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Valuations, and moduli of Goursat distributions

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Abstract.

Goursat distributions are subbundles in the tangent bundles to manifolds having the flag of consecutive Lie squares of ranks not depending on a point and growing always by 1. It is known that moduli of the local classification of these objects (distributions determine their flags, and vice versa) are not functional, only continuous numeric, and appear in codimensions two and higher; singularities of codimension one are all simple. In the present work we show that *most* of the codimension-two singularities of Goursat flags is not simple. As to the *precise* modalities of those singularities, we give them at paper's end in the conjectural mode.

§1. Introduction, main result, and infinitesimal symmetries

A distribution D of corank $r \geq 2$ on a smooth or analytic manifold M (a codimension–r subbundle of TM) is Goursat when its Lie square $[D,\, D]$ is a distribution of constant corank $r-1$, the Lie square of $[D,\, D]$ is of constant corank $r-2$, and so on until reaching the full tangent bundle TM. A Goursat flag of length r is any such D together with the nested sequence of its consecutive Lie squares through TM inclusively. (Without loss of generality, it could have been assumed that D is of rank 2. Locally it would lead to the same theory, for in any general Goursat distribution D there locally splits off an integrable subdistribution of rank $\operatorname{rk} D - 2$.)

Those distributions appear naturally, among others, as the outcome of series of so-called Cartan prolongations (see [1, 6] for details) of rank-2 distributions on different manifolds, starting from the full tangent

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bundle of a 2-dimensional surface. They generalize the well-known, also Cartan's, distributions on the jet spaces of functions $\mathbb{R} \longrightarrow \mathbb{R}$ (sometimes also called *contact systems*) in that they admit singularities, discovered in 1978 by Giaro-Kumpera-Ruiz – see their *exceptional model* (1) – and started to be systematically investigated in [5], while they retain the basic flag property of Cartan's distributions.

In fact, original Cartan structures are obtained in the prolongation procedures when at each prolongation step one avoids the vertical directions. And the vertical directions – that can be chosen in prolongations at will, either in row or intermittently with non-vertical ones – account for a rich pattern of singularities hidden in flags. Upon closer inspection there emerges, [4, 6], Jean-Montgomery-Zhitomirskii stratification of germs of flags into *geometric classes*, with strata encoded by words (of length equal to flag's length) over the alphabet $\{\mathrm{G}, \mathrm{S}, \mathrm{T}\}$: Generic, Singular, Tangent.

1.1. Geometric classes of Goursat flags.

The canonical geometric definition of them deals with a given Goursat flag

$$TM = D^0 \supset D^1 \supset D^2 \supset \cdots \supset D^{r-1} \supset D^r = D$$

($[D^j,\, D^j] = D^{j-1}$) around a fixed point $p \in M$ and firstly precises which members of it (excepting D^1 and D^2) are at p in singular positions. Namely, D^j is at p in singular position when it coincides at p with the *Cauchy characteristics* $L(D^{j-2})$ of D^{j-2}: $D^j(p) = L(D^{j-2})(p)$.

(In general, for any distribution D, $L(D)$ is the module, or sheaf of modules, of such vector fields v with values in D that preserve D, $[v,\, D] \subset D$. And one of first observations, see [6] for inst., is that for D – Goursat, $L(D)$ is a regular corank *two* subdistribution of D, $\operatorname{rk} L(D) = \operatorname{rk} D - 2$. Thus, needless to say, D^j and $L(D^{j-2})$ in the definition above have the same ranks.)

At this moment it is very useful to have under eyes the so-called *sandwich diagram* excerpted from [6]:

$$\begin{array}{ccccccccccc}
D^1 \supset & D^2 & \supset & D^3 & \supset \cdots \supset & D^{r-1} & \supset & D^r & & \\
& \cup & & \cup & & \cup & & \cup & & \\
& L(D^1) & \supset & L(D^2) & \supset \cdots \supset & L(D^{r-2}) & \supset & L(D^{r-1}) & \supset & L(D^r)\,.
\end{array}$$

All direct inclusions in this diagram are of codimension one. The squares built by [drawn] inclusions can be perceived as certain 'sandwiches': for instance, in the utmost left sandwich $L(D^1)$ and D^3 are as if fillings, while D^2 and $L(D^2)$ constitute the covers (of different dimensions, one has to admit).

For instance, the [Lie] square of the [Lie] square, i.e., D^1, of the Goursat object D^3 described on $\mathbb{R}^5(x^1, \ldots, x^5)$ by the Pfaffian equations

$$(1) \qquad dx^2 - x^3dx^1 = dx^3 - x^4dx^1 = dx^1 - x^5dx^4 = 0\,,$$

is given by $dx^2 - x^3dx^1 = 0$, and its Cauchy characteristics are $\mathrm{span}(\partial_4, \partial_5)$. This 2–plane coincides with D^3 at 0, and – more generally – at all points of the hypersurface $\{x^5 = 0\}$. Therefore, this D^3 is in singular position by far not at isolated points, but **in codomension 1**.
In general D^3, $D^4, \ldots, D^r$ can be in singular positions at a point (in different sandwiches on the diagram above) one independently of another and it gives rise to 2^{r-2} rough invariant classes of flag's germs (termed 'Kumpera–Ruiz classes' in [6]). Thus, at this moment, the local behaviour at p is encoded by a word of length r over $\{*, \mathrm{S}\}$ starting with two $*$, possibly having more $*$'s, and having S as the j-th letter ($3 \le j \le r$) precisely and only when D^j is at p in singular position.

Secondly, heading towards geometric classes and labels over $\{\mathrm{G}, \mathrm{S}, \mathrm{T}\}$, one plainly replaces all $*$ before the first S (if any) by letters G and then turns to strings of $*$ standing behind, or past, letters S (if there are such strings). Let us depart from a given letter S, being the j-th letter in the word *and* having a string of $*$'s past it. In the eventual label [of the geometric class of the flag's germ at p] this S is followed by T when $D^{j+1}(p)$ is tangent to the locus (always being a regular hypersurface in M, as in the example above) of the previous singularity 'D^j in singular position', while it is followed by G when $D^{j+1}(p)$ is not tangent to that locus (and, as a matter of course, not in the singular position at p – see the detailed discussion below).

