Advanced Studies in Pure Mathematics 30, 2001
Class Field Theory – Its Centenary and Prospect
pp. 401–414

On p-Adic Zeta Functions and Class Groups of $\mathbb{Z}_p$-Extensions of certain Totally Real Fields

Hisao Taya[1]

Abstract.

Let k be a totally real field and p an odd prime number. We assume that p splits completely in k and also that Leopoldt's conjecture is valid for k and p. In this note, focusing on Greenberg's conjecture, we will report on our recent results concerning p-adic special functions and ideal class groups in the cyclotomic $\mathbb{Z}_p$-extension of k.

§1. Introduction

For a number field k and a prime number p, we denote by k_∞ the cyclotomic $\mathbb{Z}_p$-extension of k, by k_n the n-th layer of k_∞ over k, and by A_n the p-Sylow subgroup of the ideal class group of k_n. Also, for a finite set S, we denote by $\#S$ the number of elements in S. Then Iwasawa [Iw59] has proved that there exist three integers $\lambda_p(k)$, $\mu_p(k)$ and $\nu_p(k)$, depending only on k and p, such that

$$v_p(\#A_n) = \lambda_p(k)\, n + \mu_p(k)\, p^n + \nu_p(k)$$

for sufficiently large n, where v_p denotes the p-adic valuation normalized by $v_p(p) = 1$. These integers $\lambda_p(k)$, $\mu_p(k)$ and $\nu_p(k)$ are called the Iwasawa λ-, μ- and ν-invariants, respectively, of the cyclotomic $\mathbb{Z}_p$-extension of k.

Concerning these invariants, Iwasawa mentions that it would be an important problem to find out if $\lambda_p(\mathbb{Q}(\zeta_p)^+) = \mu_p(\mathbb{Q}(\zeta_p)^+) = 0$ for any prime number p in [Iw70, page 392], or to find out when the "plus-part" of $\lambda_p(k)$ is positive for CM-fields k in [Iw73a, page 316], where $\mathbb{Q}(\zeta_p)^+$

1991 *Mathematics Subject Classification.* Primary 11R23; Secondary 11R42, 11R29.

Key words and phrases. $\mathbb{Z}_p$-extensions, Iwasawa invariants, p-adic zeta-functions, ideal class groups.

Received August 31, 1998.

[1] This research was supported in part by the Grants-in-Aid for Encouragement of Young Scientists, The Ministry of Education, Science, Sports and Culture, Japan.

is the maximal real subfield of the cyclotomic field $\mathbb{Q}(\zeta_p)$ of p-th roots of unity. Probably arising from this, the following was posed by Greenberg [Gr76] as a problem. This is now known as Greenberg's conjecture.

Conjecture 1.1 (Greenberg's conjecture). *If k is a totally real field, then $\lambda_p(k) = \mu_p(k) = 0$ for every prime number p.*

As for the μ-invariants, we know by the Ferrero-Washington theorem [FW79] (or [Wa97, Theorem 7.15]) that $\mu_p(k)$ always vanishes for every prime number p when k is abelian (not necessarily totally real) over $\mathbb{Q}$. More generally, the following is conjectured.

Conjecture 1.2. *For any number field k and any prime number p, we should have $\mu_p(k) = 0$.*

The Ferrero-Washington theorem says that Conjecture 1.2 is true in the case of abelian number fields. Besides this, we know by the theorem of Iwasawa [Iw73b, Theorem 3] that $\mu_p(k) = 0$ for any finite Galois p-extension k. Here we note that Iwasawa also constructed in [Iw73b] examples of non-cyclotomic $\mathbb{Z}_p$-extensions with arbitrarily large μ-invariant.

On the other hand, as for the λ-invariants, we know Kida's formula ([Kid80], [Kid82]) which describes the behavior of the "minus-parts" of $\lambda_p(K)$ and $\lambda_p(k)$ when K/k is a finite Galois p-extension of CM-fields. Also we have examples of k with arbitrarily large λ-invariant if k is not totally real (e.g. [Gr76, page 264]). However, very little is known about the "plus-part" of $\lambda_p(k)$ except for $k = \mathbb{Q}$. The field $\mathbb{Q}$ of rational numbers is so far the only number field that Greenberg's conjecture is known to be true for every prime number p (in fact $\lambda_p(\mathbb{Q}) = \mu_p(\mathbb{Q}) = \nu_p(\mathbb{Q}) = 0$ for any p). Recently, some efficient criteria for Greenberg's conjecture to be true are given from different points of view when k is abelian over $\mathbb{Q}$ and p is an odd prime number (see the paper [IS96-7] and the papers cited in it, especially Kraft-Schoof [22] and Kurihara [23] in part II of [IS96-7]). However, at present we do not know any algorithm to determine whether Greenberg's conjecture is true, or more precisely, to determine the order of A_n for sufficiently large n, after a finite amount of times.

