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Generalized Enriques diagrams and characteristic cones

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Abstract.

Generalized Enriques diagrams are combinatorial data associated with constellations of infinitely near points and proximity relations. Classically they were introduced to deal with linear systems of curves with base conditions. We present a survey on some aspects and new results on this diagrams, examples and applications to relative characteristic cones and Zariski's complete ideal theory.

§1. Introduction

In [6] (Libro Quarto: "Le singolarità delle curve algebriche", I. 12 et II. 17), Enriques and Chisini consider systems of plane curves passing, with assigned multiplicities, through an assigned set of points or infinitely near points to a point of the plane. They found that there exist curves with such prescribed multiplicities (with no conditions on the degree of the curves) if and only if some inequalities, on these virtual multiplicities, hold for the given points, the so-called *proximity* relations. Enriques associates a graph ("schema grafico") to the constellation of infinitely near points appearing in the desingularisation of a plane curve and equiped this graph with the data of the proximity relations which keep track of the incidence between points and the exceptional divisors obtained by blowing-up precedent points. Du Val also considers these proximities relations (see [5]) and defines the proximity matrix.

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Some years later Zariski introduces the notion of complete ideals to give a new algebraic setup of the previous geometric theory ([14], [15]), where complete (i.e. integrally closed) ideals are the local avatars of complete linear systems . One of the main results is that any complete ideal $\mathcal{I}$ in a *two* dimensional local ring has a unique factorization into simple ideals, which corresponds to the factorization of the general element of $\mathcal{I}$ into analytically irreducible factors. This fact may be reformulated as the *regularity* of the relative characteristic cone associated with the minimal blowing-up of (infinitely near) points for which $\mathcal{I}$ becomes locally principal. The condition of regularity of a cone is considered in the sense of rational cones of toric varieties, i.e. with primitive integral extremal points forming a subset of a basis of the lattice. The characteristic and the quasi-ample cone of a proper morphism have been considered by Hironaka, Mumford and Kleiman [9] in the 60's.

In higher dimension than two, as Zariski had noted, the theory is much more involved and the main results do not extend, or not in the same terms. First, one has to restrict to *finitely supported* complete ideals in order to deal with constellations of closed points. But even with this restriction, the characteristic cone is not regular, or polyhedral, in the general case (see [4] or [1] for some examples). Lipman extended this theory for two dimensional local rings with rational singularities [10], and for higher dimensional regular local rings he obtained a unique factorization result by allowing negative exponents [11].

The preceding lines do not pretend to be an exhaustive account on the history of the subject, but only a sketch to situate it; many other important contributions have been made to this "theory in search of theorems", as Lipman says.

In this work we summarize *two* generalizations of Enriques diagrams in higher dimensions; there are two natural generalizations because the *dimension* one and *codimension* one conditions that coincide in the two-dimensional classic case are not equivalent when the ambient dimension is at least three.

First we characterize the so called proximity P-Enriques diagrams and determine, in terms of numerical invariants of such a diagram, the minimal dimension of a constellation of infinitely near points which induces a given diagram.

Then, we consider a case where the characteristic and the quasi-ample cones are equal, namely the toric constellations, and characterize the P-Enriques diagrams associated with them.

Finally we consider the so called linear proximity LP-Enriques diagrams, which determine the characteristic cone in the toric constellation case, and show a converse Zariski theorem: the characteristic cone is regular if and only if the LP-Enriques diagram comes from a *two-dimensional* constellation.

> *"Qui oserait dire que ce que nous avons détruit*
> *valait cent fois mieux que ce que nous avions rêvé*
> *et transfiguré sans relâche en murmurant aux ruines ?"*
> *René Char*

§2. Constellations of infinitely near points and P-Enriques diagrams

2.1. Let X be a regular variety over an algebraically closed field $\mathbb{K}$, of dimension $d \geq 2$. Consider varieties obtained from X by a finite sequence of closed points blowing-ups. Any point in such a variety is called an infinitely near point of X.
A point P is infinitely near $Q \in X$ if Q is the image of P under the composition of the blowing-ups; $P \geq Q$ in symbol.

A *constellation of infinitely near points* (in short, a constellation, if there is no confusion with other astronomical objects) is a set $\mathcal{C} = \{Q_0, \ldots, Q_n\}$, with $Q_i \geq Q_0 \in X_0 = X$, such that $Q_i \in Bl_{Q_{i-1}} X_{i-1} =: X_i \xrightarrow{\sigma_{i-1}} X_{i-1}$, for $1 \leq i \leq n$; where $Bl_{Q_{i-1}} X_{i-1}$ denotes the blowing-up of X_{i-1} with center Q_{i-1} .

The point Q_0 is called the *origin* of the constellation $\mathcal{C}$. We call also the dimension of X the *dimension* of $\mathcal{C}$.

Let $\sigma_{\mathcal{C}} = \sigma_0 \circ \cdots \circ \sigma_n : X_{\mathcal{C}} \to X_0$ denote the composition of the blowing-ups of all the points of $\mathcal{C}$, where $X_{\mathcal{C}} = X_{n+1}$. Two constellations $\mathcal{C}$ and $\mathcal{C}'$ over X are identified if there is an automorphism π of X and an isomorphism $\pi' : X_{\mathcal{C}} \to X_{\mathcal{C}'}$ such that $\sigma_{\mathcal{C}'} \circ \pi' = \pi \circ \sigma_{\mathcal{C}}$.

The relation $Q_j \geq Q_i$, meaning that a composition of blowing-ups sends Q_j to Q_i in X_i, is a partial ordering on the points of $\mathcal{C}$. If this ordering is total, i.e. $Q_n \geq \cdots \geq Q_0$, we say that $\mathcal{C}$ is a *chain* constellation.

For example, for any constellation $\mathcal{C}$ and any $Q \in \mathcal{C}$, the set $\mathcal{C}^Q := \{P \in \mathcal{C} \mid Q \geq P\}$ of preceding points is a chain constellation. The number of points in $\mathcal{C}^Q$, different from Q, is called the *level* of Q.

For each point $Q \in \mathcal{C}$ let Q^+ be the set of points of $\mathcal{C}$ *consecutive* to Q, i.e. the points following Q such that there is no strict intermediate point; write $|Q^+|$ for the cardinal of this set. If Q^+ has only one point, it denotes this point, by a slight abuse of notation.

For each point $Q = Q_i$, call B_Q (or B_i) the *exceptional* divisor $\sigma_i^{-1}(Q)$ on X_{i+1}, and E_Q (or E_i) its *strict* (or *proper*) successive transforms on any X_j (which will be specified if necessary) with $Q_j \geq Q$, in particular on $X_{\mathcal{C}}$. The *total* transforms are denoted E_Q^* or E_i^*.

