of polar varieties, see [24], and also [7] and the references therein.

§2. Characteristic cycles and Stiefel-Whitney classes

Characteristic cycles can be also defined in real analytic and algebraic geometry for semi-algebraic and subanalytic sets cf. [19], [20], [14], or even for sets defined in any o-minimal structure [34], see also an explicit construction in [36]. More precisely, given an oriented real analytic manifold M, we have a group isomorphism

$$\mathrm{Ch} : F(M) \to \mathcal{L}(M)$$

between the group of subanalytically constructible functions $F(M)$ on M and the group of subanalytic conical Lagrangian cycles $\mathcal{L}(M)$ in T^*M. (Here by conical we mean $\mathbb{R}_{>0}$-homogeneous.) The most important difference from the complex case is that the subanalytic Lagrangian conical cycles in T^*M are not necessarily combination of conormal spaces but usually more complicated subanalytic cycles of T^*M. Moreover usually the conormal space $T^*_V M$ is not a cycle. These differences are caused by the fact that for a subanalytic continuous function $f : (V, 0) \to (\mathbb{R}, 0)$, $V \subset M$ subanalytic closed, or even for f and V real analytic, the vanishing Euler characteristic from the right (i.e., defined by the positive Milnor fiber) may not be equal to that from the left ((i.e., defined by the negative Milnor fiber). Note that in the real set-up there are more possible conventions on the sign, for instance the characteristic cycle constructed by Fu [14] corresponds to that of Kashiwara-Schapira [20] after the application of the antipodal map (multiplication by -1 in the fibers of T^*M).

Example 2.1. Let $V \subset M$ be subanalytic closed and let $\{S_i\}$ be a subanalytic Whitney stratification of V. Define

$$\Lambda^o := \bigsqcup \Lambda^o_{S_i}, \qquad \Lambda^o_{S_i} = T^*_{S_i} M \setminus \bigcup_{j \neq i} T^*_{S_j} M.$$

Decompose Λ^o into the connected components $\Lambda^o := \bigsqcup \Lambda^o_j$. Then

$$\mathrm{Ch}(\mathbb{1}_V) = \sum n_j \Lambda^o_j,$$

for some integers n_i that can be described topologically by the vanishing Euler characteristic, see the index formula below.

The analogue of the index formula (2), [20] Thm. 9.5.6, see also [19] and [37], has even more flavour of the Morse Theory. It says that for $V \subset M$ subanalytic closed, and a real analytic $f : (M, p) \to (\mathbb{R}, 0)$ such that $Gr(df)$ intersects $\mathrm{Ch}(\mathbb{1}_V)$ only at $(p, df(p))$

$$\begin{aligned}(Gr(df). \mathrm{Ch}(\mathbb{1}_V))_{(p,df(p))} &= \chi(B \cap \{x \in V, f(x) \leq +\delta\}) \\ &\quad -\chi(B \cap \{x \in V, f(x) \leq -\delta\}),\end{aligned}$$

where B denotes the ball of radius ε centered at p, $0 < \delta \ll \varepsilon \ll 1$. Given a conical Lagrangian cycle $\Lambda = \mathrm{Ch}(\alpha) \in \mathcal{L}(M)$. In order to recover the value $\alpha(p)$ at $p \in M$ it suffices to intersect Λ with $Gr(df)$, where $f : (M,p) \to (\mathbb{R}, 0)$ is a Morse function of index 0 (for instance $f(x) = x_1^2 + \cdots + x_n^2$ in local coordinates). Then

$$\alpha(p) = (Gr(df).\Lambda)_{(p,df(p))}$$

Remark 2.2. The operation inverse to Ch is related to MacPherson's Euler obstruction as follows. Let M be a complex manifold and V a complex analytic subvariety of M. Consider V as a subanalytic subset of M and M itself as an oriented real analytic manifold. Then $T^*_V M$ is a real Lagrangian cycle. Let $p \in V$ and $f : (M,p) \to (\mathbb{R}, 0)$ be a real Morse function of index 0. Then $(Gr(df).T^*_V M))_{(p,df(p))} = (-1)^{\dim_{\mathbb{C}} V} Eu_V(p)$. This formula for the Euler obstruction is essentially the definition of MacPherson, where the intersection $Gr(df)$ is replaced by the intersection with the section given by the radial vector field.