Let k be a totally real field and p an odd prime number. We denote by Γ the Galois group $\mathrm{Gal}(k_\infty/k)$, and by A_n^Γ the subgroup of A_n consisting of ideal classes which are invariant under the action of Γ. Assume that p splits completely in k and also that Leopoldt's conjecture is valid for k and p. In this note, we will report on our recent results, most of which are appearing in [Ta99a], [Ta99b] and [Ta00], concerning

p-adic special functions and ideal class groups of k_n. First, we give in §2 a simple formula for the order of A_n^Γ for sufficiently large n in terms of the residue at 1 of the p-adic zeta function of k. This enables us to calculate the order of A_n^Γ as a practical matter. When k is a real abelian field with degree prime to p, as mentioned in §3, a formula for the order of Ψ-component of A_n^Γ can be also given in terms of a special value of the p-adic L-function associated to Ψ, where Ψ is an irreducible $\mathbb{Q}_p$-character of the Galois group $\mathrm{Gal}(k/\mathbb{Q})$. In the case where p splits completely in k, the order of A_n^Γ is closely related to Greenberg's conjecture. Hence these formulas imply an alternative formulation (resp. the Ψ-component version of it) of Greenberg's criterion [Gr76, Theorem 2] on the vanishing of the Iwasawa λ- and μ-invariants of k_∞/k. In §4, as an easy application of these formulas, we give a simple proof of Ozaki's theorem [Oz97, Theorem 1] which is regarded as a totally real version of classical Kummer's criteria for p-divisibility of the class number of $\mathbb{Q}(\zeta_p)$, and also show the Ψ-component version of Ozaki's theorem. As another application, we mention in §5 the result that there exist infinitely many real quadratic fields in which the prime 3 splits and whose Iwasawa λ_3-invariant vanishes.

§2. A simple genus formula (totally real case)

Here and in what follows, we use the same notation as in the previous section. Assume that k is a totally real field and p an odd prime number. Let $\zeta_p(s,k)$ be the p-adic zeta function of k, which is continuous on $\mathbb{Z}_p - \{1\}$ and has simple pole at $s=1$ if Leopoldt's conjecture is valid for k and p (cf. [Col88]). Let us put

$$\zeta_p^*(s,k) = \frac{\zeta_p(s,k)}{\zeta_p(s,\mathbb{Q})}.$$

to cancel the simple pole at $s=1$. Then we have the following genus formula in terms of the residue of the p-adic zeta function.

Theorem 2.1 (cf. [Ta99a]). *Let k be a totally real field and p an odd prime number. Assume that p splits completely in k and also that Leopoldt's conjecture is valid for k and p. Then*

$$\#A_n^\Gamma = p^{v_p(\zeta_p^*(1,k))}$$

for sufficiently large n.

Proof. It follows from a lemma on the order of $\mathrm{Gal}(M/k_\infty)$ proved by Coates [Coa77, Lemma 8 in Appendix], a limit formula for p-adic zeta functions proved by Colmez [Col88, Main theorem], some property of a finite unramified extension over k_∞, and (local or global) class field theory, where M denotes the maximal abelian pro-p-extension of k unramified outside p. For the details, see the paper [Ta99a]. □

Remark 2.2. A limit formula for p-adic zeta functions [Col88, Main theorem] implies that the right hand side in the theorem above is given by

$$p^{v_p(\zeta_p^*(1,k))} = \#A_0\, p^{v_p(R_p(k))-[k:\mathbb{Q}]+1},$$

where $R_p(k)$ denotes the p-adic regulator of k and $[k:\mathbb{Q}]$ the degree of k over $\mathbb{Q}$. Using this formula, we can practically calculate the order of A_n^Γ for sufficiently large n, even if k is non-Galois (see [Ta99a, Example 4.2]).

Remark 2.3. The formula in Theorem 2.1 is regarded as a generalization of a formula for real quadratic fields obtained by Fukuda and Komatsu [FK86a, Proposition 1] (or [FK86b]), and also as an explicit version of a formula for real abelian fields obtained by Inatomi [In89, Proposition 2]. Comparing our formula in Remark 2.2 and Inatomi's one, we find that $v_p(R_p(k))$ is equal to the integer m, which is defined in [In89] and used to describe his formula. Therefore, it follows that

$$\#A_n^\Gamma = p^{v_p(\zeta_p^*(1,k))} \text{ for all } n \geq v_p(R_p(k)),$$

because Inatomi shows his formula holds for all $n \geq m$ and his proof works equally as well for our situation.

