Recent progress of intersection theory  
for twisted (co)homology groups  

Keiji Matsumoto and Masaaki Yoshida

§1. Introduction

Maybe you have ever seen at least one of the following formulae:

\[
B(p, q)B(-p, -q) = \frac{2\pi i(p + q)}{pq} \frac{1 - e^{2\pi i(p+q)}}{(1 - e^{2\pi ip})(1 - e^{2\pi iq})},
\]

\[
\Gamma(p)\Gamma(1 - p) = \frac{\pi}{\sin \pi p}, \quad \left(\int_{-\infty}^{\infty} e^{-t^2/2} dt\right)^2 = 2\pi,
\]

where

\[
B(p, q) := \int_{0}^{1} t^p(1 - t)^q \frac{dt}{t(1 - t)}, \quad \Gamma(p) := \int_{0}^{\infty} t^p e^{-t} \frac{dt}{t}
\]

are the Gamma and the Beta functions.

In this paper, we give a geometric meaning for these formulae: If one regards such an integral as the dual pairing between a (kind of) cycle and a (kind of) differential form, then the value given in the right hand side of each formula is the product of the intersection numbers of the two cycles and that of the two forms appeared in the left-hand side.

Of course the intersection theory is not made only to explain these well known formulae; for applications, see [CM], [KM], [Y1].

§2. Twisted (co)homology groups

Let \(l_1, \ldots, l_{n+1}\) be polynomials of degree 1 in \(t_1, \ldots, t_k\), \((n \geq k \geq 1)\) and \(\alpha_1, \ldots, \alpha_{n+1}\) be complex numbers satisfying

**Assumption 1.** \(\alpha_j \notin \mathbb{Z}, \quad \alpha_0 := -\alpha_1 - \cdots - \alpha_{n+1} \notin \mathbb{Z}.\)
Put

\[ L_j = \text{hyperplane defined by } l_j, \quad j = 1, \ldots, n + 1, \]

\[ T = \mathbb{C}^k - \bigcup_{j=1}^{n+1} L_j \]

\[ = \mathbb{P}^k - \bigcup_{j=0}^{n+1} L_j, \quad L_0 : \text{hyperplane at infinity}, \]

\[ u = \prod_{j=1}^{n+1} l_j^{\alpha_j} : \text{multi-valued function on } T, \]

\( \mathcal{L}, \tilde{\mathcal{L}} : \text{local systems caused by } u^{-1} \text{ and } u, \text{ respectively,} \)

\[ \omega = \sum_{j=1}^{n} \alpha_j \frac{d l_j}{l_j} : \text{single-valued 1-form on } T, \]

\[ \nabla = d + \omega \wedge, \quad \bar{\nabla} = d - \omega \wedge : \text{derivations}. \]

**Assumption 2.** No \( k + 1 \) hyperplanes in \( \{L_j\}_{j=0}^{n+1} \) intersect in \( \mathbb{P}^k \).

Denoting the \( k \)-dimensional cohomology groups (with compact support) and the (locally finite) homology groups by the usual symbols, we have the three natural dual pairings (explained below):

\[
\begin{align*}
H_c^k(T, \mathcal{L}) & \leftrightarrow H^k(T, \tilde{\mathcal{L}}) \\
H^{1f}_k(T, \tilde{\mathcal{L}}) & \downarrow \\
H_k(T, \mathcal{L}) & \leftrightarrow H_k(T, \tilde{\mathcal{L}}).
\end{align*}
\]

All other dimensional (co)homology groups vanish. By de Rham’s theorem, cohomology classes can be represented by smooth global forms:

\[ H_c^k(T, \mathcal{L}) \cong H^k(\mathcal{E}^\bullet, \nabla), \quad H^k(T, \tilde{\mathcal{L}}) \cong H^k(\mathcal{E}^\bullet, \bar{\nabla}), \]

where \( \mathcal{E}^p \) and \( \mathcal{E}^p_\bullet \) are spaces of smooth \( p \)-forms on \( T \) and those with compact support. Through these isomorphisms, the columns in the above diagram can be realized by the integration

\[ \langle \varphi, \delta \rangle := \int_\delta \varphi u, \quad \text{or} \quad \langle \psi, \gamma \rangle := \int_\gamma \psi u^{-1} \]

of \( k \)-forms along \( k \)-cycles, where

\[ \varphi \in H_c^k(T, \mathcal{L}), \quad \delta \in H_c^{1f}(T, \tilde{\mathcal{L}}), \quad \text{or} \quad \psi \in H^k(T, \tilde{\mathcal{L}}), \quad \gamma \in H_k(T, \mathcal{L}), \]

respectively. Such an integration is often called a hypergeometric integral (HG integral for short) because if one let the hyperplanes \( L_j \) move then the integral defines a hypergeometric function of type \((k + 1, n + 2)\). When \( k = 1, n = 2 \) this is indeed the Gauss hypergeometric function.
The first row is the *intersection form for cohomology groups*, and can be represented by the integral
\[
\varphi \cdot \psi := \int_T (\varphi u) \wedge (\psi u^{-1}) = \int_T \varphi \wedge \psi
\]
of 2k-forms over T, where \( \varphi \in H^k(\mathcal{E}^\bullet, \nabla), \psi \in H^k(\mathcal{E}^\bullet, \hat{\nabla}). \) (N.B. In [KY1], \( \psi \wedge \varphi \) is used in place of \( \varphi \wedge \psi. \))

Now these three pairings induce the Poincaré isomorphisms:
\[
H^k_c(T, \mathcal{L}) \cong H_k(T, \mathcal{L}), \quad H^k(T, \check{\mathcal{L}}) \cong H^k_{lf}(T, \check{\mathcal{L}}).
\]

Thus through these two isomorphisms the intersection form for cohomology groups induces the dual pairing, called the *intersection form for homology groups*, of the two homology groups. In this way we have the four compatible pairings:

\[
\begin{array}{ccc}
H^k_c(T, \mathcal{L}) & \leftrightarrow & H^k(T, \check{\mathcal{L}}) \\
\downarrow & & \downarrow \\
H^k_{lf}(T, \check{\mathcal{L}}) & \leftrightarrow & H_k(T, \mathcal{L})
\end{array}
\]

intersections form · for coh. intersecions form · for hom.