The sets of divisors, $\{E_Q \mid Q \in \mathcal{C}\}$ and $\{E_Q^* \mid Q \in \mathcal{C}\}$, considered in $X_{\mathcal{C}}$, are two basis of the lattice $N^1 = \bigoplus_{Q\in\mathcal{C}} \mathbb{Z}E_Q \cong \mathbb{Z}^{n+1}$ of divisorial cycles with *exceptional* support in $X_{\mathcal{C}}$.

Definition 2.2. A point $Q_j \geq Q_i$ is *proximate* to Q_i if $Q_j \in E_i$ in X_j; notation : $Q_j \to Q_i$ (or $j \to i$).

The *proximity index* of a point Q_j is defined as the number $\mathrm{ind}(Q_j)$ of points in $\mathcal{C}$ approximated by Q_j, i.e. $\mathrm{ind}(Q_j) := \#\{Q_i \in \mathcal{C} \mid Q_j \to Q_i\}$.

If $R \in Q^+$ then $R \to Q$, these are the so called trivial proximities ; if R belongs to the intersection of several exceptional divisors produced by blowing-up precedent points then R is proximate to all these points. In fact, if the dimension of C is *at least three*, then $R \to Q$ if and only $R \geq Q$ and $E_R \cap E_Q \neq \emptyset$ in $X_{\mathcal{C}}$.

If $R \to Q$ then $R \geq Q$; the converse does not hold in general. The proximity relation $(\to)$ is a binary relation on the set of points of a constellation, but not an ordering one.

Remark 2.3. For each point Q_i, the only exceptional divisors, besides E_i, appearing in the total transform E_i^*, in $X_{\mathcal{C}}$, are exactly those produced by blowing-up the points proximate to Q_i. Therefore $E_i = E_i^* - \sum_{j\to i} E_j^*$. The so called *proximity matrix* $((p_{ji}))$, with $p_{ii} = 1$, $p_{ji} = -1$ if $j \to i$ and 0 otherwise, is the basis change matrix from the E_i's to the E_j^*'s

Definition 2.4. The (proximity) *P-Enriques diagram* of a constellation $\mathcal{C}$ is the rooted tree $\Gamma_{\mathcal{C}}$ equiped with the binary relation $(\rightsquigarrow)$, whose vertices are in one to one correspondence with the points of $\mathcal{C}$, the edges with the couples of points (R, Q) such that $R \in Q^+$, the root with the origin of $\mathcal{C}$, and the relation $(\rightsquigarrow)$ with the proximity relation $(\to)$.

Any (finite) rooted tree may be obtained in this way, but not with the data of a binary relation . Next we characterize the P-Enriques diagrams, i.e. the rooted trees, equiped with a binary relation on the set of vertices, which are induced by some constellation.

Given a rooted tree Γ, denote by $(\succeq)$ the natural partial ordering on the set $\mathcal{V}(\Gamma)$ of its vertices : $p \succeq q$ if q belongs to the chain from p to the root; similarly, if $(\rightsquigarrow)$ is a binary relation on $\mathcal{V}(\Gamma)$, let $\mathrm{ind}(q) = \#\{p \in \mathcal{V}(\Gamma) \mid q \rightsquigarrow p\}$.

For each vertex q, let q^+ be the set of consecutive vertices to q with respect to the ordering $(\succeq)$.

Theorem 2.5. *Let Γ be a finite rooted tree equiped with a binary relation $(\rightsquigarrow)$ on the set of its vertices. Then Γ is the graph associated with a constellation of infinitely near points $\mathcal{C}$ and $(\rightsquigarrow)$ is induced by the the proximity relation on $\mathcal{C}$ if and only if , for any vertices p, q, r of Γ, the following conditions are satisfied:*

(a) $q \rightsquigarrow p \quad \Longrightarrow \quad q \succeq p\ ,\ q \neq p$

(b) $q \in p^+ \quad \Longrightarrow \quad q \rightsquigarrow p$

(c) $r \succeq p \succeq q$ *and* $r \rightsquigarrow q \quad \Longrightarrow \quad p \rightsquigarrow q$

If these conditions hold, then the minimum dimension $d_{\mathcal{P}}$ of a constellation whose P-Enriques diagram is the given one is at most $\max(2, \max_{q\in\mathcal{V}(\Gamma)}(\mathrm{ind}(q)) + 1)$.

Proof. The necessity of the conditions follows easily. For the sufficiency, proceed by induction on the number $|\mathcal{V}(\Gamma)|$ of vertices.

If $|\mathcal{V}(\Gamma)| > 1$, let r be a maximal vertex of Γ, and assume that a constellation $\mathcal{C}'$ of dimension d works for $\Gamma' = \Gamma \setminus \{r\}$. Let $r \in q^+$, and Q be the point of $\mathcal{C}'$ corresponding to q. The set $Y := \{P \in \mathcal{C}' \mid r \rightsquigarrow p\}$ is contained in $\mathcal{C}'^Q$ by (a) and $Q \in Y$ by (b).

By (c) one has $Q \to P$ for each $P \in Y \setminus \{Q\}$, so that $Q \in F := \bigcap_{P\in Y, P\neq Q} E_P$. It follows that $F \neq \emptyset$ and $\dim(F) = d + 1 - |Y|$, by the normal crossing of the divisors E_P, and on the other hand $\mathrm{ind}(Q) \geq |Y \setminus \{Q\}| = |Y| - 1$.

Now, we need a point R (in $X_{\mathcal{C}'}$) corresponding to r, having the corresponding proximities, i.e. a point $R \in B_Q \bigcap F$ but not in $(Q^+ \bigcup_{P\in\mathcal{C}^Q\setminus Y} E_P)$. Such a point exists if $d \geq \max_{p\in\mathcal{V}(\Gamma)} \mathrm{ind}(p) + 1$ (and at least 2), which is not less than $\max_{p\in\mathcal{V}(\Gamma')} \mathrm{ind}(p) + 1$ so the inductive hypothesis applies. This number is attained. Q.E.D.

Remark 2.6. The minimum dimension $d_{\mathcal{P}}$ of constellations inducing a given P-Enriques diagram may be one less than in the general case if there are no two maximal vertices r, with maximum indices, say r_1 and r_2, both in q^+, such that $\mathrm{ind}(r_i) = \mathrm{ind}(q) + 1$. Precisely, the minimum dimension is

$$d_{\mathcal{P}} = \max(2, \max_{q\in\mathcal{V}(\Gamma)}(\mathrm{ind}(q) + t(q)),$$

where $t(q){=}s(q)$ (resp. $t(q){=}2$) if $s(q){:=}\#\{r \in q^+ \mid \mathrm{ind}(r) > \mathrm{ind}(q)\} \leq 1$ (resp. if $s(q) \geq 2$).