Remark 2.4. In the proof of Theorem 2.1, we also use the property that M/k_∞ is an unramified extension, where M is the same as in the proof above (see [Oz97, Proposition 1] or [Ta99a, Lemma 2.3]). However, this property does not hold for $p = 2$.

Remark 2.5. The formula in Theorem 2.1 does not hold in general without the assumption on the decomposition of p in $k/\mathbb{Q}$. For example, in the case where $k = \mathbb{Q}(\sqrt{257})$ and $p = 3$ (so $p = 3$ remains prime in k), we find $v_p(R_p(k)) = 2$. Hence we have

$$p^{v_p(\zeta_p^*(1,k))} = \#A_0\, 3^{2-1} = 9$$

by a limit formula for p-adic zeta functions. On the other hand, since only one prime of k lies over p and this prime is totally ramified in k_∞/k, we have

$$\#A_n^\Gamma = \#A_0 = 3$$

for all $n \geq 0$. Therefore we see $\#A_n^\Gamma \neq p^{v_p(\zeta_p^*(1,k))}$ for $p = 3$.

In the case where p splits completely in k, A_n^Γ is an important object in the study of Greenberg's conjecture. Let D_n be the subgroup of A_n consisting of ideal classes represented by products of prime ideals of k_n lying above p. Then it is clear that $D_n \subset A_n^\Gamma$. Now we recall a theorem of Greenberg on the vanishing of the Iwasawa invariants (including $p = 2$).

Theorem 2.6 (Theorem 2 in Greenberg [Gr76]). *Let k be a totally real field and p a prime number. Assume that p splits completely in k and also that Leopoldt's conjecture is valid for k and p. Then the following two conditions are equivalent:*
(1) $\lambda_p(k) = \mu_p(k) = 0$,
(2) $\#A_n^\Gamma = \#D_n$ *for sufficiently large n.*

Now, by Theorem 2.1 and Remark 2.2, we can obtain an alternative formulation of Theorem 2.6.

Theorem 2.7 (cf. [Ta99a]). *Under the same assumptions as in Theorem 2.1, the following three conditions are equivalent:*
(1) $\lambda_p(k) = \mu_p(k) = 0$,
(2) $\#D_n = p^{v_p(\zeta_p^*(1,k))}$ *for some $n \geq 0$,*
(3) $\#D_n = \#A_0\, p^{v_p(R_p(k))-[k:\mathbb{Q}]+1}$ *for some $n \geq 0$.*

Although Theorem 2.7 seems to be only a little different from the original theorem of Greenberg, it suggests that the validity of Greenberg's conjecture can be regarded as based on a certain arithmetic relation between an analytic object and an algebraic object.

§3. A simple genus formula (real abelian case)

In this section, we assume that k is a real abelian field and p an odd prime number. Then, Leopoldt's conjecture is valid for such a case by a theorem of Brumer [Br67], and also we know $\mu_p(k) = 0$ by the Ferrero-Washington theorem [FW79]. Put $\Delta = \mathrm{Gal}(k/\mathbb{Q})$. We further assume that the order of Δ is not divisible by p. Let Ψ be an irreducible $\mathbb{Q}_p$-character of Δ, and A_n^Ψ the Ψ-component of A_n, namely,

$$A_n^\Psi = e_\Psi A_n, \quad e_\Psi = \frac{1}{\#\Delta}\sum_{\delta\in\Delta}\Psi(\delta)\delta^{-1} \in \mathbb{Z}_p[\Delta].$$

Note that e_Ψ is an idempotent of $\mathbb{Z}_p[\Delta]$.

Remark 3.1. If Ψ_0 is the trivial character of Δ, then $A_n^{\Psi_0} = \{1\}$.

Let $(A_n^\Psi)^\Gamma$ be the subgroup of A_n^Ψ consisting of ideal classes which are invariant under the action of Γ. We define the p-adic L-function associated to Ψ by

$$L_p(s, \Psi) = \prod_{\psi | \Psi} L_p(s, \psi),$$

where ψ runs over all irreducible components of Ψ over the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and $L_p(s, \psi)$ denotes Kubota-Leopoldt's p-adic L-function associated to ψ. Here we regard ψ as a primitive p-adic Dirichlet character. Note that $L_p(s, \Psi)$ has non-zero values at $s = 1$ if Ψ is non-trivial. Then we have the following genus formula which is the Ψ-part version of Theorem 2.1.

Theorem 3.2 (cf. [Ta99b]). *Let k be a real abelian field with Galois group Δ and p an odd prime number. Assume that the order of Δ is prime to p and also that p splits completely in k. Then, for each non-trivial irreducible $\mathbb{Q}_p$-character Ψ of Δ, we have*

$$\#(A_n^\Psi)^\Gamma = p^{v_p(L_p(1, \Psi))}$$

for sufficiently large n.