HG integral \quad HG integral

Let us take bases as
\[
\varphi^i \in H^k_c(T, \mathcal{L}), \quad \psi^i \in H^k(T, \check{\mathcal{L}}), \\
\delta_i \in H^k_{lf}(T, \check{\mathcal{L}}), \quad \gamma_i \in H_k(T, \mathcal{L}).
\]

Denoting the matrix \( ((\varphi^i, \delta_j))_{ij} \) by \( ((\varphi, \delta)) \) and \( (\delta_i \cdot \gamma_j)_{i,j} \) by \( (\delta \cdot \gamma) \), we have
\[
(\varphi \cdot \psi) = ((\varphi, \delta))(\gamma \cdot \delta)^{-1} t((\psi, \gamma)),
\]
which gives quadratic relations among the HG integrals.

Note that up to now we presented abstract nonsense which is valid for any complex manifold and for any local system. Our task is, for the special \( T \) and \( \mathcal{L} \) given above, to pick a suitable basis of each (co)homology group and evaluate the intersection numbers.

§3. Intersection form for cohomology groups

To pick an explicit basis of the cohomology groups, holomorphic forms or possibly algebraic forms are better. Recall the isomorphisms, due to comparison theorems,
\[
H^k(T, \mathcal{L}) \cong H^k(\mathcal{E}^\bullet, \nabla) \cong H^k(\Omega^\bullet, \nabla) \cong H^k(\Omega^\bullet(*L), \nabla) \cong H^k(\Omega^\bullet(\log L), \nabla),
\]
where $\Omega^p$, $\Omega^p(\ast L)$ and $\Omega^p(\log L)$ are spaces of holomorphic forms on $T$, algebraic forms and logarithmic forms with poles only along $\bigcup_{j=0}^{n+1} L_j$, respectively.

For a multi-index $I = (i_0, \ldots, i_k)$, $0 \leq i_0 < \cdots < i_k \leq n+1$, we define a logarithmic $k$-form

$$\varphi_I = \frac{dl_{i_0}}{l_{i_1}} \wedge \cdots \wedge \frac{dl_{i_k-1}}{l_{i_k}}.$$  

For example, the following $\binom{n}{k}$ forms give a basis of $H^k(\Omega^\ast(\log L), \nabla)$:

$$\varphi_I, \quad i_0 = 0 < i_1 < \cdots < i_k \leq n.$$  

It is known (e.g. [DM]) and easy to prove, under Assumption 1, the isomorphism

$$H^k_c(T, \mathcal{L}) \cong H^k(T, \mathcal{L}).$$

Thus together with the isomorphism $H^k(T, \mathcal{L}) \cong H^k(\Omega^\ast(\log L), \nabla)$ above, we can let $\varphi_I$ represent also an element of $H^k_c(T, \mathcal{L})$. We wish to evaluate the intersection numbers of these forms. The key point is to represent the isomorphism

$$\iota : H^k(\Omega^\ast(\log L), \nabla) \xrightarrow{\cong} H^k(\mathcal{L}^\ast, \nabla) \quad (\cong H^k_c(T, \mathcal{L})),$$

explicit enough so that the $2k$-dimensional integral

$$\int \iota(\varphi_I) \wedge \varphi_J$$

is computable. This can be done and we get

**Theorem 1.** The intersection number $\varphi_I \cdot \varphi_J$ of

$$\varphi_I \in H^k_c(T, \mathcal{L}) \text{ and } \varphi_J \in H^k(T, \mathcal{L}),$$

where $I = \{i_0, \ldots, i_k\}$, $0 \leq i_0 < \cdots < i_k \leq n+1$, $J = \{j_0, \ldots, j_k\}$, $0 \leq j_0 < \cdots < j_k \leq n+1$, is equal to the $(I, J)$-minor of the tri-diagonal symmetric matrix

$$\text{Int}_{\text{coh}}(\alpha) = 2\pi \sqrt{-1} \begin{pmatrix} 1/\alpha_0 + 1/\alpha_1 & 1/\alpha_1 & 0 & \cdots \\ 1/\alpha_1 & 1/\alpha_1 + 1/\alpha_2 & 1/\alpha_2 & \cdots \\ 0 & 1/\alpha_2 & 1/\alpha_2 + 1/\alpha_3 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$
Actual value of $\varphi_I \cdot \varphi_J$ is given as follows:

$$(2\pi\sqrt{-1})^k \frac{\sum_{i \in I} \alpha_i}{\prod_{i \in I} \alpha_i} \quad \text{if} \quad I = J,$$

$$(2\pi\sqrt{-1})^k \frac{(-1)^{\mu+\nu}}{\prod_{i \in I \cap J} \alpha_i} \quad \text{if} \quad \#(I \cap J) = k,$$

0, otherwise, where $\mu$ and $\nu$ are determined by $\{i_\mu\} = I - J$ and $\{j_\nu\} = J - I$.

Though there are technical difficulties for general $k$, the essential idea of the proof can be seen from that of the case $k = 1$. So we prove this theorem only when $k = 1$, and when $k \geq 2$ we describe where the difficulty lies and how we can manage.

**3.1. Proof of Theorem 1 when $k = 1$.** We express the image $\iota(\varphi_I)$ explicitly. We find a smooth function $f$ on $T$ such that $\varphi_I - \nabla f$ is compactly supported. This means that $\varphi_I - \nabla f$ represents the class $\iota(\varphi_I)$ of $H^1_c(T, \mathcal{L})$.

