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Weighted homogeneous polynomials and blow-analytic equivalence

Ould M Abderrahmane

Abstract.

Based on the T. Fukui invariant and the recent motivic invariants proposed by S. Koike and A. Parusiński we give a simple classification of two variable quasihomogeneous polynomials by the blow-analytic equivalence.

§1. INTRODUCTION

Unlike the topological triviality of real algebraic germs, the C^1-equisingularity admits continuous moduli. For instance, the Whitney family $W_t(x,y) = xy(x-y)(x-ty)$, $t > 1$, has an infinite number of different C^1-types. Nevertheless, as was noticed by Tzee-Char Kuo, this family is blow-analytically trivial, that is, after composing with the blowing-up $\beta\colon M^2 \to \mathbf{R}^2$, $W_t \circ \beta$ becomes analytically trivial. T.-C. Kuo proposed new notions of blow-analytic equisingularity and the blow-analytic function (see [6, 3] for survey). Let $f\colon U \to \mathbf{R}$, U open in $\mathbf{R}^n$, be a continuous function. We say that f is blow-analytic, if there exists a sequence of blowing-up β such that the composition $f \circ \beta$ is analytic (for instance $f(x,y) = \frac{x^2y}{x^2+y^2}$ is blow-analytic but not C^1). A local homeomorphism $h\colon (\mathbf{R}^n,0) \to (\mathbf{R}^n,0)$ is called blow-analytic if so are all coordinate functions of h and h^{-1}. Two function germs f_1, $f_2\colon (\mathbf{R}^n,0) \to (\mathbf{R},0)$ are blow-analytically equivalent if there is a blow-analytic homeomorphism h such that $f_1 = f_2 \circ h$.

Observation. Let f, $g\colon (\mathbf{C}^n,0) \to (\mathbf{C},0)$ be weighted homogeneous polynomials with isolated singularities. It is known, for $n=2$, 3, that if $(\mathbf{C}^n, f^{-1}(0))$ and $(\mathbf{C}^n, g^{-1}(0))$ are homeomorphic as germs at $0 \in \mathbf{C}^n$, then, their systems of weights coincide.

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We will consider real singularities. We can easily see that the notion of topological equivalence is too weak to consider the same problem for real analytic singularities. For example, consider $f(x,y) = x^3 + xy^6$ and $g(x,y) = x^3 + y^8$, they are topologically equivalent by Kuiper-Kuo Theorem (see [7, 8]). However, f and g have different weights. We replace the topological equivalence by the blow-analytic equivalence, and we will consider the following problem suggested by T. Fukui.

Problem 1 (T. Fukui, [2], Conjecture 9.2). *Let f, g: $(\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ be weighted homogeneous polynomials with isolated singularities. Suppose that f and g are blow-analytically equivalent. Then, do their systems of weights coincide?*

The purpose of this paper is to establish this conjecture for two variables. Namely, we will prove the following:

Theorem 1. *Let f_i: $(\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$ $(i = 1, 2)$ be non-degenerate quasihomogeneous polynomials of type $(1; r_{i1}, r_{i2})$ such that $0 < r_{i2} \leq r_{i1}$. If f_1 and f_2 are blow-analytically equivalent, then either both f_1 and f_2 are nonsingular, or both are analytically equivalent to xy, or $(r_{11}, r_{12}) = (r_{21}, r_{22})$.*

We call a polynomial f quasihomogeneous of type $(d; w_1, \ldots, w_n) \in \mathbf{Q}^{n+1}$ if $i_1 w_1 + \cdots + i_n w_n = d$ for any monomial $\alpha x_1^{i_1} \ldots x_n^{i_n}$ of f. We say that a polynomial $f(x)$ is non-degenerate if $\{\frac{\partial f}{\partial x_1}(x) = \cdots = \frac{\partial f}{\partial x_n}(x) = 0\} \subset \{0\}$ as germs at the origin of $\mathbf{R}^n$.

We will next recall some important results on blow-analytic equivalence.

Theorem 2 (T. Fukui - L. Paunescu [4]). *Given a system of weights $w = (w_1, \ldots, w_n)$, let f_t: $(\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ be an analytic function for $t \in I = [0, 1]$. Suppose that for each $t \in I$, the weighted initial form of f_t with respect to w is the same weighted degree and has an isolated singularity at $0 \in \mathbf{R}^n$. Then $\{f_t\}_{t \in I}$ is blow-analytically trivial over I.*

T. Fukui ([2]) gave some invariants for blow-analytic equivalence. One of them is defined as follows:

For an analytic function f: $(\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$, set

$$A(f) = \{O(f \circ \lambda) \mid \lambda \colon (\mathbf{R}, 0) \to (\mathbf{R}^n, 0)\ C^w arc\}.$$

Then we have

Theorem 3 (Fukui's invariant). *Suppose that analytic functions* f, g: $(\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ *are blow-analytically equivalent, then* $A(f) = A(g)$.