2.1. Stiefel-Whitney classes

In 1935 Stiefel defined a characteristic class $w_i(X) \in H_i(X; \mathbb{Z}_2)$ for any smooth compact manifold. He conjectured that $w_i(X)$ is represented by the sum of all the i-simplices of the first barycentric subdivision of a triangulation of X. Stiefel's Conjecture was proved by Whitney in 1939. In 1969 Sullivan observed that Stiefel's definition can be applied to real analytic spaces since they are (mod 2) Euler spaces, that is to say, the link of each point has even Euler characteristic. Then, for a triangulated Euler space, the sum of all the i-simplices of the first barycentric subdivision is a $\mathbb{Z}_2$-cycle.

It was noticed in [15] that the Stiefel-Whitney classes of subanalytic sets can be defined via the characteristic cycles. We give below just a short account, for details the reader is refered to [15].

Remark 2.3. ([33], [20]) Verdier Duality on sheaves induces a duality on constructible functions. This duality can be written as

$$D\alpha(p) = \alpha(p) - \int_{S_p^\varepsilon} \alpha \, d\chi,$$

where S_p^ε is a small sphere centered at p. The corresponding duality on the conical Lagrangian cycles is given by the antipodal map that is by the multiplication by (-1) in the fibres of T^*M. Note that in the complex case the duality on constructible function and the one on conical Lagrangian cycles are the identity maps.

Let M be an oriented real analytic manifold. A subanalytically constructible function $\alpha \in F(M)$ is called *(mod 2) Euler* if it is self dual modulo 2 (equivalently its Euler integral along any small sphere is even). For such a function the projectivization of its characteristic cycle

$$\mathbb{P}Ch(\alpha) \subset \mathbb{P}T^*M$$

is a (mod 2)-cycle.

For a (mod 2) Euler constructible function $\alpha \in F(M)$ one may define its ith Stiefel-Whitney class by a formula corresponding to (3)

$$w_i(\alpha) = \pi_*(\gamma_M^{n-i-1} \cap [\mathbb{P}Ch(\alpha)])$$

where $\pi : \mathbb{P}T^*M \to M$ is the projection and

$$\gamma_M^k = \sum_j \pi^*(w^i(TM)) \cap \zeta_M^{k-j},$$

where $\zeta_M \in H^1(\mathbb{P}T^*M; \mathbb{Z}_2)$ is the first Stiefel-Whitney class of the tautological line bundle on $\mathbb{P}T^*M$.

Defined this way, Stiefel-Whitney homological classes satisfy the axioms analogous to the Deligne-Grothendieck axioms for the CSM-classes and the Verdier specialization property.

§3. Hypersurface case

Let M be a nonsingular compact complex analytic variety of pure dimension n and let L be a holomorphic line bundle on M. Take $f \in H^0(X, L)$ a holomorphic section of L such that the variety X of zeros of f is a reduced hypersurface in M.

Consider the constructible function $\chi : X \to \mathbb{Z}$ defined for $x \in X$ by $\chi(x) := \chi(F_x)$, where F_x denotes the Milnor fibre at x and $\chi(F_x)$ its Euler characteristic. Also, define $\mu := (-1)^{n-1}(\chi - 1\!\!1_X)$, that is the signed vanishing Euler characteristic.

In this section, following [1], [2], [29], [3], we give the common descriptions of the CSM and Fulton classes of X as well as the results that present the contribution of the singularities of X to the difference of these two classes. Similarly to [29] our approach is based on the computation of characteristic cycle of X. For an account on different possible approaches see [4].

3.1. Local description of characteristic cycle

The characteristic cycle of X was calculated in [5] and [23] in terms of the blow-up of the Jacobian ideal of a local equation of X in M. More precisely, let $X \subset U \subset \mathbb{C}^n$ be the zero set of a holomorphic function $f : U \to \mathbb{C}$ and denote by $\pi : \mathrm{Bl}_{\mathcal{J}_f} U \to U$ the blowing-up of the Jacobian ideal of $\mathcal{J}_f = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)$.