Proof. It follows from Iwasawa main conjecture proved by Mazur and Wiles [MW84], some property of a finite unramified extension over k_∞ and (local or global) class field theory. For the details, see the paper [Ta99b]. □

We denote by $\lambda_p^\Psi(k)$ the Ψ-component of $\lambda_p(k)$, namely, the Iwasawa λ-invariant associated to A_n^Ψ. Note that $\lambda_p(k) = \sum_\Psi d_\Psi \lambda_p^\Psi(k)$, where Ψ runs over all irreducible $\mathbb{Q}_p$-characters of Δ and d_Ψ denotes the degree of Ψ over $\mathbb{Q}_p$. In the case of real abelian fields, Greenberg's conjecture is equivalent to the statement called Greenberg's conjecture for Ψ-components that $\lambda_p^\Psi(k) = 0$ for any (non-trivial) Ψ. Let D_n^Ψ be the Ψ-component of D_n. Since Theorem 2.6 holds by replacing each object with its Ψ-component, we obtain by Theorem 3.2 the following Ψ-part version of an alternative formulation of Greenberg's theorem.

Theorem 3.3 (cf. [Ta99b]). *Under the same assumptions as in Theorem 3.2, for a non-trivial irreducible $\mathbb{Q}_p$-character Ψ of Δ, the following two conditions are equivalent*:

(1) $\lambda_p^\Psi(k) = 0$,

(2) $\#D_n^\Psi = p^{v_p(L_p(1, \Psi))}$ *for some $n \geq 0$.*

Remark 3.4. Let $\zeta_p^*(s,k)$ be as in the previous section. Since

$$\zeta_p^*(s,k) = \prod_{\Psi \neq 1} L_p(s,\Psi)$$

where the product runs over all non-trivial irreducible $\mathbb{Q}_p$-characters Ψ of Δ, we see by Remark 3.1 that Theorems 3.2 and 3.3 imply a special case of Theorems 2.1 and 2.7, respectively.

§4. A simple proof of Ozaki's theorem

Ozaki showed the following theorem which can be regarded as a "totally real" analogue of classical Kummer's criterion for p-divisibility of the class number of $\mathbb{Q}(\zeta_p)$. In this section, we give its simple proof by using Theorem 2.1, and further, show the Ψ-component version of his theorem.

Theorem 4.1 (Theorem 1 in Ozaki [Oz97]). *Let k be a totally real field and p an odd prime number. Let K_∞ be a $\mathbb{Z}_p$-extension of k, with n-th layer K_n, in which the primes of k lying over p are totally ramified. Assume that p splits completely in k. Then the following two statements are equivalent:*

(1) *The class number of K_n is divisible by p for all $n \geq 1$,*

(2) $p\zeta_p(0,k) \equiv 0 \pmod{p}$.

Remark 4.2. In the theorem above, since K_∞/k is totally ramified at p, the negative proposition of statement (1) is equivalent to the following one:

(a) The class number of K_n is not divisible by p for all $n \geq 0$.

Indeed, if statement (a) does not hold, namely, if the class number of K_n is divisible by p for some $n \geq 0$ (so, for every n sufficiently large), then $\mathrm{Gal}(L_\infty/K_\infty) \neq \{1\}$, where L_∞ is the maximal unramified abelian pro-p-extension of K_∞. Note that $\mathrm{Gal}(L_\infty/K_\infty)$ is a finitely generated torsion $\mathbb{Z}_p[[T]]$-module (cf. [Iw73a, Theorem 5]). Let $\nu_n = ((1+T)^{p^n} - 1)/T \in \mathbb{Z}_p[[T]]$. By Nakayama's lemma, we have $\mathrm{Gal}(L_\infty/K_\infty) / \nu_n \mathrm{Gal}(L_\infty/K_\infty) \neq \{1\}$ for all $n \geq 1$. It is known that $\mathrm{Gal}(L_\infty/K_\infty) / \nu_n \mathrm{Gal}(L_\infty/K_\infty L_0)$ is isomorphic to the p-Sylow subgroup of the ideal class group of K_n, where L_0 is the maximal unramified abelian p-extension of k (cf. [Iw73a, Theorem 6]). Hence, this implies statement (1) in Theorem 4.1, because $\mathrm{Gal}(L_\infty/K_\infty L_0) \subset \mathrm{Gal}(L_\infty/K_\infty)$. The converse is obvious. Therefore we conclude the desired equivalence.