We can find a convergent power series $f_p$ centered at the point $L_p$ satisfying $\nabla f_p = \varphi_I$. Let $h_p$ be a smooth real function on $\mathbb{P}^1$ such that $h_p(t) = 0$ ($t \notin U_p$), $0 < h_p(t) < 1$ ($t \in U_p \setminus V_p$), $h_p(t) = 1$ ($t \in V_p$), where $L_p \in V_p \subset U_p$, and $U_p$ is a small neighborhood of $L_p$. Regarding $f := \sum_{p=0}^{n+1} h_p f_p$ as defined on $T$, we have

$$\varphi_I - \nabla f = \varphi_I - \sum_{p=0}^{n+1} [h_p \nabla(f_p) + f_p dh_p] = \sum_{p=0}^{n+1} [(1 - h_p)\varphi_I - f_p dh_p],$$

which is of compact support on $T$. The Stokes theorem and the residue theorem yields

$$\int_T \iota(\varphi_I) \wedge \varphi_J = \sum_{p=0}^{n+1} \int_T [(1 - h_p)\varphi_I - f_p dh_p] \wedge \varphi_J$$

$$= \sum_{p=0}^{n+1} \int_{U_p \setminus V_p} -f_p dh_p \wedge \varphi_J = \sum_{p=0}^{n+1} \int_{\partial(U_p \setminus V_p)} h_p f_p \varphi_J$$

$$= \sum_{p=0}^{n+1} \int_{\partial V_p} f_p \varphi_J = 2\pi\sqrt{-1} \sum_{p=0}^{n+1} \text{Res}_{L_p}(f_p \varphi_J).$$

Completion of the proof is now immediate.
3.2. Strategy for \( k \geq 2 \). We prepare some notation. Let \( L_{P^q} \) be the intersection of \( L_{P_1}, L_{P_2}, \ldots, L_{P_q} \), and let \( U_{P^q} \) be a small tubular neighborhood of \( L_{P^q} \) in \( \mathbb{P}^k \), where \( P^q \) is a multi-index with cardinality \( q \), say,

\[
P^q = \{ p_1, p_2, \ldots, p_q \}, \quad 0 \leq p_1 < p_2 < \cdots < p_q \leq n + 1.
\]

For multi-indices \( P^{q-1} \) and \( P^q \), if \( P^{q-1} \subset P^q \), then we put

\[
\delta(P^{q-1}; P^q) = (-1)^r, \quad \text{where } \{ p_r \} := P^q \setminus P^{q-1}.
\]

Step 1. Construct a system of holomorphic \((k-q)\)-forms \( f_{P^q} \) on \( U_{P^q} \cap T \) such that

\[
\nabla(f_{P^1}) = \varphi_I, \quad \nabla(f_{P^q}) = \sum_{P^{q-1} \subset P^q} \delta(P^{q-1}; P^q)f_{P^{q-1}} \quad (2 \leq q \leq k);
\]

these can be obtained as convergent power series. Complexity lies on the fact that the singularities \( \cup L_{P=0} \) are not isolated.

Step 2. By patching \( f_{P^q} \) inductively by the help of partition of the unity on \( \cup_{j=0}^{n+1} U_j \), we get a smooth \((k-1)\)-form \( f \) on \( T \) such that

\[
\nabla f = \varphi_I \quad \text{in } \cup_{j=0}^{n+1} U_j.
\]

Since \( \varphi_I - \nabla f \) is of compact support on \( T \) and is cohomologous to \( \varphi_I \) in \( H^k(\mathcal{E}, \nabla) \), it represents \( \iota(\varphi_I) \).

Step 3. Repeated use of the Stokes theorem and the residue theorem leads to

\[
\int_T \iota(\varphi_I) \wedge \varphi_J = \int_T -df \wedge \varphi_J = (2\pi\sqrt{-1})^k \sum_{P^k} \text{Res}_{L_{P^k}}(f_{P^k} \varphi_J),
\]

which will imply the theorem.

§4. Intersection form for homology groups

Since we assumed that our hyperplane arrangement is in general position (Assumption 2), we can continuously deform the arrangement, keeping its intersection pattern, into a real arrangement, by which we mean all the linear forms \( l_j \) are defined over the real numbers. So we assume that our arrangement is real.
Note that there are many arrangements not in general position that one can not deform into a real one.

Let $T_{\mathbb{R}}$ be the real locus of $T$. $\binom{n}{k}$ bounded chambers support cycles forming a basis of $H^T_k(T, \mathcal{L})$. One can load any branch of $u$ on the chambers; too much freedom annoys us. In order to make it in a systematic way, we further deform the arrangement and put the hypersurfaces in a specially nice way. Then the $k$-dimensional cases can be reduced to the simplest case $k = 1$.

**Loaded cycles:** We represent elements of $H_p(T, \mathcal{L})$ by *loaded $p$-cycles*, which is convenient here and will be indispensable in §8.2. A loaded $p$-chain is a formal sum of loaded $p$-simplexes. A loaded $p$-simplex is a topological simplex on which a branch of $u$ is assigned. The boundary operator is naturally defined. For example, the boundary of a loaded path (1-chain) is given by

\[
\text{(ending point loaded with the value of the function there)} - \text{(starting point loaded with the value of the function there)}.
\]

The boundary of a higher dimensional loaded chain is defined in an obvious way. A loaded $p$-chain is called a *loaded $p$-cycle* if its boundary vanishes.