Let $\mathcal{X} = \pi^{-1}(X)$ denote the total transform of X by π and we denote the irreducible components of $\mathcal{X}$ by D_i and by C_i their projections onto U. Denote by $\mathcal{I}_{C_i}$ the ideal defining C_i and the multiplicities of $\mathcal{I}_{C_i}$, f, and $\mathcal{J}_f$ along D_i by n_i, m_i and p_i respectively. Note that D_i is contained in the exceptional divisor of π if and only if $p_i = 0$. It is known by the transversality of polar varieties that $m_i = n_i + p_i$, see [40], [5], and [31].

By [5], [23], we have the following explicit formulas

$$\mathrm{Ch}(\mathbb{1}_X) = (-1)^{n-1} \sum_i n_i T^*_{C_i} U\,;$$

$$\mathrm{Ch}(\chi) = \mathrm{Ch}(\mathrm{R}\,\Psi_f \mathbb{C}_U) = (-1)^{n-1} \sum_i m_i T^*_{C_i} U\,;$$

$$\mathrm{Ch}(\mu) = (-1)^{n-1}\,\mathrm{Ch}\big(\mathrm{R}\,\Phi_f \mathbb{C}_U\big) = \sum_i p_i T^*_{C_i} U\,.$$

(Here $\mathrm{R}\,\Psi_f$ and $\mathrm{R}\,\Phi_f$ denote the complexes of nearby and vanishing cycles respectively.)

$\mathrm{Bl}_{\mathcal{J}_f} U$ can be interpreted geometrically by means of the *relative conormal space* $T^*_f \subset T^*U$

$$T^*_f := \text{ Closure } \{(x, \eta) \subset T^*U; df(x) \neq 0, \exists_\lambda \text{ such that } \eta = \lambda df(x)\}.$$

Let $\tilde f : T^*_f \to \mathbb{C}$ denote the composition of the projection $T^*_f \to U$ and f. Then $\tilde f^{-1}(c)$, for a regular value c, equals the conormal space to $f^{-1}(c)$. Thus by Lagrangian specialization, cf. [25], [18], $\tilde f^{-1}(0)$ is a conical Lagrangian subvariety of T^*U. It is equal to $\mathrm{Ch}(\chi)$ since Ch commutes with specialization. Moreover the total transform $\mathcal{X}$ of X by π, is the set of limits of the direction of the gradient $[\frac{\partial f}{\partial z_1}(x) : \dots : \frac{\partial f}{\partial z_n}(x)]$ and hence equals, at least as a set, the projectivization of $\tilde f^{-1}(0)$. Thus $\tilde f^{-1}(0)$ is the union of conormals $T^U_{C_i}$ and each $D_i = \mathbb{P}T^*_{C_i} U$. In particular, we may rewrite the above formulas as follows

$$(5)\qquad \begin{aligned} [\mathbb{P}\,\mathrm{Ch}(\mathbb{1}_X)] &= (-1)^{n-1}\left([\mathcal{X}] - [\mathcal{Y}]\right)\,; \\ [\mathbb{P}\,\mathrm{Ch}(\chi)] &= (-1)^{n-1}\,[\mathcal{X}]\,; \\ [\mathbb{P}\,\mathrm{Ch}(\mu)] &= [\mathcal{Y}] \end{aligned}$$

where $\mathcal{Y}$ denote the exceptional divisor of π.

Remark 3.1. The computation of coefficients n_i, m_i and p_i can be done by a topological argument based on the Morse theory and generic polar curves, see for instance [23]. In particular, in the isolated singularity case, $SingX = \{p_0\}$, the coefficients at $T^*_{p_0}U$ are equal to $(-1)^{n-1}\mu_{n-1}$, $(-1)^{n-1}(\mu_n+\mu_{n-1})$, and μ_n respectively. Here μ_n denotes the Milnor number of f at p_0 and μ_{n-1} the Milnor number of the generic hyperplane section of f at p_0. One may show that $1-(-1)^{n-1}\mu_{n-1}$ equals the Euler characteristic of the complex link of X at p_0.

3.2. Global description of characteristic cycle

The singular scheme of X, that we denote by Y, is defined in local coordinates by $(f, \mathcal{J}_f) = \left(f, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$. Since f belongs to the integral closure of $\mathcal{J}_f$, the normalizations of blow-ups of $\mathcal{J}_f$ and $(f, \mathcal{J}_f)$ are equal. Hence the formulas (5) hold true locally if we replace the blow-up of the former ideal by the blow-up of the latter one. We shall see that they hold true globally.

