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Helmholtz-Type Equation on Non-compact Two-Dimensional Riemannian Manifolds

Reiji Konno

§1. Introduction

We shall consider the existence, or rather non-existence of square integrable solutions of the equation $-\triangle f = \lambda f$ on a non-compact Riemannian manifold which is homeomorphic to $\mathbf{R}^n$ minus a ball, where $\triangle$ is the Laplace-Beltrami operator and λ is an arbitrary positive constant. The source of this problem is the study of the non-existence of positive eigenvalues of the Schrödinger operator $-\triangle + q$ in a region of $\mathbf{R}^n$, and the method used there was found to be applicable to problems of the above type.

There may be several ways of physical interpretation of the equation $-\triangle f = \lambda f$ on manifolds. But probably the most essential one is as follows: Let a Riemannian manifold $\mathcal{M}$ represent a non-Euclidean space which is filled up with a medium whose displacement on some quantity, e.g. pressure, electric field etc., obeys Hooke's law isotropically and homogeneously in each small portion of the medium. We suppose further that the displacement is transferred entirely to the neighboring portions without influence of the curvature. (This situation occurs, for example, if $\mathcal{M}$ is a surface and the medium is distributed on and moving along $\mathcal{M}$ without friction or obstruction.) Then, the displacement D should enjoy the "wave equation" $D_{tt} = \triangle D$ (by taking an appropriate scale), therefore $-\triangle f = \lambda f$ describes the standing wave $D = e^{i\sqrt{\lambda}t} f(x)$.

We notice that the *total energy* $\int_{\mathcal{M}} (|D_t|^2 + |\nabla D|^2) d\mathcal{M}$ is finite if and only if $\int_{\mathcal{M}} |f|^2 d\mathcal{M}$ is finite. Therefore, what we are asking is the conditions for $\mathcal{M}$ not to admit a standing wave of finite energy.

Before describing the general statement, let us see examples of $\mathcal{M}$ which have L^2-solutions.

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Examples. (a) $\mathcal{M}$ is the semi-infinite cylinder whose metric $ds^2 = dr^2 + \rho_0^2 d\theta^2$, $r \in (r_0, \infty)$, $\theta \in \mathbf{S}^1$, where ρ_0 is a positive constant. Since $\triangle = \partial^2/\partial r^2 + \rho_0^{-2}\partial^2/\partial\theta^2$, the function $f = e^{-ar+2\pi ni\theta}$ is an L^2-solution for $\lambda = -a^2 + 4\pi^2 n^2/\rho_0^2$ if the constant a and the integer n satisfy $0 < a < 2\pi n/\rho_0$. (b) Let $ds^2 = dr^2 + e^{2ar}d\theta^2$, $r \in (r_0, \infty)$, $\theta \in \mathbf{S}^1$. If a and b are constants such that $0 < b < a < 2b$, then $f = e^{-br}$ is a solution for $\lambda = b(a-b)$ which is square integrable since $d\mathcal{M} = e^{ar}drd\theta$.

The above examples suggest that, in so far as 2-dimensional *rotationally symmetric* manifolds are concerned, the following theorem is in some sense a satisfactory one.

Theorem 1 ([2],[4]). *If $\mathcal{M}$ is a two-dimensional manifold whose metric has the form*

$$ds^2 = dr^2 + \rho(r)^2 d\theta^2, \quad r \in (r_0, \infty), \ \theta \in \mathbf{S}^1$$

where $\rho(r)$ is a positive absolutely continuous nondecreasing function of r which enjoys (i) $\rho(r) \to \infty$ $(r \to \infty)$ *and* (ii) $\displaystyle\int_{r_0}^{\infty} \frac{dr}{\rho(r)} = \infty$, *then for any constant $\lambda > 0$ and any nontrivial locally square integrable solution of $-\triangle f = \lambda f$, there exist constants $C > 0$ and $r_1 \geq r_0$ such that*

$$\int_{r_0<r<R} |f|^2 d\mathcal{M} \geq C \int_{r_0}^{R} \frac{dr}{\rho(r)} \quad (R \geq r_1)$$

holds where $d\mathcal{M} = \rho(r)drd\theta$. (Therefore $f \notin L^2(\mathcal{M})$ unless $f \equiv 0$.)

The previous example (a) does not satisfy (i), while (b) breaks (ii).

Corollary. *Let $\mathcal{M}$ be a surface of revolution in $\mathbf{R}^3$ obtained by rotating the graph of an arbitrary absolutely continuous function $z = z(\rho)$, $\rho_0 < \rho < \infty$, around the z-axis. Then $\mathcal{M}$ has the same property with respect to the natural metric. In particular, any non-vanishing solution of $-\triangle f = \lambda f$, $\lambda > 0$ can not be square integrable.*

As to the higher dimensional cases, we have the following theorem.