**4.1. Case $k = 1$.** Let $x_1, \ldots, x_{n+1}$ be distinct real points on $\mathbb{P}^1$ satisfying $x_1 < \cdots < x_{n+1}$. Then the multi-valued function

\[
u = \prod_{j=1}^{n} l_j^\alpha_j, \quad l_j = t - x_j
\]

is defined on $T = \mathbb{P}^1 - \{x_1, \ldots, x_{n+1}, x_0 = \infty\}$. On each oriented interval $(x_p, x_{p+1})$, we load a branch of the function $u$ determined by

\[
\arg(t - x_j) = \begin{cases} 0 & j \leq p, \\ -\pi & p + 1 \leq j, \end{cases}
\]

and call this loaded path $\tilde{I}_p$. Note that if you analytically continue the branch of $u$ corresponding to some loaded path $\tilde{I}_j$ through the lower half part of the $t$-plane $T$, then you get the branches of $u$ corresponding to other loaded paths $\tilde{I}_j$. But if you do the same starting from a point in $(x_j, x_{j+1})$, passing through the upper part and ending at a point in $(x_j-1, x_j)$, you get

\[
c_j := e^{2\pi i \alpha_j}
\]

times the branch $u$ corresponding to the loaded paths $\tilde{I}_{j-1}$. 

Anyway, \( I_j \) represent elements of \( H^1_{lf}(T, \mathcal{L}) \). For example, \( n \) non-compact loaded cycles \( I_1, \ldots, I_n \) form a basis. Loading \( u^{-1} \) in place of \( u \), we get non-compact loaded cycles \( I_j \); for example, \( I_1, \ldots, I_n \) form a basis of \( H^1_{lf}(T, \mathcal{L}) \).

As we did in §3, to define intersection numbers, we must make a compact counterpart \( \text{re}g I_j \), regularization of \( I_j \). This can be done by attaching two circles at the ends:

\[
-\frac{c_j}{d_j} C^j_\varepsilon + \left( x_j + \varepsilon, x_{j+1} - \varepsilon \right) + \frac{c_{j+1}}{d_{j+1}} C^{j+1}_{-\varepsilon}, \quad d_j := c_j - 1,
\]

where \( C^j_\varepsilon \) is the positively oriented circle of radius \( \varepsilon > 0 \) center at \( x_j \) starting at \( x_j \pm \varepsilon \) (see Figure 1), and by loading \( u^{-1} \) along the three paths, where the branch of \( u^{-1} \) at each starting point is that of \( I_j \). Note that \( \text{re}g I_j \) is homologous to \( I_j \) in \( H^1_{lf}(T, \mathcal{L}) \). \( \text{re}g I_1, \ldots, \text{re}g I_n \) form a basis of \( H_1(T, \mathcal{L}) \).

Let us evaluate the intersection number \( \text{re}g I_i \cdot \tilde{I}_j \). As is explained in §2, the definition is made through the intersection number of cohomology groups; it is a, so to speak, indirect analytic definition. In the following, we give a direct it topological definition, by which one can evaluate intersection numbers explicitly. These two definitions agree (see [KY1]); this fact will be referred to the compatibility of intersection forms for homology and cohomology groups.

Deform the support of \( \tilde{I}_j \) so that it intersects transversally with that of \( \text{re}g I_i \); any deformation will do. At each intersection point of the two supports, multiply the values of the two functions loaded to make the local intersection number at this point. Then sum up all the local intersection numbers, and finally change the sign to get \( \text{re}g I_i \cdot \tilde{I}_j \) (see Figure 1). Here is an actual computation:

\[
(\text{re}g I_j) \cdot \tilde{I}_j = \left( \frac{c_j}{d_j} - 1 + \frac{c_{j+1}}{d_{j+1}} \right) = - \left( \frac{d_{j,j+1}}{d_j d_{j+1}} \right),
\]

\[
(\text{re}g I_j) \cdot \tilde{I}_{j-1} = \frac{1}{d_j}, \quad (\text{re}g I_{j-1}) \cdot \tilde{I}_j = \frac{c_j}{d_j},
\]
0, otherwise, where \( d_{ij} = c_i c_j - 1 \). Therefore the intersection matrix \( \text{Int}_{\text{hom}}(\alpha) = (\text{reg}_i \cdot \text{I}_j)_{ij} \) is given by the following tri-diagonal matrix

\[
\text{Int}_{\text{hom}}(\alpha) = \begin{pmatrix}
\frac{d_{12}}{d_1} & d_2 / d_2 & 0 & \cdots \\
-1 / d_2 & \frac{d_{23}}{d_2} & d_3 / d_3 & \cdots \\
0 & -1 / d_3 & \frac{d_{34}}{d_3} & \cdots \\
& & & & \ddots
\end{pmatrix}.
\]

(N.B. The intersection matrix in [KY1] is given by \(-^t \text{Int}_{\text{hom}}(\alpha) = \text{Int}_{\text{hom}}(-\alpha)\) according to the definition of the intersection form for cohomology groups made there (cf. §2).)

**4.2. Case \( k \geq 2 \).** For given \( n + 1 \) real points on \( \mathbb{C} \)

\[
x_1 < \cdots < x_j < \cdots < x_{n+1}, \quad x_0 = \infty,
\]

we define \( n + 1 \) real hyperplanes \( L_1, \ldots, L_n \) in \( t = (t_1, \ldots, t_k) \)-space by

\[
l_j := t_r + (-x_j)t_{r-1} + \cdots + (-x_j)^{r-1}t_1 + (-x_j)^r, \quad 1 \leq j \leq n,
\]

and \( L_0 \) the hyperplane at infinity. This arrangement \( \{L_0, \ldots, L_n\} \) is called a Veronese arrangement, since an embedding of \( \mathbb{P}^1 \) into \( \mathbb{P}^k \) by

\[
t_0 = s^k, \quad t_1 = s^{k-1}, \ldots, t_{k-1} = s, \quad t_k = 1
\]

is called the Veronese embedding. When \( k = 2 \) and \( n = 4 \), the arrangement is illustrated in Figure 2. Set

\[
U = \prod_{j=1}^{n} l_j(t)^{a_j},
\]

where \( l_j(t) \) is the linear form in \( t \) just defined above. For a multi-index,

\[
I = (i_1, \ldots, i_k), \quad 1 \leq i_1 < \cdots < i_k \leq n,
\]

we define loaded cycles \( D_I \in H^f_k(T, \mathcal{L}) \) and \( \check{D}_I \in H^f_k(T, \check{\mathcal{L}}) \) with support on the chamber (see Figure 2)

\[
|D_I| = \{t \in T_R \mid (-1)^{P(j)} l_j(t) > 0, \quad 1 \leq j \leq n\},
\]

loaded with \( U^{-1} \) and \( U \), respectively, with

\[
\arg l_j = -P(j)\pi, \quad 1 \leq j \leq n,
\]

where \( P(j) \) denotes the cardinality of \( \{p \mid i_p < j\} \). Since each loaded
cycle is locally a direct product of 1-dimensional cycles, the regularizations $\text{reg}D_I \in H_k(T, \mathcal{L})$ are naturally defined. We now state the result, which is very similar to Theorem 1.