Let $B = \mathrm{Bl}_Y M \to M$ be the blow-up of M along Y. Let $\mathcal{X}$ and $\mathcal{Y}$ denote the total transform of X and the exceptional divisor in B respectively. To get a convenient description of B, we use the bundle $\mathcal{P}^1_M L$ of principal parts of L over M, as in [2], [31]. The differentials and the sections of L take values in $\mathcal{P}^1_M L$ and also $\mathcal{P}^1_M L$ fits in an exact sequence

$$0 \to T^*M \otimes L \to \mathcal{P}^1_M L \to L \to 0.$$

Thus f determines a section of $\mathcal{P}^1_M L$ that is written locally as $(df, f) = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}, f\right)$. The closure of the image of the meromorphic map $M \dashrightarrow \mathbb{P}\mathcal{P}^1_M L$ induced by this section is the blow-up $B \to M$. Thus we may treat B as a subvariety of $\mathbb{P}\mathcal{P}^1_M L$. Clearly, the total transform $\mathcal{X}$ of X equals $B \cap \mathbb{P}(T^*M \otimes L)$, that we identify with a subvariety of $\mathbb{P}(T^*M \otimes L)$. Since $\mathbb{P}(T^*M \otimes L) = \mathbb{P}(T^*M)$ we see that the formulas (5) hold globally.

By an elementary computation on $\mathbb{P}(T^*M \otimes L)$, see [31], this gives

$$\begin{aligned} c_*(X) &= c(TM|_X) \cap \pi_* \left(\frac{[\mathcal{X}] - [\mathcal{Y}]}{1 + \mathcal{X} - \mathcal{Y}}\right) \\ c_*(\chi) &= c(TM|_X) \cap \pi_* \left(\frac{[\mathcal{X}]}{1 + \mathcal{X} - \mathcal{Y}}\right); \\ c_*(\mu) &= (-1)^{n-1} c(TM|_X) \cap \pi_* \left(\frac{[\mathcal{Y}]}{1 + \mathcal{X} - \mathcal{Y}}\right). \end{aligned} \tag{6}$$

The first of the above formulas was obtained by Aluffi [2] by means of resolution of singularities and a detailed description of how the formula changes under a blowing-up. He has also got the following formula for the Chern-Mather class of X.

$$c_M(X) = c(TM|_X) \cap \pi_* \left(\frac{[\mathcal{X}']}{1 + \mathcal{X} - \mathcal{Y}} \right),$$

where $\mathcal{X}'$ is the proper transform of X. In local coordinates $[\mathcal{X}'] = \mathbb{P}T_X U$ and hence this result follows from (4).

3.3. Aluffi's formulas. Milnor class of a hypersurface.

From (6) we derive the formulas obtained by Aluffi in [1]. First note that in our case

$$c^F(X) = c^{FJ}(X) = c(TM|_X - L|_X) \cap [X].$$

By birational invariance of Segre classes [16], Chap.4:

$$\begin{aligned} c^F(X) = c(TM|_X) \cap s(X, M) &= c(TM|_X) \cap \pi_* s(\mathcal{X}, B) \\ &= c(TM|_X) \cap \pi_* \left(\frac{[\mathcal{X}]}{1 + \mathcal{X}} \right). \end{aligned}$$

In [1] Aluffi defines a "thickening" of X along its singular subscheme Y: X^k is the subscheme of M defined by the ideal $\mathcal{I}_X \mathcal{I}_Y^k$. He shows that the Fulton class of X^k is a polynomial in k with the CSM class being equal to $c^F(X^{-1})$. Indeed, as above,

$$(7)\quad c^F(X^k) = c(TM|_X) \cap s(X^k, M) = c(TM|_X) \cap \pi_* \left(\frac{[\mathcal{X}] + k[\mathcal{Y}]}{1 + \mathcal{X} - k\mathcal{Y}} \right).$$

