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Some Remarks on the Infinitesimal Rigidity of the Complex Quadric

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Introduction

Let (X, g) be a compact Riemannian symmetric space. We say that a symmetric 2-form h on X satisfies the zero-energy condition if for all closed geodesics γ of X the integral

$$\int_\gamma h = \int_0^L h(\dot\gamma(s), \dot\gamma(s))ds$$

of h over γ vanishes, where $\dot\gamma(s)$ is the tangent vector to the geodesic γ parametrized by its arc-length and L is the length of γ. A Lie derivative of the metric g always satisfies the zero-energy condition. The space (X, g) said to be infinitesimally rigid if the only symmetric 2-forms on X satisfying the zero-energy condition are the Lie derivatives of the metric g.

Michel introduced the notion of infinitesimal rigidity in the context of the Blaschke conjecture, and proved that the real projective spaces $\mathbb{RP}^n$, with $n \geq 2$, and the flat tori of dimension ≥ 2 are infinitesimally rigid (see [17], [18] and [2]). Michel and Tsukamoto demonstrated the infinitesimal rigidity of the complex projective space $\mathbb{CP}^n$ of dimension $n \geq 2$ (see [17], [21], [6] and [7]); in fact, they proved that all the projective spaces which are not isometric to a sphere are infinitesimally rigid.

In [7] and [9], we showed that the complex quadric Q_n of dimension n is infinitesimally rigid when $n \geq 4$. In the monograph [12], we shall give a complete proof of the infinitesimal rigidity of the complex quadric Q_3 of dimension 3, which relies on the Guillemin rigidity of the Grassmannian of 2-planes in $\mathbb{R}^{n+2}$ proved in [10] and on results of Tela Nlenvo [20].

In this note, we present outlines of some new proofs of the infinitesimal rigidity of the complex quadric Q_n of dimension $n \geq 4$; the

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complete proofs shall appear in [12]. In particular, we show that the infinitesimal rigidity of the quadric Q_3 implies that all the quadrics Q_n, with $n \geq 4$, are infinitesimally rigid. The new proof of the infinitesimal rigidity of the complex quadric Q_n of dimension $n \geq 5$ presented here is quite different from the one found in [7] and follows some of the lines of the proof for the infinitesimal rigidity of the complex quadric Q_4 given in [9].

§1. Symmetric spaces

Let (X, g) be a Riemannian manifold. We denote by T and T^* its tangent and cotangent bundles. By $\bigotimes^k T^*$, $S^l T^*$, $\bigwedge^j T^*$, we shall mean the k-th tensor product, the l-th symmetric product and the j-th exterior product of the vector bundle T^*. If α, $\beta \in T^*$, we identify the symmetric product $\alpha \cdot \beta$ with the element $\alpha \otimes \beta + \beta \otimes \alpha$ of $\bigotimes^2 T^*$. If E is a vector bundle over X, we denote by $E_{\mathbb{C}}$ its complexification, by $\mathcal{E}$ the sheaf of sections of E over X and by $C^\infty(E)$ the space of global sections of E over X. If ξ is a vector field on X and β is a section of $\bigotimes^k T^*$ over X, we denote by $\mathcal{L}_\xi \beta$ the Lie derivative of β along ξ. Let $g^\sharp \colon T^* \to T$ be the isomorphism determined by the metric g.

Let $B = B_X$ be the sub-bundle of $\bigwedge^2 T^* \otimes \bigwedge^2 T^*$ consisting of those tensors $u \in \bigwedge^2 T^* \otimes \bigwedge^2 T^*$ satisfying the first Bianchi identity

$$u(\xi_1, \xi_2, \xi_3, \xi_4) + u(\xi_2, \xi_3, \xi_1, \xi_4) + u(\xi_3, \xi_1, \xi_2, \xi_4) = 0,$$

for all ξ_1, ξ_2, ξ_3, $\xi_4 \in T$. Let H denote the sub-bundle of $T^* \otimes B$ consisting of those tensors $v \in T^* \otimes B$ which satisfy the relation

$$v(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) + v(\xi_2, \xi_3, \xi_1, \xi_4, \xi_5) + v(\xi_3, \xi_1, \xi_2, \xi_4, \xi_5) = 0,$$

for all ξ_1, ξ_2, ξ_3, ξ_4, $\xi_5 \in T$.

Let

$$\mathrm{Tr} \colon S^2 T^* \to \mathbb{R}, \qquad \mathrm{Tr} \colon \bigwedge^2 T^* \otimes \bigwedge^2 T^* \to \bigotimes^2 T^*$$

be the trace mappings defined by

$$\mathrm{Tr}\, h = \sum_{j=1}^n h(t_j, t_j), \qquad (\mathrm{Tr}\, u)(\xi, \eta) = \sum_{j=1}^n u(t_j, \xi, t_j, \eta),$$

for $h \in S^2 T^*_x$, $u \in \bigwedge^2 T^* \otimes \bigwedge^2 T^*_x$ and ξ, $\eta \in T_x$, where $x \in X$ and $\{t_1, \ldots, t_n\}$ is an orthonormal basis of T_x. It is easily seen that

$$\mathrm{Tr}\, B \subset S^2 T^*.$$

We denote by $S^2_0T^*$ the sub-bundle of S^2T^* equal to the kernel of the trace mapping $\mathrm{Tr}\colon S^2T^* \to \mathbb{R}$.