**Theorem 2.** For multi-indices $I = (i_1 \ldots i_k), 1 \leq i_1 < \cdots < i_k \leq n$, $J = (j_1 \ldots j_k), 1 \leq j_1 < \cdots < j_k \leq n$, the intersection number $\text{reg}D_I \cdot \tilde{D}_J$ is equal to the $(I, J)$-minor of the matrix $\text{Int}_{\text{hom}}(\alpha)$.

For rigorous proofs, see [KY2]. This theorem can be naturally understood if you write

$$D_J = I_{j_1} \wedge \cdots \wedge I_{j_k}, \quad J = (j_1, \ldots, j_k)$$

which is justified in [IK2].
§5. Quadratic relations

As we pointed out at the end of §2 (see also the middle of §4.1), the compatibility of the intersection forms for homology groups and cohomology groups, which is a general, universal and abstract equality, produces explicit quadratic relations among hypergeometric integrals — twisted analogues of the Riemann equality for periods.

The simplest example is the one in §1

\[ B(p, q)B(-p, -q) = \frac{2\pi i(p + q)}{pq} \cdot \frac{1 - e^{2\pi i(p+q)}}{(1 - e^{2\pi i p})(1 - e^{2\pi i q})}. \]

Now we know the meaning of the right-hand side: It is the product of the intersection number of the forms

\[ \frac{dt}{t(1 - t)} \in H^1(\Omega^\bullet(\log L), \nabla) \quad \text{and} \quad \frac{dt}{t(1 - t)} \in H^1(\Omega^\bullet(\log L), \hat{\nabla}), \quad L = \{0, 1, \infty\} \]

and that of the cycles

\[(0, 1) \otimes u^{-1} \in H_1(T, \mathcal{L}) \quad \text{and} \quad (0, 1) \otimes u \in H_1(T, \hat{\mathcal{L}}), \quad u := t^p(1 - t)^q.\]

Here is another example due to Gauss:

\[ F(a, b, c; x)F(1 - a, 1 - b, 2 - c; x) = F(a + 1 - c, b + 1 - c, 2 - c; x)F(c - a, c - b, c; x), \]

where \( F \) is the hypergeometric function (cf. [CM], [Mat1]).

**Twisted analogues of Riemann inequality.** When \( \alpha_j \in \mathbb{R} \), we can speak about the Hodge structure on the cohomology groups, and get twisted analogues of Riemann inequality. [HY] studies these when \( k = 1 \).

§6. Further study

So far, we worked on the projective spaces \( \mathbb{P}^k \), linear forms \( l_j \), function \( u = \prod l_j^{\alpha_j} \), 1-form \( \omega = du/u \), etc, under Assumption 1: \( \alpha_j \notin \mathbb{Z} \), and Assumption 2: no \( k + 1 \) hyperplanes in \( \{L_j\} \) intersect.

For a general arrangement, without Assumption 2 but with a genericity for \( \alpha_j \) corresponding to Assumption 1, the structure of the cohomology group can be described in terms of the so-called Orlik-Solomon
algebra, and an explicit basis of the homology group is known, if the arrangement is real. By successive blowing-up one can make the proper transform of the arrangement normally crossing — there is a systematic way to do this — then one can, in principle, evaluate the intersection numbers (cf. [KY2], [Yos2]). We expect that these intersection numbers can be expressed combinatorially in a closed form.

For imaginary arrangements, $k \geq 2$, or non-linear arrangements (cf. [KY2]), little is known about explicit cycles.

Motivated by an integral whose integrand involves hypergeometric functions, Hanamura, Ohara and Takayama study intersection theory when the rank of the local system $L$ is larger than 1 (cf. [Oha1,2], [OT]). They use hyperplane-section method, which is expected to be effective also to the previous problem.

Recall the famous limit formula:

$$(1 + \lambda t)^{1/\lambda} \to e^t, \quad \text{as } \lambda \to 0$$

and a less famous one

$$(1 + \lambda t)^{1/\lambda(\mu-\lambda)}(1 + \mu t)^{1/\mu(\lambda-\mu)} \to e^{t^2/2}, \quad \text{as } \lambda, \mu \to 0.$$ 

In §1, starting from the Beta integral you find two ‘limit’ integrals, one of them is the Gamma function. These formulae suggest another direction of generalization of the theories stated above, that is, to consider for example

$$u = \prod_{j=1}^{m}(t-x_j)^{\alpha_j} \exp f, \quad \omega = d\log u = \sum_{j=1}^{m} \alpha_j \frac{dt}{t-x_j} + df, \quad \nabla = d + \omega \wedge,$$

where $f$ is a polynomial in $t$. The corresponding hypergeometric integrals represent various confluent hypergeometric functions; the extreme ones are those without $l_j$; such integrals are called generalized Airy integrals, because

$$\int \exp(-t^3/3 + xt)dt$$

represents the Airy function.