This can be expressed in the following suggestive form, cf. [1],

$$c_*(X) = c^F(X^{-1}) = c(TM|_X) \cap s(X \setminus Y, M).$$

The *Milnor class* was first defined by Yokura [44] as

$$\mathcal{M}(X) := (-1)^{n-1} \big(c^{FJ}(X) - c_*(X) \big).$$

As follows from (6), (7)

$$(8)\quad \begin{aligned} \mathcal{M}(X) &= (-1)^{n-1} c(TM|_X) \cap \pi_* \left(\frac{[\mathcal{Y}]}{(1 + \mathcal{X})(1 + \mathcal{X} - \mathcal{Y})} \right) \\ &= (-1)^{n-1} c(TM|_X) \cap \pi_* \left(\frac{[\mathbb{P}\,\mathrm{Ch}(\mu)]}{(1 + \mathcal{X})(1 + \mathcal{X} - \mathcal{Y})} \right) \\ &= c(L|_X)^{-1} c_*(\mu),. \end{aligned}$$

Let $\mathcal{S} = \{S\}$ be any stratification of X such that μ is constant on the strata of $\mathcal{S}$. (One may take, for instance, any Whitney stratification of X.) Denote the value of μ on the stratum S by μ_S and let

$$\alpha(S) := \mu_S - \sum_{S' \neq S, S \subset \overline{S'}} \alpha(S')$$

be the numbers defined inductively on descending dimension of S. Then $\mu = \sum_{S \in \mathcal{S}} \alpha(S) \mathbb{1}_{\overline{S}}$ and

$$\mathrm{Ch}(\mu) = \sum_{S \in \mathcal{S}} \alpha(S)\, \mathrm{Ch}(\mathbb{1}_{\overline{S}}).$$

This gives the following formula on the Milnor class, see [29],

$$\mathcal{M}(X) = \sum_{S \in \mathcal{S}} \alpha(S)\, c(L|_X)^{-1} \cap (i_{\overline{S},X})_* \, c_*(\overline{S}), \tag{9}$$

where $i_{\overline{S},X} : \overline{S} \to X$ denotes the inclusion. This formula was first conjectured by Yokura in [44] and proven in [31]. If Y is smooth and μ is constant on Y, equal say μ_Y, then it reads

$$\mathcal{M}(X) = \mu_Y (c(L|_X)^{-1} \cap (i_{Y,X})_* \, c_*(Y)).$$

If the singular set of X is finite, then we get

$$\mathcal{M}(X) = \sum_{x \in Sing(X)} \mu(x)[x]. \tag{10}$$

This formula was obtained also by Suwa [39] in a more general set-up of isolated singularities of (global) complete intersections.

The Milnor class is related to the μ-class supported on Y, introduced by Aluffi in [1],

$$\mu_L(Y) = c(T^*M \otimes L) \cap s(Y, M).$$

As shown in [2]

$$\mathcal{M}(X) = (-1)^n c(L)^{n-1} (\mu_L(Y)^\vee \otimes L).$$

For the notation $\mu_L(Y)^\vee$ and the proof we refer the reader to [2]. We just note that the above formula can be derived from (8).

Following [3] we give another interpretation of (8). Let Y be a subvariety of M. Let $\pi : C_Y M \to Y$ denote the normal cone to Y in M and let C_i be the irreducible components of $C_Y M$. Denote by m_i their multiplicities and let $Y_i = \pi(C_i)$. Then the *weighted Mather class of* Y is defined by

$$c_{wM}(Y) := \sum (-1)^{\dim Y_i} m_i c_M(Y_i).$$

(here the factor $(-1)^{\dim Y}$ is removed from the original Aluffi's definition). Note that $c_{wM}(Y)$ depends on the scheme structure of Y, in particular it is sensitive to the presence of embedded components. An important property is that $c_{wM}(Y)$ is intrinsic to Y (independent of the ambient nonsingular variety). In the particular case when Y is the singular scheme of a hypersurface X

$$c_{wM}(Y) = c_*(\mu) = c(L|_Y) \cap \mathcal{M}(X).$$

Thus (8) takes the following form

$$c^{FJ}(X) - c_*(X) = (-1)^{n-1} c(L|_{Sing(X)})^{-1} \cap c_{wM}(Sing(X)) \tag{11}$$

with the right hand side depending only on the scheme structure of $Sing(X)$ and on $c(L|_{Sing(X)})$.

Remark 3.2. Many of the results presented above were motivated by their corresponding formulas for the Euler characteristic. The generalized Milnor number was first defined in [28] as

$$\mu(X) := (-1)^{n-1}(c(TM|_X - L|_X) \cap [X] - \chi(X))).$$

If X has only isolated singularities then the generalized Milnor number of X equals the sum of their local Milnor numbers. Yokura's Conjecture, i.e. formula (9), was motivated by a similar formula for the Euler characteristic established in [29].