Recently in [5], S. Koike and A. Parusiński have defined motivic zeta functions (inspired by the work of Denef and Loser [1]) which are invariant for blow-analytic equivalence. We will briefly recall their definition of the zeta functions.

Denote by $\mathcal{L}$ the space of analytic arcs at the origin $0 \in \mathbf{R}^n$:

$$\mathcal{L} = \{\gamma \colon (\mathbf{R}, 0) \to (\mathbf{R}^n, 0) \mid \gamma \text{ is analytic } \}$$

and by $\mathcal{L}_k$ the space of truncated arcs:

$$\mathcal{L}_k = \{\gamma \in \mathcal{L} \mid \gamma(t) = v_1 t + \cdots + v_k t^k,\ v_i \in \mathbf{R}^n\}.$$

Given an analytic function $f \colon (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$. For $k \geq 1$ we denote

$$A_k(f) = \{\gamma \in \mathcal{L}_k \mid f \circ \gamma(t) = ct^k + \cdots,\ c \neq 0\}.$$

We define the zeta function of f by

$$Z_f(T) = \sum_{k \geq 1} (-1)^{-kn} \chi^c(A_k(f)) T^k$$

where χ^c denotes the Euler characteristic with compact support.

Then we have

Theorem 4 (S. Koike - A. Parusiński [5]). *Suppose that analytic functions* f, g: $(\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ *are blow-analytically equivalent, then* $Z_f = Z_g$.

Before starting the proof of Theorem 1, we will make one more remark, as follows.

Remark 5. *Let* $f \colon (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ *be a non-degenerate quasihomogeneous polynomial of type* $(d; w_1, \ldots, w_n)$. *Taking a new representative of the blow-analytic class of* f *if necessary we can suppose that, for each* $\alpha \in \mathbf{N}^n$ *such that* $\langle \alpha, w \rangle = \alpha_1 w_1 + \cdots + \alpha_n w_n = d$, *the coefficient term* $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ *is not zero in* $f(x)$.

Our remark is a simple consequence of Theorem 2 (we omit the details).

§2. PROOF OF THEOREM 1

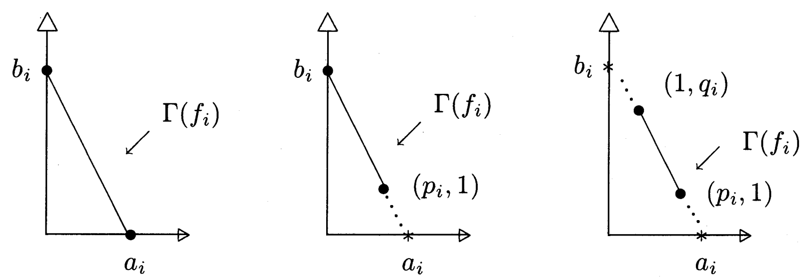
Let $f_i\colon (\mathbf{R}^2,0) \to (\mathbf{R},0)$ $(i=1,2)$ be non-degenerate quasihomogeneous polynomials of type $(1;\, r_{i1}, r_{i2})$. Setting

$$a_i = \frac{1}{r_{i1}} \text{ and } b_i = \frac{1}{r_{i2}} \quad \text{for } i = 1,2.$$

Modulo a permutation coordinate of $\mathbf{R}^2$, we may assume that $a_i \le b_i$. Moreover, if $a_i < 2$, then f_i is analytically equivalent to $g(x,y) = x$ or xy by the Implicit Function Theorem. But $0 \in \mathbf{R}^2$ is a regular point of x and the polynomial xy is a weighted homogeneous of type $(1;\, \frac{1}{2}, \frac{1}{2})$. Given this, we can assume that

$$2 \le a_i \le b_i \quad \text{for } i = 1,2. \tag{2.1}$$

Since f_i are non-degenerate quasihomogeneous polynomials, we have the following cases for Newton boundary $\Gamma(f_i)$ as in the following figure:



$a_i,\ b_i \in \mathbf{N}$ $\qquad$ $a_i \in \mathbf{N}$ or $b_i \in \mathbf{N}$ $\qquad$ $a_i \notin \mathbf{N}$, $b_i \notin \mathbf{N}$

These figures suggest that the proof of Theorem 1 should be divided into several steps, according to the possible cases for a_i and b_i:

Case 1. In this case, we suppose $a_i,\ b_i \in \mathbf{N}$ (i.e., f_i nearly convenient). Here $\mathbf{N}$ denotes the set of positive integers and let for any $a \in \mathbf{N}$, $\mathbf{N}_{\ge a} = \{k \in \mathbf{N} \mid k \ge a\}$. We first remark that the Fukui invariant of f_i can be computed easily as follows:

Assertion 6.

$$A(f_i) = \begin{cases} a_i\mathbf{N} \cup b_i\mathbf{N} \cup \{\infty\} & \textit{if } f_i^{-1}(0) = \{0\}, \\ a_i\mathbf{N} \cup b_i\mathbf{N} \cup \mathbf{N}_{\ge [a_i, b_i]} \cup \{\infty\} & \textit{otherwise.} \end{cases} \tag{2.2}$$

Where $[a_i, b_i] = LCM(a_i, b_i)$.

Proof. Let $\lambda\colon (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be an analytic arc. Then $\lambda(t) = (X(t), Y(t))$ can be expressed in the following way:

$$X(t) = \alpha_u t^u + \alpha_{u+1} t^{u+1} + \cdots, \quad Y(t) = c_v t^v + c_{v+1} t^{v+1} + \cdots,$$

where α_u, $c_v \neq 0$ and u, $v \geq 1$. By the above Remark 5, we may assume that there exist the terms X^{a_i} and Y^{b_i} with non-zero coefficients in $f_i(X, Y)$.

We will first consider the case whereby $f_i^{-1}(0) = \{0\}$. If $u\, a_i \neq v\, b_i$, we have

$$f_i(X(t), Y(t)) = d_i\, t^{\min\{u\, a_i\, , v\, b_i\}} + \cdots, \ d_i \neq 0$$

then $O(f_i \circ \lambda) = \min\{u\, a_i\, , v\, b_i\} \in a_i\mathbf{N} \cup b_i\mathbf{N} \cup \{\infty\}$. Thus it remains for us to consider the case $u\, a_i = v\, b_i$. In this case, we have

$$f_i(X(t), Y(t)) = f_i(\alpha_u\, , c_v)\, t^{u\, a_i} + \cdots,$$

since $f_i(\alpha_u\, , c_v) \neq 0$. Therefore $A(f_i) \subseteq a_i\mathbf{N} \cup b_i\mathbf{N} \cup \{\infty\}$. Any integer $s \in a_i\mathbf{N} \cup b_i\mathbf{N}$, for instance $s = k\, a_i$, is attained by the arc $\gamma(t) = (t^k, 0)$. Hence we have

$$A(f_i) = a_i\mathbf{N} \cup b_i\mathbf{N} \cup \{\infty\}.$$

We will next consider the case whereby $f_i^{-1}(0) \neq \{0\}$. Similarly we have

$$a_i\mathbf{N} \cup b_i\mathbf{N} \cup \{\infty\} \subseteq A(f_i) \subseteq a_i\mathbf{N} \cup b_i\mathbf{N} \cup \mathbf{N}_{\geq [a_i, b_i]} \cup \{\infty\}.$$

Obviously we only have to prove that $\mathbf{N}_{\geq [a_i, b_i]} \subseteq A(f_i)$. Suppose that $k \in \mathbf{N}_{\geq [a_i, b_i]}$. Then there exists an arc γ through $0 \in \mathbf{R}^2$ such that $O(f \circ \gamma) = k$. Setting $[a_i, b_i] = n_i\, a_i = m_i\, b_i$, since f_i is non-degenerate and $f_i^{-1}(0) \neq \{0\}$, there exists a $(\alpha, c) \in f_i^{-1}(0)$ such that $(\frac{\partial f_i}{\partial X}(\alpha, c), \frac{\partial f_i}{\partial Y}(\alpha, c))$ $\neq (0, 0)$, we may assume that $\frac{\partial f_i}{\partial X}(\alpha, c) \neq 0$. Then it is easy to see that for any positive integers $[a_i, b_i] + s \in A(f)$, $s \in \mathbf{N}$, is attained by an arc $\gamma(t) = (\alpha t^{n_i} + t^{s+n_i}, c t^{m_i})$.

Evidently, this completes the proof of the Assertion. Q.E.D.