At this point it is important to explain that this tangent position of D^{j+1} with respect to the hypersurface

$$(2) \qquad H \;=\; \{\, q \in M :\; D^j(q) = L(D^{j-2})(q)\,\},$$

when materializing at p, implies that $D^{j+1}(p)$ itself is not in the singular position. Or, in other words, that the presently being defined meaning of '**S**T' does not *conflict* with the previously introduced meaning of 'SS'. Indeed, choosing any fixed local vector field V with values in $L(D^{j-1})$ and independent of $L(D^j)$, $L(D^{j-1}) = (V) \oplus L(D^j)$, the flow φ_V^t of V preserves D^{j-1}, hence also D^{j-2} and $L(D^{j-2})$: $\left(\varphi_V^t\right)_* L(D^{j-2}) = L(D^{j-2})$ for small $|t| > 0$. On the other hand, recalling, V takes values in D^j ($\supset L(D^{j-1})$) but not in $L(D^j)$. Consequently, $\left(\varphi_V^t\right)_* D^j \neq D^j$ for $|t| > 0$ small and, altogether, the flow of V does not preserve the defining equation (2) of H. In fact, upon analyzing more carefully the *speed* of $\left(\varphi_V^t\right)_* D^j$ deviating from D^j, V is not tangent to H at p. Hence, H being of codimension one, V is transverse, and all the more so is $L(D^{j-1})(p)$. A new singular position at p (i.e., the second S in row in a code) would then imply the transversality to the locus (H) of the previous singular position, while the [$(j+1)$-st member'] position encoded by T is tangent to H.

In turn, T can follow ST, further selecting the germs with $D^{j+2}(p)$ tangent to the locus $\widetilde{H}$ (now being regular of codimension 2 in M) of the geometry ...ST. Such a new ST**T** does not conflict with a hypothetical ST**S** – because the latter would imply the relative (within H) transversality of $D^{j+2}(\cdot)$ to $\widetilde{H}$. When there are sufficiently many $*$'s after the reference letter S, another T can follow ...STT, again causing no conflict with a potential ...STT**S** in view of another tangency as opposed to relative transversality, and so on. Since the moment of interruption of such sequence of tangencies, one plainly replaces all remaining $*$'s by G's, till the next letter S.

All thus emerging labels are geometrically realizable; the only restriction in them, clear from the geometric behaviours being encoded, are the two necessary G's in the beginning and that T cannot go directly after G. Therefore, for length 2 there is but one class GG, for length 3 – only GGG and GGS (the latter having as a unique local model (1)), for length 4 – GGGG, GGSG, GGST, GGSS, GGGS. A straightforward recurrence yields that there exist u_{2r-3} (Fibonacci number) geometric classes of the germs of flags of length r. They are, obviously, pairwise disjoint and invariant under the action of local diffeomorphisms between manifolds. The class GG...G is the fattest single orbit,[1] as established in now classical papers [2, 13] (those contributions were widely popularized in the 1920s by Goursat in his book *Leçons sur le problème de Pfaff*). The well-known local representative is, for the length r, the *chained model* of that length – the germ at 0 of

$$dx^2 - x^3dx^1 = dx^3 - x^4dx^1 = \cdots = dx^{r+1} - x^{r+2}dx^1 = 0\,, \tag{3}$$

still actively in use in control theory and differential geometry.

Clearly, the geometric classes approximate from above orbits of the local classification of Goursat germs. As a matter of record, we just note that this approximation is 100% precise for lengths $\leq$ 6, but too rough from length 7 onwards ([7]). Before passing to finer issues, let us note one remarkable property of geometric classes' stratification.

Namely, the materialization of any given stratum is, if non-empty, an embedded submanifold of **codimension** equal to the **number** of letters S and T in its code. For inst., without these letters, there comes the unique open stratum of generic germs; all of them, recalling, equivalent to the relevant chained model (3).

Notwithstanding so regular a definition, there exist continuous numerical moduli of the local classification of Goursat distributions. In the first turn, in [12, 7], such moduli were found in codimension three. Later, in [11], real invariants were produced already in codimension two. While they are absent in codimension one, for it turns out, [8], that the codimension-one strata, i. e., all GG...GSG..G, are single orbits of the local classification. Moreover, because

[1] and the only one open, cf. a remark on codimensions below

these strata are adjacent only to the generic stratum, the codimension-one singularities of Goursat are all simple.

The present work extends substantially [11] and, at the same time, contributes to a vast project (first stated in 1999 by the authors of [6]) of finding all *simple* singularities of Goursat flags.

1.2. Codimension–two singularities of Goursat.

Codimension-two singularities are of two essentially different types.

• Either precisely two different members of a flag are simultaneously in singular positions at the same point; this is the concatenation SGG...GS of two singularities S separated by a number $j \geq 0$ of intermediate flag's members being in generic positions G.

•• Or else only one flag's member, say of corank k, is at a point in singular position, and its 'Lie square root' of corank $k+1$ is tangent at that point to the locus (a hypersurface in M) of the corank–k member being in singular position. Those latter singularities, as we know from sec. 1.1, are labelled ST. In virtue of [9], the ST singularities are simple, possibly modulo subvarieties of codimension three.

Passing to the class •, it is generally conjectured that the singularities SS (that is, with $j = 0$ letters G in between the S's) are all simple, too. However, the situation in geometric classes having two singularities S really separated by a number of G's, is different. The results presented in this paper, together with those of [11], justify that, excepting possibly the classes with the first S at the earliest admissible position No 3, for all *positive* numbers j of intermediate positions G, a modulus hides already in the flag' member three steps past the second singularity S. In fact, we are going to prove that

Theorem 1. *Fix any segment* [G...G] *of* $j \geq 1$ *letters* G. *Excepting (possibly) only the geometric classes starting with exactly two* G*'s before that segment, in each geometric class* GG...GS[G...G]SGGG... *with at least three* G*'s in the end, there sits at least one modulus of the local classification. Therefore, such classes' modalities are all not smaller than one.*

Therefore, in codimension two there are but *few* simple geometric classes: only 'ST', conjecturally 'SS', GG...GS[G...G]S... with at most two G's past the second S, and plausibly all GGS[G...G]S..., with, this time, any number of G's at the end.

This statement becomes more precise when one fixes flag's length r. Then the number of these simple and supposedly simple classes is $2(r-3) + (r-7)4 + 3 + 2 + 1 = 6r - 28$, while the number of all codimension-two geometric classes of length r is $(r-3) + (r-3) + (r-4) + \cdots + 2 + 1 = \frac{r^2-3r}{2}$. The remaining codimension-two classes, that assuredly are not simple by Theorem 1, constitute therefore the

$$1 - \frac{(6r-28)2}{r^2-3r} = \frac{r^2-15r+56}{r^2-3r}$$

fraction of all codimension-two classes in that length, an overwhelming majority.

Remark 1. The assertions of Thm. 1 can be equivalently stated in terms of the mentioned systematization of prolongations of Goursat germs from [6]. In the language of that reference work these reformulations go as follows.
For any $k \geq 4$, $j \geq 1$, and any germ, at a point p, of a Goursat flag

$$\cdots \supset D^{k+1+j} \supset D^{k+2+j} \supset D^{k+3+j} \supset D^{k+4+j} \tag{4}$$

sitting in the class $\mathrm{G}_{k-1}\mathrm{SG}_j\mathrm{SGGG}$, **firstly**, the prolongation pattern of the germ of D^{k+2+j} at p is 1: there is only one fixed point $L(D^{k+1+j})(p)/L(D^{k+2+j})(p)$ on the circle $S^1(D^{k+2+j})(p)$, and only two orbits in it: this fixed point and all the remaining of the circle; the prolonged germ D^{k+3+j} sits in that second orbit (this corresponds to all values b in (5) below being equivalent, hence equivalent to the value 0). And **secondly** – the main property being established in the present paper – the prolongation pattern of the germ at p of D^{k+3+j} is either 2c or 3, and, consequently, there appears a modulus in the local classification of the one-step prolongations of D^{k+3+j} (the value c cannot be moved by *flows* of symmetries of (5) understood not as a germ).