§3. Toric constellations and proximity

We begin by recalling some definitions and fixing notations for toric varieties (for a detailed treatement see some of the basic references on this subject, chapter 1 of [13] or [8]).

3.1. Let $N \cong \mathbb{Z}^d$ be a lattice of dimension $d \geq 2$ and Σ a *fan* in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, i.e. a finite set of strongly convex polyhedral cones such that every face of a cone of Σ belongs to Σ and the intersection of two cones of Σ is a face of both. Denote by X_Σ the toric variety over a field $\mathbb{K}$ associated with Σ, equiped with the action of an algebraic torus $T \cong (\mathbb{K}^*)^d$. There is a one to one canonical correspondence between the T-orbits in X_Σ and the cones of Σ. Two basic facts of this correspondence are that the dimension of a T-orbit is equal to the codimension of the corresponding cone, and that a T-orbit is contained in the *closure* of another T-orbit if and only if the cone associated with the first one contains the cone associated with the second one.

The *morphisms* of toric varieties are the equivariant maps induced by the maps of fans $\varphi : (N', \Sigma') \to (N, \Sigma)$ such that $\varphi : N' \to N$ is a $\mathbb{Z}$-linear homomorphism whose scalar extension $\varphi : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ has the property that for each $\sigma' \in \Sigma'$ there exists $\sigma \in \Sigma$ such that $\varphi(\sigma') \subset \sigma$; (see [13], 1.5).

Let $X_0 := X_{\Sigma_0} \cong \mathbb{K}^d$ be the d-dimensional affine toric variety associated with the fan Σ_0 formed by all the faces of a regular d-dimensional rational cone Δ in $N_{\mathbb{R}}$. Recall that a rational cone is called *regular* (or nonsingular) if the primitive integral extremal points form a subset of a basis of the lattice.

A *toric constellation* of infinitely near points is a constellation $\mathcal{C} = \{Q_0, \ldots, Q_n\}$ such that each Q_j is a fixed point for the action of the torus in the toric variety X_j obtained by blowing-up X_{j-1} with center Q_{j-1}, $1 \leq j \leq n$. If a toric constellation is a chain, it is called a *toric chain.* The identification of constellations defined in 2.1 is the same in the toric case, but considering equivariant isomorphisms.

3.2. By choosing a fixed *ordered basis* $\mathcal{B} = \{v_1, ..., v_d\}$ of the lattice N we obtain a *codification* of the toric constellations, as well as criteria for proximity and (as shown in the following section) linear proximity.

Let $\Delta = \langle \mathcal{B} \rangle$ be the (regular) cone generated by the basis $\mathcal{B}$. The blowing-up $\sigma_i : X_i \to X_{i-1}$ of the closed orbit Q_{i-1}, is described as an elementary subdivision of a fan, as follows.

The variety X_1 is the toric variety associated with the fan Σ_1, obtained as the minimal subdivision of Σ_0 which contains the ray through

$u = \sum_{1 \le j \le d} v_j$.
For each integer i, $1 \le i \le d$, let $\mathcal{B}_i$ be the ordered basis of N obtained by replacing v_i by u in the basis $\mathcal{B}$; and let $\Delta_i := \langle \mathcal{B}_i \rangle$. The exceptional divisor B_0 is the closure in X_1 of the T-orbit defined by the ray through u, and each T-fixed point in X_1 corresponds to a maximal cone Δ_i of the fan Σ_1, $1 \le i \le d$.

The choice of the point $Q_1 \ge Q_0$ is thus equivalent to the choice of an integer a_1, $1 \le a_1 \le d$, which determines a cone Δ_{a_1} of the fan Σ_1.

The subdivision Σ_2 of Σ_1 corresponding to the blowing-up of Q_1 is obtained by replacing Δ_{a_1} (and its faces) in Σ_1 by the cones $\Delta_{a_1 i} := \langle \mathcal{B}_{a_1 i} \rangle$ (and their faces), where $\mathcal{B}_{a_1 i}$ is the ordered basis of N obtained from $\mathcal{B}_{a_1}$ by the substitution of its i-th vector by $\sum_{v \in \mathcal{B}_{a_1}} v$.

The choice of $Q_2 \in B_1$ is equivalent to the choice of an integer a_2 ,$1 \le a_2 \le d$, which determines a (regular) cone $\Delta_{a_1 a_2}$.

Proceeding by induction on n we obtain a *codification* of toric chains and also constellations, since for each $Q \in \mathcal{C}$, the constellation $\mathcal{C}^Q$ is a chain.

The codification is given by trees with weighted edges, where the weights are integers a, $1 \le a \le d$, which give the *direction* in which the following blowing-up is done. The precise description follows.

Definition 3.3. Let Γ be a tree, $\mathcal{E}(\Gamma)$ the set of edges of Γ, d an integer, d ≥ 2.
A *d-weihgting* of Γ is a map $\alpha : \mathcal{E}(\Gamma) \to \{1, \dots ,\mathrm{d}\}$ which associates to each edge of Γ a positive integer not greater than d, such that two edges with a common origin have different weights. A couple (Γ, α) is called a *d-weighted tree.*

Proposition 3.4. *Let $\mathcal{B}$ be an ordered basis of the lattice N and n a positive integer.*

(a) *The map which associates to each sequence of integers $\{a_1, \dots, a_n\}$ such that $1 \le a_i \le d$, $1 \le i \le n$, the toric chain $\{Q_0, \dots , Q_n\}$ where Q_0 is the T-orbit corresponding to the cone $\Delta = \langle \mathcal{B} \rangle$, and where Q_i, $1 \le i \le n$, is the T-orbit in X_i corresponding to the cone $\Delta_{a_1 \dots a_i}$ of the fan Σ_i, is a bijection between the set of such sequences and the set of d-dimensional toric* chains *with $n+1$ points.*

(b) *A natural bijection between the set of d-dimensional toric constellations and the set of d-weighted trees is induced by the correspondence* (a).

Remark 3.5. Note that in a d-weighted tree each vertex is the origin of at most d edges. A d-weighting of a tree Γ induces a partition of the set $\mathcal{E}(\Gamma)$ of edges, where two edges are in the same class if they have the same weight. To each class of isomorphism of d-dimensional toric constellations is associated a unique class of isomorphism of trees equiped with a partition of the set of edges, partition with at most d classes of edges [7].

3.6. Given a toric constellation by a d-weighted graph, a vertex following q through a chain with edges weighted by a sequence $(a_1, \dots, a_k)$ is denoted by $q(a_1, \dots, a_k)$; if Q is the point corresponding to q, then the point corresponding to $q(a_1, \dots, a_k)$ is written in a similar way $Q(a_1, \dots, a_k)$.