In the following sections we study the confluent cases. Since the above limit formulae are delicate, if you know what I mean, the above theories in §§2 – 5 do not directly imply those for confluent cases; we must establish it independently. Of course you can expect some limit relations among them (see [KHT2], [Ha2]).
§7. Confluent cases, general frame

Let $n_1 \geq \cdots \geq n_m$ be natural numbers and $L_j$ ($1 \leq j \leq m$) be hyperplanes in $\mathbb{P}^k$ defined by linear forms $l_j$ of $t_1, \ldots, t_k$; put $T = \mathbb{P}^k \setminus \bigcup_{j=1}^m L_j$. We define a rational exact 1-form $\omega_j'$ with $n_j$-fold poles along $L_j$; this is explicitly given in §9. Put

$$\omega = \sum_{j=1}^m \left( \alpha_j \frac{dl_j}{l_j} + \omega_j' \right), \quad \nabla = d + \omega \wedge$$

and consider the following complex

$$0 \to \Omega^0(*L) \xrightarrow{\nabla} \Omega^1(*L) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^k(*L) \to 0.$$

We want to define the intersection pairing between $H^k(\Omega^\bullet(*L), \nabla)$ and $H^k(\Omega^\bullet(*L), \nabla)$ as we did in non-confluent cases. However, we can easily see that

$$H^k(\mathcal{E}^\bullet, \nabla) \not\cong H^k(\Omega^\bullet(*L), \nabla) \not\cong H^k(\mathcal{E}^\bullet, \nabla)$$

in general. So we need to introduce a reasonable cohomology theory on which a perfect pairing can be naturally defined. We also want to have a suitable homology theory and Poincaré isomorphisms to get intersection numbers for homology groups. Up to now only two extreme cases are studied:

Case $k = 1$,
Case $T = \mathbb{C}^k$, i.e. $\omega$ admits poles only along the hyperplane at infinity.

§8. Confluent cases $k = 1$

8.1. Twisted de Rham cohomology groups.

A smooth function $f$ defined in a neighborhood $U$ of the point $x$ is said to be rapidly decreasing at $x$ if $f$ satisfies

$$\frac{\partial^{p+q}}{\partial t^p \partial \mathbf{u}^q} f(x) = 0, \quad p, q = 0, 1, 2, \ldots,$$

Let $S^p$ be the vector space of smooth $p$-forms on $\mathbb{P}^1$ which are rapidly decreasing at $x_i (= L_i)$ for every $i$. A smooth function $f$ defined in $U \setminus \{x\}$ is said to be polynomially growing at $x$ if there exists $r \in \mathbb{N}$ such that $(t - x)^r f$ is smooth on $U$. Let $\mathcal{P}^p$ be the vector space of smooth $p$-forms $f$ on $T$ which are polynomially growing at $x_i$ for every $i$. 
We consider two complexes with differential $\nabla$:

$$(\mathcal{S}^\bullet, \nabla) : \mathcal{S}^0 \xrightarrow{\nabla} \mathcal{S}^1 \xrightarrow{\nabla} \mathcal{S}^2 \xrightarrow{\nabla} 0,$$

$$(\mathcal{P}^\bullet, \nabla) : \mathcal{P}^0 \xrightarrow{\nabla} \mathcal{P}^1 \xrightarrow{\nabla} \mathcal{P}^2 \xrightarrow{\nabla} 0.$$  

The cohomology groups $H^k(\mathcal{S}^\bullet, \nabla)$ and $H^k(\mathcal{P}^\bullet, \nabla)$ are called rapidly decreasing and polynomially growing twisted de Rham cohomology groups with respect to $\nabla$, respectively. The inclusions

$$(\Omega^\bullet(*L), \nabla) \subset (\mathcal{P}^\bullet, \nabla), \quad (\mathcal{S}^\bullet, \nabla) \subset (\mathcal{P}^\bullet, \nabla)$$

of complexes induce the following isomorphisms among twisted de Rham cohomology groups.

**Theorem 3.** $H^p(\Omega^\bullet(*L), \nabla) \simeq H^p(\mathcal{P}^\bullet, \nabla) \simeq H^p(\mathcal{S}^\bullet, \nabla), \quad p = 0, 1, 2.$

The first isomorphism can be proved by the help of $\bar{\partial}$-calculus. Since the injectivity of the natural map $H^p(\mathcal{S}^\bullet, \nabla) \rightarrow H^p(\mathcal{P}^\bullet, \nabla)$ is easy, we mention briefly its surjectivity when $p = 1$. For a $\nabla$-closed form $\varphi \in \Omega^1(*L)$, there exists a unique formal meromorphic Laurent series $F_i$ around $x_i$ satisfying $\nabla F_i = \varphi$. If $n_i \geq 2$, $F_i$ is divergent in general, however, there exists a polynomially growing smooth function $f_i$ with the same expansion as $F_i$. Thus the form

$$\varphi - \sum_{i=0}^{m} \nabla(h_i f_i)$$

is in $\mathcal{S}^1$, where $h_i$ is a smooth function defined in §3.1. This implies the surjectivity.

**8.2. Twisted homology groups.** Let $\Delta$ be a singular $p$-simplex in $T$, define a function $u_\Delta$ on $\Delta$ by

$$u_\Delta(t) = \exp \left( \int_0^t \omega \right),$$

where the path of the integration is in $\Delta$. We consider only chains $\rho$ such that if $x_i$ belongs to the closure of $\rho = \sum_j b_j \Delta_j$ in $\mathbb{P}^1$ then

$$\lim_{t \to x_i, t \in \rho} (t - x_i)^r u_\rho(t) = 0, \quad r = 0, 1, 2, \ldots,$$

where $u_\rho(t) = u_{\Delta_j}(t)$ ($t \in \Delta_j$). Let $C_p(T, \omega)$ be the space of loaded $p$-chains $\sum_j b_j \Delta_j \otimes u_{\Delta_j}$ for all such $p$-chains $\rho = \sum_j b_j \Delta_j$. The boundary
operator $\partial_\omega$ on $C_\bullet(T, \omega)$ is naturally defined, and we get the $p$-th homology group $H_p(C_\bullet(T, \omega), \partial_\omega)$ as we did in §4. There is a natural pairing between $H^1(S^\bullet, \nabla)$ and $H_1(C_\bullet(T, \omega), \partial_\omega)$ through the (confluent) hypergeometric integral

$$\langle \varphi, \gamma \rangle = \sum_j b_j \int_{\Delta_j} u_{\Delta_j}(t) \varphi,$$

where $\varphi \in S^1, \gamma = \sum_j b_j \Delta_j \otimes u_{\Delta_j}(t) \in C_1(T, \omega)$.

