# Brieskorn Manifolds and Metrics
# of Positive Scalar Curvature

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**Abstract.**

Using relevant properties of the Seiberg-Witten monopole invari-
ants, a non-existence theorem of metrics of positive scalar curvature
on certain series of Brieskorn 3-manifolds is proved. This gives a
short proof without the help of Gromov-Lawson's theorem.

## §1.  Introduction

In this note, we will study a non-existence theorem of Riemann-
ian metrics of positive scalar curvature on certain series of Brieskorn
3-manifolds by using relevant properties of the Seiberg-Witten mono-
pole invariants. By contrast, this theorem can be regarded as a char-
acterization of Bireskorn 3-manifolds obtained by the so-called simple
singularites of type $A_n, D_n, E_6, E_7$ and $E_8$.

Let $p, q$ and $r$ be integers grater than or equal to 2. Let $\Sigma(p, q, r)$ be a
smooth oriented compact 3-dimensional manifold defined by the intersec-
tion of the complex algebraic surface $\{(z_1, z_2, z_3) \in \mathbf{C}^3 \mid z_1^p + z_2^q + z_3^r = 0\}$
with the unit sphere $\{(z_1, z_2, z_3) \in \mathbf{C}^3 \mid |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$. We
call $\Sigma(p, q, r)$ the Brieskorn 3-manifold of type $(p, q, r)$. These manifolds
have a non-trivial $S^1$-action. Roughly speaking, the family of Brieskorn
3-manifolds can be devided geometrically into three classes, according
as the rational number $1/p + 1/q + 1/r - 1$ is positve, zero or negative.
Milnor [Mi] showed that in the positive case, $\Sigma(p, q, r)$ can be described
as a quotient space of $S^3$ by a certain finite subgroup $\Gamma$ of $SU(2)$. It
is well-known that finite subgroups of $SU(2)$ can be classified by the
Dynkin diagrams of type $A_n, D_n, E_6, E_7$ and $E_8$. Of course, since $\Gamma$
acts on $S^3$ isometrically, $\Sigma(p, q, r)$ actually carries a metric of positive

scalar curvature in these cases. In [O-O], in collaboration with Ono the present author studied this positive case from the point of view of a relation with singularity theory and contact/symplectic geometry. In this note, we consider the negative case. By making use of the same technique as that employed in the paper [O-O], we can prove the following theorem.

**Theorem.**    *Let* $\Sigma(p,q,r)$ *be a Brieskorn rational homology 3-sphere. If* $1/p + 1/q + 1/r < 1$, *then* $\Sigma(p,q,r)$ *carries no Riemannian metric of positive scalar curvature.*

*Remark.*    (1)    The assumption that $\Sigma(p,q,r)$ is a rational homology 3-sphere is not necessary. In the proof of Theorem, we will use a vanishing theorem on the Seiberg-Witten invariant. Although it is known as folklore that the vanishing theorem can be applied to the case when $\Sigma(p,q,r)$ is not a rational homology 3-sphere, there is no published proof of the vanishing theorem for general cases. We have a proof for the ratioanl homology 3-shpere case in [O-O]. If we admit the vanishing theorem for general cases, we might be able to prove Theorem without the assuming that $\Sigma(p,q,r)$ is a rational homology 3-sphere, by slightly modifing the argument below. It should be also remarked that when $\Sigma(p,q,r)$ is a rational homology 3-sphere, it is a total space of an $S^1$-bundle over $S^2$ with three singular fibres. Of course, a total space of an $S^1$-bundle over $S^2$ with no singular fibres admits a Riemannian metric of positive scalar curvature. Thus this theorem implies that singular fibres of the $S^1$-bundle is related to an obstruction for the total space to admit a metric of positive scalar curvature. The similar phenomenon can be seen in the 4-dimensional manifolds which have an elliptic fibration structure.

(2)    A celebrated theorem deu to Gromov and Lawson [G-L] shows that any 3-dimensional $K(\pi,1)$-space does not admit a metric of positive scalar curvature. On the other hand, Milnor showed in [Mi] that in the case $1/p+1/q+1/r < 1$, the universal covering space of the Brieskorn 3-manifold $\Sigma(p,q,r)$ is the universal cover of $SL(2,\mathbf{R})$. His proof is based on a study of automorphic forms of fractional degree. Combing these two deep theorems, we can show Theorem. In this sense, this theorem is not new. But we will give an independent short proof of Theorem without the help of Gromov-Lanson's theorem and Milnor's theorem. We use relevant properties of the Seiberg-Witten monopole invariants.

(3)    For the case $1/p+1/q+1/r = 1$, Milnor showed that $\Sigma(p,q,r)$ is a nil-manifold and a circle bundle over a torus. (The triple $(p,q,r)$ must be either $(2,3,6)$, $(2,4,4)$ or $(3,3,3)$.) By a result of [G-L] again, we find that $\Sigma(p,q,r)$ carries no metric of positive scalar curvature in

this case. In this note we do not discuss this case from through the monopole invariants.

