Minor Summation Formulas of Pfaffians,
Survey and A New Identity

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Abstract.

In this paper we treat the minor summation formulas of pfaffians presented in [IW1] and derive several basic formulas concerning pfaffians from it. We also present a pfaffian version of the Plücker relation and give a new pfaffian identity as its application.

Chapter I. Introduction

In this short note we treat the minor summation formulas of pfaffians presented in [IW1] and derive several basic formulas concerning pfaffians. We also present a pfaffian version of the Plücker relations and give a new pfaffian identity as its application in Chapter III.

The minor summation formula we call here is an identity which involves pfaffians for a weighted sum of minors of a given matrix. The first appearance of this kind of minor sum is when one tries to count the number of the totally symmetric plane partitions (see [O1]). Once we establish the minor summation formula full in general, one gets various applications (see, e.g., [IOW], [KO], [O2]). Indeed, for example, using the minor summation formula we obtained quite a number of generalizations of theLittlewood formulas concerning various generating functions of the Schur polynomials (see [IW2,3,4]).

Though the notion of pfaffians is less familiar than that of determinants it is also known by a square root of the determinant of a skew
symmetric matrix. We recall now a more combinatorial definition of pfaffians. Let \( S_n \) be the symmetric group on the set of the letters \( 1, 2, \ldots, n \) and, for each permutation \( \sigma \in S_n \), let \( \text{sgn} \sigma \) stand for \((-1)^{\ell(\sigma)}\), the sign of \( \sigma \), where \( \ell(\sigma) \) is the number of inversions of \( \sigma \).

Let \( n = 2r \) be even. Let \( H \) be the subgroup of \( S_n \) generated by the elements \( (2i - 1, 2i) \) for \( 1 \leq i \leq r \) and \( (2i - 1, 2i + 1)(2i, 2i + 2) \) for \( 1 \leq i < r \). We set a subset \( \mathcal{F}_n \) of \( S_n \) to be

\[
\mathcal{F}_n = \left\{ \sigma = (\sigma_1, \ldots, \sigma_n) \in S_n \mid \begin{array}{l}
\sigma_{2i-1} < \sigma_{2i} \quad (1 \leq i \leq r) \\
\sigma_{2i-1} < \sigma_{2i+1} \quad (1 \leq i \leq r - 1)
\end{array} \right\}.
\]

An element of \( \mathcal{F}_n \) is called a perfect matching or a 1-factor. For each \( \pi \in S_n \), \( H \pi \cap \mathcal{F}_n \) has a unique element \( \sigma \). Let \( B = (b_{ij})_{1 \leq i, j \leq n} \) be an \( n \) by \( n \) skew-symmetric matrix with entries \( b_{ij} \) in a commutative ring. The pfaffian of \( B \) is then defined as follows:

\[
\text{pf}(B) = \sum_{\sigma \in \mathcal{F}_n} \text{sgn} \sigma \, b_{\sigma(1)} b_{\sigma(2)} \ldots b_{\sigma(n-1)} b_{\sigma(n)}.
\]  

(1.1)

Chapter II. Pfaffian Identities

Let us denote by \( \mathbb{N} \) the set of nonnegative integers, and by \( \mathbb{Z} \) the set of integers. Let \([n]\) denote the subset \( \{1, 2, \ldots, n\} \) of \( \mathbb{N} \) for a positive integer \( n \).

Let \( n, M \) and \( N \) be positive integers such that \( n \leq M, N \) and let \( T \) be any \( M \) by \( N \) matrix. For \( n \)-element subsets \( I = \{i_1 < \cdots < i_n\} \subseteq [M] \) and \( J = \{j_1 < \cdots < j_n\} \subseteq [N] \) of row and column indices, let \( T^I_J = T^i_{j_1 \cdots j_n} \) denote the sub-matrix of \( T \) obtained by picking up the rows and columns indexed by \( I \) and \( J \). In the case that \( n = M \) and \( I \) contains all row indices, we omit \( I = [M] \) from the above expression and simply write \( T^I_J = T^I_J \). Similarly we write \( T^I_J \) for \( T^I_J \) if \( n = N \) and \( J = [N] \).