We now introduce various differential operators associated to the Riemannian manifold (X, g). First, let ∇ be the Levi-Civita connection of (X, g). The Killing operator

$$D_0\colon \mathcal{T} \to S^2\mathcal{T}^*$$

of (X, g) sends $\xi \in \mathcal{T}$ into $\mathcal{L}_\xi g$. The Killing vector fields of (X, g) are the solutions $\xi \in C^\infty(T)$ of the equation $D_0\xi = 0$. Consider the first-order differential operator

$$\mathrm{div}\colon S^2\mathcal{T}^* \to \mathcal{T}^*$$

and the Laplacian

$$\overline{\Delta}\colon S^2\mathcal{T}^* \to S^2\mathcal{T}^*$$

defined by

$$(\mathrm{div}\, h)(\xi) = -\sum_{j=1}^{n} (\nabla h)(t_j, t_j, \xi),$$

$$(\overline{\Delta} h)(\xi, \eta) = -\sum_{j=1}^{n} (\nabla^2 h)(t_j, t_j, \xi, \eta),$$

for $h \in C^\infty(S^2T^*)$, $\xi,\ \eta \in T_x$, where $x \in X$ and $\{t_1, \dots, t_n\}$ is an orthonormal basis of T_x. The formal adjoint of D_0 is equal to $2g^\sharp \cdot \mathrm{div}\colon S^2\mathcal{T}^* \to \mathcal{T}$. Since D_0 is elliptic, if X is compact, we therefore have the orthogonal decomposition

$$\text{(1.1)} \qquad C^\infty(S^2T^*) = D_0C^\infty(T) \oplus \{h \in C^\infty(S^2T^*) \mid \mathrm{div}\, h = 0\}$$

(see [1]).

Let $\mathcal{R}(h)$ be the Riemann curvature tensor, as defined in [5, §4], and $\mathrm{Ric}(h)$ be the Ricci tensor of a metric h on X, which is are sections of B and S^2T^*, respectively. We set $R = \mathcal{R}(g)$ and $\mathrm{Ric} = \mathrm{Ric}(g)$; we have $\mathrm{Ric} = -\mathrm{Tr}\, R$. We also consider the curvature tensor $\widetilde{R}$ which is the section of $\bigwedge^2 T^* \otimes T^* \otimes T$ related to R by

$$g(\widetilde{R}(\xi_1, \xi_2, \xi_3), \xi_4) = R(\xi_1, \xi_2, \xi_3, \xi_4),$$

for ξ_1, ξ_2, ξ_3, $\xi_4 \in T$. Let

$$\mathcal{R}'_g\colon S^2\mathcal{T}^* \to \mathcal{B}$$

be the linear differential operator of order 2 which is the linearization along g of the non-linear operator $h \mapsto \mathcal{R}(h)$, where h is a Riemannian metric on X. The invariance of the operator $h \mapsto \mathcal{R}(h)$ leads us to the formula

$$\mathcal{R}'_g(\mathcal{L}_\xi g) = \mathcal{L}_\xi R, \tag{1.2}$$

for all $\xi \in \mathcal{T}$.

We now suppose that (X, g) is an Einstein manifold and we write $\mathrm{Ric} = \lambda g$, with $\lambda \in \mathbb{R}$. We consider the morphism of vector bundles $L\colon S^2T^* \to S^2T^*$ determined by

$$L(\alpha \cdot \beta)(\xi, \eta) = 2\big(R(\xi, g^\sharp\alpha, \eta, g^\sharp\beta) + R(\xi, g^\sharp\beta, \eta, g^\sharp\alpha)\big),$$

for α, $\beta \in T^*$ and ξ, $\eta \in T$, and the Lichnerowicz Laplacian

$$\Delta\colon S^2\mathcal{T}^* \to S^2\mathcal{T}^*$$

of [16] defined by

$$\Delta h = \overline{\Delta} h + 2\lambda h + Lh,$$

for $h \in S^2\mathcal{T}^*$. If X is compact, in [1] Berger-Ebin define the space $E(X)$ of infinitesimal Einstein deformations of the metric g by

$$E(X) = \{h \in C^\infty(S^2T^*) \mid \operatorname{div} h = 0,\ \operatorname{Tr} h = 0,\ \Delta h = 2\lambda h\}$$

(see also Koiso [14]); by definition, the space $E(X)$ is contained in an eigenspace of the Lichnerowicz Laplacian Δ, which is a determined elliptic operator, and is therefore finite-dimensional.

For the remainder of this section, we shall suppose that (X, g) is a connected locally symmetric space. We consider the sub-bundle $\tilde{B} = \tilde{B}_X$ of B, which is the infinitesimal orbit of the curvature and whose fiber at $x \in X$ is

$$\tilde{B}_x = \{(\mathcal{L}_\xi R)(x) \mid \xi \in \mathcal{T}_x \text{ with } (\mathcal{L}_\xi g)(x) = 0\}.$$

We denote by $\alpha\colon B \to B/\tilde{B}$ the canonical projection and we consider the second-order differential operator

$$D_1\colon S^2\mathcal{T}^* \to \mathcal{B}/\tilde{\mathcal{B}}$$

introduced in [5] and determined by

$$(D_1 h)(x) = \alpha(\mathcal{R}'_g(h - \mathcal{L}_\xi g))(x),$$

for $x \in X$ and $h \in S^2\mathcal{T}^*_x$, where ξ is an element of $\mathcal{T}_x$ satisfying $h(x) = (\mathcal{L}_\xi g)(x)$. Using (1.2), it is easily seen that this operator is well-defined and that

$$D_1 \cdot D_0 = 0.$$

Thus we may consider the complex

$$C^\infty(T) \xrightarrow{D_0} C^\infty(S^2T^*) \xrightarrow{D_1} C^\infty(B/\tilde{B}). \tag{1.3}$$

In [5] and [12], we prove the following result:

Theorem 1.1. *Suppose that (X, g) is a symmetric space of compact type. If the equality*

$$H \cap (T^* \otimes \tilde{B}) = \{0\} \tag{1.4}$$

holds, the sequence (1.3) is exact.