Proposition 3.7 (*Criterion* for proximity in terms of a codification). *$Q(a_1, \dots, a_k) \to Q$ if and only if $a_1 \neq a_j$ for $2 \leq j \leq k$.*

Proof. The criterion follows from the fact that this is the condition to obtain, by elementary subdivisions of a regular fan, an adjacent maximal cone $\Delta_{a_1 \dots a_k}$ (corresponding to a 0-dimensional orbit) to the central ray of Δ_{a_1} (corresponding to the exceptional divisor) of the first subdivision of the cone Δ corresponding to Q. This is equivalent to saying that $Q(a_1, \dots, a_k) \in E_Q$, i.e. $Q(a_1, \dots, a_k) \to Q$. Q.E.D.

Theorem 3.8. *A P-Enriques diagram $(\Gamma, (\rightsquigarrow))$ is toric, i.e. may be induced by a toric constellation, if and only if:*

(a) *The proximity index is non-decreasing, i.e.* $\mathrm{ind}(r) \geq \mathrm{ind}(q)$ *if* $r \succeq q$.

(b) *If r is proximate to q, then there is at most one vertex s consecutive to r and not proximate to q, i.e. if $r \rightsquigarrow q$ then $\#\{s \in r^+ \mid s \not\rightsquigarrow q\} \leq 1$.*

If these conditions hold, then the minimum dimension $dt_{\mathcal{P}}(\Gamma, (\rightsquigarrow))$ of a toric constellation inducing the given P-Enriques diagram $(\Gamma, (\rightsquigarrow))$ is

$\max(2, \max_{q \in \Gamma}(\mathrm{ind}(q) + s(q)))$, *where* $s(q) := \#\{r \in q^+ \mid \mathrm{ind}(r) > \mathrm{ind}(q)\}$ *is the number of consecutive points to q whose proximity index is greater than the proximity index of q.*

Proof. In fact, if $R \in Q^+$, then $\mathrm{ind}(R) \leq \mathrm{ind}(Q) + 1$ for any constellation, since, by (c) of Theorem 2.5, $R \to P$ implies $Q \to P$; in the toric case one also has $\mathrm{ind}(Q) \leq \mathrm{ind}(R)$ because R may loose at most one proximity to a point approximated by Q, but on the other hand $R \to Q$. Indeed, restricting to the chain from the origin to R and by using the codification and the criterion of proximity 3.7, let $R = Q(a)$ and assume $Q \to P$, $Q \to P'$, $R \not\to P$; then $P^+ = P(a)$. Let $P'^+ = P'(a')$. If $P \geq P'$ (resp. if $P' \geq P$), then $a \neq a'$, because $Q \to P'$ (resp. $Q \to P$); then $R \to P'$. This shows the necessity of (a).

To prove (b), recall that two edges with origin R have necessarily different weights, so there is at most one whose weight is equal to the weight of the edge, with origin Q, in the chain from Q to R.

To prove the sufficiency, proceed by induction on the number of vertices and use the proximity criterion. Assume $|\mathcal{V}(\Gamma)| > 1$, let r be a maximal vertex of Γ, say $r \in q^+$. By the inductive hypothesis, the full subgraph $\Gamma' = \Gamma \setminus q^+$ equiped with the binary relation restricted to $\mathcal{V}(\Gamma')$, is also a P-Enriques diagram satisfying the conditions (a) and (b), so may be induced by a toric constellation $\mathcal{C}'$, codified by a d-weighted tree with d $\leq dt_{\mathcal{P}}(\Gamma', (\rightsquigarrow))$. Let $q^+ = \{r_1, \ldots, r_s, \ldots, r_t\}$; by (i) one has $\mathrm{ind}(r_j) \geq \mathrm{ind}(q)$, for $1 \leq j \leq t$. Let $r_1, \ldots, r_s$ be vertices, in q^+, whose index is greater than $\mathrm{ind}(q)$; one needs $s = s(q)$ new weights to codify these vertices; but for each r_j with $s+1 \leq j \leq t$, the weight is determined by the lost proximity among the vertices approximated by q, and these weights are all different by the condition (ii). This shows the inductive step and the existence of a toric constellation associated with the given $(\Gamma, (\rightsquigarrow))$ with the dimension $dt_{\mathcal{P}}$. This is the minimum dimension since for any such constellation of dimension d, one has $d \geq \mathrm{ind}(q) + s(q)$, for each $q \in \mathcal{V}(\Gamma)$. Q.E.D.

Remark 3.9. The minimum dimension $dt_{\mathcal{P}}$ may be greater than $d_{\mathcal{P}}$, the dimension in the not necessarily toric case (Theorem 2.5), because there are less points available, so one needs to add $s(q)$ to the proximity index, not just 1 as in the general case.

Corollary 3.10. *A P-Enriques diagram* $(\Gamma, (\rightsquigarrow))$ *whose graph* Γ *is a* chain, *is toric if and only if the proximity index is not decreasing. In this case, the minimum dimension of an associated constellation is the index of the terminal point* (*and at least* 2).

Proof. In the toric chain case the condition (b) of the theorem is automatically satisfied and $\max_{q \in \Gamma}(\mathrm{ind}(q) + s(q))) = \max_{q \in \Gamma}(\mathrm{ind}(q))$ holds. Q.E.D.

Examples 3.11. (1) The simplest example of a non-toric P-Enriques diagram is a chain with four vertices, say q_0, q_1, q_2, q_3 such that, besides the trivial proximities of consecutive vertices, the only other proximity is $q_2 \rightarrow q_0$. In this example one has $\mathrm{ind}(q_2) = 2$ and $\mathrm{ind}(q_3) = 1$; condition (a) fails.

(2) Another example of a non-toric case is a graph of type $\mathbb{D}_4$, with a non-central vertex as the root, and with only the proximities of consecutive vertices. In this case condition (b) fails.

Remark that both cases may be induced by two dimensional constellations.

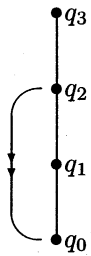

Figure (1).

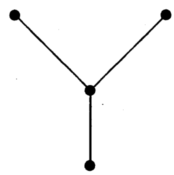

Figure (2).

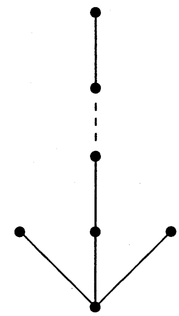

Figure (3).

(3) If the central vertex is the root in a graph of type $\mathbb{D}_n$, with $n \geq 4$, and if the only proximities are those of consecutive vertices, then conditions (a) and (b) hold; the minimal dimension of a constellation inducing this P-Enriques diagram is three for toric constellations and two for non-toric ones. If q_0 is the root, then $\operatorname{ind}(q_0) = 0$, $s(q_0) = 3$, $t(q_0) = 2$, and $\operatorname{ind}(q) = 1$, $s(q) = 0$ for each $q \neq q_0$.