Proof of Theorem 4.1. Assume that Leopoldt's conjecture for k and p is valid. Then K_∞ is the cyclotomic $\mathbb{Z}_p$-extension of k. By Remark 4.2, it suffices to prove that

$$A_n = \{1\} \quad \text{for all} \quad n \geq 0 \Longleftrightarrow p\zeta_p(0,k) \not\equiv 0 \pmod{p}.$$

Since $A_n = \{1\}$ if and only if $A_n^\Gamma = \{1\}$ and since K_∞/k is totally ramified at p, it follows from Theorem 2.1 that

$$\begin{aligned} A_n = \{1\} \quad \text{for all} \quad n \geq 0 &\Longleftrightarrow A_n = \{1\} \quad \text{for sufficiently large } n \\ &\Longleftrightarrow A_n^\Gamma = \{1\} \quad \text{for sufficiently large } n \\ &\Longleftrightarrow \zeta_p^*(1,k) \not\equiv 0 \pmod{p}. \end{aligned}$$

On the other hand, we can see that

$$\zeta_p^*(1,k) = \frac{\zeta_p(0,k)}{\zeta_p(0,\mathbb{Q})} \quad \text{and} \quad p\zeta_p(0,\mathbb{Q}) \equiv 1 \pmod{p}.$$

Hence it follows from this that

$$\zeta_p^*(1,k) \equiv p\zeta_p(0,k) \pmod{p}.$$

Therefore we get the desired result.

If Leopoldt's conjecture for k and p is not valid, then we can find that statement (1) holds by the fact that the maximal abelian pro-p-extension of k unramified outside p is unramified over K_∞ [Oz97, Proposition 1] (or [Ta99a, Lemma 2.3]), and also that statement (2) holds by the Iwasawa main conjecture proved by Mazur and Wiles [MW84]. □

The following is one of immediate consequences of Theorem 4.1.

Corollary 4.3. *Let p be a fixed odd prime number and q an odd prime number satisfying $p \equiv 1 \pmod{q}$. Put $k = \mathbb{Q}(\zeta_q)^+$ and $\Delta = \mathrm{Gal}(k/\mathbb{Q})$. For the cyclotomic $\mathbb{Z}_p$-extension of k, let k_n be its n-th layer and A_n the p-Sylow subgroup of the ideal class group of k_n. Then the following two statements are equivalent:*

(1) $A_n \neq \{1\}$ *for all* $n \geq 1$,

(2) $\prod_{\chi \neq 1} B_{1,\chi\omega^{-1}} \equiv 0 \pmod{p}$,

where the product runs over all non-trivial p-adic Dirichlet characters χ of Δ, ω the Teichmüller character for p, and $B_{1,\chi\omega^{-1}}$ the generalized Bernoulli number.

Proof. Since p splits completely in k, this easily follows from Theorem 4.1 because

$$p\zeta_p(0,k) \equiv \prod_{\chi \neq 1} (-B_{1,\chi\omega^{-1}}) \pmod{p}$$

in this situation. □

Remark 4.4. Using the theory of cyclotomic units, Kim proved in [Kim95, Theorem 2] the assertion that (2) implies (1) in Corollary 4.3. Also, another result of Kim [Kim95, Theorem 3] is regarded as a consequence of the contraposition of the assertion that (1) implies (2) in the corollary.

Similarly as in the proof of Theorem 4.1, we can show the following theorem which is the Ψ-part version of Theorem 4.1 in the case of real abelian fields.

Theorem 4.5. *Let k be a real abelian field with Galois group Δ, p an odd prime number and k_∞ the cyclotomic $\mathbb{Z}_p$-extension of k with n-th layer k_n. Assume that the order of Δ is prime to p and also that p splits completely in k. Then, for a non-trivial irreducible $\mathbb{Q}_p$-character Ψ of Δ, the following two statements are equivalent:*
(1) $A_n^\Psi \neq \{1\}$ *for all* $n \geq 1$,
(2) $L_p(0, \Psi) \equiv 0 \pmod{p}$.

Again, the following is an immediate consequence of Theorem 4.5.

Corollary 4.6. *Let p, q, k, Δ and A_n be as in Corollary 4.3. Then, for a non-trivial irreducible $\mathbb{Q}_p$-character Ψ of Δ, the following two statements are equivalent:*
(1) $A_n^\Psi \neq \{1\}$ *for all* $n \geq 1$,
(2) $\prod_{\psi|\Psi} B_{1,\psi\omega^{-1}} \equiv 0 \pmod{p}$,
where the product runs over all irreducible components ψ of Ψ over $\overline{\mathbb{Q}}_p$, and ω and $B_{1,\psi\omega^{-1}}$ be the same as in Corollary 4.3.