**Theorem 4.** The pairing between $H^1(S^\bullet, \nabla)$ and $H_1(C_\bullet(T, \omega), \partial_\omega)$ is perfect.

8.3. Intersection pairings.

There is a natural pairing between $S^1$ and $P^1$ by

$$\int_{P^1} \varphi \wedge \psi, \quad \varphi \in S^1, \psi \in P^1.$$

This pairing descends to the perfect pairing $\cdot$ between $H^1(S^\bullet, \nabla)$ and $H^1(P^\bullet, \hat{\nabla})$. Theorem 3 yields the isomorphism $\iota : H^1(\Omega^\bullet(\ast L), \nabla) \to H^1(S^\bullet(\ast L), \nabla)$, which induces the intersection pairing of $H^1(\Omega^\bullet(\ast L), \nabla)$ and $H^1(\Omega^\bullet(\ast L), \hat{\nabla})$ by

$$\varphi \cdot \psi = \int_{P^1} \iota(\varphi) \wedge \psi.$$

**Theorem 5.** The intersection number $\varphi \cdot \psi$ of $\varphi \in H^1(\Omega^\bullet(\ast L), \nabla)$ and $\psi \in H^1(\Omega^\bullet(\ast L), \hat{\nabla})$ is given by

$$\varphi \cdot \psi = 2\pi i \sum_{j=0}^m \text{Res}_{t = x_j}(F_j \psi),$$

where $F_j$ is the meromorphic formal Laurent series around $x_j$ satisfying $\nabla F_j = \varphi$.

Note that we can evaluate the intersection number $\varphi \cdot \psi$ by this theorem; see examples in the next subsection.

So far in this section, we defined three pairings:

$$\begin{align*}
H^1(\Omega^\bullet(\ast L), \nabla) &\cong H^1(S^\bullet, \nabla) &\cong H^1(P^\bullet, \hat{\nabla})
\end{align*}$$

These pairings define a pairing between the two homology groups.
Theorem 6. Suppose for two loaded cycles
\[ \rho^+ = \sum_i b_i \Delta^+_i \otimes u^+_\Delta_i(t) \in H_1(C_\bullet(T, \omega), \partial_\omega) \]
and
\[ \rho^- = \sum_j b_j \Delta^-_j \otimes u^-_{\Delta_j}(t) \in H_1(C_\bullet(T, -\omega), \partial_{-\omega}), \]
\(\Delta_i^+\) and \(\Delta_j^-\) meet transversally at finitely many points. Then the intersection number \(\rho^+ \cdot \rho^-\) is equal to
\[ \rho^+ \cdot \rho^- = \sum_{i,j} \sum_{v \in \Delta_i^+ \cap \Delta_j^-} b_i b_j [u^+_{\Delta_i^+}(t)]_{t=v} [u^-_{\Delta_j^-}(t)]_{t=v} I_v(\Delta^+_i, \Delta^-_j), \]
where \(I_v(\Delta^+_i, \Delta^-_j)\) is the topological intersection number of \(\Delta^+_i\) and \(\Delta^-_j\) at \(v \in T\).

8.4. Examples. The compatibility of theparings yields quadratic relations among confluent hypergeometric functions.

Let \(\omega = -tdt\), so \(u(t) = e^{-t^2/2}\). The (co)homology groups in question are 1-dimensional. Put
\[ \rho^+ = [-\infty, \infty] \otimes e^{-t^2/2}, \quad \rho^- = [-i\infty, i\infty] \otimes e^{t^2/2}. \]
Let us compute the intersection number \(dt \cdot dt\) applying Theorem 5. Since the pole of \(\omega\) is at \(\infty\) only, we solve the equation \(\nabla F = dt\) at \(\infty\). By a straightforward calculation, we have
\[ F = -s + s^3 - 2s^5 + 2 \cdot 4s^7 - 2 \cdot 4 \cdot 6s^9 + \cdots, \quad s = 1/t. \]
Since \(\text{Res}_{s=0}(F(s)(-ds/s^2)) = 1\), \(dt \cdot dt\) equals \(2\pi i\). One can easily see that Theorem 6 implies \(\rho^+ \cdot \rho^- = 1\). Since
\[ \langle dt, \gamma^+ \rangle = \int_{-\infty}^{+\infty} e^{-t^2/2} dt, \quad \langle dt, \gamma^- \rangle = \int_{-i\infty}^{+i\infty} e^{t^2/2} dt = i \int_{-\infty}^{+\infty} e^{-t^2/2} dt, \]
we have the formula announced in §1:
\[ \left( \int_{-\infty}^{\infty} e^{-t^2/2} dt \right) \cdot 1 \cdot \left( i \int_{-\infty}^{\infty} e^{-t^2/2} dt \right) = 2\pi i. \]

We present two more examples: the inversion formula for the gamma function
\[ \Gamma(\alpha)\Gamma(1 - \alpha) = \frac{\pi}{\sin \pi \alpha}. \]
and Lommel's formula

\[ J_a(z)J_{-a+1}(z) + J_{a-1}(z)J_{-a}(z) = \frac{2\sin(\pi a)}{\pi z}, \]

which holds for the Bessel function with parameter \( a \in \mathbb{C} \setminus \mathbb{Z} \)

\[ J_a(z) = \left(\frac{z}{2}\right)^a \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(a+k+1)} \left(\frac{z}{2}\right)^k, \]

where \( z \in \{ z \in \mathbb{C} \mid \Re(z) > 0 \} \) and the argument of \( z \) is in \((-\pi/2, \pi/2)\). For details including proofs, refer to [MMT].