3.4. Specialization

Suppose now that the line bundle L admits a section $g \in H^0(M, L)$ such that $X' = g^{-1}(0)$ is smooth and transverse to the strata of a Whitney stratification of X. This is for instance the case when L is very ample. The Milnor class, and so the Fulton class, of X equals the CSM class of a simple constructible function on X.

For $t \in \mathbb{C}$, denote $f_t = f - tg$. In this paragraph by $\mathcal{X}$ we denote

$$\mathcal{X} := \left\{(x,t) \in M \times \mathbb{C} \mid f_t(x) = 0\right\}.$$

Let $p : \mathcal{X} \to \mathbb{C}$ be the restriction to $\mathcal{X}$ of the projection onto the second factor of $M \times \mathbb{C}$. Then $p^{-1}(t) = \{x \in M \mid f_t(x) = 0\}$ for $t \in \mathbb{C}$. Denote by

$$\mathrm{sp} : F(\mathcal{X}) \to F(X)$$

the specialization by p. By Proposition 5.1 of [31]

$$\mathrm{sp}\,1\!\!1_{\mathcal{X}}(x) = \begin{cases} \chi(x) = 1 + (-1)^{n-1}\mu(x) & \text{for } x \notin X \cap X', \\ 1 & \text{for } x \in X \cap X'. \end{cases}$$

Then by the commutativity of the CSM class with specialization

$$c^F(X) = c_*(\mathrm{sp}\,1\!\!1_{\mathcal{X}}), \quad \mathcal{M}(X) = (-1)^{n-1}c_*(\mathrm{sp}\,1\!\!1_{\mathcal{X}} - 1\!\!1_X),$$

see [31] for details. In particular the last equality gives the formula (8) for the Milnor class.

Remark 3.3. One may show easily that in the algebraic case $c^F(X)$ or $\mathcal{M}(X)$, or indeed any algebraic cycle, is of the form $c_*(\alpha)$ for a constructible function α. Note that the above formulas give such α's explicitely (under the assumption of ampleness of L).

§4. Milnor classes

One would like to extend the results described in the previous section to the local complete intersection case. To be more precise, let M be a nonsingular variety and let $i : X \hookrightarrow M$ be a regular embedding (cf. [16] Appendix B7). Then

$$c^F(X) = c^{FJ}(X) = c(\tau_X) \cap [X],$$

Let $N_X M$ denote its normal bundle. The question is whether there is a formula so that

$$\mathcal{M}(X) = c(N_X M)^{-1} \cap class(Sing X) \quad ?$$

and $class(Sing X)$ is a characteristic class depending only on some data given by $Sing(X)$, see also [46] for similar questions.

By a result of Suwa [39] this is the case if X has only isolated singularities. Then

$$\mathcal{M}(X) = \sum_{x \in Sing(X)} \mu_x\,[x],$$

where μ_x denotes the Milnor number at x.

But the very first obstacle to extend the hypersurface case is the absence of a good candidate for constructible function μ. For $f : (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}^k, 0)$, $k \geq 2$, the Milnor fibration (or the nearby cycles functor) is not well-defined in general unless f is "sans éclatement en codimension 0", see [18], that is the case, for instance, in the isolated singularity case.

Nevertheless there are some partial answers that we describe below.

4.1. Milnor class by the obstruction theory

In [8] the authors assume that M is a complex manifold and $X \subset M$ is a local complete intersection. We assume moreover that X is the zero set of a holomorphic section, generically transverse to the zero section, of a holomorphic vector bundle E on M (this case is sometimes called a global complete intersection). X is also assumed to be compact. Let $n = \dim X$. It is showed in [8] that $\mathcal{M}(X)$ can be localized at a connected component S of the singular set Y of X. For such a component S and the following data: a tubular neighbourhood U of S in X, a positive integer r, and an r frame $v^{(r)}$ of vectors tangent to X defined on $\partial U \cap D$, D being the $2(n-r+1)$-skeleton of a cellular decomposition of X, the authors define two classes: the localized Schwartz class $Sch(v^{(r)}, S)$ and the localized virtual class $Vir(v^{(r)}, S)$ both living in $\in H_{2(r-1)}(S)$. The former class contributes to $c_{r-1}(X)$ and the latter to the homology characteristic class of the virtual tangent bundle $\tau_X = TM|_X - N_X M$. Then, as shown in [8],