(See Figures (1), (2) and (3)).

§4. Linear proximity and characteristic cones

4.1. In dimension two, the exceptional divisors appearing in the definition of the proximity relations are (rational) curves; in higher dimension we introduce, in the toric case, a condition involving curves, which will be finer, in general, than the proximity.

Definition 4.2. Let $\mathcal{C} = \{Q_0, \ldots, Q_n\}$ be a toric constellation. A point Q_j is *linear proximate* to a point Q_i with respect to a one dimensional T-orbit $\ell \subset B_i$ if Q_j belongs to the strict transform in X_j of the closure of ℓ.

This relation is denoted by $Q_j \twoheadrightarrow Q_i$, or $Q_j \overset{\ell}{\twoheadrightarrow} Q_i$ if we need to specify the line ℓ involved.

If $R \twoheadrightarrow Q$ then $R \longrightarrow Q$, but the converse does not hold in general.

Proposition 4.3 (*Criterion* for the linear proximity in terms of a codification). *Let Q be a point in a toric constellation of dimension* *d. Each 1-dimensional orbit ℓ in the exceptional divisor B_Q contains in its closure only two fixed points, say $Q(a)$ and $Q(b)$, which determine uniquely ℓ.*

$R \overset{\ell}{\twoheadrightarrow} Q$ if and only if there are integers a, b and m such that $a \neq b$, $1 \leq a \leq d$, $1 \leq b \leq d$, $0 \leq m$ and $R = Q(a, b^{[m]})$ or $R = Q(b, a^{[m]})$, where $x^{[m]}$ means x repeated m times.

Proof. The wall running between the cones corresponding to $Q(a)$ and $Q(b)$ is the cone corresponding to the line defined by this two points in B_Q. The only maximal cones, obtained by elementary subdivisions, having this wall as a face are those corresponding to the points $Q(a, b^{[m]})$ or $Q(b, a^{[m]})$ for some $m \geq 0$. Q.E.D.

In dimension two, proximity and linear proximity are equivalent. One inclusion may be generalized for toric *chains* in any dimension.

Proposition 4.4. *If C is toric chain (in any dimension), the proximity relation determines the linear proximity relation.*

Proof. If $R \twoheadrightarrow Q$, then $P \to Q$ for any P such that $R \geq P \geq Q$, $P \neq Q$, and these are the only proximities, for the intermediate points in the chain from Q to R, besides the proximities of consecutive points. Conversely, assuming this property, then $R \overset{\ell}{\twoheadrightarrow} Q$ for the line ℓ determined by the point Q^+ and the direction Q^{++} in the projective space B_Q, if Q^{++} is defined and precedes R, or any line through Q otherwise. Indeed, this assumption forces the code of R to be $Q(a, b^{[m]})$ for some weights a and b, $m \geq 0$. Q.E.D.

On the other hand, in general the linear proximity does not determine the proximity, even for chains.

4.5. We introduce now some definitions leading to the notion of the so called (linear proximity) LP-Enriques diagrams.

Given a rooted tree Γ, a sub graph formed by two chains with a common root and no common edge is called a *bi-chain.*

If Γ is the rooted tree associated with a toric constellation $\mathcal{C}$, q the vertex corresponding to $Q \in \mathcal{C}$ and ℓ is a 1-dimensional orbit in B_Q, then $\Gamma_q(\ell)$ denotes the full subgraph of Γ with vertices corresponding to Q and to the points $R \in \mathcal{C}$ such that $R \overset{\ell}{\twoheadrightarrow} Q$. Let $\Gamma(q)$ be the family of the maximal $\Gamma_q(\ell)$ when ℓ describes the set of one dimensional orbits in B_Q.

A vertex $q \in \Gamma$ is called *simple* (resp. *ramified*) if $|q^+| = 1$ (resp. if $|q^+| > 1$).

The following properties are easily checked with the linear proximity criterion (Proposition 4.3).

Proposition 4.6. *Let $\mathcal{C}$ be a toric constellation, Γ the associated tree.*

1. (a) *For each $q \in \Gamma$, the family $\Gamma(q)$ is non-empty and the elements of $\Gamma(q)$ are chains or bi-chains with root q.*
   (b) *If γ, $\gamma' \in \Gamma(q)$ and $\gamma \subset \gamma'$, then $\gamma = \gamma'$.*
2. (a) *Two distinct elements of $\bigcup_q \Gamma(q)$ have at most one common edge.*
   (b) *Two edges with common ramification root vertex q (resp. the edge with the simple root vertex q) belong (resp. belongs) to one and only one element of $\Gamma(q)$.*
3. (a) *For each $q \in \Gamma$ and $r \in q^+$ there is at most one vertex $s \in r^+$ such that the chain (q, r, s) is not contained in any element of $\Gamma(q)$.*
   (b) *If $(p, \ldots, q, r)$ is a chain contained in a $\gamma \in \Gamma(p)$ and $s \in r^+$ satisfies* 3. (a), *then the chain $(p, \ldots, q, r, s)$ is contained in γ.*

Definition 4.7. The *LP-Enriques diagram* of a toric constellation $\mathcal{C}$ is the associated graph $\Gamma_{\mathcal{C}}$ equiped with the linear proximity structure formed by the family of full subgraphs $\{\Gamma_{\mathcal{C}}(q) \mid q \in \Gamma_{\mathcal{C}}\}$.

Theorem 4.8. *The couple $(\Gamma, \{\Gamma(q) \mid q \in \Gamma\})$, given by a tree Γ and a family of full subgraphs $\Gamma(q)$, is the LP-Enriques diagram of a toric constellation $\mathcal{C}$ if and only if the properties* 1, 2 *and* 3 *hold.*

The minimum dimension of the constellations with given PL-Enriques diagram is $d_{\mathcal{PL}} = \max(2, \max_{q \in \Gamma}(|q^+| + n_q))$, where

$$n_q = \max_{r \in q^+} \#\{\gamma \in \Gamma(q) \mid r \in \gamma \textit{ and } \gamma \textit{ is a chain of length } > 1\}$$

Proof. The proof uses the codification of toric constellations, the linear proximity criterion 4.3 and proceeds by induction on the level of the vertices. Let's check that the dimension given in the statement is the minimum possible dimension; this will also show the essential part of the inductive step. Recall that the length of a chain graph is the number of its edges.

Now assume that the LP-Enriques diagram of a d-dimensional toric constellation $\mathcal{C}$ is the given one, and consider a d-weighting of Γ defining $\mathcal{C}$. For each $q \in \Gamma$ one needs d distinct weights for the $|q^+|$ edges with root q, hence d$\geq |q^+|$. Furthermore, for each $r \in |q^+|$ one needs another weight for the second edge of each chain γ of length at least 2 such that

$r \in \gamma \in \Gamma(q)$, and this weight must be different from the $|q^+|$ weights of the edges with root q in order not to get a bi-chain and a contradiction with the conditions 2(a) and 2(b), and also different from the weights of the second edge of the other chains of the same type containing r. This shows that d$\geq (|q^+| + n_q)$.