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## §2. Proof of Theorem

We put the polynomial $f_{p,q,r}(z_1, z_2, z_3) = z_1^p + z_2^q + z_3^r$. Let $M(p,q,r)$ be the Milnor fibre defined by $\{(z_1, z_2, z_3) \in \mathbf{C}^3 \mid f_{p,q,r}(z_1, z_2, z_3) = \epsilon\}$, where $\epsilon$ is a non-zero number. By the argument similar to that in [O-O], we use the compactification of the Milnor fibre in a certain weighted projective space. (In this case we put $\epsilon = 1$.) To do this, we regard $f_{p,q,r}(z_1, z_2, z_3)$ as a weighted homogeneous polynomial. We denote by $l(p,q,r)$ the least common multiple of $p, q$ and $r$, which gives the degree of the weighted homogenous polynomial. The weights are given by $l(p,q,r)/p$, $l(p,q,r)/q$ and $l(p,q,r)/r$, respectively. Let us consider a weighted projective space

$$\mathbf{P}(l(p,q,r)/p, l(p,q,r)/q, l(p,q,r)/r, 1) = (\mathbf{C}^4 \setminus 0)/\sim,$$

where $(z_1, z_2, z_3, z_4) \sim (t^{l(p,q,r)/p} z_1, t^{l(p,q,r)/q} z_2, t^{l(p,q,r)/r} z_3, t z_4)$ for $t \in \mathbf{C}^*$. By a map $\iota$ from $\mathbf{C}^3$ to $\mathbf{P}(l(p,q,r)/p, l(p,q,r)/q, l(p,q,r)/r, 1)$ defined by $\iota(z_1, z_2, z_3) = [z_1 : z_2 : z_3 : 1]$, $\mathbf{C}^3$ can be embedded as an open dense subset such that

$$\mathbf{P}(l(p,q,r)/p, l(p,q,r)/q, l(p,q,r)/r, 1)$$
$$= \mathbf{C}^3 \cup \mathbf{P}(l(p,q,r)/p, l(p,q,r)/q, l(p,q,r)/r).$$

Here

$$\mathbf{P}(l(p,q,r)/p, l(p,q,r)/q, l(p,q,r)/r)$$
$$= \{[z_1 : z_2 : z_3 : 0] \in \mathbf{P}(l(p,q,r)/p, l(p,q,r)/q, l(p,q,r)/r, 1)\}.$$

We denote by $\overline{M(p,q,r)}$ the closure of $\iota(M(p,q,r))$ in

$$\mathbf{P}(l(p,q,r)/p, l(p,q,r)/q, l(p,q,r)/r, 1).$$

It can be identified with a hypersurface $f_{p,q,r}(z_1, z_2, z_3) = z_4^{l(p,q,r)}$ in the weighted projective space $\mathbf{P}(l(p,q,r)/p, l(p,q,r)/q, l(p,q,r)/r, 1)$. The singular points on $\overline{M(p,q,r)}$ lie on the complement:

$$\overline{M(p,q,r)} \setminus \iota(M(p,q,r))$$
$$= \overline{M(p,q,r)} \cap \mathbf{P}(l(p,q,r)/p, l(p,q,r)/q, l(p,q,r)/r).$$

Taking the minimal resolution $\widetilde{M(p,q,r)} \to \overline{M(p,q,r)}$, we have a compact smooth projective complex surface $\widetilde{M(p,q,r)}$. (See, for example, [S]). This contains the Milnor fibre $M(p,q,r)$ as an open dense subset.

We put $M(p,q,r)^{\mathrm{out}} = \widetilde{M(p,q,r)} \setminus \iota(M(p,q,r) \cap B^6)$, where we denote by $B^6$ the unit open ball around the origin in $\mathbf{C}^3$. We denote by $b_2^+(M(p,q,r)^{\mathrm{out}})$ the number of positive eigenvalues of the intersection form of $M(p,q,r)^{\mathrm{out}}$, which is a non-degenerate quadratic form because the boundary of $M(p,q,r)^{\mathrm{out}}$ is a rational homology 3-sphere. Then we can show the following.