From Theorem 3, $A(f_1) = A(f_2)$. Thus, by the above Assertion, we have the following result:

$$\begin{aligned} &a_1 = a_2 \ \text{ same multiplicity for } f_i, \\ &b_1 = b_2 \ \text{ if } b_1 \notin a_1\mathbf{N} \text{ or } b_2 \notin a_2\mathbf{N}, \\ &b_1 = b_2 \ \text{ if } f_i^{-1}(0) \neq \{0\}. \end{aligned}$$

Manifestly, the Fukui invariant determines the weights except in the following case:

$$b_1 = k_1 a,\ b_2 = k_2 a \text{ and } f_i^{-1}(0) = \{0\},$$

where $a = a_1 = a_2$ is the smallest number in $A(f_i)$, and there remains to prove $k_1 = k_2$. In fact, assume that $k_1 \neq k_2$, for example $k_2 > k_1$. We will show that this gives rise to a contraduction by comparing the coefficients of the zeta functions. If $k_2 > k_1$ then we may write

$$\begin{aligned} A_{b_1}(f_2) &= \{\gamma(t) = (c_{k_1} t^{k_1} + \cdots + c_{b_1} t^{b_1}\ ,\ d_1 t^1 + \cdots + d_{b_1} t^{b_1}) \mid c_{k_1} \neq 0\} \\ &\simeq \mathbf{R}^* \times \mathbf{R}^{b_1 - k_1} \times \mathbf{R}^{b_1}. \end{aligned}$$

That is

$$\chi^c(A_{b_1}(f_2)) = (-2)\chi^c(\mathbf{R}^{b_1 - k_1 + b_1}) = (-2)(-1)^{2b_1 - k_1}. \tag{2.3}$$

Also, since $f_1^{-1}(0) = \{0\}$, we obtain

$$\begin{aligned} A_{b_1}(f_1) &= \{\gamma = (u_{k_1} t^{k_1} + \cdots + u_{b_1} t^{b_1}, v_1 t^1 + \cdots + v_{b_1} t^{b_1}) \mid (u_{k_1}, v_1) \neq 0\} \\ &\simeq (\mathbf{R}^2 - \{0\}) \times \mathbf{R}^{b - k_1} \times \mathbf{R}^{b_1 - 1} \end{aligned}$$

which means

$$\chi^c(A_{b_1}(f_1)) = \chi^c(\mathbf{R}^2 - \{0\})\, \chi^c(\mathbf{R}^{2b_1 - k_1 - 1}).$$

Since $\chi^c(\mathbf{R}^2 - \{0\}) = 0$ we get by (2.3) that $\chi^c(A_{b_1}(f_1)) \neq \chi^c(A_{b_1}(f_2))$. Therefore $Z_{f_1} \neq Z_{f_2}$, which contradicts Theorem 4. This ends the proof of Theorem 1 in the first case.

Case 2. In this case, we suppose $a_i \notin \mathbf{N}$, $b_i \in \mathbf{N}$ for $i = 1, 2$. Since f_i is non-degenerate, then there exists the term $x^{p_i} y$ for some integers $p_i \geq 1$ with non-zero coefficients in $f_i(x, y)$. By Theorem 2 and (2.1), it is easy to see that for any integers $s \geq 1$, $f_i(x, y) + x^{p_i + s}$ is blow-analytically equivalent to $f_i(x, y)$. Then the Fukui invariant of f_i is determined by

$$A(f_i) = \{p_i + 1,\ p_i + 2,\ p_i + 3, \cdots\} \cup \{\infty\}. \tag{2.4}$$

Moreover $A(f_1) = A(f_2)$, and it follows that $p_1 = p_2$. Consequently it is sufficient to prove that $b_1 = b_2$. Indeed, suppose that $b_1 < b_2$. Then, we let

$$p = p_1 = p_2, \quad \Re_n = \{(r, s) \in \mathbf{N}^2 \mid rp + s = n\}$$

and

$$C^n_{r,s} = \{\gamma(t) = (u_r t^r + \cdots + u_n t^n\,,\, v_s t^s + \cdots + v_n t^n) \mid u_r, v_s \neq 0\}$$
$$\simeq (\mathbf{R}^*)^2 \times \mathbf{R}^{2n-r-s}.$$

Let us first compute $\chi^c(A_{b_1}(f_i))$. It is easy to see that for any positive integers $n < b_i$, we have that $A_n(f_i) = \bigcup_{(r,s)\in\Re_n} C^n_{r,s}$ (Remark that the union is disjoint). Thus, by the additivity of χ^c, we have

$$\chi^c(A_{b_1}(f_2)) = \sum_{(r,s)\in\Re_{b_1}} (-2)^2(-1)^{2b_1-r-s}. \tag{2.5}$$