1.3. Basic preliminaries needed in the proof.

It follows from the works [5, 6] and previous contributions by the author that any Goursat germ D^{k+4+j} sitting in the geometric class $\mathrm{G}_{k-1}\mathrm{SG}_j\mathrm{SG}_3$ can be written down in certain *Kumpera-Ruiz* coordinates $x^1, x^2, \ldots, x^{k+6+j}, \ldots, x^n$ as the germ at $0 \in \mathbb{R}^n$ of a system of Pfaffian equations

$$
\begin{aligned}
dx^2 - x^3 dx^1 &= 0,\\
dx^3 - x^4 dx^1 &= 0,\\
* \qquad & *\\
dx^k - x^{k+1} dx^1 &= 0,\\
dx^1 - x^{k+2} dx^{k+1} &= 0,\\
dx^{k+2} - \left(1 + x^{k+3}\right) dx^{k+1} &= 0,\\
dx^{k+3} - x^{k+4}\, dx^{k+1} &= 0,\\
* \qquad & *\\
dx^{k+1+j} - x^{k+2+j} dx^{k+1} &= 0,\\
dx^{k+1} - x^{k+3+j} dx^{k+2+j} &= 0,\\
dx^{k+3+j} - \left(1 + x^{k+4+j}\right) dx^{k+2+j} &= 0,\\
dx^{k+4+j} - \left(b + x^{k+5+j}\right) dx^{k+2+j} &= 0,\\
dx^{k+5+j} - \left(c + x^{k+6+j}\right) dx^{k+2+j} &= 0,
\end{aligned}
\tag{5}
$$

with *certain* real constants b and c. (In these constants resides much of the difficulty and geometric complication of D^{k+4+j}, and we will cope with

them rather heavily.) The directions of the remaining, invisible in (5) variables $x^{k+7+j}, \dots, x^{n-1}, x^n$ do span the Cauchy-characteristic subdistribution $L(D^{k+4+j})$ (see sec. 1.1 for the definition) by which one can always factor out.
In all the sequel we assume this done; in other words, in what follows, $n = k+6+j$. This factoring out simplification, that loses – this is critical – no local geometry of a Goursat flag, amounts to saying that, without loss of generality, we assume the smallest member D^{k+4+j} of a flag to be of **rank two**. The ambient dimension after factoring out $(k+6+j)$ exceeds then just by two the flag's length $(k+4+j)$.

We will strive to move the parameters b, c. That is, to conjugate germs (5) displaying different values of b and c. The work [7] shows how involved it is to do on the level of *finite* symmetries. Only after that contribution we realized that the level of *infinitesimal* symmetries, of a non-local object encompassing nearby germs, was more promising.
How to describe the vector fields infinitesimally preserving a given Goursat distribution like D^{k+4+j} above? And so, each rank two Goursat germ is equivalent to the result of a sequence of certain projective extensions (called Cartan prolongations, described in detail in [1, 6]) started from the differential system (a contact structure) $\omega^1 = dx^2 - x^3dx^1 = 0$ living on $\mathbb{R}^3(x^1, x^2, x^3)$. And the infinitesimal symmetries of $\omega^1 = 0$ are generated by all C^∞ (or analytic, depending on the chosen category) functions $f(x^1, x^2, x^3)$ – a deep and basic thing observed long time ago by S. Lie. Those generating functions are nowadays called *contact hamiltonians.*

In view of the mentioned stepwise extensions yielding D^{k+4+j}, the i. s.'s of D^{k+4+j} turn out, fortunately if not unexpectedly, to be sequences of fairly simple parallel prolongations of the i. s.'s of the relevant sequence of Cartan prolongations of *that* Darboux structure. Consequently, they inherit the property of being locally 1–1 parametrized by C^∞ or C^ω functions in three variables.
However, the parametrization depends sensitively on the distribution of inversions of differentials in the pseudo-normal form for D (i. e., depends on the word over $\{*, S\}$ preliminarily encoding the germ' local geometry – see sec. 1.1 – or still else, depends on which members of the flag of D are in singular positions at the reference point). Therefore, one has to deal in general with a vast *binary tree* of different parametrizations. This is a disadvantage, yet for D^{k+4+j} in a concrete pseudo-normal form as above one can advance rather far.

These 'infinitesimal' tools are given in more detail in, for inst., [8] or [9]. Here we just recapitulate that, having a D of rank two, in a Kumpera-Ruiz pseudo-normal form (originating from [5]) in the ambient dimension $r+2$, one denotes by $\mathcal{Y}_f$ its infinitesimal symmetry induced by a function $f(x^1, x^2, x^3)$

and deliberately puts in relief in $\mathcal{Y}_f$ the first three components,

$$(6) \qquad \mathcal{Y}_f = A\partial_1 + B\partial_2 + C\partial_3 + \sum_{l=4}^{r+2} F^l \partial_l$$

– because, understandingly, the vector field $A\partial_1 + B\partial_2 + C\partial_3$ is an infinitesimal symmetry of $dx^2 - x^3 dx^1 = 0$. Hence the classical expressions of Lie: $A = -f_3$, $B = f - x^3 f_3$, $C = f_1 + x^3 f_2$.

Prior to write the infinitesimal symmetries of D^{k+4+j} in our situation, we need the following three vector fields

$$(7) \qquad \begin{aligned} y &= \partial_1 + x^3\partial_2 + x^4\partial_3 + \cdots + x^{k+1}\partial_k\,, \\ Y &= x^{k+2}y + \partial_{k+1} + X^{k+3}\partial_{k+2} + x^{k+4}\partial_{k+3} + \cdots + x^{k+2+j}\partial_{k+1+j}\,, \\ \widehat{Y} &= x^{k+3+j}Y + \partial_{k+2+j} + X^{k+4+j}\partial_{k+3+j} + X^{k+5+j}\partial_{k+4+j} + X^{k+6+j}\partial_{k+5+j}\,. \end{aligned}$$

With these notations, the first group of components of $\mathcal{Y}_f$ contains, on top of functions A, B, C,

$$(8) \qquad F^4 = yC - x^4 yA\,, \quad F^l = yF^{l-1} - x^l yA \qquad \text{for } 5 \le l \le k+1\,.$$