If (X, g) has constant curvature, according to [5] we have

$$\tilde{B} = \{0\}; \tag{1.5}$$

in this case, the operator D_1 is equal to the one introduced by Calabi [3].

Let Y be a connected totally geodesic submanifold of X; we denote by i the natural imbedding of Y into X. Let $g_Y = i^*g$ be the Riemannian metric on Y induced by g. Then (Y, g_Y) is a connected locally symmetric space. For $x \in Y$, we consider the mapping $i^*\colon B_x \to B_{Y,x}$; in [7] and [12], we show that

$$i^*\tilde{B}_x \subset \tilde{B}_{Y,x}.$$

If Y has constant curvature, by (1.5) we know that $\tilde{B}_Y = \{0\}$, and so we infer that

$$i^*\tilde{B} = \{0\}. \tag{1.6}$$

The following lemma is proved in [12] (see also Lemma 1.2 of [7]).

Lemma 1.1. *Assume that (X, g) is a connected locally symmetric space. Let Y, Z be totally geodesic submanifolds of X; suppose that Z is a submanifold of Y of constant curvature. Let h be a section of S^2T^* over X. Let $x \in Z$ and u be an element of B_x such that $(D_1h)(x) = \alpha u$. If the restriction of h to the submanifold Y is a Lie derivative of the metric on Y induced by g, then the restriction of u to the submanifold Z vanishes.*

§2. Criteria for infinitesimal rigidity

Let (X, g) be a compact locally symmetric space. As we remarked in the introduction, if ξ is a vector field on X, the symmetric 2-form $\mathcal{L}_\xi g$ on X satisfies the zero-energy condition. From this fact and the decomposition (1.1), we obtain:

Proposition 2.1. *Let X be a compact locally symmetric space. Assume that any symmetric 2-form h, which satisfies the zero-energy condition and the relation* $\operatorname{div} h = 0$, *vanishes. Then the space X is infinitesimally rigid.*

We now assume that (X, g) is a symmetric space of compact type. Then there is a Riemannian symmetric pair (G, K) of compact type, where G is a compact, connected semi-simple Lie group and K is a closed subgroup of G such that the space X is isometric to the homogeneous space G/K endowed with a G-invariant metric. We identify X with G/K.

Let $\mathcal{F}$ be a family of closed connected totally geodesic surfaces of X which is invariant under the group G. Then the set $N_{\mathcal{F}}$ consisting of those elements of B, which vanish when restricted to the submanifolds belonging to $\mathcal{F}$, is a sub-bundle of B. According to formula (1.6), we see that

$$\tilde{B} \subset N_{\mathcal{F}};$$

we shall identify $N_{\mathcal{F}}/\tilde{B}$ with a sub-bundle of $B/\tilde{B}$. If $\beta\colon B/\tilde{B} \to B/N_{\mathcal{F}}$ is the canonical projection, we consider the differential operator

$$D_{1,\mathcal{F}} = \beta D_1 \colon S^2\mathcal{T}^* \to \mathcal{B}/\mathcal{N}_{\mathcal{F}}.$$

Let $\mathcal{F}'$ be a family of closed connected totally geodesic submanifolds of X. We denote by $\mathcal{L}(\mathcal{F}')$ the subspace of $C^\infty(S^2T^*)$ consisting of all symmetric 2-forms h satisfying the following condition: for all submanifolds $Z \in \mathcal{F}'$, the restriction of h to Z is a Lie derivative of the metric of Z induced by g. If every submanifold of X belonging to $\mathcal{F}'$ is infinitesimally rigid, then a symmetric 2-form h on X satisfying the zero-energy condition belongs to $\mathcal{L}(\mathcal{F}')$; indeed, the restriction of h to a submanifold $Z \in \mathcal{F}'$ also satisfies the zero-energy condition.

From Lemma 1.1, we obtain:

Proposition 2.2. *Let (X, g) be a symmetric space of compact type. Let $\mathcal{F}$ be a family of closed connected totally geodesic surfaces of X which is invariant under the group G, and let $\mathcal{F}'$ be a family of closed connected totally geodesic submanifolds of X. Assume that each surface of X belonging to $\mathcal{F}$ is contained in a submanifold of X belonging to $\mathcal{F}'$. A symmetric 2-form h on X belonging to $\mathcal{L}(\mathcal{F}')$ satisfies the relation $D_{1,\mathcal{F}}h = 0$.*

Theorem 2.1. *Let (X, g) be a symmetric space of compact type. Let $\mathcal{F}$ be a family of closed connected totally geodesic surfaces of X which is invariant under the group G, and let $\mathcal{F}'$ be a family of closed connected totally geodesic submanifolds of X. Assume that every submanifold of*

X belonging to $\mathcal{F}'$ is infinitesimally rigid; assume that each surface of X belonging to $\mathcal{F}$ is contained in a submanifold of X belonging to $\mathcal{F}'$. Suppose that the relation (1.4) and the equality

$$N_{\mathcal{F}} = \tilde{B} \tag{2.1}$$

hold. Then the symmetric space X is infinitesimally rigid.

Proof. Let h be a symmetric 2-form h on X satisfying the zero-energy condition. According to our hypothesis on the family $\mathcal{F}'$, we know that h belongs to $\mathcal{L}(\mathcal{F}')$. From Proposition 2.1, we obtain the relation $D_{1,\mathcal{F}}h = 0$. According to the equality (2.1), we therefore see that $D_1 h = 0$. By the relation (1.4) and Theorem 1.1, the sequence (1.3) is exact, and so h is a Lie derivative of the metric g.

We now assume that (X, g) is an irreducible symmetric space of compact type; then X is an Einstein manifold and we have $\mathrm{Ric} = \lambda g$, where λ is a positive real number. The following result appears in [12].