Let \( B \) be an arbitrary \( N \) by \( N \) skew symmetric matrix; that is, \( B = (b_{ij}) \) satisfies \( b_{ij} = -b_{ji} \). In Theorem 1 of the paper [IW1], we obtained a formula concerning a certain summation of minors which we call the minor summation formula of pfaffians:

**Theorem 2.1.** Let \( n \leq N \) and assume \( n \) is even. Let \( T = (t_{ij})_{1 \leq i \leq n, 1 \leq j \leq N} \) be any \( n \) by \( N \) matrix, and let \( B = (b_{ij})_{1 \leq i, j \leq N} \) be any \( N \) by \( N \) skew symmetric matrix. Then

\[
\sum_{I \subseteq [N], \#I = n} \text{pf}(B^I_J) \det(T^I_I) = \text{pf}(Q),
\]  

(2.1)
where $Q$ is the $n$ by $n$ skew-symmetric matrix defined by $Q = TB^tT$, i.e.

$$Q_{ij} = \sum_{1 \leq k < l \leq N} b_{kl} \det(T_{kl}^{ij}), \quad (1 \leq i, j \leq n). \tag{2.2}$$

We note that another proof of this minor summation formula and some other extensions using the so-called lattice path methods will be given in the forthcoming paper [IW5].

We now add on one useful formula which relates to the skew symmetric part of a general square matrix. Actually the following type of pfaffians may arise naturally when we consider the imaginary part of a Hermitian form.

**Corollary 2.1.** Fix positive integers $m, n$ such that $m \leq 2n$. Let $A$ and $B$ be arbitrary $n \times m$ matrices, and $X$ be an $n \times n$ symmetric matrix. (i.e. $^tX = X$). Let $P$ be the skew symmetric matrix defined by $P = ^tAXB - ^tBXA$. Then we have

$$\text{pf}(P) = \sum_{K \subseteq [2n]} \text{pf} \left( \begin{pmatrix} O_n & X \\ -X & O_n \end{pmatrix}^K \right) \det \left( \begin{pmatrix} A \\ B \end{pmatrix}^K \right).$$

In particular, when $m = 2n$ we have

$$\text{pf}(P) = \det(X) \det \left( \begin{pmatrix} A \\ B \end{pmatrix} \right).$$

**Proof.** Apply the above theorem to the $2n \times 2n$ skew symmetric matrix $\begin{pmatrix} O_n & X \\ -X & O_n \end{pmatrix}$ and the $2n \times m$ matrix $\begin{pmatrix} A \\ B \end{pmatrix}$. Then the elementary identity

$$^t \begin{pmatrix} A \\ B \end{pmatrix} \begin{pmatrix} O_n & X \\ -X & O_n \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = ^tAXB - ^tBXA$$

immediately asserts the corollary.

As a corollary of the theorem above we have the following expansion formula (cf. [Ste], [IW1]):

**Corollary 2.2.** Let $A$ and $B$ be $m$ by $m$ skew symmetric matrices. Put $n = \lfloor \frac{m}{2} \rfloor$, the integer part of $\frac{m}{2}$. Then

$$\text{pf}(A + B) = \sum_{r=0}^{n} \sum_{\frac{m}{2}}^{n} (-1)^{|I|-r} \text{pf}(A_I^T) \text{pf}(B_I^T), \quad (2.3)$$
where we denote by $\bar{I}$ the complement of $I$ in $[m]$ and $|I|$ is the sum of the elements of $I$ (i.e. $|I| = \sum_{i \in I} i$).

In particular, we have the expansion formula of pfaffian with respect to any column (row): For any $i, j$ we have

$$
\delta_{ij} \text{pf}(A) = \sum_{k=1}^{m} a_{ki} \gamma(k, j),
$$

(2.4)

$$
\delta_{ij} \text{pf}(A) = \sum_{k=1}^{m} a_{ik} \gamma(j, k),
$$

(2.5)

where

$$
\gamma(i, j) = \begin{cases} 
(-1)^{i+j-1} \text{pf}(A_{ij}) & \text{if } i < j, \\
0 & \text{if } i = j, \\
(-1)^{i+j} \text{pf}(A_{ij}) & \text{if } j < i.
\end{cases}
$$

(2.6)

and $A_{ij}$ stands for the $(m-2)$ by $(m-2)$ skew symmetric matrix which is obtained from $A$ by removing both the $i, j$-th rows and $i, j$-th columns for $1 \leq i \neq j \leq m$.

We close this chapter by noting the fact that one may give a proof of the fundamental relation; $\text{pf}(A)^2 = \det(A)$, for a skew symmetric matrix $A$ without any use of a process of the "diagonalization" by employing the expansion formula above and the Lewis-Carroll formula for determinants discussing below.

\section*{Chapter III. The Lewis-Carroll formula, etc.}

In this chapter we provide a Pfaffian version of Lewis-Carroll’s formula and Plücker’s relations. The latter relations are also treated in [DW], and in [Kn] it is called the (generalized) basic identity. First of all we recall the so-called Lewis-Carroll formula, or known as the Jacobi formula among minor determinants.