In the second group of components,

$$(9) \qquad \begin{aligned} F^{k+2} &= x^{k+2}\big(yA - YF^{k+1}\big)\,, \qquad F^{k+3} = YF^{k+2} - X^{k+3}YF^{k+1}\,, \\ F^l &= YF^{l-1} - x^l YF^{k+1} \qquad \text{for } k+4 \le l \le k+2+j; \\ F^{k+3+j} &= x^{k+3+j}\big(YF^{k+1} - \widehat{Y}F^{k+2+j}\big)\,, \\ F^{k+4+j} &= \widehat{Y}F^{k+3+j} - X^{k+4+j}\,\widehat{Y}F^{k+2+j}\,, \\ F^{k+5+j} &= \widehat{Y}F^{k+4+j} - X^{k+5+j}\,\widehat{Y}F^{k+2+j}\,, \\ F^{k+6+j} &= \widehat{Y}F^{k+5+j} - X^{k+6+j}\,\widehat{Y}F^{k+2+j}\,. \end{aligned}$$

These formulas confirm and re-establish a basic property of Kumpera-Ruiz coordinates (for whatever local Goursat object of corank r) that, in such coordinates, for $4 \le \nu \le r+2$, the component F^ν depends only on $x^1, x^2, \dots, x^{\nu-1}, x^\nu$. In our situation, they also help to quickly find the first $k+3$ components of $\mathcal{Y}_f$ **at zero**,

$$(10) \quad \mathcal{Y}_f\,|0 = -f_3\partial_1 + f\partial_2 + \sum_{j=3}^{k+1} f_{1_{j-2}}\partial_j - \big(2f_2 + (2k-1)f_{13}\big)\partial_{k+3} + \cdots$$

(the ∂_{k+3} component is also standard, cf., for inst., the formula (5) in [11]). Note the absence of the ∂_{k+2} component in (10), explained by the fact that the hypersurface $x^{k+2} = 0$ is invariant under all symmetries of D^{k+4+j}, let alone those embeddable in flows (for the same reason, cf. the proof of Prop. 1 below, each such i. s. $\mathcal{Y}_f\,|0$ has no ∂_{k+3+j} component as well). The next component ∂_{k+4} at 0 is computationally more delicate (albeit simple in the outcome).

1.4. Computation of $F^{k+4}\,|\,0$.

During all the computation we use the recursive formulas (9):

$$\begin{aligned} F^{k+4}\,|\,0 &= YF^{k+3} - x^{k+4}YF^{k+1}\,|\,0 = Y^2F^{k+2} - X^{k+3}Y^2F^{k+1}\,|\,0 \\ &= 2X^{k+3}Y\Big(yA - YF^{k+1}\Big) - X^{k+3}Y^2F^{k+1}\,|\,0 \\ &= 2X^{k+3}YyA - 3X^{k+3}Y^2F^{k+1}\,|\,0\,. \end{aligned} \tag{11}$$

To proceed, an operational expression for F^{k+1} is needed,

Lemma 1. $F^{k+1} = y^{k-2}C - (k-2)x^{k+1}yA \mod (x^k,\, x^{k-1}, \dots,\, x^4)\,.$

This lemma follows from the following formula than can easily be proved by induction on k, departing from F^4, already expressed in (8):

$$F^{k+1} = y^{k-2}C - \binom{k-2}{1}x^{k+1}yA - \binom{k-2}{2}x^ky^2A - \cdots - \binom{k-2}{k-2}x^4y^{k-2}A\,.$$

Differentiating the RHS in Lem. 1 twice with respect to Y,

$$Y^2F^{k+1} = Y^2y^{k-2}C - 2(k-2)YyA \mod (x^{k+2},\, x^{k+1}, \dots,\, x^4)\,.$$

After evaluating the RHS here at 0 and substituting to (11),

$$F^{k+4}\,|\,0 = \big(2 + 6(k-2)\big)X^{k+3}YyA - 3X^{k+3}Y^2y^{k-2}C\,|\,0\,. \tag{12}$$

The first summand on the RHS in (12) is being made transparent immediately,

$$YyA\,|\,0 = \begin{cases} -f_{33}\,|\,0 & \text{when } k = 3\,, \\ 0 & \text{when } k \geq 4\,. \end{cases}$$

As for the second summand, a more careful approach is needed, because the derivative $y^{k-2}C$ consists, for k big, of a rich array of terms.[2] Trying to see through them, we note that, in C, there clearly is the term f_1, in yC there is f_{11} (cf. Ex. 1 below) and likewise there is $f_{1_{k-1}}$ in $y^{k-2}C$. In fact, the structure of terms building up $y^{k-2}C$, together with the particular form of the vector field Y, imply that the exemplified term $f_{1_{k-1}}$ is the *only* one that contributes under Y^2 to the value of $Y^2y^{k-2}C$ at 0,

$$f_{1_{k-1}} \xrightarrow{Y} x^{k+2}f_{1_k} + \cdots \xrightarrow{Y} X^{k+3}f_{1_k} + \cdots\,,$$

[2] A deeper analysis of $y^{k-2}C$ is given in sec. 6.1 of the preprint No 39, *Simple codimension–two singularities of Goursat flags, I*, available at http://www.mimuw.edu.pl/english/research/reports/imat

with the last $\cdots$ vanishing at 0. Thus $Y^2 y^{k-2} C\,|\,0 = X^{k+3} f_{1_k}\,|\,0$. Substituting all these data to (12),

$$F^{k+4}\,|\,0 \;=\; \begin{cases} -3\big(X^{k+3}\big)^2 f_{1_k} - 8X^{k+3} f_{33}\,|\,0 & \text{when } k=3\,, \\ -3\big(X^{k+3}\big)^2 f_{1_k}\,|\,0 & \text{when } k \ge 4\,. \end{cases} \tag{13}$$

This information will be instrumental later in the proof of Thm. 1.

§2. Annihilation of the constant b

For transparency reasons it is useful to work, instead of the object (5), with a 'universal' distribution, E, that displays *no* constants shifting the last two variables, $E =$

$$\begin{aligned}
&\big(dx^2 - x^3 dx^1,\ dx^3 - x^4 dx^1,\ \ldots,\ dx^k - x^{k+1} dx^1,\ dx^1 - x^{k+2} dx^{k+1},\\
&\ dx^{k+2} - X^{k+3} dx^{k+1},\ dx^{k+3} - x^{k+4} dx^{k+1},\ \ldots,\ dx^{k+1+j} - x^{k+2+j} dx^{k+1},\\
&\ dx^{k+1} - x^{k+3+j} dx^{k+2+j},\ dx^{k+3+j} - X^{k+4+j} dx^{k+2+j},\\
&\qquad dx^{k+4+j} - x^{k+5+j} dx^{k+2+j},\ dx^{k+5+j} - x^{k+6+j} dx^{k+2+j}\big)\,,
\end{aligned} \tag{14}$$

$X^{k+3} = 1 + x^{k+3}$, $X^{k+4+j} = 1 + x^{k+4+j}$. The reason is that the symmetries under consideration will keep all but the last two coordinates of $0 \in \mathbb{R}^{k+6+j}$, while the two last ones will be moved.