Theorem 2.2. *Let (X, g) be an irreducible symmetric space of compact type. Let $\mathcal{F}$ be a family of closed connected totally geodesic surfaces of X which is invariant under the group G, and let $\mathcal{F}'$ be a family of closed connected totally geodesic submanifolds of X. Let E be a G-invariant sub-bundle of $S^2_0 T^*$. Assume that each surface of X belonging to $\mathcal{F}$ is contained in a submanifold of X belonging to $\mathcal{F}'$, and suppose that the relation*

$$\mathrm{Tr}\, N_{\mathcal{F}} \subset E \tag{2.2}$$

holds. Let h be a symmetric 2-form on X satisfying the relations

$$\mathrm{div}\, h = 0, \qquad D_{1,\mathcal{F}}h = 0.$$

Then we may write

$$h = h_1 + h_2,$$

where h_1 is an element of $E(X)$ and h_2 is a section of E; moreover, if h also satisfies the zero-energy condition, we may require that h_1 and h_2 satisfy the zero-energy condition.

Proof. Since $\mathrm{Tr}\, E = \{0\}$ and since the relation (2.2) holds, by Lemma 2.1 of [11], with $N = N_{\mathcal{F}}$, we see that $\mathrm{Tr}\, h = 0$ and that

$$\Delta h - 2\lambda h \in C^\infty(E).$$

A variant of Proposition 4.2 of [11], with $\mu = 2\lambda$, gives us the desired result.

§3. The complex quadric

We suppose that X is the complex quadric Q_n, with $n \geq 2$, which is the complex hypersurface of complex projective space $\mathbb{CP}^{n+1}$ defined by the homogeneous equation

$$\zeta_0^2 + \zeta_1^2 + \cdots + \zeta_{n+1}^2 = 0,$$

where $\zeta = (\zeta_0, \zeta_1, \ldots, \zeta_{n+1})$ is the standard complex coordinate system of $\mathbb{C}^{n+2}$. Let g be the Kähler metric on X induced by the Fubini-Study metric $\tilde{g}$ on $\mathbb{CP}^{n+1}$ of constant holomorphic curvature 4. We denote by J the complex structure of X or of $\mathbb{CP}^{n+1}$.

The group $SU(n+2)$ acts on $\mathbb{C}^{n+2}$ and $\mathbb{CP}^{n+1}$ by holomorphic isometries. Its subgroup $G = SO(n+2)$ leaves the submanifold X of $\mathbb{CP}^{n+1}$ invariant; in fact, the group G acts transitively and effectively on the Riemannian manifold (X, g) by holomorphic isometries. It is easily verified that X is isometric to the homogeneous space

$$SO(n+2)/SO(2) \times SO(n)$$

of the group $SO(n+2)$, which is a Hermitian symmetric space of compact type; when $n \geq 3$, this space is irreducible. We also know that (X, g) is an Einstein manifold; its Ricci tensor is given by

$$\text{Ric} = 2ng. \tag{3.1}$$

We now recall some results of Smyth [19]. The second fundamental form C of the complex hypersurface X of $\mathbb{CP}^{n+1}$ is a symmetric 2-form with values in the normal bundle of X in $\mathbb{CP}^{n+1}$. We denote by S the bundle of unit vectors of this normal bundle.

Let x be a point of X and ν be an element of S_x. We consider the element h_ν of $S^2T_x^*$ defined by

$$h_\nu(\xi, \eta) = \tilde{g}(C(\xi, \eta), \nu),$$

for all ξ, $\eta \in T_x$. Since $\{\nu, J\nu\}$ is an orthonormal basis for the fiber of the normal bundle of X in $\mathbb{CP}^{n+1}$ at the point x, we see that

$$C(\xi, \eta) = h_\nu(\xi, \eta)\nu + h_{J\nu}(\xi, \eta)J\nu,$$

for all ξ, $\eta \in T_x$. If μ is another element of S_x, we have

$$\mu = \cos\theta \cdot \nu + \sin\theta \cdot J\nu, \tag{3.2}$$

with $\theta \in \mathbb{R}$. We consider the symmetric endomorphism K_ν of T_x determined by

$$h_\nu(\xi, \eta) = g(K_\nu \xi, \eta),$$

for all $\xi,\ \eta \in T_x$. Since our manifolds are Kähler, we have

$$C(\xi, J\eta) = JC(\xi, \eta),$$

for all $\xi,\ \eta \in T_x$; from this relation, we deduce the equalities

$$K_{J\nu} = JK_\nu = -K_\nu J. \tag{3.3}$$

It follows that h_ν and $h_{J\nu}$ are linearly independent. By (3.3), we see that h_ν belongs to $(S^2T^*)^-$. If μ is the element (3.2) of S_x, it is easily verified that

$$K_\mu = \cos\theta \cdot K_\nu + \sin\theta \cdot JK_\nu. \tag{3.4}$$

From the Gauss equation, the expression for the Riemann curvature tensor of $\mathbb{CP}^{n+1}$ (endowed with the metric $\tilde{g}$) and the relation (3.3), we obtain the equality

$$\begin{aligned}\tilde{R}(\xi,\eta)\zeta = g(\eta,\zeta)\xi - g(\xi,\zeta)\eta + g(J\eta,\zeta)J\xi - g(J\xi,\zeta)J\eta \\ - 2g(J\xi,\eta)J\zeta + g(K_\nu\eta,\zeta)K_\nu\xi - g(K_\nu\xi,\zeta)K_\nu\eta \\ + g(JK_\nu\eta,\zeta)JK_\nu\xi - g(JK_\nu\xi,\zeta)JK_\nu\eta,\end{aligned} \tag{3.5}$$

for all $\xi,\ \eta,\ \zeta \in T_x$. From (3.3), we infer that the trace of the endomorphism K_ν of T_x vanishes. According to this last remark and formulas (3.3) and (3.5), we see that

$$\mathrm{Ric}(\xi,\eta) = -2g(K_\nu^2\xi,\eta) + 2(n+1)g(\xi,\eta),$$

for all $\xi,\ \eta \in T_x$. From (3.1), it follows that K_ν is an involution. We call K_ν the *real structure* of the quadric associated to the unit normal ν.