Proof. Since p splits completely in k and the order of Δ is not divisible by p, this easily follows from Theorem 4.5 because $L_p(0, \psi) = -B_{1,\psi\omega^{-1}}$ in this case. □

§5. A weaker problem on the λ-invariants

Althought there are some efficient criteria for Greenberg's conjecture to be true (see the paper [IS96-7] and the papers cited in it), we do not have general results on Iwasawa λ-invariants of totally real fields, like as the Ferrero-Washington theorem, except for $k = \mathbb{Q}$. So, in this section, we concentrate on the simplest case where k is a real quadratic field, and consider the following weaker problem than Greenberg's conjecture.

Problem 5.1. *For a given prime number p, do there exist infinitely many real quadratic fields k satisfying $\lambda_p(k) = 0$ and "some additional conditions"?*

Needless to say, if there exist infinitely many real quadratic fields satisfying such "some additional conditions", then we clearly have an affirmative answer to Problem 5.1 under the validity of Greenberg's conjecture. Nevertheless, it seems that little is known about this problem, though we already have some results on it for $p = 2, 3$. Before recalling such known results, we describes the following theorem of Iwasawa which is often useful for this kind of problem.

Theorem 5.2 (Iwasawa [Iw56]). *Let p be a prime number and L a finite extension of $\mathbb{Q}$. Let L_∞ be the cyclotomic $\mathbb{Z}_p$-extension of L and L_n its n-th layer. Assume that p dose not split in L and that the class number of L is not divisible by p. Then the class number of L_n is not divisible by p for all $n \geq 0$. In particular, we have $\lambda_p(L) = \mu_p(L) = \nu_p(L) = 0$.*

First, in the case where $p = 2$, genus theory implies that there exist infinitely many real quadratic fields k such that the class number of k is odd and 2 does not split in k. Hence, it follows from Theorem 5.2 that there exist infinitely many real quadratic fields k such that $\lambda_2(k) = 0$ and 2 does not split in k. Also, in [OT97, Theorem], we gave several infinite families of real quadratic fields k with $\lambda_2(k) = 0$. According to this, we immediately see the following:

(i) There exist infinitely many real quadratic fields k such that $\lambda_2(k) = 0$, the prime 2 splits in k and the class number of k is odd (or even),

(ii) There exist infinitely many real quadratic fields k such that $\lambda_2(k) = 0$, the prime 2 does not split in k and the class number of k is even,

(iii) Let N be a given positive integer. Then there exist infinitely many real quadratic fields k such that $\lambda_2(k) = 0$, the class number of k is even (or the prime 2 splits in k) and the "minus" λ-invariant corresponding to k is greater than N, where the "minus" λ-invariant corresponding to k means $\lambda_2(\mathbb{Q}(\sqrt{-d}))$ if $k = \mathbb{Q}(\sqrt{d})$.

Further, Ozaki recently constructed a new infinite family of real quadratic fields k with $\lambda_2(k) = 0$ (see [Oz98, Theorem 6]). The following is a conclusion from his result.

(iv) There exist infinitely many real quadratic fields k such that $\lambda_2(k) = 0$, the prime 2 splits in k (resp. does not split in k) and the 2-rank of the ideal class group of k is equal to 2 (resp. 3).

Here we note that we cannot apply Theorem 5.2 to cases (i)$\sim$(iv).

Next, we recall the case where $p = 3$. For a positive integer N, we denote by $\mathcal{K}(N)$ the set of real quadratic fields with discriminant less than N. Then, Nakagawa and Horie showed in [NH88] the following, using a theorem of Davenport and Heilbronn [DH71].

Theorem 5.3 (Theorem 3 in Nakagawa and Horie [NH88]). *For a real quadratic field k, let $d(k)$ be the discriminant of k and $h(k)$ the class number of k. Then*

$$\liminf_{N\to\infty} \frac{\#\{\ k \in \mathcal{K}(N) \mid d(k) \not\equiv 1 \pmod 3,\ 3 \nmid h(k)\ \}}{\#\mathcal{K}(N)} \geq \frac{25}{48} = 0.5208\dot{3}.$$

From Theorems 5.2 and 5.3, it follows that

$$\liminf_{N\to\infty} \frac{\#\{\ k \in \mathcal{K}(N) \mid d(k) \not\equiv 1 \pmod 3,\ \lambda_3(k) = \mu_3(k) = \nu_3(k) = 0\ \}}{\#\mathcal{K}(N)} \geq \frac{25}{48}.$$

Therefore we obtain the following corollary.

Corollary 5.4. *There exist infinitely many real quadratic fields k such that $\lambda_3(k) = 0$ and the prime 3 does not split in k.*

On the other hand, we have not known whether there exist infinitely many real quadratic fields k with $\lambda_p(k)$ vanishing and p splitting there, for a given odd prime number p. But, we can now give an answer to this problem in the case where $p = 3$ splits in k as an application of Theorem 4.1 (or Theorem 4.5). We introduce this in the rest of this note.