Theorem 2 ([3],[4]). *Let $\mathcal{M} = \{(r, \omega) \mid r_0 < r < \infty, \ \omega \in \mathbf{S}^{n-1}\}$ $(n \geq 2)$ with the metric $ds^2 = dr^2 + \rho(r)^2 d\tilde{s}^2$ where ρ is a positive function and $d\tilde{s}$ is the line element of the $(n-1)$-sphere $\mathbf{S}^{n-1}$. Suppose that*

(i) $\rho \in C^2(r_0, \infty)$, $\rho'(r) > 0$ *and* $\rho(r) \to \infty$ $(r \to \infty)$.
(ii) $\rho'(r)/\rho(r) \to 0$ $(r \to \infty)$.

(iii) $\rho''(r)/\rho'(r) \to 0 \ (r \to \infty)$.
(iv) *There exists a number* $\alpha > 0$ *such that*

$$\int_{r_0}^{\infty} \frac{dr}{\rho(r)^{\alpha}} = \infty.$$

Then for any $\lambda > 0$ *and any non-zero solution of* $-\triangle f = \lambda f$ *and for an arbitrary* $\varepsilon > 0$, *we can take* $C > 0$ *and* $r_1 \geq r_0$ *so that*

$$\int_{r_0<r<R} |f|^2 d\mathcal{M} \geq C \int_{r_0}^{R} \frac{dr}{\rho(r)^{\varepsilon}} \quad (R \geq r_1).$$

We see that Theorem 2 assumes weaker growth of $\rho(r)$ than Theorem 1. Moreover, the obtained estimate is better. But it requires higher smoothness of ρ and restricts the magnitude of ρ'' in return.

§2. Not symmetric manifolds

T. Tayoshi's work [6] treated the case in which the metric itself is not rotationally symmetric but approaches such one asymptotically. His theorem is a generalization of Theorem 2 above, though not completely. Here we want to have an extension of Theorem 1.

Let $\mathcal{M}$ be a two-dimensional manifold whose metric has the form

$$ds^2 = a(r,\theta)dr^2 + 2b(r,\theta)\rho(r)drd\theta + c(r,\theta)\rho(r)^2 d\theta^2,$$

where a, b, c and ρ are real-valued functions. To describe the conditions altogether, let us begin with definitions.

Definition 1. (i) $t(r) = \exp(-\int_{r_0}^{r} \frac{ds}{\rho(s)})$.

(ii) For each number $m > 0$, the quantity $h(r;m)$ is the one that satisfies

$$\int_{r}^{r+h(r;m)} \frac{ds}{\rho(s)} = mt(r).$$

(iii) $$\varphi(r;m) = \operatorname*{ess\,inf}_{r \leq s \leq r+h(r;m)} \rho(s)^2 \rho'(s).$$

Assumption on ρ.

(i) $\rho(r)$ is positive, nondecreasing and absolutely continuous with $\rho'(r) > 0$ a.e.

(ii) $\rho(r) \to \infty \ (n \to \infty)$.

(iii) $\displaystyle\int_{r_0}^{\infty} \frac{dr}{\rho(r)} = \infty.$

(iv) $t(r)/\rho'(r) \to 0 \quad (r \to \infty).$

(v) $\displaystyle\int_{r_0}^{\infty} \frac{\varphi(r;m)}{\rho(r+h(r;m))} dr = \infty.$

Remark. If $\rho(r)t(r)$ is bounded and $\rho'(r) \leq 1$, and moreover $\rho(r)^2\rho'(r)$ is nondecreasing or nonincreasing, then the condition (v) is fulfilled.

Definition 2. $g = \sqrt{ac-b^2}, \quad A = a/g, \quad B = b/g, \quad C = c/g.$

Definition 3. A function $f(t,\theta)$ is said to satisfy *the condition of Definition* 3 if it enjoys the inequality

$$|f(\text{point } 1) - f(\text{point } 2)| \leq \psi(\text{distance})$$

where $\psi(x)$ is a positive continuous nondecreasing function of $x > 0$ which fulfills $\int_{\to +0} \{\psi(x)/x\}dx < \infty$. By the way, if the two points are (t_1, θ_1) and (t_2, θ_2) then the distance is $\sqrt{t_1^2 + t_2^2 - 2t_1t_2\cos(\theta_1 - \theta_2)}$.

Remark. This condition is a generalization of the uniform Hölder continuity, the latter corresponding to $\psi(x) = Kx^{\alpha}$.

Assumptions on a, b, c.

(i) $a, b, c \in C^1((r_0,\infty) \times \mathbf{S}^1)$, $\quad a > 0$, $a/c \to 1$, and $b \to 0$ as $r \to \infty$ and there exist numbers k, l and r_1 $(k > 0,\ 0 < l < 2,\ r_1 \geq r_0)$ such that

$$g \geq k, \quad g_r/g \geq -l\rho'/\rho \quad (r \geq r_1,\, \theta \in \mathbf{S}^1).$$

(ii) $$g_\theta b/(g^2\rho') \to 0 \quad (r \to \infty),$$
$$g_\theta t/(g^2\rho') \to 0 \quad (r \to \infty).$$

(iii) As functions of t and θ,

$$\rho t^{-1}A_r, \quad \rho t^{-1}B_r, \quad \rho t^{-2}C_r, \quad t^{-1}A_\theta, \quad t^{-2}B_\theta, \quad t^{-1}C_\theta$$

have the limits at $t = 0$ (i.e., $r = \infty$), and satisfy the condition of Definition 3 near $t = 0$.