[Ha2] shows that such quadratic relations are indeed obtained from these in §5 by confluence process.

§9. Confluent cases, generalized Airy \( k \geq 2 \)

Let \( \omega \) be an exact 1-form on \( T = \mathbb{C}^k \), with parameters \( \alpha_1, \ldots, \alpha_n \), defined as

\[ \omega = d\theta_{n+1}(t) + \sum_{j=1}^{n} \alpha_j d\theta_j(t), \]

where \( \theta_j \) are polynomials in \( t = (t_1, \ldots, t_k) \) of degree \( j \) defined by

\[ \log(1 + t_1X + t_2X^2 + \cdots + t_kX^k) = \sum_{j \geq 1} \theta_j(t)X^j; \]

for example, \( \theta_1(t) = t_1, \ \theta_2(t) = t_2 - t_1^2/2, \ \theta_3(t) = t_3 - t_1t_2 + t_1^3/3. \)

Note that the form \( \omega \) has poles of order \( n+2 \) along the hyperplane \( L \) at infinity. Let \( H^p(\Omega^\bullet, \nabla) \) be the \( p \)-th cohomology group of the complex

\[ (\Omega^\bullet, \nabla) : 0 \longrightarrow \Omega^0 \xrightarrow{\nabla} \Omega^1 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^k \longrightarrow 0, \]

where \( \Omega^p \) the vector space of polynomial \( p \)-forms. In [Kim2], it is shown that only \( H^k(\Omega^\bullet, \nabla) \) survives and is \( \binom{n}{k} \)-dimensional, further it is conjectured that there exists a basis expressed in terms of Schur polynomials. This conjecture is established in [IM]. In order to state this, we consider the map

\[ \phi : \mathbb{C}^k \ni s = (s_1, \ldots, s_k) \mapsto t = (t_1, \ldots, t_k) = (e_1(s), \ldots, e_k(s)) \in \mathbb{C}^k, \]

where \( e_j(s) \) is the elementary symmetric polynomial of degree \( j \).
**Theorem 7.** \( H^k(\Omega^*, \nabla) \) can be spanned by

\[
\Theta_I = d\theta_{i_1} \wedge \cdots \wedge d\theta_{i_k}, \quad I = (i_1, \ldots, i_k), \quad 1 \leq i_1 < \cdots < i_k \leq n.
\]

The pull back \( \phi^*(\Theta_I) \) of \( \Theta_I \) by \( \phi \) is given by

\[
\phi^*(\Theta_I) = \text{Sc}_\lambda(s) \Delta(s) ds_{i_1} \wedge \cdots \wedge ds_k,
\]

where \( \text{Sc}_\lambda(s) \) is the Schur polynomial attached to the Young diagram \( \lambda = (i_k - k, \ldots, i_1 - 1) \) and \( \Delta(s) \) is the difference product of \( s_1, \ldots, s_k \).

Let us define the intersection pairing \( H^k(\Omega^*, \nabla) \) and \( H^k(\Omega^*, \nabla) \). Note that the map \( \phi \) induces the biholomorphic map from the quotient variety \( (\mathbb{P}^1)^k / S_k \) to \( \mathbb{P}^k \). We can easily see that

\[
\phi^*(d\theta_i(t_1, \ldots, t_k)) = \sum_{j=1}^k d\theta_i(s_j, 0, \ldots, 0).
\]

We regard \( \phi^*(\omega) \) as a meromorphic 1-form on \( (\mathbb{P}^1)^k / S_k \). We can deform \( \phi^*(\Theta_I) \) into a \( S_k \)-invariant rapidly decreasing \( k \)-form \( \iota(\phi^*(\Theta_I)) \) on \( \mathbb{C}^k \) by adding \( dF + \phi^*(\omega) \wedge F \), where \( F \) is a \( S_k \)-invariant polynomially growing \( (k-1) \)-form on \( \mathbb{C}^k \). Since \( (\mathbb{P}^1)^k \) is the \( k! \)-fold covering of \( (\mathbb{P}^1)^k / S_k \), we define the intersection number \( \Theta_I \cdot \Theta_J \) for \( \Theta_I \in H^k(\Omega^*, \nabla) \) and \( \Theta_J \in H^k(\Omega^*, \nabla) \) as

\[
\langle \Theta_I, \Theta_J \rangle = \frac{1}{k!} \int_{\mathbb{C}^k} \iota(\phi^*(\Theta_I)) \wedge \phi^*(\Theta_J).
\]

**Theorem 8.** The intersection number \( \langle \Theta_I, \Theta_J \rangle \) is equal to the skew Schur polynomial \( \text{Sc}_{\lambda/\mu}(\alpha) \) with elementary symmetric polynomials as variables, where \( \mu \) is the complement of the Young diagram \( \mu = (j_k - k, \ldots, j_1 - 1) \) in the \( k \times (n-k) \) rectangle, and \( \lambda/\mu \) is the skew Young diagram of \( \lambda = (i_k - k, \ldots, i_1 - 1) \) and \( \mu \).

The cohomology theory introduced in this section will be presented in full in [IM]. The homological counter part is still unsettled.

**References**


Intersection theory


[Ha2] Y. Haraoka, Quadratic relations for confluent hypergeometric functions on $\mathbb{Z}_{2,n+1}$, to appear in Funkcial. Ekvac.


[Oha1] K. Ohara, Computation of the monodromy for the generalized hypergeometric function \(_pF_{p-1}(a_1, \ldots, a_p; b_1, \ldots, b_p; z)\), Kyushu J. Math. 51(1997), 101–124.


Keiji Matsumoto
Department of Mathematics,
Hokkaido University,
Sapporo 060-0810
Japan

Masaaki Yoshida
Graduate School of Mathematics,
Kyushu University,
6-10-1 Hakozaki, Higashi-ku
Fukuoka 812-8581
Japan