We denote by T_ν^+ and T_ν^- the eigenspaces of K_ν corresponding to the eigenvalues $+1$ and -1, respectively. Then by (3.3), we infer that J induces isomorphisms of T_ν^+ onto T_ν^- and of T_ν^- onto T_ν^+, and that

$$T_x = T_\nu^+ \oplus T_\nu^- \tag{3.6}$$

is an orthogonal decomposition. If ϕ is an element of the group G, we have

$$C(\phi_*\xi, \phi_*\eta) = \phi_* C(\xi,\eta),$$

for all $\xi,\ \eta \in T$. Thus, if μ is the tangent vector $\phi_*\nu$ belonging to $S_{\phi(x)}$, we see that

$$h_\mu(\phi_*\xi, \phi_*\eta) = h_\nu(\xi,\eta),$$

for all $\xi,\ \eta \in T_x$, and hence that

$$K_\mu\phi_* = \phi_* K_\nu \tag{3.7}$$

on T_x. Therefore ϕ induces isomorphisms

$$\phi_*: T_\nu^+ \to T_\mu^+, \qquad \phi_*: T_\nu^- \to T_\mu^-.$$

We now decompose the homogeneous bundle S^2T^* of symmetric 2-forms on X into G-invariant sub-bundles following [8]. The complex structure of X induces a decomposition

$$S^2T^* = (S^2T^*)^+ \oplus (S^2T^*)^-$$

of the bundle S^2T^*, where $(S^2T^*)^+$ is the sub-bundle of Hermitian forms and $(S^2T^*)^-$ is the sub-bundle of skew-Hermitian forms. We consider the sub-bundle L of $(S^2T^*)^-$ introduced in [8], whose fiber at $x \in X$ is equal to

$$L_x = \{h_\mu \mid \mu \in S_x\};$$

according to (3.4), this fiber L_x is generated by the elements h_ν and $h_{J\nu}$ and so the sub-bundle L of $(S^2T^*)^-$ is of rank 2. We denote by $(S^2T^*)^{-\perp}$ the orthogonal complement of L in $(S^2T^*)^-$.

For $h \in (S^2T^*)^+_x$, we define an element $K_\nu(h)$ of $S^2T^*_x$ by

$$K_\nu(h)(\xi, \eta) = h(K_\nu\xi, K_\nu\eta),$$

for all ξ, $\eta \in T_x$. Using (3.3) and (3.5), we see that $K_\nu(h)$ belongs to $(S^2T^*)^+$ and does not depend on the choice of the unit normal ν. We thus obtain a canonical involution of $(S^2T^*)^+$ over all of X, which gives us the orthogonal decomposition

$$(S^2T^*)^+ = (S^2T^*)^{++} \oplus (S^2T^*)^{+-}$$

into the direct sum of the eigenbundles $(S^2T^*)^{++}$ and $(S^2T^*)^{+-}$ corresponding to the eigenvalues $+1$ and -1, respectively, of this involution. We easily see that

$$(S^2T^*)^{++}_x = \{h \in (S^2T^*)^+_x \mid h(\xi, J\eta) = 0, \text{ for all } \xi, \eta \in T_\nu^+\},$$
$$(S^2T^*)^{+-}_x = \{h \in (S^2T^*)^+_x \mid h(\xi, \eta) = 0, \text{ for all } \xi, \eta \in T_\nu^+\}.$$

The metric g is a section of $(S^2T^*)^{++}$ and generates a line bundle $\{g\}$, whose orthogonal complement in $(S^2T^*)^{++}$ is the sub-bundle $(S^2T^*)^{++}_0$ consisting of the traceless symmetric tensors of $(S^2T^*)^{++}$. We thus obtain the G-invariant orthogonal decomposition

$$S^2T^* = L \oplus (S^2T^*)^{-\perp} \oplus \{g\} \oplus (S^2T^*)^{++}_0 \oplus (S^2T^*)^{+-}; \tag{3.8}$$

using the relation (3.7), we easily see that this decomposition is G-invariant.

Let x_0 be a fixed point of X and let K be the subgroup of G equal to the isotropy group of the point x_0. Let $\mathfrak{g}$ denote the complexification of the Lie algebra $\mathfrak{so}(n+2)$ of G. The fibers at x_0 of the sub-bundles of S^2T^* appearing in the decomposition (3.8) and their complexifications are K-modules.

We write

$$E_1 = (S^2T^*)^{++}_{0,\mathbb{C}}, \qquad E_2 = L_{\mathbb{C}}, \qquad E_3 = (S^2T^*)^{-\perp}_{\mathbb{C}}.$$

In [12], we prove the following result:

Lemma 3.1. *Let X be the complex quadric Q_n, with $n \geq 3$.*

(i) *We have*

$$\operatorname{Hom}_K(\mathfrak{g}, E_{j,x_0}) = \{0\},$$

for $j = 1, 2, 3$.