On the other hand, the bi-chains with root q are automatically weighted once the first edges are weighted, and the second edges of chains with second vertex in q^+ different from r may have the same weights as those of the second vertices of chains through r. The maximality of the elements of $\Gamma(q)$ for each $q \in \Gamma$ and the conditions 3(a) and 3(b) insure that the codification in the inductive step is coherent with the preceding weights, and that the minimum dimension is attained. Q.E.D.

Remark 4.9. A PL-Enriques diagram may be induced by two non-isomorphic constellations. In some cases, for instance if for each vertex q the family $\Gamma(q)$ has only bi-chains or is reduced to the vertex, then the constellation inducing the given PL-Enriques diagram is unique (up to isomorphism of constellations), and its dimension is $|q_0^+|$ if q_0 denotes the root.

The maximum posible linear proximity dimension d_{LP} of a fixed tree, by changing its LP structure, is the number of edges. In this case all the chains (resp. bi-chains) have only one edge (resp. two edges) or are reduced to a vertex, for the maximal ones.

4.10. We recall now some definitions and resume relevant facts on *complete ideal theory* and *characteristic cones* (see [15], [10], [11], [9]), then we give some examples and applications of the results on Enriques diagrams.

Let $\mathcal{I}$ be an ideal in any commutative ring $\mathcal{R}$; an element $x \in \mathcal{R}$ is *integral over* $\mathcal{I}$ if x satisfies a condition of the form $x^n + r_1x^{n-1} + r_2x^{n-2} + \cdots + r_n = 0$ for some $n > 0$ and some $r_j \in \mathcal{I}^j$, $1 \leq j \leq n$. The set of all such x, denoted $\overline{\mathcal{I}}$, is called the *integral closure* or *completion* of $\mathcal{I}$; it is itself an ideal and we have $\mathcal{I} \subset \overline{\mathcal{I}} = \overline{\overline{\mathcal{I}}}$.

The ideal $\mathcal{I}$ is *integrally closed* or *complete* if $\mathcal{I} = \overline{\mathcal{I}}$.

For any two ideals $\mathcal{I}$, $\mathcal{J}$ in $\mathcal{R}$, define the $*$-product $\mathcal{I} * \mathcal{J} := \overline{\mathcal{I}\mathcal{J}}$ (the completion of the product $\mathcal{I}\mathcal{J}$). Then we have $\mathcal{I} * \mathcal{J} = \overline{\mathcal{I}} * \overline{\mathcal{J}}$.

The set of non-zero complete ideals in $\mathcal{R}$ with the $*$-product form a commutative monoid with cancellation (i.e. $\mathcal{I}_1 * \mathcal{J} = \mathcal{I}_2 * \mathcal{J} \Rightarrow \mathcal{I}_1 = \mathcal{I}_2$).

In the following let $\mathcal{R}$ be the local ring $\mathcal{O}_{X,Q_0}$ of the regular variety X at the origin $Q_0 \in X$ of the constellations of infinitely points to consider. In this context, the ideals in $\mathcal{R}$ that we consider are the $\mathcal{M}$-primary ideals , where $\mathcal{M}$ denotes the maximal ideal of R.

An ideal $\mathcal{I}$ is *finitely supported* if $\mathcal{I} \neq (0)$ and if there exists a constellation $\mathcal{C}$ such that $\mathcal{I}\mathcal{O}_{X_{\mathcal{C}}}$ is a locally principal ideal. The points Q_i, $0 \leq i \leq n$, of the *minimal* constellation $\mathcal{C}_{\mathcal{I}}$ with this property are called the *base points* of the ideal $\mathcal{I}$.

If $\mathcal{I}$ is a non-zero ideal in $\mathcal{O}_{X_i,Q_i}$, let $\mathrm{ord}_{Q_i}\mathcal{I} = \mathrm{ord}_{Q_i} f$ for a general $f \in \mathcal{I}$.

Define recursively the *weak transform* $\mathcal{I}_i = \mathcal{I}_{Q_i}$ and *the strict multiplicity* (called *point base* in [11]) $m_i = m_{Q_i}$ of $\mathcal{I}$ at a base point Q_i:
$\mathcal{I}_0 = \mathcal{I}$, $m_0 = \mathrm{ord}_{Q_0}\mathcal{I}$ and for $R \in Q^+$ let $\mathcal{I}_R = (x)^{-m_Q}\mathcal{I}_Q\mathcal{O}_{X_R,R}$ and $m_R = \mathrm{ord}_R\mathcal{I}_R$, where $x = 0$ is a local equation of B_Q at R.

The divisor $D_{\mathcal{I}} = \sum_i m_i E_i^*$ is the divisor defined by $\mathcal{I}$ in $X_{\mathcal{C}}$, i.e.the divisor $D_{\mathcal{I}}$ such that $\mathcal{I}\mathcal{O}_{X_{\mathcal{C}}} = \mathcal{O}_{X_{\mathcal{C}}}(-D_{\mathcal{I}})$.

If $\sigma = \sigma_{\mathcal{C}} : X_{\mathcal{C}} \to X$ is the composition of the blowing-ups, then the completion $\overline{\mathcal{I}}$ of $\mathcal{I}$ is nothing but the stalk at Q_0 of $\sigma_*(\mathcal{O}_{X_{\mathcal{C}}}(-D_{\mathcal{I}}))$. Therefore, if we consider the set of finitely supported *complete* ideals $\mathcal{I}$ with base points contained in $\mathcal{C}$, the map $\mathcal{I} \stackrel{\alpha}{\longmapsto} D_{\mathcal{I}}$ from this set into the set of exceptional divisors of $X_{\mathcal{C}}$, is injective.

On the other hand, the image of the map α is the set of effective exceptional divisors D in $X_{\mathcal{C}}$, such that $\mathcal{O}_{X_{\mathcal{C}}}(-D)$ is generated by its global sections in a neighborhood of the support of D (such divisors are called σ-generated), i.e. such that the natural morphism $\sigma^*\sigma_*\mathcal{O}_{X_{\mathcal{C}}}(-D) \to \mathcal{O}_{X_{\mathcal{C}}}(-D)$ is surjective.

The map α is actually a monoid isomorphism onto its image, with respect to the $*$-product and the addition of divisors, i.e. $\alpha(\mathcal{I}_1 * \mathcal{I}_2) = \alpha(\mathcal{I}_1) + \alpha(\mathcal{I}_2)$.