Similarly if $b_1 - 1 \notin p\mathbf{N}$, we obtain

$$\chi^c(A_{b_1}(f_1)) = (-2)(-1)^{2b_1-d} + \sum_{(r,s)\in\Re_{b_1}} (-2)^2(-1)^{2b_1-r-s} \tag{2.6}$$

where d is the smallest number in $\{1,\ldots,b_1\}$ such that $dp+1 > b_1$. It follows from (2.5) and (2.6) that $\chi^c(A_{b_1}(f_2)) \neq \chi^c(A_{b_1}(f_1))$. But this implies a contradiction, by comparing the coefficients of the zeta functions. Hence we have $b_1 - 1 \in p\mathbf{N}$. Now assume $b_1 = kp+1$. Then by elementary computation, we have

$$A_{b_1}(f_1) = C_{f_1} \cup (\cup_{(r,s)\in\Re_{b_1}\setminus\{(k,1)\}} C^{b_1}_{r,s}),$$

where

$$C_{f_1} = \{\gamma(t) = (u_k t^k + \cdots + u_{b_1} t^{b_1}\,,\, v_1 t^1 + \cdots + v_{b_1} t^{b_1}) \mid f_1(u_k, v_1) \neq 0\}$$
$$\simeq \{f_1 \neq 0\} \times \mathbf{R}^{2b_1-k-1},$$

Also, by the additivity of the Euler characteristic with compact support, we obtain

$$\chi^c(A_{b_1}(f_1)) = \chi^c(\{f_1 \neq 0\})(-1)^{2b_1-k-1} + \sum_{(r,s)\in\Re_{b_1}\setminus\{(k,1)\}} 4(-1)^{2b_1-r-s}.$$

Together with (2.5), it follows that

$$\chi^c(\{f_1 = 0\}) = -3. \tag{2.7}$$

We will next compute the $\chi^c(A_{b_1+1}(f_i))$, $(i = 1,2)$. Setting $m = kp+2 = b_1+1$. Then, by the above, $m-1 \notin p\mathbf{N}$ and $m \leq b_2$, we can easily see the following

(2.8)

$$\chi^c(A_m(f_2)) = \begin{cases} \sum_{(r,s)\in\Re_m} 4(-1)^{2m-r-s} & \text{if } m < b_2, \\ -2(-1)^{2m-k-1} + \sum_{(r,s)\in\Re_m} 4(-1)^{2m-r-s} & \text{if } m = b_2 \end{cases}$$

Now we compute $\chi^c(A_m(f_1))$. Let $\lambda(t) = (X(t), Y(t))$ be an analytic arc defined by

$$X(t) = u_k t^k + \cdots + u_m t^m,$$
$$Y(t) = v_1 t + \cdots + v_m t^m.$$

We can write

$$f_1(X(t), Y(t)) = f_1(u_k, v_1)t^{m-1} + \langle \nabla f_1(u_k, v_1)\,;\, (u_{k+1}, v_2) \rangle\; t^m + \cdots,$$

where

$$\langle \nabla f_1(u_k, v_1)\,;\, (u_{k+1}, v_2) \rangle = \frac{\partial f_1}{\partial x}(u_k, v_1)\, u_{k+1} + \frac{\partial f_1}{\partial y}(u_k, v_1)\, v_2.$$

Moreover, if $f_1(u_k, v_1) = 0$ and $\langle \nabla f_1(u_k, v_1)\,;\, (u_{k+1}, v_2) \rangle \neq 0$, then we have $O(f_1 \circ \lambda) = m$. Let us put

$$\begin{aligned} B_1 &= \{(u, v, w, z) \in (f_1^{-1}(0) - \{0\}) \times \mathbf{R}^2 \mid \langle \nabla f_1(u, v)\,;\, (w, z) \rangle \neq 0\}, \\ B_2 &= \{(u, v, w, z) \in (f_1^{-1}(0) - \{0\}) \times \mathbf{R}^2 \mid \langle \nabla f_1(u, v)\,;\, (w, z) \rangle = 0\}, \\ C_{\nabla f_1} &= \{(u_k t^k + \cdots + u_m t^m, v_1 t^1 + \cdots + u_m t^m) \mid (u_k, u_{k+1}, v_1, v_2) \in B_1\} \\ &\simeq B_1 \times \mathbf{R}^{2m-k-3}, \end{aligned}$$

Then, by the above, the $A_m(f_1)$ given by

$$A_m(f_1) = C_{\nabla f_1} \cup (\cup_{(r,s)\in\Re_m} C^m_{r,s}).$$

Thus the Euler characteristic with support compact of $A_{b_m}(f_1)$ equals

$$\chi^c(A_m(f_1)) = \chi^c(B_1)(-1)^{2m-k-3} + \sum_{(r,s)\in\Re_m} (-2)^2(-1)^{2m-r-s}. \tag{2.9}$$

By identification of the m-coefficients of both zeta functions of f_i for $i = 1, 2$, it follows from (2.8) and (2.9) that $\chi^c(B_1) = 0$ or -2. On the other hand, $(f_1^{-1}(0) - \{0\}) \times \mathbf{R}^2 = B_1 \cup B_2$. Therefore

$$\chi^c(f_1^{-1}(0) - \{0\}) = \chi^c(B_1) + \chi^c(B_2),$$

but $B_2 \simeq (f_1^{-1}(0) - \{0\}) \times \mathbf{R}$. This is clear because f_1 is non-degenerate, then we have

$$\chi^c(f_1^{-1}(0) - \{0\}) = \chi^c(f_1^{-1}(0) - \{0\})(-1) + \chi^c(B_1).$$

Since $\chi^c(B_1) = 0$ or -2, this yields

$$\chi^c(f_1^{-1}(0)) = 1 \text{ or } 0,$$

which contradicts (2.7). This ends the proof of Theorem 1 in the second case.