First, in the case where k is a real quadratic fields and $p = 3$, we can rewrite Theorem 4.1 (or Theorem 4.5) as follows, by using Remark 4.2 and a relationship between the class numbers of imaginary quadratic fields and generalized Bernoulli numbers (or special values of p-adic L-functions).

Proposition 5.5 (cf. [Ta00]). *Let d be a square-free positive integer with $d \equiv 1 \pmod 3$. Put $k = \mathbb{Q}(\sqrt{d})$ and $k^* = \mathbb{Q}(\sqrt{-3d})$. For the cyclotomic $\mathbb{Z}_3$-extension k_∞ of k, let k_n be its n-th layer and A_n the 3-Sylow subgroup of the ideal class group of k_n. Then the following two statements are equivalent:*

(1) *$A_n = \{1\}$ for all $n \geq 0$,*

(2) *The class number $h(k^*)$ of k^* is not divisible by 3.*

Remark 5.6. Proposition 5.5 can be directly proved by a purely algebraic argument. For the details, see [Ta00, Remark 2].

Proposition 5.5 says that the problem of finding infinite families of real quadratic fields k with $\lambda_3(k)$ vanishing and the prime 3 splitting is reduced to the problem of finding infinite families of imaginary quadratic fields k whose class number is not divisible by 3. And the latter problem can be solved by results of Nakagawa and Horie [NH88]. Consequently, we obtain the following theorem.

Theorem 5.7 (cf. [Ta00]). *For a real quadratic field k, let $d(k)$ be the discriminant of k, k_n the n-th layer of the cyclotomic $\mathbb{Z}_3$-extension of k, and h_n the class number of k_n. Then*

$$\liminf_{N\to\infty} \frac{\#\{\, k \in \mathcal{K}(N) \mid d(k) \equiv 1 \pmod 3,\ 3 \nmid h_n \text{ for all } n \geq 0 \,\}}{\#\mathcal{K}(N)} \geq \frac{3}{16} = 0.1875.$$

Proof. This follows from Proposition 5.5 and [NH88, Theorem 1 and Proposition 2]. For the detail, see the paper [Ta00]. □

Now, we can give an affirmative answer in the case where $p = 3$ splits in k. Namely, we obtain the following.

Corollary 5.8. *There exist infinitely many real quadratic fields k such that $\lambda_3(k) = 0$ and the prime 3 splits in k.*

Combining Theorems 5.3 and 5.7, we get the following.

Theorem 5.9. *We have*

$$\liminf_{N\to\infty} \frac{\#\{\, k \in \mathcal{K}(N) \mid \lambda_3(k) = \mu_3(k) = \nu_3(k) = 0 \,\}}{\#\mathcal{K}(N)} \geq \frac{17}{24} = 0.708\dot{3}.$$

Remark 5.10. Recently, K. Ono [On99] showed by Theorem 5.2 that for each prime number between $3 < p < 5000$, there exist infinitely many real quadratic fields k such that $\lambda_p(k) = \mu_p(k) = \nu_p(k) = 0$ and p does not split in k. In fact, he succeeded in estimating the number of real quadratic fields k with discriminant $d(k)$ less than a given positive integer, such that its class number is not divisible by p, the prime p ramifies in k, and $v_p\left(R_p(k)/\sqrt{d(k)}\right) = 0$.

Remark 5.11. We do not know any infinite families of real quadratic fields k with $\lambda_p(k) = 0$ and $\nu_p(k) \neq 0$, for a give prime number $p \geq 3$.

Remark 5.12. We do not know any infinite families of real quadratic fields k such that $\lambda_p(k) = 0$ and the p-rank of the ideal class group of k is (arbitrarily) large, for a give prime number $p \geq 2$.

References

[Br67] A. Brumer, On the units of algebraic number fields, *Mathematika*, **14** (1967), 121–124.

[Coa77] J. Coates, *p-adic L-functions and Iwasawa's theory*, in Algebraic Number Fields, Durham Symposium, 1975, ed. by A. Fröhlich, Academic Press, 1977, 269–353.

[Col88] P. Colmez, *Résidu en $s = 1$ des functions zêta p-adiques*, Invent. Math., **91** (1988), 371–389.

[DH71] H. Davenport and H. Heilbronn, *On the density of discriminants of cubic fields II*, Proc. Roy. Soc. London Ser. A, **322** (1971), 405–422.

[FW79] B. Ferrero and L. C. Washington, *The Iwasawa invariant μ_p vanishes for abelian number fields*, Ann. of Math., **109** (1979), 377–395.