Remark 4.11. This map is analyzed in [2] in terms of *clusters*, i.e. weighted constellations, where the weights are the strict multiplicities $\underline{m} = (m_i \mid Q_i \in \mathcal{C})$ defined for a finitely supported (complete) ideal. The clusters corresponding to images of α are called idealistic clusters.

4.12. Let $N_1 = N_1(X_{\mathcal{C}}/X)$ (resp.$N^1 = N^1(X_{\mathcal{C}}/X)$) be the abelian group of exceptional – i.e. whose support contracts to Q_0 – one dimensional cycles on $X_{\mathcal{C}}$ (resp. (Cartier) divisors on $X_{\mathcal{C}}$) modulo numerical equivalence. A one dimensional cycle C (resp. a divisor D) is numerically equivalent to 0 if the intersection number $(C \cdot D) = 0$, for all divisors D (resp. all exceptional complete curves) on $X_{\mathcal{C}}$.

Set $A_1 = A_1(X_{\mathcal{C}}/X) = N_1 \otimes_{\mathbb{Z}} \mathbb{R}$, $A^1 = A^1(X_{\mathcal{C}}/X) = N^1 \otimes_{\mathbb{Z}} \mathbb{R}$. The exceptional fiber $\sigma^{-1}(Q_0)$ of σ is a projective scheme over $\mathbb{K}$ and the vector space A^1 maps injectively into $A^1(\sigma^{-1}(Q_0))$, so the dimension of A^1 is finite and the intersection pairing makes A_1 and A^1 dual vector spaces.

Now let $NE(X_\mathcal{C}/X)$ be the convex cone generated by the effective exceptional curves in A_1. Consider, in the dual space A^1, two cones:

Definition 4.13. Let $P(X_\mathcal{C}/X)$ be the dual cone of $-NE(X_\mathcal{C}/X)$, i.e. the cone formed by the classes d such that $(c\cdot d) \leq 0$ for every class c of exceptional effective curve in $X_\mathcal{C}$. In other words, $P(X_\mathcal{C}/X)$ is minus the *semiample relative cone* for the morphism σ at Q_0 (see [9]).

Let $\tilde{P}(X_\mathcal{C}/X)$ be the convex cone generated by the classes of divisors D such that $\mathcal{O}_{X_\mathcal{C}}(-D)$ is generated by their global sections; this cone is called the *characteristic cone* for σ at Q_0; this terminology has been introduced by H. Hironaka. We'll say that this cone is associated with $\mathcal{C}$.

Remark 4.14. The set of lattice points of the characteristic cone $\tilde{P}(X_\mathcal{C}/X)$ (i.e. its intersection with the lattice N^1) is the monoid of the classes of σ-generated divisors on $X_\mathcal{C}$. This monoid generates the characteristic cone and is canonically isomorphic to the monoid of finitely supported complete ideals in $\mathcal{O}_{X,Q_0}$ with base points contained in the constellation $\mathcal{C}$, via the map α introduced above. (In [2] this monoid is called the *galaxy* of the constellation).

Definition 4.15. Following Lipman [11], define a $*$-simple ideal as an ideal $\mathcal{P}$ in $\mathcal{R}$ if $\mathcal{P} \neq \mathcal{R}$ and if $\mathcal{P}$ does not have a non-trivial $*$-factorization, i.e. if $\mathcal{P} = \mathcal{I}_1 * \mathcal{I}_2$ then either $\mathcal{I}_1 = \mathcal{R}$ or $\mathcal{I}_2 = \mathcal{R}$.

4.16. The central result on unique factorization in [11] may be formulated as follows.

For each point $Q \in \mathcal{C}$ there is a (unique complete) $*$-simple ideal $\mathcal{P}_Q$ whose base points are the points of the chain from Q to the origin Q_O, such that the set of strict multiplicities is minimal for the inverse lexicographic order with respect to the natural ordering ($\geq$) in $\mathcal{C}$, and with $m_Q = 1$. Let's call $\mathcal{P}_Q$ the *special $*$-simple ideal* associated with Q

It follows from this fact, that every finitely supported complete $\mathcal{I}$ with base points contained in $\mathcal{C}$ has a *unique factorization* as a $*$-product of the special $*$-simple ideals $\mathcal{P}_Q$, allowing negative exponents. This means that there is a $*$-product of $\mathcal{I}$ by special $*$-simple ideals, equal to a product of special $*$-simple ideals, with different factors appearing in each side of the equality; i.e. there are unique integers r_Q such that $\mathcal{I} = \overset{*}{\prod}_{Q\in\mathcal{C}} \mathcal{P}_Q^{r_Q}$.

Indeed, it is straightforward to see, from the properties of the strict multiplicities, of the $\mathcal{P}_Q$, that the set $\alpha(\mathcal{P}_Q)$, $Q \in \mathcal{C}$, is a basis of the lattice N^1.

4.17. The above factorisation for any finitely generated complete ideal with base points in $\mathcal{C}$ has non negative exponents if and only if the $\alpha(\mathcal{P}_Q)$, with $Q \in \mathcal{C}$ generate the characteristic cone of $\sigma_{\mathcal{C}}$, and this is equivalent to saying that this cone is *regular*, in the sense of toric varieties, i.e. rational and with primitive integral extremal points forming (a subset of) a basis of the lattice.

The determination of the constellations with *regular* characteristic cone is an interesting open question.

The result of Zariski (see [14], [15] and [12] for a recent presentation) is formulated in this language by saying that in dimension two the characteristic cone is always regular. Furthermore, the $*$-product is just the product of ideals, and the $*$-simple ideals are the simple ideals.

Examples 4.18. There are chain constellations whose characteristic cone is polyhedral but not simplicial, or even non polyhedral, i.e. the simplicial cone is not closed (see for instance [4] example 3, [1] examples 4.1, 4.2, 4.3).

Even for chain constellations in dimension three the characteristic cone may be non regular. One could hope that for this "simple" case life would be easy, since for the toric chains it is always regular (see 4.25), as it follows from the linear proximities.

For example, a chain constellation consisting of six points with the second point Q_1 in a non degenerate conic in the plane B_0 and the following points in the strict transform of the conic, has a characteristic cone (in dimension six) with seven maximal faces and nine edges generated by the six integral points associated to the special $*$-simple ideals and three others.

Another example is given by a regular non inflexion point on a rational plane cubic curve in B_0 and eight following points on the strict transform of the cubic; in this case the monoid of integral points of the characteristic cone in dimension ten is not finitely generated.

4.19. For toric constellations the characteristic cone may be explicitely obtained (see [2], theorem 2.10). Note that in this case the characteristic cone coincides with the semiample cone (see [8], page 47).

The natural ideals to consider are the invariant ideals for the toric action, so that the constellations of base points are toric.