In the annihilation of b there will be used certain concrete contact hamiltonians. Namely, $f(x^1, x^2, x^3) = \big(x^1\big)^l x^2$ when $j = 2l$ is even, and $f = \big(x^1\big)^{k+l}$ when $j = 2l+1$ is odd. This dependence on the parity of a distance parameter j should not be surprising, comparing, for inst., with the arguments in the codimension-one situation (Sec. 4 in [8]). We are going to reduce the constant b to 0, changing also – this is inevitable, cf. [11] – the value of c, but preserving the normalizations already achieved in (5). The value of b will be moved to 0 gradually, using the indicated flow of symmetries of the *non-local* object (14).

An important auxiliary question is why this would not perturb the zero constants standing by $x^{k+5}, x^{k+6}, \ldots, x^{k+2+j}$, as well as the constants one standing by x^{k+3} and x^{k+4+j}. It is so because

Proposition 1. *In either case of j even or odd, the corresponding infinitesimal symmetry $\mathcal{Y}_f$, with f proposed above, has at 0 the first $k+4+j$ components zero.*

Most of the present section is devoted to a proof of this statement. Concerning the first $k+4$ components of $\mathcal{Y}_f$, it is clear, in view of (10) and (13). Concerning $F^{k+3+j}\,|\,0$, it is also clear, for the $(k+1+j)$-th letter in the code is S and the hypersurface $x^{k+3+j} = 0$ is, naturally, invariant. Likewise, as regards $F^{k+4+j}\,|\,0$, this component corresponds to a place in the code going directly after a letter S and, as such, is a multiple of the additive constant one standing by the variable x^{k+4+j}. (In the absence of that constant, the germ would represent an 'ST' singularity that should be preserved by all symmetries of any its representative, and thus the $(k+4+j)$-th component would vanish.) It equals 1 times an integer combination of the basic partials $f_2\,|\,0$ and $f_{13}\,|\,0$. In fact, after a straightforward computation like in [8, 10],

$$F^{k+4+j}\,|\,0 \;=\; (2j+3)f_2 + \big(2+(k-1)(2j+3)\big)f_{13}\,|\,0\,. \tag{15}$$

Yet, trapped in between, there are $j-2$ remaining components and one should cope with them, too. In order to avoid many separate (and laborious) formulas, we propose a particular *valuation* $w(\cdot)$ assigning integer values, or *multiplicities*, to all Kumpera-Ruiz variables[3] and allowing to assign integer *abstract weights* to all terms in the polynomial expansion for $F^{\nu}(f)$, whichever is ν. In such a way, moreover, that each $F^{\nu}(f)$, a polynomial in $x^3,\, x^4, \dots,\, x^{k+2},\, X^{k+3}, \dots$ with coefficients – integer combinations of partials of f, gets its own abstract weight; cf. also sec. 2.3 in [10].
For it turns out that all terms of a given polynomial expansion have the *same* weights. Such is a surprising 'additional value' brought in by that valuation. There is, however, a price to it: the polynomials are to be understood particularly: variables shifted by a constant, like X^{k+3}, should be treated as indivisible entities; a valuation is assigned to a letter, irrespectively of its being small or capital.
In the proof of Prop. 1, upon getting the abstract weights of F^{k+5} through F^{k+2+j}, we will be in a position to rule out the presence of certain terms in their expansions, and that will do.

2.1. Definition of the valuation w.

An algebraic machinery underlying this definition concerns the auxiliary vector fields $y,\, Y,\, \widehat{Y}$ defined in (7) that are crucial in the formulas for the components of $\mathcal{Y}_f$. Not entering into details, roughly speaking we stipulate their being quasihomogeneous of order $-(2j+3)$, -2, and -1, respectively. We underline, however, that it is rather an algebraic, not analytic, quasihomogeneity; this is constantly being reminded of in the adjective 'abstract' (abstract weights). Here is that instrumental valuation.

$$w(x^1) = 2j+3\,, \quad w(x^2) = 2+(k-1)(2j+3)\,, \quad w(x^3) = 2+(k-2)(2j+3)\,,$$

$$w(x^4) = 2+(k-3)(2j+3)\,, \;\dots\dots,\; w(x^k) = 2+2j+3, \quad w(x^{k+1}) = 2\,,$$

$$w(x^{k+2}) = 2j+1,\; w\big(X^{k+3}\big) = 2j-1,\; w(x^{k+4}) = 2j-3, \dots,\; w(x^{k+1+j}) = 3,$$

$$w(x^{k+2+j}) = 1\,, \quad w(x^{k+3+j}) = 1\,, \quad w\big(X^{k+4+j}\big) = 0\,,$$

[3] not to be confused with nonholonomic orders of variables as functions! These arithmetic, not analytic, tools are in use since 1999.

$$w(x^{k+5+j}) = -1\,, \qquad w(x^{k+6+j}) = -2\,.$$

Having these multiplicities of letters, to each formal monomial $x^I X^J f_K$ (multiindices vary within their pertinent ranges; for J it is $\{k+3,\ k+4+j\}$, for K it is $\{1,\ 2,\ 3\}$, etc.) we attach its abstract weight

$$(16) \qquad w\big(x^I X^J f_K\big) \stackrel{\text{def}}{=} \sum_{k\in K} w(x^k) - \sum_{j\in J} w(X^j) - \sum_{i\in I} w(x^i)\,.$$

Note that these abstract weights are attached irrespectively of the concrete nature of f which could even be, say, identically zero.

Example 1. $F^4 = yC - x^4 yA = y(f_1 + x^3 f_2) - x^4 y(-f_3) =$

$$f_{11} + 2x^3 f_{12} + 2x^4 f_{13} + \big(x^3\big)^2 f_{22} + 2x^3 x^4 f_{23} + \big(x^4\big)^2 f_{33}$$

has, as one instantly checks, all displayed monomial terms of abstract weight $2(2j+3)$.

Example 2. Let us analyze the terms appearing on the RHS of (13). The monomial $\big(X^{k+3}\big)^2 f_{1_k}$ has weight $kw(x^1) - 2w(X^{k+3}) = k(2j+3) - 2(2j-1)$. The auxiliary monomial appearing for $k=3$, $X^{k+3} f_{33}$, has weight $2(2+(3-2)(2j+3)) - (2j-1) = 3(2j+3) - 2(2j-1)$ coinciding, for this value of k, with the previous one.

These examples are instances of a more general, already invoiced, fact.