(ii) *If $n \neq 4$, we have*

$$\dim \operatorname{Hom}_K(\mathfrak{g}, (S^2T^*)^{+-}_{\mathbb{C},x_0}) = 1.$$

(iii) *If $n = 4$, we have*

$$\dim \operatorname{Hom}_K(\mathfrak{g}, (S^2T^*)^{+-}_{\mathbb{C},x_0}) = 2.$$

From Lemma 3.1 and the decomposition (3.8), we deduce that

$$\dim \operatorname{Hom}_K(\mathfrak{g}, S^2_0T^*_{\mathbb{C},x_0}) = 1 \tag{3.9}$$

when $n \neq 4$, and that

$$\dim \operatorname{Hom}_K(\mathfrak{g}, S^2_0T^*_{\mathbb{C},x_0}) = 2 \tag{3.10}$$

when $n = 4$.

In [12], it is shown that the following proposition is a consequence of Lemma 3.1 and the equalities (3.9) and (3.10).

Proposition 3.1. *Let X be the complex quadric Q_n, with $n \geq 3$. If $n \neq 4$, we have*

$$E(X) = \{0\}.$$

If $n = 4$, we have

$$E(X) \subset C^\infty((S^2T^*)^{+-}).$$

When $n \neq 4$, the vanishing of the space $E(X)$ was first proved by Koiso (see [14] and [15]).

§4. Totally geodesic submanifolds of the quadric

In this section, we suppose that X is the complex quadric Q_n, with $n \geq 3$. We first introduce various families of closed connected totally geodesic submanifolds of X. Let x be a point of X and ν be an element of S_x.

If $\{\xi, \eta\}$ is an orthonormal set of vectors of T_ν^+, according to formula (2.5) we see that the set $\mathrm{Exp}_x F$ is a closed connected totally geodesic surface of X, whenever F is the subspace of T_x generated by one of following families of vectors:

(A_1) $\{\xi, J\eta\}$;
(A_2) $\{\xi + J\eta, J\xi - \eta\}$;
(A_3) $\{\xi, J\xi\}$;
(A_4) $\{\xi, \eta\}$.

Let $\{\xi, \eta\}$ be an orthonormal set of vectors of T_ν^+. According to [4], if F is generated by the family (A_2) (resp. the family (A_3)) of vectors, the surface $\mathrm{Exp}_x F$ is isometric to the complex projective line $\mathbb{CP}^1$ with its metric of constant holomorphic curvature 4 (resp. curvature 2). Moreover, if F is generated by the family (A_1), the surface $\mathrm{Exp}_x F$ is isometric to a flat torus. In [12], we verify that, if F is generated by the family (A_4), the surface $\mathrm{Exp}_x F$ is isometric to a sphere of constant curvature 2.

For $1 \leq j \leq 4$, we denote by $\tilde{\mathcal{F}}^{j,\nu}$ the set of all closed totally geodesic surfaces of X which can be written in the form $\mathrm{Exp}_x F$, where F is a subspace of T_x generated by a family of vectors of type (A_j).

If ε is a number equal to ± 1 and if ξ, η, ζ are unit vectors of T_ν^+ satisfying

$$g(\xi, \eta) = g(\xi, \zeta) = 3g(\eta, \zeta) = \varepsilon \frac{3}{5},$$

and if F is the subspace of T_x generated by the vectors

$$\{\xi + J\zeta, \eta + \varepsilon J(\xi - \eta) - J\zeta\},$$

according to (2.5) we also see that the set $\mathrm{Exp}_x F$ is a closed connected totally geodesic surface of X. Moreover, according to [4] this surface is isometric to a sphere of constant curvature $2/5$. We denote by $\tilde{\mathcal{F}}^{5,\nu}$ the set of all such closed totally geodesic surfaces of X.

If $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ is an orthonormal set of vectors of T_ν^+ and if F is the subspace of T_x generated by the vectors

$$\{\xi_1 + J\xi_2, \xi_3 + J\xi_4\},$$

according to (2.5) we see that the set $\mathrm{Exp}_x F$ is a closed connected totally geodesic surface of X. Moreover, according to [4] this surface is isometric to the real projective plane $\mathbb{RP}^2$ of constant curvature 1. Clearly such submanifolds of X only occur when $n \geq 4$. We denote by $\tilde{\mathcal{F}}^{6,\nu}$ the set of all such closed totally geodesic surfaces of X.

If $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ is an orthonormal set of vectors of T_ν^+ and if F is the subspace of T_x generated by the vectors

$$\{\xi_1 + J\xi_2, J\xi_1 - \xi_2, \xi_3 + J\xi_4, J\xi_3 - \xi_4\},$$

according to (2.5) we see that the set $\mathrm{Exp}_x F$ is a closed connected totally geodesic submanifold of X. Moreover, this submanifold is isometric to the complex projective plane $\mathbb{CP}^2$ of constant holomorphic curvature 4. Clearly such submanifolds of X only occur when $n \geq 4$. We denote by $\tilde{\mathcal{F}}^{7,\nu}$ the set of all such closed totally geodesic submanifolds of X.

When $n \geq 4$, clearly a surface belonging to the family $\tilde{\mathcal{F}}^{2,\nu}$ or to the family $\tilde{\mathcal{F}}^{6,\nu}$ is contained in a closed totally geodesic submanifold of X belonging to the family $\tilde{\mathcal{F}}^{7,\nu}$. In fact, the surfaces of the family $\tilde{\mathcal{F}}^{2,\nu}$ (resp. the family $\tilde{\mathcal{F}}^{6,\nu}$) correspond to complex lines (resp. to linearly imbedded real projective planes) of the submanifolds of X belonging to the family $\tilde{\mathcal{F}}^{7,\nu}$ viewed as complex projective planes.

Let W be a subspace of T_ν^+ of dimension $k \geq 2$; by (3.6), we may consider the subspace $F = W \oplus JW$ of T_x of dimension $2k$, which is stable under J. The set $\mathrm{Exp}_x F$ is a closed connected totally geodesic complex submanifold of X; in [12], we show that it isometric to the quadric Q_k of dimension k. Let $\mathcal{F}'$ be the G-invariant family of all closed connected totally geodesic submanifolds of X which are isometric to the quadric Q_3 of dimension 3.