**Proposition 3.1.** Let $A$ be an $n$ by $n$ matrix and $\bar{A}$ be the matrix of its cofactors. Let $r \leq n$ and $I, J \subseteq [n]$, $\#I = \#J = r$. Then

$$
\det \bar{A}_I^J = (-1)^{r(|I|+|J|)} (\det A)^{r-1} \det A_{\bar{I}}^{\bar{J}},
$$

(3.1)

where $\bar{I}, \bar{J} \subseteq [n]$ stand for the complementary of $I, J$, respectively.
Example 1. We give here a few examples of Lewis-Carroll’s formula for matrices of small degree.

\[
\begin{vmatrix}
  a_{11} & a_{13} \\
  a_{31} & a_{33}
\end{vmatrix}
\begin{vmatrix}
  a_{11} & a_{12} \\
  a_{31} & a_{32}
\end{vmatrix}
= \begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix}.
\]

(3.2)

We give one more;

\[
\begin{vmatrix}
  a_{11} & a_{14} \\
  a_{21} & a_{24}
\end{vmatrix}
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{31} & a_{32} & a_{33} \\
  a_{41} & a_{42} & a_{43}
\end{vmatrix}
= \begin{vmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & a_{22} & a_{23} & a_{24} \\
  a_{31} & a_{32} & a_{33} & a_{34} \\
  a_{41} & a_{42} & a_{43} & a_{44}
\end{vmatrix}.
\]

(3.3)

Hereafter we write \( A_I \) for \( A^T_I \) for short. We hope that it doesn’t cause the reader any confusion since we only treat square matrices. Let \( m \) be an even integer and \( A \) be an \( m \) by \( m \) skew symmetric matrix. Assume that pf(A) is nonzero, that is, \( A \) is non-singular.

Let \( \Delta(i, j) = (-1)^{i+j} \det A^{ij} \) denote the \((i, j)\)-cofactor of \( A \). If we multiply the both sides of (2.6) by pf(A) and use the fundamental relation between determinants and pfaffians: \( \det A = [\text{pf}(A)]^2 \), we obtain

\[
\sum_{i=1}^{m} a_{ij} \gamma(i, k) \text{pf}(A) = \delta_{jk} [\text{pf}(A)]^2 = \delta_{jk} \det A.
\]

(3.4)

Comparing this with the cofactor expansion of \( \det A \), we obtain the following relation between \( \Delta(i, j) \) and \( \gamma(i, j) \):

\[
\Delta(i, j) = \gamma(i, j) \text{pf}(A).
\]

(3.5)

The following relation is considered as a pfaffian version of the Lewis-Carroll formula.

**Theorem 3.1.** Let \( m \) be an even integer and \( A \) be an \( m \) by \( m \) skew symmetric matrix. Let \( \hat{A} = (\gamma(j, i)) \). Then, for any \( I \subseteq [m] \) such that \( \# I = 2r \), we have

\[
\text{pf} \left[ (\hat{A})_I \right] = (-1)^{|I|} [\text{pf}(A)]^{r-1} \text{pf}(A_I).
\]

(3.6)
Example 2. Taking \( m = 6 \), \( t = 1 \) and \( I = \{1, 2, 3, 4\} \) in the above theorem, we see

\[
\gamma(1, 2)\gamma(3, 4) - \gamma(1, 3)\gamma(2, 4) + \gamma(1, 4)\gamma(2, 3) = \text{pf}(A)\text{pf}(A_{\{5, 6\}}).
\]

Hence by definition, we see that this turns out to be

\[
\text{pf}(A_{\{3,4,5,6\}})\text{pf}(A_{\{1,2,5,6\}}) - \text{pf}(A_{\{2,4,5,6\}})\text{pf}(A_{\{1,3,5,6\}}) + \text{pf}(A_{\{2,3,5,6\}})\text{pf}(A_{\{1,4,5,6\}}) = \text{pf}(A)\text{pf}(A_{\{5,6\}}),
\]