Proposition 2. *All monomials building up each fixed F^ν have one and the same abstract weight that becomes, by definition, the abstract weight $w(F^\nu)$ of that F^ν. In particular, $w(F^{k+4+\nu}) = w(F^{k+4}) + 2\nu = k(2j+3) - 2(2j-1) + 2\nu$ for $\nu = 1,\ 2, \dots, j-2$.*
Moreover, $w(F^{k+3+j}) = w(x^2) - 1$, $w(F^{k+4+j}) = w(x^2)$, $w(F^{k+5+j}) = w(x^2) + 1$, $w(F^{k+6+j}) = w(x^2) + 2$.

One proves this proposition constantly using the recursive formulas (9) for the components of an i.s., plus the quasihomogeneity of the fields y, Y, $\widehat{Y}$. For the indicated particular $j-2$ components, in the recurrences there appears uniquely the field Y which is quasihomogeneous of order -2, whence that arithmetical progression with step 2 on the level of abstract weights. □

2.2. Proof of Proposition 1.

Assume first $j = 2l$ even. In view of Prop. 2, it is a matter of arithmetics that, for $\nu = 1,\ 2, \dots, j-2$,

$$w\big(F^{k+4+\nu}\big) = w\big(f_{1_l 2}\big) - (l+1)(2j-1) - (2j-2-2\nu)\,.$$

Note that $2j-2-2\nu$ takes values in the set $\{2,\ 4, \dots, 2j-4\}$. Therefore, no term $\big(X^{k+3}\big)^\mu f_{1_l 2}$, having its abstract weight equal to $w\big(f_{1_l 2}\big) - \mu(2j-1)$, may appear in the expansion of $F^{k+4+\nu}$.

Assume now $j = 2l + 1$ odd. It is also only simple arithmetics that, for $\nu = 1, 2, \dots, j-2$,

$$w\big(F^{k+4+\nu}\big) \;=\; w\big(f_{1_{k+l}}\big) - (l+2)(2j-1) - (2j-2-2\nu)\,.$$

And, this time, no term $\big(X^{k+3}\big)^{\mu} f_{1_{k+l}}$, having its abstract weight $w\big(f_{1_{k+l}}\big) - \mu(2j-1)$, appears in the expansion of $F^{k+4+\nu}$.
The proof of Prop. 1 is now complete. $\square$

2.3. The reduction at work.

Recalling, we are going to analyze $F^{k+5+j}(f)\,|\,0$ in two different situations. Firstly for $j = 2l$, when $f = \big(x^1\big)^l x^2$, and later for $j = 2l+1$, when $f = \big(x^1\big)^{k+l}$. The value at 0 is important, but we will also need the values of this component of the infinitesimal symmetry at all points of the x^{k+5+j}–axis (as it was the case in codimension one in [8], and codimension two in ([9])).

Observation 1. *In either case of j even or odd, $F^{k+5+j}(f)\,|\,0$ is a positive integer $N(k,j)$.*

Idea of proof. In the even $j = 2l$ case, one directly indicates, in the formal polynomial $F^{k+5+j}(f)$, a term $(\cdot)\big(X^{k+3}\big)^{l+1}\big(X^{k+4+j}\big)^2 f_{1_l\,2}$, with $(\cdot)$ being a positive integer. Moreover, as one can also check, this is the only term in $F^{k+5+j}(f)$ using this concrete partial of f. And, clearly, only this partial does not vanish at 0, when evaluated on the proposed contact hamiltonian. Whence the statement with $N(k,j) = l!(\cdot)$.
In the odd $j = 2l+1$ case, in the polynomial in question there is a term $(\cdot)\big(X^{k+3}\big)^{l+2}\big(X^{k+4+j}\big)^2 f_{1_{k+l}}$ with, again, a positive integer $(\cdot)$, and this is the unique term displaying this partial of f – the only partial sensitive to (or: not-vanishing on) the proposed hamiltonian, implying the statement with, this time, $N(k,j) = (k+l)!(\cdot)$. $\square$

Dealing with the infinitesimal symmetries of (14), we need to know the values of F^{k+5+j} not only at 0, but also at all points of the x^{k+5+j}-axis. Yet this causes no complications. It is a matter of course that this function is affine in x^{k+5+j}. In fact, after standard computations that we omit here (compare with (15)),

$$\begin{aligned} &F^{k+5+j}\big(0, 0, \dots, 0, x^{k+5+j}\big) \;= \\ (17)\qquad &\quad F^{k+5+j}(0) + \big((3j+4)f_2 + \big(3 + (k-1)(3j+4)\big)f_{13}\big)(0)\cdot x^{k+5+j} \\ &\;= \; N(k,j) \end{aligned}$$

by Obs. 1, for f in either case: the coefficient standing by x^{k+5+j} vanishes because so do $f_2\,|\,0$ and $f_{13}\,|\,0$ (the partial f_{13} vanishes identically; f_2 vanishes identically in the odd case, and equals $\big(x^1\big)^l$ in the even case).
Moreover, with our final objective being the next variable x^{k+6+j} (supposed to conceal a modulus) we need to know the next, and last, component F^{k+6+j}

at all its arguments of the form $(0,0,\dots,0,x^{k+5+j},\,x^{k+6+j})$, similarly to the situation $j=1$ discussed in [11]. Much like in that reference, nearly repeating the arguments from there, one sees that the latter function does not depend on x^{k+6+j} and is just affine in x^{k+5+j},

$$F^{k+6+j}(0,\,0,\,\dots,\,0,\,x^{k+5+j},\,x^{k+6+j}) \;=\; A(k,j)+B(k,j)x^{k+5+j}\,,$$

for certain real (even integer) constants A, B (in the present work, unlike in [11], we are not interested in their precise values). Let us recapitulate our informations. With the above chosen f, we have the first $k+4+j$ components of $\mathcal{Y}_f$ vanishing at 0, the next component effectively known on the x^{k+5+j}–axis, and the still next effectively known on the $(x^{k+5+j},\,x^{k+6+j})$-plane. Therefore, the integral curve $\gamma(\cdot)$ of $\mathcal{Y}_f$ passing at time $t=0$ through $(0,0,\dots,0,b,c)\in\mathbb{R}^{k+6+j}$ is easily tractable. Indeed, it reads

$$\gamma(t) \;=\; \Big(0,\;0,\;\dots,\;0,\;\;b+N\,t,\;\;c+\big(A+bB\big)t+\frac{BN}{2}t^2\Big) \tag{18}$$

and is defined for all values of t. We are interested in the time $t=-\frac{b}{N}$ when the one before last coordinate of γ vanishes. This point on the curve equals, after a short computation,

$$\gamma\Big(-\frac{b}{N}\Big) \;=\; \Big(0,\,0,\,0,\dots,\,0,\,0,\;c-\frac{2bA+b^2B}{2N}\Big)\,.$$

Retreating now from the universal object E back to the Goursat germs (5), we summarize the present section in

Corollary 1. *It is possible to annihilate the constant b in (5) at the expense of passing from the value c of the next (and, in the occurrence, last) constant to a new value $c-\frac{2bA+b^2B}{2N}$, with A, B, and $N>0$ depending only on k and j.*

Attention. For the sake of simplicity, we will use the same letter c for the last constant in (5) also after the annihilation of b.