$$\mu_{r-1}(X, S) := (-1)^{n-1}((Sch(v^{(r)}, S) - Vir(v^{(r)}, S))$$

is independent on the choices and the total Milnor class is the sum over the connected components S_α

$$\mathcal{M}(X) = \sum_\alpha (i_\alpha)_* \mu_*(X, S_\alpha)$$

where $i_\alpha : S_\alpha \hookrightarrow X$ and $\mu_*(X, S_\alpha) = \sum \mu_i(X, S_\alpha)$.

4.2. On Verdier-type Riemann-Roch for CSM classes

Let $f : X \to Y$ be a local complete intersection morphism (an l.c.i. for short), that is the composition of a regular embedding i and a smooth morphism p, cf. [16] Appendix B. 7. Guided by the Riemann-Roch theorem and by the bivariant theory of Chern classes, Yokura [43] posed the question of commutativity (or rather of understanding the non-commutativity) of the following diagram

$$\begin{array}{ccc} F(Y) & \xrightarrow{c_*} & A_*(Y) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle c(T_f)\cap f^*} \\ F(X) & \xrightarrow{c_*} & A_*(X) \end{array}$$

where T_f is the virtual tangent bundle, f^* on the left-hand side is the pull-back of constructible functions, f^* on the right-hand side is the Gysin homomorphism, i.e., the composition of the smooth pull-back p^*

and the Gysin i^* for regular embeddings. The non-commutativity of the above diagram is related to the Milnor class as follows, cf. [43] Example (3.1). Let $i : X \hookrightarrow M$ be a regular embedding as above and let $p : M \to pt$ be the projection to a point. Then $f = p \circ i : X \to pt$ is an l.c.i. morphism and applying the morphisms of the diagram to $\mathbb{1}_{pt}$ we get

$$c_*(f^*\mathbb{1}_{pt}) = c_*(X),$$
$$c(T_f) \cap f^*c_*(\mathbb{1}_{pt}) = c(T_f) \cap [X] = c^{FJ}(X).$$

Thus, in this case, the non-commutativity of the diagram is measured exactly by the Milnor class.

Actually, only the regular embeddings contribute to the non-commutativity of the diagram. Yokura [43] shows that the diagram is commutative for smooth morphisms. Indeed, let us verify it for X and Y non-singular on $\mathbb{1}_Y \in F(Y)$:

$$\begin{aligned} c(T_f) \cap f^*c_*(\mathbb{1}_Y) &= c(T_f) \cap f^*(c(TY) \cap [Y]) \\ &= (c(T_f) \cup c(f^*TY)) \cap f^*[Y] \\ &= (c(T_f) \cup f^*c(TY)) \cap [X] \\ &= c(TX) \cap [X]. \end{aligned}$$

The general case can be reduced to the above one by the resolution of singularities.

4.3. Schürmann's formula

We present the main result of [35] that generalizes the results on the hypersurface case to the case of the regular embedding. Recall that for a general holomorphic map $f : (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}^k, 0)$, $k \geq 2$, the Milnor fibration and the nearby Euler characteristic are not well-defined. The main idea of Schürmann is to overcome this difficulty by replacing X by a hypersurface using the classical construction of the deformation to the normal cone, cf. [16] Ch. 5.

Let $i : X \hookrightarrow Y$ be a regular embedding, and we do not have to assume that Y is smooth. Let C_XY be the normal cone of X in Y and let $\pi : C_XY \to X$ and $k : X \to C_XY$ denote the projection and the embedding as the zero section respectively. Since i is a regular embedding, C_XY is equal to the normal bundle N_XY. In paricular, π is smooth. Denote by $M_XY \to \mathbb{C}$ the deformation of Y to the normal cone $C_X Y$. We have the commutative diagram

$$\begin{array}{ccccc} C_X Y & \hookrightarrow & M_X Y & \hookleftarrow & Y \times \mathbb{C}^* \\ \downarrow & & \downarrow{\scriptstyle flat} & & \downarrow \\ \{0\} & \hookrightarrow & \mathbb{C} & \hookleftarrow & \mathbb{C}^* \end{array}$$