The conditions that such an ideal $\mathcal{I}$ is finitely generated and complete are formulated in terms of the Newton polyhedron $\mathcal{N}$ of $\mathcal{I}$ relative to the local system of parameters of the local ring, induced by a basis of the lattice where the fan lives.

The first condition is that the fan associated to the Newton polyhedron (which gives the normalized blowing-up of center $\mathcal{I}$) admits a regular subdivision obtained by elementary subdivisions of the regular cone Δ corresponding to Q_0; and the second one is that every monomial corresponding to an integral point of $\mathcal{N} + \Delta^\vee$ is in $\mathcal{I}$, where $\Delta^\vee$ denotes the dual cone of Δ.

The following result generalizes, for toric constellations in any dimension, the two dimensional proximity inequalities found by Enriques. Recall Proposition 4.3.

Theorem 4.20. *Let C be a toric constellation of dimension d. The characteristic cone associated with C is the cone generated by the classes of the divisors $D_{\underline{m}} = \sum_{Q \in C} m_Q E_Q^*$ such that $\underline{m}$ verifies the linear proximity inequalities $m_Q \geq \sum_{P \overset{\ell}{\twoheadrightarrow} Q} m_P$ for each $Q \in C$ and each $\ell = \ell(Q(a), Q(b))$, $a \neq b$ $1 \leq a \leq d$, $1 \leq b \leq d$.*

Proof. The linear proximity inequalities are necessary, since they are equivalent to $(D_{\underline{m}} \cdot \bar{\ell}) \leq 0$ for a semiample divisor $-D_{\underline{m}}$ and the closure $\bar{\ell}$ of each one dimensional orbit $\ell(Q(a), Q(b))$. Conversely, if these inequalities hold, then $-D_{\underline{m}}$ is semiample since the classes of the closures of the one dimensional orbits generate the cone of the numerically effective curves NE, and then the divisor is σ-generated because σ is a toric morphism. Q.E.D.

Remark 4.21. A constructive proof giving the Newton polyhedron of the unique complete ideal associated to such a divisor $D_{\underline{m}}$ (or the corresponding *idealistic cluster*) is presented in [2] theorem 2.10 (ii).

Corollary 4.22. *We keep the notations of the theorem. Let $C = \{Q_0, \dots, Q_n\}$ be a toric* chain.

(*a*) *The characteristic cone asociated with C is given by*

$$m_i \geq \sum_{j \twoheadrightarrow i} m_j \ , \ 0 \leq i \leq n.$$

(*b*) *The divisor $D_n = \sum_{0 \leq i \leq n} m_{i,n} E_i^*$ associated to the special $*$-simple ideal $\mathcal{P}_{Q_n}$ is given by $m_{n,n} = 1$, $m_{i,n} = \sum_{j \twoheadrightarrow i} m_{j,n}$, for $0 \leq i \leq n$.*

Proof. (a) follows from the Theorem and the fact that for each point there is only one relevant inequality, since C is a chain.

(b) follows from (a) since the minimality property of $\underline{m}$ is obtained if $m_{n,n} = 1$ and if every inequality involving an index $i \neq n$ becomes an equality. Q.E.D.

The special $*$-simple ideals, and the exponents of the factorizations are determined by the linear proximities:

Theorem 4.23. *Let $\mathcal{C}$ be a toric constellation.*

(a) Let $(D_Q)_{Q\in\mathcal{C}}$, be the basis of N_1 corresponding to the special $*$-simple ideals with base points in $\mathcal{C}$. Then $D_Q = \sum_{P\in\mathcal{C}} m_{PQ} E_P^*$, where $m_{PQ} = 0$ if $P \not\leq Q$, $m_{QQ} = 1$ and $m_{PQ} = \sum_{R\in\mathcal{C} \ \mid \ Q\geq R\twoheadrightarrow P} m_{RQ}$ if $P \leq Q$.

(b) Let $\mathbb{P}_L = ((l_{PQ}))$ be the linear proximity matrix *defined by $l_{PP} = 1$, $l_{PQ} = -1$ if $P \twoheadrightarrow Q$ and 0 otherwise.*
Then ${}^t\mathbb{P}_L$ is the basis change matrix from (E_Q^*) to (D_Q).

(c) Let $\mathcal{I}$ be a toric finitely generated ideal with base points in $\mathcal{C}$.
Then the exponents of its factorisation in terms of special $*$-simple ideals are:
$r_Q = m_Q - \sum_{P\twoheadrightarrow Q} m_P$.

Proof. (a) follows from 4.22, (b).
(b) and (c) follow from (a) and linear algebra. Q.E.D.

Recall the definition of the LP structure of the tree Γ associated with $\mathcal{C}$ (Proposition 4.6).

Corollary 4.24. *Let $P_{\mathcal{C}} = P(X_{\mathcal{C}}/X)$ be the characteristic cone associated with $\mathcal{C}$. The following conditions are equivalent:*
(a) The cone $P_{\mathcal{C}}$ is regular.
(b) $(D_Q)_{Q\in\mathcal{C}}$ is a basis of the semigroup $P_{\mathcal{C}} \bigcap N^1$.
(c) The cone $P_{\mathcal{C}}$ is simplicial.
(d) The special $*$-simple factorizations have only non negative exponents.
(e) For each $Q \in \mathcal{C}$ there is only one (maximal) chain or bichain in $\Gamma(q)$.

Proof. The conditions (a), (b), (c) and (d) are equivalent since the divisors D_Q form a basis of N^1. The equivalence between (e) and (c) follows from the preceding theorem, and the fact that the supporting hyperplanes of the maximal faces of the cone $P_{\mathcal{C}}$ are those associated with the maximal elements of Γ_Q for each $Q \in \mathcal{C}$. Q.E.D.

Remark 4.25. In particular, every toric chain constellation in any dimension has a regular characteristic cone. There are also non-chain constellations with this property.

We conclude with an application of the LP-Enriques diagrams for a converse Zariski theorem for toric constellations. Recall the definition of the minimal LP-dimension, $d_{\mathcal{LP}}$ of a LP-Enriques diagram (Theorem 4.8).

Theorem 4.26. *The characteristic cone of a toric constellation is* regular *if and only if its LP-Enriques diagram is induced by a two dimensional constellation.*

Proof. The characteristic cone of any two dimensional constellation is regular, by Zariski. Conversely, assume that the characteristic cone is regular. Then $\Gamma(q)$ has only one element for each $q \in \Gamma$, by the last Corollary. It follows necessarily that $0 \leq |q^+| \leq 2$. Now, $0 \leq |q^+| \leq 1$ implies that $0 \leq n_q \leq 1$ and $|q^+| = 2$ implies that $n_q = 0$. It follows that the minimal dimension $d_{\mathcal{LP}}$ of a constellation inducing the given LP-Enriques diagram is two. Q.E.D.

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