Remark 7. *If we drop the assumption that b_2 is an integer, then the above proof still holds.*

Case 3. In this case, we suppose $a_i \in \mathbf{N}$, $b_i \notin \mathbf{N}$ for $i = 1, 2$. Since f_i is non-degenerate, then there exists the term xy^{q_i} for some integers $q_i \geq 1$ with non-zero coefficients in $f_i(x, y)$. For any real α we denote by $e(\alpha)$ the minimum positive integer n such that $n \geq \alpha$. By an argument similar to that of Assertion 6 and (2.4), we can compute the Fukui invariant of f_i as follows:

$$A(f_i) = a_i\mathbf{N} \cup \{e(b_i),\, e(b_i) + 1, \cdots\} \cup \{\infty\}.$$

By Theorem 3, $A(f_1) = A(f_2)$. Then we have the following result:

$$a_1 = a_2 \text{ and } e(b_1) = e(b_2) \tag{2.10}$$

Suppose now $b_1 \neq b_2$. Then $q_1 \neq q_2$, but $\mid b_1 - b_2 \mid \geq \mid q_1 - q_2 \mid \geq 1$. It follows that $e(b_1) \neq e(b_2)$, which contradicts (2.10). This complete the proof of Theorem 1 in the third case.

Case 4. In this case, we suppose a_i, $b_i \notin \mathbf{N}$ for $i = 1, 2$. Since f_i is non-degenerate, then there exist the terms $x^{p_i}y$ and xy^{q_i} for some integers $p_i \geq 1$ and $q_i \geq 1$ with non-zero coefficients in $f_i(x, y)$. Thus, the Fukui invariant of f_i can be written as

$$A(f_i) = \{p_i + 1,\, p_i + 2,\, p_i + 3, \cdots\} \cup \{\infty\},$$

which implies $p_1 = p_2$. Thus we only have to prove that $b_1 = b_2$. Indeed, let us assume that $b_1 < b_2$. Then we have $q_1 < q_2$ which implies $b_1 < e(b_1) < b_2$. Let us put

$$p = p_1 = p_2,\ m = e(b_1) \text{ and } \Re_m = \{(r, s) \in (\mathbf{N} - \{0\})^2 \mid rp + s = m\}.$$

We first observe that $m - 1 \notin p\mathbf{N}$. Otherwise, if $m - 1 = r\,p$, then we have:

$$b_1 < q_1 + r < r\,p + 1 < r\,a_1. \tag{2.11}$$

This is a consequence of $b_1 < m = r\,p + 1$ and also $(1\,,\, q_1)$ and $(p\,,\, 1)$ are vertices of $\Gamma(f_1)$. But $m = \min\{n \in \mathbf{N} \mid n > b_1\}$, which contradicts (2.11). Hence we have $m - 1 \notin p\mathbf{N}$. Using this observation and by elementary computation we obtain the following result:

$$\begin{aligned} \chi^c(A_m(f_2)) &= \sum_{(r,s)\in\Re_m} (-2)^2(-1)^{2m-r-s}, \\ \chi^c(A_m(f_1)) &= (-2)^2(-1)^{m+q_1-1} + \sum_{(r,s)\in\Re_m} (-2)^2(-1)^{2m-r-s}. \end{aligned} \tag{2.12}$$

This means that $Z_{f_1} \neq Z_{f_2}$, which contradicts Theorem 4. This complete the proof of Theorem 1 in the fourth case.

In order to finish the proof of Theorem 1, it suffices to show the following lemmas.

Lemma 8. $a_1 \in \mathbf{N}$ *if and only if* $a_2 \in \mathbf{N}$.