Denote by $\tilde{\pi} : Y \times \mathbb{C}^* \to Y$ the projection to the first factor. Schürmann defines the "constructible function version" of Verdier specialization as

$$\mathrm{sp} = \mathrm{sp}_{X \backslash Y} = \psi_h \circ \tilde{\pi}^* : F(Y) \to F_{mon}(C_X Y),$$

whose image is in monodromic, i.e. conical, constructible functions on $C_X Y$. By Verdier specialization, the CSM class commutes with the analogously defined specialization on homology $\mathrm{Sp} : H_*(Y) \to H_*(C_X Y)$: $\mathrm{Sp} \circ c_* = c_* \circ \mathrm{sp}$. The "vanishing Euler characteristic" transformation associated to the embedding i is defined by $\Phi_i = \mathrm{sp} - \pi^* i^*$. Thus

$$(12) \quad c_*(\Phi_i(\cdot)) \quad = c_*(\mathrm{sp}(\cdot) - \pi^* i^*(\cdot)) = \mathrm{Sp}(c_*(\cdot)) - c_*(\pi^* i^*(\cdot)).$$

This formula holds in $H_*(C_X Y)$. To go down to $H_*(X)$, we use the Gysin isomorphism $k^* = (\pi^*)^{-1} : H_*(C_X Y) \to H_*(X)$ (recall that $k : X \to C_X Y$ denotes the embedding on the zero section)

$$k^* c_*(\Phi_i(\alpha)) = k^* \, \mathrm{Sp}(c_*(\alpha)) - k^* c_*(\pi^* i^*(\alpha)).$$

The Gysin homomorphism i^* is defined by $i^* = k^* \circ \mathrm{Sp}$. Since π is smooth, by Yokura's theorem $c_*(\pi^*(\cdot)) = c(T_\pi) \cap \pi^*(c_*(\cdot))$ and hence

$$k^* c_*(\pi^*(\cdot)) = c(k^* T_\pi) \cap k^* \pi^*(c_*(\cdot)) = c(N_X Y) \cap c_*(\cdot).$$

Consequently

$$k^* c_*(\Phi_i(\alpha)) = i^*(c_*(\alpha)) - c(N_X Y) \cap c_*(i^*(\alpha)).$$

Thus applying $c(N_X Y)^{-1} \cap k^*$ to both sides of (12)

$$(13) \qquad c(N_X Y)^{-1} \cap k^* c_*(\Phi_i(\cdot)) = c(N_X Y)^{-1} \cap i^* c_*(\cdot) - c_*(i^*(\cdot)).$$

This is the formula of Schürmann [35].

If Y is smooth, this formula applied to $\mathbb{1}_Y$ reads

$$(14) \qquad c(N_X Y)^{-1} \cap k^* c_*(\Phi_i(\mathbb{1}_Y)) = c^F(X) - c_*(X).$$

If $i : X \hookrightarrow Y$ is a regular embedding of codimension 1, Y arbitrary, then the specialization in local coordinates defines the vanishing Euler characteristic functor $\mu : F(Y) \to F(X)$. It satisfies $\Phi_i = \pi^* \circ \mu - k_* \circ \mu$ and Schürmann's formula takes the following simple form

(15) $c(N_XY)^{-1} \cap c_*(\mu(\alpha)) = c(N_XY)^{-1} \cap i^*c_*(\alpha) - c_*(i^*(\alpha)),$

for $\alpha \in F(Y)$. If Y is smooth and $\alpha = 1\!\!1_Y$, we recover the formula of Yokura's Conjecture (8).

Suppose that Y is smooth. Since the geometric construction of deformation onto the normal cone can be localized, Schürmann's formula for the Milnor class can be also localized at each connected component of $Sing(X)$. For such a connected component S denote: $i_S : S \hookrightarrow X$ the inclusion, $n_S : N_XY|S \to N_XY$ the induced inclusion of normal cones, $\mu_S := n_S^*\Phi_i(1\!\!1_Y)$, and $k_S : S \to N_XY|S$ is the inclusion on the zero section. Then

$$c^{FJ}(X) - c_*(X) = \sum_S (i_S)_*(c(N_XY|S)^{-1} \cap k_S^*c_*(\mu_S)).$$

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