§3. Proof of Theorem 1

From now on we assume that $j\ge 2$, since the geometric classes with the segment 'SGS' ($j=1$) have already been treated in the work [11]. We assume also, cf. the wording of Theorem, that the first letter S in the class' code has number $k\ge 4$; the second S has thus number $k+j+1$. It is very important that the cases $k=3$ are put aside.
The moduli emerging in the situations with $j=1$ are understood to the end: it is known that they are of the type 3 of the systematization of [6], cf. Rem. 1. Recalling after that remark, in the classes with $j\ge 2$ we will only show that the modulus' type is either 2c or 3.

We will explore the possibility of changing only the last constant c in (5) when keeping previously secured simplifications *and* having, after Sec. 2, $b = 0$. Prior to that, however, we are to work more with the distribution E given by (14). Recalling, that object has no additive constant next to x^{k+6+j}, and the only variables in its description that are shifted by constants are X^{k+3} and X^{k+4+j}. This is advantageous; to the distribution that does display the constant c we will come back only in sec. 3.3.

Lemma 2. *Concerning the infinitesimal symmetries of E, for any contact hamiltonian f, $F^{k+6+j}(f)\,|0 \;=\; (\cdot)X^{k+3}\big(X^{k+4+j}\big)^2 f_{1_k}\,|0 \;= (\cdot)f_{1_k}\,|0$, where $(\cdot)$ is a certain integer.*

This lemma is critical for the paper and its proof will be long. Starting it now, one notes that only the terms of the form $\big(X^{k+3}\big)^\mu\big(X^{k+4+j}\big)^\nu f_K$, with certain $\mu,\,\nu \geq 0$, *can* contribute to the value of F^{k+6+j} at 0. It is also clear that $K \neq \{3\}$ (because $w(F^{k+6+2}) = w(x^2) + 2 > w(f_3)$), and that any possible $2 \in K$ is arithmetically equivalent to 1 and 3 taken together, as $w(x^2) = w(x^1) + w(x^3)$, and the same will hold for the other valuation, v, to be defined below in sec. 3.1, see (21)).
Therefore, without loss of generality, one can assume that either

- $\bullet$ K contains uniquely $d \geq 1$ integers 1, or else
- $\bullet\bullet$ K contains one 3 and $d \geq 1$ integers 1, or else
- $\bullet\bullet\bullet$ K contains $d \geq 0$ integers 1 and $e \geq 2$ integers 3.

The alternative $\bullet$ is the easiest one and we can handle it right now. Suppose that such a term shows up in the polynomial expansion of F^{k+6+j} and look at the difference $w(f_{1_d}) - w(F^{k+6+j})$ which has to be non-negative, hence necessarily $d \geq k$, and which has to be a multiple of $w\big(X^{k+3}\big) = 2j - 1$, that is to say,

$$(19)\qquad (d-k+1)(2j+3) - 4 \;\equiv\; 0 \mod (2j-1)\,.$$

For $d_1 = k$ we get a first solution, yielding the associated-to-it value $\mu_1 = 1$. What are the subsequent solutions? For $d = k+1$ the LHS of (19) is congruent to $4\cdot 1$, for $d = k+2$ — congruent to $4\cdot 2$, and so on. Since 4 and $2j-1$ are coprime, not sooner than for $d_2 = k + 2j - 1$ we get the second solution, with the associated to it $\mu_2 = 2j + 4$. Such value of μ is far too high, the highest theoretically conceivable power of X^{k+3} being $j + 4$: it is only $\big(X^{k+3}\big)^1$ in F^{k+3}, it is at most $\big(X^{k+3}\big)^2$ in F^{k+4}, and so on, at most $\big(X^{k+3}\big)^{j+4}$ (or, only in terms contributing at 0, a lower power of X^{k+3}) in F^{k+6+j}. The subsequent solutions have still bigger μ's, and we thus end $\bullet$ with the unique possibility

$$(20)\qquad d = k\,, \qquad \mu = 1\,, \qquad \nu = ?\,.$$

Before tackling (and, in the outcome, disposing of) the alternative $\bullet\bullet\bullet$ we need to introduce another valuation. Its necessity is felt already in (20): the valuation w is sensitive to X^{k+3}, but insensitive to X^{k+4+j}, attributing it the multiplicity zero. Because of that its weak point, at this moment we do not know the value(s) of ν in (20). Our next valuation v, not surprisingly, will be sensitive to X^{k+4+j} and insensitive to X^{k+3}.

3.1. Definition of the valuation v.

This new valuation is entirely abstract, or: algebraic; it was found much time after the valuation w. Together with w, it allows to catch the moduli of the geometric classes 'SG...GS' in a crossing fire.

$$v(x^1) = -2\,, \quad v(x^2) = -2k+1\,, \quad v(x^3) = -2k+3\,,$$

$$v(x^4) = -2k+5\,, \ldots\ldots, \quad v(x^k) = -3, \quad v(x^{k+1}) = -1\,,$$

$$v(x^{k+2}) = -1,\ v\big(X^{k+3}\big) = 0,\ v(x^{k+4}) = 1,\ v(x^{k+5}) = 2\ \ldots\,,$$

$$v(x^{k+2+j}) = j-1\,, \quad v(x^{k+3+j}) = -j\,, \quad v\big(X^{k+4+j}\big) = -2j+1\,,$$

$$v(x^{k+5+j}) = -3j+2\,, \qquad v(x^{k+6+j}) = -4j+3\,.$$

With this valuation, the vector fields y, Y, $\widehat{Y}$ are quasihomogeneous of orders, respectively, 2, 1, and $-j+1$. Note also the fundamental property

$$v(x^2) \ = \ v(x^1) + v(x^3)\,, \tag{21}$$

as well as the fact that the analogue of Prop. 2 holds. That is, the v-abstract weights of the components of infinitesimal symmetries are well-defined. In particular, $v(F^{k+4+j}) = 2j-2k$, $v(F^{k+5+j}) = 3j-2k-1$,

$$v(F^{k+6+j}) \ = \ 4j-2k-2\,. \tag{22}$$

3.2. Dispensing with the alternatives • • • and •• – the end of proof of Lemma 2.

The alternative • • • becomes now tractable, modulo an elementary (if not completely trivial)

Observation 2. *Any term* $\big(X^{k+3}\big)^{\mu}\big(X^{k+4+j}\big)^{\nu} f_K$ *that contributes to the value* $F^{k+6+j}\,|0$ *has the exponent* $\nu < 3$.

Idea of proof. What is needed, is just to express the function F^{k+6+j} recursively back via F^{k+4+j} (the first component with the factor X^{k+4+j} present) and F^{k+2+j}, and then analyze carefully all the appearing terms, focusing on those contributing at 0.