that is, in more familiar form we see

\[
\begin{align*}
\text{pf} \begin{pmatrix} 0 & a_{34} & a_{35} & a_{36} \\ -a_{34} & 0 & a_{45} & a_{46} \\ -a_{35} & -a_{45} & 0 & a_{56} \\ -a_{36} & -a_{46} & -a_{56} & 0 \\ \end{pmatrix} & \text{pf} \begin{pmatrix} 0 & a_{12} & a_{15} & a_{16} \\ -a_{12} & 0 & a_{25} & a_{26} \\ -a_{15} & -a_{25} & 0 & a_{56} \\ -a_{16} & -a_{26} & -a_{56} & 0 \\ \end{pmatrix} \\
-\text{pf} \begin{pmatrix} 0 & a_{24} & a_{25} & a_{26} \\ -a_{24} & 0 & a_{45} & a_{46} \\ -a_{25} & -a_{45} & 0 & a_{56} \\ -a_{26} & -a_{46} & -a_{56} & 0 \\ \end{pmatrix} & \text{pf} \begin{pmatrix} 0 & a_{13} & a_{15} & a_{16} \\ -a_{13} & 0 & a_{35} & a_{36} \\ -a_{15} & -a_{35} & 0 & a_{56} \\ -a_{16} & -a_{36} & -a_{56} & 0 \\ \end{pmatrix} \\
+\text{pf} \begin{pmatrix} 0 & a_{23} & a_{25} & a_{26} \\ -a_{23} & 0 & a_{35} & a_{36} \\ -a_{25} & -a_{35} & 0 & a_{56} \\ -a_{26} & -a_{36} & -a_{56} & 0 \\ \end{pmatrix} & \text{pf} \begin{pmatrix} 0 & a_{14} & a_{15} & a_{16} \\ -a_{14} & 0 & a_{45} & a_{46} \\ -a_{15} & -a_{45} & 0 & a_{56} \\ -a_{16} & -a_{46} & -a_{56} & 0 \\ \end{pmatrix} \\
= \text{pf} \begin{pmatrix} 0 & a_{56} \\ -a_{56} & 0 \\ \end{pmatrix} & \text{pf} \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} & a_{26} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} & a_{36} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} & a_{46} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 & a_{56} \\ -a_{16} & -a_{26} & -a_{36} & -a_{46} & -a_{56} & 0 \\ \end{pmatrix}.
\end{align*}
\]

We next state a pfaffian version of the Plücker relations (or known as the Grassmann-Plücker relations) for determinants which is a quadratic relations among several subpfaffians. This identity is also proved in the book [Hi] and a recent paper [DW] in the framework of an exterior algebra.

Theorem 3.2. Suppose \( m, n \) are odd integers. Let \( A \) be an \((m + n) \times (m + n)\) skew symmetric matrices of odd degrees. Fix a sequence of integers \( I = \{i_1 < i_2 < \cdots < i_m\} \subseteq [m + n] \) such that \( \#I = m \). Denote the complement of \( I \) by \( \bar{I} = \{k_1, k_2, \ldots, k_n\} \subseteq [m + n] \) which has the
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cardinality \( n \). Then the following relation holds.

\[
\sum_{j=1}^{m} (-1)^{j-1} \text{pf}(A_{I \setminus \{i_j\}}) \text{pf}(A_{(i_j) \cup \{j\}}) = \sum_{j=1}^{n} (-1)^{j-1} \text{pf}(A_{I \cup \{k_j\}}) \text{pf}(A_{I \setminus k_j}).
\]

(3.8)

The following assertion, which is called by the basic identity in \([Kn]\) is a special consequence of the formula above.

**Corollary 3.1.** Let \( A \) be a skew symmetric matrix of degree \( N \). Fix a subset \( I = \{i_1, i_2, \ldots, i_{2k}\} \subseteq [N] \) such that \( |I| = 2k \). Take an integer \( l \) which satisfies \( 2k + 2l \leq N \). Then

\[
\text{pf}(A_{1,2,\ldots,2l}) \text{pf}(A_{i_1, i_2, \ldots, i_{2k}, 1, \ldots, 2l}) = \sum_{j=1}^{2k-1} (-1)^{j-1} \text{pf}(A_{i_1,1,2,\ldots,2l,i_{j+1}}) \text{pf}(A_{i_2, \ldots, i_{j+1}, \ldots, i_{2k}, 1, \ldots, 2l}).
\]

(3.9)

The theorem stated below is proved by induction using this basic identity. Its proof will be given in the forthcoming paper \([IW5]\).