Let Z be a surface belonging to the family $\tilde{\mathcal{F}}^{j,\nu}$, with $1 \leq j \leq 5$. We may write $Z = \mathrm{Exp}_x F$, where F is an appropriate subspace of T_x. Clearly, this space F is contained in a subspace of T_x which can be written in the form $W \oplus JW$, where W is a subspace of T_ν^+ of dimension 3. Therefore Z is contained in a submanifold of X belonging to $\mathcal{F}'$.

For $1 \leq j \leq 7$, we consider the G-invariant families

$$\tilde{\mathcal{F}}^j = \bigcup_{\substack{\nu \in S_x \\ x \in X}} \mathcal{F}^{j,\nu}$$

of closed connected totally geodesic submanifolds of X. When $n \geq 4$, we know that a surface belonging to the family $\tilde{\mathcal{F}}^2$ is contained in a closed totally geodesic submanifold of X belonging to the family $\tilde{\mathcal{F}}^7$. We write

$$\mathcal{F}_1 = \tilde{\mathcal{F}}^1 \cup \tilde{\mathcal{F}}^3 \cup \tilde{\mathcal{F}}^4, \qquad \mathcal{F}_2 = \tilde{\mathcal{F}}^1 \cup \tilde{\mathcal{F}}^2 \cup \tilde{\mathcal{F}}^6,$$
$$\mathcal{F}_3 = \tilde{\mathcal{F}}^1 \cup \tilde{\mathcal{F}}^2 \cup \tilde{\mathcal{F}}^4 \cup \tilde{\mathcal{F}}^5.$$

We have seen that a surface belonging to the family $\tilde{\mathcal{F}}^j$, with $1 \leq j \leq 5$, is contained in a closed totally geodesic submanifold of X belonging to the family $\mathcal{F}'$.

In [4], Dieng classifies all closed connected totally geodesic surfaces of X and proves the following:

Proposition 4.1. *If $n \geq 3$, then the family of all closed connected totally geodesic surfaces of X is equal to $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.*

In fact, the family $\tilde{\mathcal{F}}^1$ is equal to the set of all maximal flat totally geodesic tori of X.

We now describe some of the relationships between the families of closed totally geodesic surfaces of X introduced above, the G-invariant sub-bundles of S^2T^* and the infinitesimal orbit of the curvature $\tilde{B}$. If $\mathcal{F}$ is a G-invariant family of closed connected totally geodesic surfaces of X, we denote by $N_{\mathcal{F}}$ the sub-bundle of B consisting of those elements of B which vanish when restricted to the submanifolds of $\mathcal{F}$.

For $j = 1, 2, 3$, we set

$$N_j = N_{\mathcal{F}_j}.$$

According to formula (1.6), we see that

$$\tilde{B} \subset N_j,$$

for $j = 1, 2, 3$.

The following lemma, proved in [12], will not be required here.

Lemma 4.1. *For $n \geq 3$, we have*

$$\operatorname{Tr} N_1 \subset (S^2T^*)^{+-}.$$

In [12], we prove Proposition 4.2; on the other hand, Proposition 4.3 is given by Proposition 5.1 of [8].

Proposition 4.2. *For $n \geq 5$, we have*

$$\operatorname{Tr} N_2 = L.$$

Proposition 4.3. *For $n = 4$, we have*

$$\operatorname{Tr} N_2 \subset L \oplus (S^2T^*)^{+-}.$$

In [6], Dieng shows that an element of N_3 vanishes when restricted to a surface of X belonging to the family $\tilde{\mathcal{F}}^3$ and proves the following result:

Proposition 4.4. *For $n \geq 3$, we have*

$$N_3 = \tilde{B}.$$

When $n \geq 3$, Dieng [4] shows that

$$H \cap (T^* \otimes N_3) = \{0\},$$

and then deduces the relation (1.4) for the complex quadric X from Proposition 4.4; thus, we have the following result:

Proposition 4.5. *For $n \geq 3$, we have*

$$H \cap (T^* \otimes \tilde{B}) = \{0\}.$$

From Proposition 4.5 and Theorem 1.1, we deduce the exactness of the sequence (1.3) for the complex quadric $X = Q_n$, with $n \geq 3$.

§5. Infinitesimal rigidity of the quadric

The sub-bundle $L_{\mathbb{C}}$ of $S^2T^*_{\mathbb{C}}$ is a homogeneous bundle over X; thus $C^\infty(L_{\mathbb{C}})$ is a G-module. Let γ be an element of the set $\hat{G}$ of equivalence classes of irreducible G-modules over $\mathbb{C}$, and let V_γ be an irreducible G-module which is a representative of γ. In [12], we show that the isotypic component $C^\infty_\gamma(L_{\mathbb{C}})$ of the G-module $C^\infty(L_{\mathbb{C}})$ corresponding to γ is a G-submodule of $C^\infty(L_{\mathbb{C}})$ isomorphic to k copies of V_γ, where k is equal either to 0 or 2. When $k = 2$, we also describe an explicit basis for the subspace W_γ of dimension 2 generated by the highest weight vectors of the G-module $C^\infty_\gamma(L_{\mathbb{C}})$; we then consider the action of the differential operator $\operatorname{div}\colon S^2T^*_{\mathbb{C}} \to T^*_{\mathbb{C}}$ on the elements of W_γ and prove that the induced mapping $\operatorname{div}\colon W_\gamma \to C^\infty(T^*_{\mathbb{C}})$ is injective. Since the restriction $\operatorname{div}\colon L_{\mathbb{C}} \to T^*_{\mathbb{C}}$ is a homogeneous differential operator, from these facts we deduce the following result:

Proposition 5.1. *Let X be the complex quadric Q_n, with $n \geq 3$. A section h of L over X, which satisfies the relation* $\operatorname{div} h = 0$, *vanishes identically.*

The essential aspects of the proof of following proposition were first given by Dieng in [4].