Let us suppose now the presence in F^{k+6+j} of a term $\big(X^{k+3}\big)^{\mu}\big(X^{k+4+j}\big)^{\nu} f_{1_d\,3_e}$, with $d \geq 0$ and $e \geq 2$. This term has its v-abstract weight equal to (22),

$$-\,2d - 2ek + 3e - \nu(-2j+1) \ = \ 4j-2k-2\,,$$

or else

$$-2d - 2(e-1)k + 3e \;=\; (2-\nu)(2j-1)\,. \tag{23}$$

The RHS of this equation is, in view of Obs. 2, non-negative, whereas the LHS is negative, for $3e < 8(e-1) \le 2(e-1)k$. This contradiction shows that the alternative $\bullet\bullet\bullet$ is void.

Remark 2. The LHS of (23) is often negative also for $e = 1$ (and not only for $e \ge 2$). It equals then $-2d + 3$ and is negative for $d \ge 2$, while the RHS is always non-negative in virtue of Obs. 2.

In turn, passing to the alternative $\bullet\bullet$, suppose that such a term $\big(X^{k+3}\big)^{\mu}\big(X^{k+4+j}\big)^{\nu} f_{1_d\,3}$ is being present and look at the 'old' quantity $w(f_{1_d\,3}) - w(F^{k+6+j})$ which must be a non-negative multiple of $w\big(X^{k+3}\big) \;=\; 2j-1$. That is,

$$2 + (k+d-2)(2j+3) - 4 - (k-1)(2j+3) \;\equiv\; 0 \mod (2j-1)\,,$$

or else

$$(d-1)(2j+3) - 2 \;\equiv\; 0 \mod (2j-1)\,. \tag{24}$$

For $d = 1$ the LHS of this congruence is -2. For $d = 2$ it is congruent to $2\cdot 1$, for $d = 3$ — congruent to $2\cdot 3$, for $d = 4$ — to $2\cdot 5$, and so on. Since 2 and $2j-1$ are coprime, it is straightforward to see that the smallest natural integer solution of (24) is $d = j+1$; the LHS of (24) is then congruent to $2(2j-1)$, that is congruent to 0. In view of Rem. 2 above, this, and all the remaining (bigger) solutions of (24) lead to no contributing term in $F^{k+6+j}\,|\,0$. The alternative $\bullet\bullet$ is void.

Knowing that much, we are now in a position to find the unknown ν in (20). Indeed, the v-abstract weights say that, for a term $X^{k+3}\big(X^{k+4+j}\big)^{\nu} f_{1_k}$ to appear in the polynomial F^{k+6+j}, there must hold $4j - 2k - 2 = -2k - \nu(-2j+1)$, yielding $\nu = 2$. At long last, Lemma 2 is proved. $\square$

3.3. The end of proof of Theorem 1.

Now that the last component at 0 of the infinitesimal symmetries of the distribution E, (14), is known, we ask the same question for the specific D written down in (5) and, recalling, having the constant $b = 0$. That is, differing from E only by the constant c standing in the last Pfaffian equation.
The situation is similar to those leading earlier to the formulas (15) and (17). To the value ascertained in Lem. 2 one should add a correction term involving only the partials $f_2\,|\,0$ and $f_{13}\,|\,0$. The computation of that correction is fully algorithmized and yields

$$F^{k+6+j}(f)\,|\,0 \;=\; (\cdot) f_{1_k} + c\big((4j+5)f_2 + \big(4 + (k-1)(4j+5)\big)f_{13}\big)\;|\,0\,. \tag{25}$$

Consider now any infinitesimal symmetry, of a *non*-local object given by the equations (5) now with $b = 0$, such that its first $k+5+j$ components vanish at 0. That is, on the level of germs, stipulated is the preservation of all normalizations made until now save, hypothetically, the preservation of the value of c: the finite symmetries that emerge by integration preserve all but the last coordinate of the point $0 \in \mathbb{R}^{k+6+j}$. 'But the last' is in the hypothetical mode and we will instantly see that this possibility is void.

In fact, we consider the contact hamiltonians f such that all but the last components of $\mathcal{Y}_f\,|0$ vanish and look at the formulas (10) and (15). The RHS in the latter is zero and so is the coefficient standing next to ∂_{k+3} in the former. On top of that, the matrix of integer coefficients creating the mentioned expressions is invertible,

$$\begin{vmatrix} -2 & -(2k-1) \\ & \\ 2j+3 & 2+(k-1)(2j+3) \end{vmatrix} = 2j-1\,.$$

Hence $f_2\,|0 = f_{13}\,|0 = 0$. In view of (13), also $f_{1_k}\,|0 = 0$. By (25), therefore, $F^{k+6+j}(f)\,|0 = 0$. The preservation of all but the last coordinates of 0 implies the same for the last one – the preservation of the point 0 as such!

It is impossible to perturb the value of c in (5) by means of the *embeddable* symmetries of (5) understood as a finite object. The prolongation pattern at the last step No $k+4+j$, visualised in (4), is thus – out of the five alternatives of [6] – either 2c or 3, meaning the emergence of a modulus of the local classification. Theorem 1 is now proved. □

§4. Conjectured precise modalities of the 'SG...GS' classes

In this section, in contradistinction to the preceding part, $j \geq 1$ (we merge now with the domain of [11]. That is, in the codes of codimension-two geometric classes of Goursat flags there stand $j \geq 1$ letters G in between the two letters S.) The classes with the two S just neighbouring are consequently not discussed in the present work.

We **conjecture** since long, and recently even more so in the light of partially related to this field works [14, 3, 15] of Ishikawa and Zhitomirskii, that [but see also Rem. 3 below]

⋆ the modality of all classes $\mathrm{GGSG}_j\mathrm{SG}_l$, $l \geq 1$, is zero — they are all simple;

⋆ the modality of classes $\mathrm{GGGSG}_j\mathrm{SG}_l$ is one from $l = 3$ onwards;

⋆ the modality of classes $\mathrm{GGGGSG}_j\mathrm{SG}_l$ is one for $l = 3, 4, 5, 6$, and is two from $l = 7$ onwards;

⋆ the modality of classes $\mathrm{GGGGGSG}_j\mathrm{SG}_l$ is one for $l = 3, 4, 5, 6$, is two for $l = 7, 8, 9, 10$, and is three from $l = 11$ onwards,

and so on.

Remark 3. Precisely speaking, not 100% of these statements is in the conjectural mode. Namely, these for $l = 3$, with at least three G's at the code' beginning, are proved: for $j = 1$ in [11] and for $j \geq 2$ in the present work.

The moduli exemplified in [11] are of type 3/[6], and the same we *suppose* for those of Thm. 1 related to $j \geq 2$. Yet, by the infinitesimal methods alone, it is not possible to distinguish between the module types 2c and 3 of [6] (cf. in this respect also Sec. 4 in [10]).

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