Proof. Suppose that this is not the case. Namely, $a_1 \in \mathbf{N}$ and $a_2 \notin \mathbf{N}$. Since f_2 is non-degenerate, then there exists the term $x^{p_2}y$ for some integers $p_2 \geq 1$ with non-zero coefficients in $f_2(x,y)$. Again using the same argument in (2.4) one gets

$$A(f_2) = \{p_2+1,\, p_2+2,\, p_2+3, \cdots, \infty\},$$

Since $A(f_1) = A(f_2)$, then we have $a_1 = b_1 = p_2+1$, set $m = p_2+1$. We shall compute the $\chi^c(A_m(f_i))$ for $i = 1,2$, that is

$$\begin{aligned} A_m(f_2) &= \{\gamma(t) = (u_1t + \cdots + u_mt^m,\, v_1t + \cdots + v_mt^m) \mid u_1, v_1 \neq 0\} \\ &\simeq (\mathbf{R}^*)^2 \times \mathbf{R}^{2m-2}, \end{aligned}$$

so

$$\begin{aligned} A_m(f_1) &= \{\gamma(t) = (u_1t + \cdots + u_mt^m,\, v_1t + \cdots + v_mt^m) \mid f_1(u_1,v_1) \neq 0\} \\ &\simeq \{f_1 \neq 0\} \times \mathbf{R}^{2m-2}, \end{aligned}$$

and hence to

$$\chi^c(A_m(f_i)) = \begin{cases} (-2)^2(-1)^{2m-2} & \text{if } i = 2, \\ \chi^c(\{f_1 \neq 0\})(-1)^{2m-2} & \text{if } i = 1. \end{cases} \tag{2.13}$$

Since $\chi^c(A_m(f_1)) = \chi^c(A_m(f_2))$, then we have

$$\chi^c(\{f_1 = 0\}) = -3. \tag{2.14}$$

Using the same argument as Case 2, the $(m+1)$-coefficients of Z_{f_i} for $i = 1,2$ can be computed as follows:

$$\chi^c(A_{m+1}(f_1)) = \chi^c(B_1) \ \text{ and } \ \chi^c(A_{m+1}(f_2)) = \begin{cases} -4 & \text{if } m \neq b_2, \\ -6 & \text{if } m = b_2. \end{cases}$$

We recall that:

$$\begin{aligned} B_1 &= \{(u,v,w,z) \in (f_1^{-1}(0) - \{0\}) \times \mathbf{R}^2 \mid \langle \nabla f_1(u,v)\,;\, (w,z) \rangle \neq 0\}, \\ B_2 &= \{(u,v,w,z) \in (f_1^{-1}(0) - \{0\}) \times \mathbf{R}^2 \mid \langle \nabla f_1(u,v)\,;\, (w,z) \rangle = 0\}. \end{aligned}$$

Finally, by comparing the $(m+1)$-coefficients of both zeta functions Z_{f_i}, it is evident that $\chi^c(B_1) = -4$ or -6, but $(f_1^{-1}(0)-\{0\})\times\mathbf{R}^2 = B_1 \cup B_2$. It follows from the additivity of the Euler characteristic that $\chi^c(f_1^{-1}(0)-\{0\}) = \chi^c(B_1)+\chi^c(B_2)$. On the other hand, by $B_2 \simeq (f_1^{-1}(0)-\{0\})\times\mathbf{R}$ (because f_1 is non-degenerate), then we have

$$\chi^c(f_1^{-1}(0)) = -1 \text{ or } -2,$$

which contradicts (2.14). This proves the lemma. Q.E.D.

Lemma 9. *$b_1 \in \mathbf{N}$ if and only if $b_2 \in \mathbf{N}$.*

Proof. Suppose now that $b_1 \in \mathbf{N}$ and $b_2 \notin \mathbf{N}$. Since f_2 is non-degenerate, then there exists the term xy^{q_2} for some integers $q_2 \geq 1$ with non-zero coefficients in $f_2(x,y)$.

We first consider $a_i \in \mathbf{N}$ for $i = 1, 2$. Then, by the same reason as above, we can compute the Fukui invariant of f_i as follows:

$$A(f_1) = a_1\mathbf{N} \cup b_1\mathbf{N} \cup \mathbf{N}_{\geq[a_1,b_1]} \cup \{\infty\},$$
$$A(f_2) = a_2\mathbf{N} \cup \mathbf{N}_{\geq e(b_2)} \cup \{\infty\}.$$

Since $A(f_1) = A(f_2)$, then we have the following result:

$$(2.15) \qquad a_1 = a_2, \ \ b_1 = k\,a_1, \ \ \text{and} \quad e(b_2) = b_1 \text{ or } b_1 + 1.$$

Since $b_1 = k\,a_1$, we may assume by Remark 5 that there exists the term $xy^{k(a_1-1)}$ with non-zero coefficients in $f_1(x,y)$. But $|b_2 - b_1| \geq |q_2 - k(a_1-1)| \geq 1$, which implies $b_2 \geq b_1 + 1$ or $b_1 \geq b_2 + 1$. It follows that $e(b_2) > b_1 + 1$ or $e(b_2) < b_1$, which contradicts (2.15), and ends the first part of the lemma.