**Lemma.**   $b_2^+(M(p,q,r)^{\mathrm{out}}) \geq 1$.

*Proof.*   We put

$$B(p,q,r) := \{(z_1, z_2, z_3) \in \mathbf{C}^3 \mid f_{p,q,r}(z_1, z_2, z_3) = 0\} \cap \overline{B^6}.$$

We have an isolated normal singularity at the origin. Taking the minimal resolution $\widetilde{B(p,q,r)} \to B(p,q,r)$, we have a non-singular Kähler surface with boundary $\Sigma(p,q,r)$. The intersection form of $\widetilde{B(p,q,r)}$ is negative definite [Mu]. Note that we have a symplectically fillable contact structure $\xi$ on the boundary $\Sigma(p,q,r)$, for which $\widetilde{B(p,q,r)}$ is a symplectically filling 4-manifold. On the other hand, we put

$$M(p,q,r)^{\mathrm{in}} := M(p,q,r) \cap \overline{B^6}.$$

By the map $\iota$, we can embed $M(p,q,r)^{\mathrm{in}}$ into $\widetilde{M(p,q,r)}$ so that

$$\widetilde{M(p,q,r)} = M(p,q,r)^{\mathrm{in}} \cup_{\Sigma(p,q,r)} M(p,q,r)^{\mathrm{out}}.$$

Here we identify $\iota(M(p,q,r)^{\mathrm{in}})$ with $M(p,q,r)^{\mathrm{in}}$. Note that $M(p,q,r)^{\mathrm{in}}$ is also a non-singular Kähler surface with boundary $\Sigma(p,q,r)$. We have a symplectically fillable contact structure $\xi_\epsilon$ on the boundary, for which $M(p,q,r)^{\mathrm{in}}$ is a symplectically filling 4-manifold. The contact structure $\xi$ is nothing but $\xi_0$, so we have a smooth family of contact structures on $\Sigma(p,q,r)$. Hence Gray's stability theorem [G] implies that $\xi$ and $\xi_\epsilon$ with $\epsilon \neq 0$ is isotopic. So we have two symplectically filling 4-manifolds $\widetilde{B(p,q,r)}$ and $M(p,q,r)^{\mathrm{in}}$ of $\Sigma(p,q,r)$ with the specific fillable contact structure. Then by the argument similar to Lemma 2.1 in [O-O], we can

glue $\widetilde{B(p,q,r)}$ and $M(p,q,r)^{\text{out}}$ to obtain a closed symplectic 4-manifold

$$Z(p,q,r) = \widetilde{B(p,q,r)} \cup_{\Sigma(p,q,r)} M(p,q,r)^{\text{out}},$$

by replacing $M(p,q,r)^{\text{in}}$ in $\widetilde{M(p,q,r)}$ by $\widetilde{B(p,q,r)}$. Since we have a symplectic form on it, we have $b_2^+(Z(p,q,r)) \geq 1$. We recall that $b_2^+(\widetilde{B(p,q,r)}) = 0$. Therefore we have $b_2^+(M(p,q,r)^{\text{out}}) \geq 1$, which proves Lemma. □

Now, let us consider the compact non-singular Kähler surface with the following decomposition:

$$\widetilde{M(p,q,r)} = M(p,q,r)^{\text{in}} \cup_{\Sigma(p,q,r)} M(p,q,r)^{\text{out}}.$$

The signature of $M(p,q,r)^{\text{in}}$ is computed by E. Brieskorn [B]. According to his result, $b_2^+(M(p,q,r)^{\text{in}})$ can be calculated by counting the number of triples of integers $(k,l,m)$ satisfying the follwing inequality:

$$0 < \frac{k}{p} + \frac{l}{q} + \frac{m}{r} < 1 \quad (\text{mod } 2).$$

Therefore, if $1/p+1/q+1/r < 1$, we have $b_2^+(M(p,q,r)^{\text{in}}) \geq 1$ obviously. (Note that if $1/p + 1/q + 1/r > 1$, we have $b_2^+(M(p,q,r)^{\text{in}}) = 0$). Thus, by taking Lemma into account, we have $b_2^+(\widetilde{M(p,q,r)}) \geq 2$. Hence, by Taubes' theorem [T], we find that the Siberg-Witten monopole invariant for the canonical Spin$^c$ structure of $\widetilde{M(p,q,r)}$ is non-trivial. Now suppose that $\Sigma(p,q,r)$ carries a Riemannian metric of positive scalar curvature. Then we have the following vanishing theorem on the Seiberg-Witten monopole invariants.

**Proposition.** *Let $X$ be a closed oriented connected smooth 4-manifold. Suppose that $X$ has a decomposition $X = X_1 \cup_Y X_2$ such that $b_2^+(X_i) \geq 1$, ($i = 1, 2$) and $Y$ is a closed oriented 3-manifold with a metric of positive scalar curvature. Then all the Seiberg-Witten invariants of $X$ vanish.*

See [O-O] for the proof in the case when $Y$ is a rational homology 3-sphere. Since we have $b_2^+(M(p,q,r)^{\text{in}}) \geq 1$ and $b_2^+(M(p,q,r)^{\text{out}}) \geq 1$ by Lemma, we can apply the proposition to the case $X = \widetilde{M(p,q,r)}$. Hence we have a contradiction to Taubes' theorem. Thus $\Sigma(p,q,r)$ carries no Riemannian metric of positive scalar curvature, whenever $1/p + 1/q + 1/r < 1$.

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