[FK86a] T. Fukuda and K. Komatsu, *On the λ invariants of $\mathbb{Z}_p$-extensions of real quadratic fields*, J. Number Theory, **23** (1986), 238–242.

[FK86b] T. Fukuda and K. Komatsu, *On $\mathbb{Z}_p$-extensions of real quadratic fields*, J. Math. Soc. Japan, **38** (1986), 95–102.

[Gr76] R. Greenberg, *On the Iwasawa invariants of totally real number fields*, Amer. J. Math., **98** (1976), 263–284.

[IS96-7] H. Ichimura and H. Sumida, *On the Iwasawa λ-invariants of certain real abelian fields*, Tôhoku Math. J., **49** (1997), 203–215; part II: International J. Math., **7** (1996), 721–744.

[In89] A. Inatomi, *On $\mathbb{Z}_p$-extensions of real abelian fields*, Kodai. Math. J., **12** (1989), 420–422.

[Iw56] K. Iwasawa, *A note on class numbers of algebraic number fields*, Abh. Math. Sem. Univ. Hamburg, **20** (1956), 257–258.

[Iw59] K. Iwasawa, *On Γ-extensions of algebraic number fields*, Bull. Amer. Math. Soc., **65** (1959), 183–226.

[Iw70] K. Iwasawa, *On some infinite abelian extensions of algebraic number fields*, Actes, Congrés. Intern. Math. (Nice, 1970), Tome 1, 391–394.

[Iw73a] K. Iwasawa, *On $\mathbb{Z}_l$-extensions of algebraic number fields*, Ann. of Math., **98** (1973), 246–326.

[Iw73b] K. Iwasawa, *On the μ-invariants of $\mathbb{Z}_l$-extensions*, Number theory, Algebraic Geometry and Commutative algebra (in honor of Y. Akizuki), ed. by Y. Kusunoki, et al., Kinokuniya, Tokyo, 1973, 1–11.

[Kid80] Y. Kida, *l-extensions of CM-fields and cyclotomic invariants*, J. Number Theory, **12** (1980), 519–528.

[Kid82] Y. Kida, *Cyclotomic $\mathbb{Z}_2$-extensions of J-fields*, J. Number Theory, **14** (1982), 340–352.

[Kim95] J. M. Kim, *Class numbers of certain real abelian fields*, Acta Arith., **72** (1995), 335–345.

[MW84] B. Mazur and A. Wiles, *Class fields of abelian extensions of* $\mathbb{Q}$, Invent. Math., **76** (1984), 179-330.

[NH88] J. Nakagawa and K. Horie, *Elliptic curves with no rational points*, Proc. Amer. Math. Soc., **104** (1988), 20–24.

[On99] K. Ono, *Indivisibility of class numbers of real quadratic fields*, Compositio Math., **119** (1999), 1–11.

[Oz97] M. Ozaki, *The class group of* $\mathbb{Z}_p$*-extensions over totally real number fields*, Tohoku Math. J., **49** (1997), 431–435.

[Oz98] M. Ozaki, *On Greenberg's conjecture* (Japanese), Sûrikaisekikenkyûsho Kôkyuroku, **1026** (1998), Res. Inst. Math. Sci. Kyoto Univ. 20–27.

[OT97] M. Ozaki and H. Taya, *On the Iwasawa* λ_2*-invariants of certain families of real quadratic fields*, Manuscripta Math., **94** (1997), 437–444.

[Ta99a] H. Taya, *On p-adic zeta functions and* $\mathbb{Z}_p$*-extensions of certain totally real number fields*, Tohoku Math. J., **51** (1999), 21–33.

[Ta99b] H. Taya, *On p-adic L-functions and* $\mathbb{Z}_p$*-extensions of certain real abelian number fields*, J. Number Theory, **75** (1999), 170–184.

[Ta00] H. Taya, *Iwasawa invariants and class numbers of quadratic fields for the prime* 3, Proc. Amer. Math. Soc., **128** (2000), 1285–1292.

[Wa97] L. C. Washington, *Introduction to Cyclotomic Fields*, Second edition, Graduate Texts in Math. Vol. 83. Springer-Verlag, New York, Heidelberg, Berlin 1997.

Added in proof. Concerning Problem 5.1, D. Byeon recently showed by refining Ono's ideal that for a given prime number $p \geq 5$, there exist infinitely many real quadratic fields k such that $\lambda_p(k) = 0$ and p splits in k. For the details, see his paper "Indivisibility of class numbers and Iwasawa λ-invariants of real quadratic fields", which is to appear in Compositio Math.

Graduate School of Information Sciences
Tohoku University
Katahira, Sendai,
980–8577, Japan
E-mail address: `taya@math.is.tohoku.ac.jp`