Proposition 5.2. *The infinitesimal rigidity of the quadric Q_3 implies that all the quadrics Q_n, with $n \geq 3$, are infinitesimally rigid.*

Proof. We consider the G-invariant family $\mathcal{F}_3$ of closed connected totally geodesic surfaces of X and the family $\mathcal{F}'$ of closed connected totally geodesic submanifolds of X isometric to the quadric Q_3 of §4. We have seen that each surface belonging to the family $\mathcal{F}_3$ is contained in a totally geodesic submanifold of X belonging to the family $\mathcal{F}'$. Assume that we know that the quadric Q_3 is infinitesimally rigid; then every submanifold of X belonging to $\mathcal{F}'$ is infinitesimally rigid; moreover, by Propositions 4.4 and 4.5, the families $\mathcal{F} = \mathcal{F}_3$ and $\mathcal{F}'$ satisfy the hypotheses of Theorem 2.1. From this last theorem, we deduce the infinitesimal rigidity of X.

We consider the families $\tilde{\mathcal{F}}^1$, $\tilde{\mathcal{F}}^2$, $\tilde{\mathcal{F}}^6$ and $\tilde{\mathcal{F}}^7$ of closed connected totally geodesic submanifolds of X. We set

$$\mathcal{F}'' = \tilde{\mathcal{F}}^1 \cup \tilde{\mathcal{F}}^6 \cup \tilde{\mathcal{F}}^7.$$

We consider the G-invariant family

$$\mathcal{F} = \mathcal{F}_2 = \tilde{\mathcal{F}}^1 \cup \tilde{\mathcal{F}}^2 \cup \tilde{\mathcal{F}}^6$$

of totally geodesic surfaces of X and the sub-bundle $N_2 = N_{\mathcal{F}_2}$ of B, introduced in §4, and the corresponding differential operator

$$D_{1,\mathcal{F}} \colon S^2\mathcal{T}^* \to \mathcal{B}/\mathcal{N}_2.$$

We recall that a submanifold of X belonging to $\tilde{\mathcal{F}}^1$ (resp. to $\tilde{\mathcal{F}}^6$) is a surface isometric to the flat 2-torus (resp. to the real projective plane $\mathbb{RP}^2$), while a submanifold of X belonging to $\tilde{\mathcal{F}}^7$ is isometric to the complex projective space $\mathbb{CP}^2$. Each surface belonging to $\tilde{\mathcal{F}}^2$ is contained in a submanifold of X belonging to the family $\tilde{\mathcal{F}}^7$; therefore each surface of X belonging to $\mathcal{F}$ is contained in a submanifold of X belonging to the family $\mathcal{F}''$. In the introduction, we mentioned that a flat 2-tori, the real projective plane $\mathbb{RP}^2$ and the complex projective space $\mathbb{CP}^2$ are infinitesimally rigid symmetric spaces. Thus every submanifold of X belonging to $\mathcal{F}''$ is infinitesimally rigid. Hence a symmetric 2-form h on X satisfying the zero-energy condition belongs to $\mathcal{L}(\mathcal{F}'')$; by Proposition 2.2, the 2-form h verifies the relation

$$D_{1,\mathcal{F}} h = 0.$$

Proposition 5.3. *Let h be a symmetric 2-form on quadric $X = Q_n$, with $n \geq 4$, satisfying the zero-energy condition and the relation* $\operatorname{div} h = 0$. *Then when $n \geq 5$, the symmetric form h is a section of the vector bundle L; when $n = 4$, it is a section of the vector bundle* $L \oplus (S^2T^*)^{+-}$.

Proof. We know that h belongs to $\mathcal{L}(\mathcal{F}'')$. We suppose that $n \geq 5$ (resp. that $n = 4$). According to Proposition 4.2 (resp. to Proposition 4.3), we see that the hypotheses of Theorem 2.2 hold, with $E = L$ (resp. with $E = L \oplus (S^2T^*)^{+-}$). By Proposition 3.1, we know that $E(X) = \{0\}$ (resp. that $E(X) \subset C^\infty((S^2T^*)^{+-})$). Then Theorem 2.2 tells us that h is a section of L (resp. of $L \oplus (S^2T^*)^{+-}$).

The following result is proved in [9] (see also [12]):

Proposition 5.4. *Let X be the quadric Q_4. A section h of the vector bundle $L \oplus (S^2T^*)^{+-}$ satisfying the relations*

$$\operatorname{div} h = 0, \qquad D_{1,\mathcal{F}} h = 0$$

vanishes identically.

We now prove the infinitesimal rigidity of the quadric $X = Q_n$, with $n \geq 4$, using Propositions 5.1, 5.3 and 5.4. In the case $n = 4$, this proof appears in [9]. Let h be a symmetric 2-form on the quadric $X = Q_n$, with $n \geq 4$, satisfying the zero-energy condition and the relation $\operatorname{div} h = 0$. When $n \geq 5$, Proposition 5.3 tells us that h is a section of L; by Proposition 5.1, we see that h vanishes identically. When $n = 4$, Proposition 5.3 tells us that h is a section of $L \oplus (S^2T^*)^{+-}$, and, as we saw above, Proposition 2.2 gives us the relation $D_{1,\mathcal{F}} h = 0$; by Proposition 5.4, we see that h vanishes. Then Proposition 2.1 gives us the infinitesimal rigidity of X.

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