Now we consider the case where $a_i \notin \mathbf{N}$ for $i = 1, 2$. Since f_i is non-degenerate, then there exists the term $x^{p_i}y$ for some integers $p_i \geq 1$ with non-zero coefficients in $f_i(x,y)$. It is easy to see that

$$A(f_i) = \{p_i + 1,\, p_i + 2,\, p_i + 3, \cdots\} \cup \{\infty\}.$$

Moreover $A(f_1) = A(f_2)$, and we get $p_1 = p_2$. Set

$$p = p_1 = p_2, \ m = e(b_2) \ \text{ and } \quad \Re_m = \{(r,s) \in \mathbf{N}^2 \mid rp + s = m\}.$$

As stated in Remark 7, we can exclude the case where $b_1 < b_2$ (because this is proved in exactly the same way as Case 2). Thus it remains to consider the case $b_2 < b_1$.

We next compute the m-coefficients of both zeta functions Z_{f_i} for $i = 1, 2$. For this, we can assert that $m - 1 \notin p\mathbf{N}$. Indeed, suppose that

$m-1=\alpha\, p$ for some positive integer α. Since $b_2 < m = \alpha\, p + 1$ which implies $b_2 < q_2 + \alpha < \alpha p + 1$. This is clear because $(1\,,\, q_2) \in \Gamma(f_2)$. But $m = e(b_2)$ is equal to the smallest integer greater than b_2, which is a contradiction. Therefore we obtain that $m-1 \notin p\mathbf{N}$, and so on by elementary computation, we have the following result:

$$(2.16)\quad \chi^c(A_m(f_2)) = (-2)^2(-1)^{m+q_2-1} + \sum_{(r,s)\in\Re_m} (-2)^2(-1)^{2m-r-s}.$$

And

$$\chi^c(A_m(f_1)) = \sum_{(r,s)\in\Re_m} (-2)^2(-1)^{2m-r-s} \qquad \text{if } m < b_1,$$

$$\chi^c(A_m(f_1)) = (-2)(-1)^{m+q_2} + \sum_{(r,s)\in\Re_m} (-2)^2(-1)^{2m-r-s} \qquad \text{if } m = b_1.$$

Now it suffices to note by the above equalities that $Z_{f_1} \neq Z_{f_2}$, which contradicts Theorem 4. This completes the proof. Q.E.D.

Theorem 1 is therefore proved.

Example 10. *Let k be an arbitrary integer greater than or equal to 4. We consider quasihomogeneous polynomial functions f_k, $g_k\colon (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$ defined by*

$$f_k(x,y) = x^5 + x\, y^{2\,k}, \qquad g_k(x,y) = x^5 - y^{2\,k\,+\,2}.$$

Note that the weights of f_k and g_k are $(\frac{1}{5}\,,\,\frac{2}{5\,k})$ and $(\frac{1}{5}\,,\,\frac{1}{2\,k+2})$ respectively. Since f_k and g_k have different weights for $k > 4$, they are not blow-analytically equivalent by Theorem 1. However, f_k and g_k are topologically equivalent. In fact, the above $f_k(x,y) = x^5 + x\, y^{2\,k} \in J^{2\,k+1}_{\mathbf{R}}(2,1)$ is C^0-sufficient by the Kuiper-Kuo Theorem (see [7, 8]). Therefore, f_k is topologically equivalent to $f_k - y^{2\,k\,+\,2}$. On the other hand, g_k and $g_k + x\, y^{2\,k}$ are blow-analytically equivalent by Theorem 2. Besides $f_k - y^{2\,k\,+\,2} = g_k + x\, y^{2\,k}$, hence the conclusion holds. Consequently, $f_k \in J^{2\,k+1}_{\mathbf{R}}(2,1)$ is not blow-analytically sufficient for $k > 4$.

In the case $k = 4$, the weights of f_4 and g_4 are equal to $(\frac{1}{5}\,,\,\frac{1}{10})$. Furthermore, f_4 is blow-analytically equivalent to g_4. Indeed, consider the family $H_t\colon (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$ $(t \in [0,1])$ defined by $H_t(x,y) = (1-t) f_4(x,y) + t\, g_4(x,y)$. It is easy to see that for each $t \in [0,1]$, H_t has an isolated singularity at $0 \in \mathbf{R}^2$. Therefore, it follows from Theorem 2 that $\{H_t\}_{0 \le t \le 1}$ is blow-analytically trivial over $[0,1]$. In particular, $H_0 = f_4$ is blow-analytically equivalent to $H_1 = g_4$.

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Department of Mathematics
Faculty of Science
Saitama University, 255 Shimo-Okubo
Urawa, 338-8570 Japan
E-mail: vould@rimath.saitama-u.ac.jp