**Theorem 3.3.**

\[
\text{pf} \left\{ \frac{y_i - y_j}{a + b(x_i + x_j) + cx_i x_j} \right\} \prod_{1 \leq i < j \leq 2n} \{a + b(x_i + x_j) + cx_i x_j\}
\]

\[
= (ac - b^2)^{\frac{n(n-1)}{2}} \sum_{I \subseteq [2n], \#I = n} (-1)^{n-\frac{n(n+1)}{2}} y_I \Delta_I(x) \Delta_{\bar{I}}(x) J_I(x) J_{\bar{I}}(x),
\]

where the sum runs over all \( n \)-element subset \( I = \{i_1 < \cdots < i_n\} \) of \([2n]\) and \( \bar{I} = \{j_1 < \cdots < j_n\} \) is the complementary subset of \( I \) in \([2n]\).

Further we write

\[
\Delta_I(x) = \prod_{i,j \in I, i < j} (x_i - x_j),
\]

\[
J_I(x) = \prod_{i,j \in I, i < j} \{a + b(x_i + x_j) + cx_i x_j\},
\]

\[
y_I = \prod_{i \in I} y_i.
\]
As a corollary of this theorem we obtain the following identity in [Su2]. Indeed, if we put $a = c = 1, b = 0$ in the theorem, then we have the

**Corollary 3.2.**

$$
\text{pf} \left( \frac{y_i - y_j}{1 + x_i x_j} \right)_{1\leq i, j \leq 2n} \times \prod_{1 \leq i < j \leq 2n} (1 + x_i x_j) = \sum_{\lambda, \mu} a_{\lambda + \delta_n, \mu + \delta_n} (x, y),
$$

where the sums runs over pairs of partitions

$$
\lambda = (\alpha_1 - 1, \ldots, \alpha_p - 1 | \alpha_1, \ldots, \alpha_p), \mu = (\beta_1 - 1, \ldots, \beta_p - 1 | \beta_1, \ldots, \beta_p)
$$

in Frobenius notation with $\alpha_1, \beta_1 < n - 1$. Also, for $\alpha$ and $\beta$ partitions (compositions, in general) of length $n$, we put

$$
a_{\alpha, \beta}(x, y) = \sum_{\sigma \in S_{2n}} \epsilon(\sigma) \sigma(x_1^{\alpha_1} y_1 \cdots x_n^{\alpha_n} y_n \cdots x_{2n}^{\beta_n}),
$$

where $\sigma \in S_{2n}$ acts on each of two sets of variables $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ by permuting indices, and $\delta_n = (n-1, n-2, \ldots, 0)$.

**Proof.** Recall that

$$
\sum_{\lambda=(\alpha_1-1,\ldots,\alpha_p-1 | \alpha_1,\ldots,\alpha_p)} s_\lambda(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (1 + x_i x_j), \quad (3.10)
$$

where $s_\lambda = s_\lambda(x_1, \ldots, x_n) = a_{\lambda + \delta_n}/a_{\delta_n}$ and $a_\alpha = \det(x_i^{\alpha_j})_{1 \leq i, j \leq n}$ for a composition $\alpha$. We write $a_\alpha(I) = a_\alpha(x_{i_1}, \ldots, x_{i_n})$ for $I = \{i_1 < \cdots < i_n\} \subseteq [2n]$. By the theorem and (3.10) we see

$$
\text{pf} \left( \frac{y_i - y_j}{1 + x_i x_j} \right)_{1 \leq i, j \leq 2n} \times \prod_{1 \leq i < j \leq 2n} (1 + x_i x_j)
$$

$$
= \sum_{I \subseteq [2n]} \sum_{\lambda, \mu} (-1)^{|I| - \frac{n(n+1)}{2}} y_I \prod a_{\lambda + \delta_n}(I) a_{\mu + \delta_n}(\bar{I})
$$

$$
= \sum_{\lambda, \mu} \sum_{i_1 < \cdots < i_n} \sum_{\sigma, \tau \in S_n} (-1)^{|I| - \frac{n(n+1)}{2}} \epsilon(\sigma) \epsilon(\tau)
$$

$$
\times \sigma(x_1^{\lambda_1+n-1} y_{i_1} \cdots x_n^{\lambda_n} y_{i_n}) \tau(x_1^{\mu_1+n-1} \cdots x_n^{\mu_n}),
$$
where $\bar{I} = \{j_1, \ldots, j_n\}$. Thus, the last sum is turned to be
\[
\sum_{\lambda, \mu} \sum_{\sigma, \tau \in S_n} \epsilon(\sigma) \sigma(x_1^{\lambda_1+n-1} y_1 \ldots x_n^{\lambda_n} y_n x_{n+1}^{\mu_1+n-1} \ldots x_{2mn})
\]
\[
= \sum_{\lambda, \mu} a_{\lambda+\delta_n, \mu+\delta_n}(x, y).
\]
This completes the proof of the corollary.

References


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