Our main theorem is as follows:

Theorem 3. *Under the above assumptions, the equation $-\triangle f = \lambda f$ on $\mathcal{M}$ has no non-trivial solution of integrable square, provided $\lambda > 0$.*

It should be noted that the conditions do not make reference to the second order derivatives of the metric.

This theorem is proved by combining the following two lemmas.

Lemma 1 (Estimate by isothermal coordinates).

Let the two-dimensional Riemannian manifold $\mathcal{M}$ admit a global system of coordinates u, v, $u_0 < u < \infty$, $v \in \mathbf{S}^1$ so that they are isothermal, that is, the metric has the form

$$ds^2 = \tau(u,v)\big(du^2 + dv^2\big)$$

by a positive function $\tau(u,v)$. We suppose that τ is absolutely continuous with respect to u for a.e. $v \in \mathbf{S}^1$ and of class C^1 with respect to v for a.e. u. Moreover let

$$\varphi(u) = \operatorname*{ess\,inf}_{v\in\mathbf{S}^1} \frac{\partial}{\partial u}\tau(u,v)$$

satisfy

$$\int_{u_0}^{\infty} \varphi(u)du = \infty.$$

Then for every non-trivial solution of $-\triangle f = \lambda f$ on $\mathcal{M}$ $(\lambda > 0)$, we can find numbers $C > 0$ and $u_1 \geq u_0$ such that

$$\int_{u_0<u<U} |f|^2 du \geq CU \quad (U \geq u_1)$$

holds ($d\mathcal{M} = \tau du dv$).

Lemma 2 (Existence of suitable isothermal coordinates).

If a two-dimensional Riemannian manifold satisfies the assumptions of Theorem 3, there exist a number r_1 and C^1-functions $u(r,\theta)$ and $v(r,\theta)$ defined for $r \geq r_1$, $\theta \in \mathbf{S}^1$, which satisfy

$$v_r = Bu_r - A\rho^{-1}u_\theta,$$

$$v_\theta = C\rho u_r - Bu_\theta.$$

Here (i) *for each θ, $u(r,\theta)$ is strictly increasing with r, $u_r(r,\theta)$ is absolutely continuous and $u(r,\theta) \to \infty$ as $r \to \infty$. On the other hand $v_\theta > 0$ and the value of $v(r,\theta)$ is determined up to the difference of $2k\pi$ $(k \in \mathbf{Z})$.*

(ii) *In terms of u and v, the metric is expressed as*

$$ds^2 = \tau(u,v)\big(du^2 + dv^2\big),$$

$$\tau = \frac{g}{Cu_r^2 - 2B\rho^{-1}u_r u_\theta + A\rho^{-2}u_\theta^2}.$$

(iii) $\varphi(u) = \operatorname*{ess\,inf}_{v\in\mathbf{S}^1} \partial\tau/\partial u$ *enjoys*

$$\int_{u_0}^{\infty} \varphi(u)du = \infty.$$

Lemma 1 together with Lemma 2 claims that if the solution f is square integrable over $\mathcal{M}$ then $f(r,\theta) \equiv 0$ for sufficiently large r, say, $r \geq r_1$. But, in our situation, we can easily verify the unique continuation property so that $f \equiv 0$ holds throughout $\mathcal{M}$. (So far as the unique continuation applies, $\mathcal{M}$ itself need not be of the shape described before. If only a part of $\mathcal{M}$ has that shape, we must have the same conclusion again.)

§3. Sketch of the proof of Lemma 2

The proof of Lemma 1 can be got by a standard argument. Therefore we will leave it to the full paper [5].

The main point of the proof of Lemma 2 is to obtain the solution of $\triangle u = 0$ which has the asymptotic form $u \sim \int dr/\rho$. To this end we change the variables from r, θ to t, θ and look for the solution of $\triangle u = 0$ having the form $u = -\log t + \xi(t,\theta)$, $\xi \in C^2$ in the neighborhood of $t = 0$. In fact, ξ enjoys the equation

$$(\tilde{C}\xi_x + \tilde{B}\xi_y)_x + (\tilde{B}\xi_x + \tilde{A}\xi_y)_y = t^{-1}C_t + t^{-2}B_\theta$$

where $x = t\cos\theta$, $y = t\sin\theta$ and $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are quadratic forms of $\cos\theta$ and $\sin\theta$ whose coefficients are linear combinations of A, B and C. Thus we can apply the classical theory of Korn and Lichtenstein or its extended version by Hartman and Wintner. We cite here a part of their theorem.

Theorem (Hartman & Wintner [1]). *Suppose $A_1(x,y)$, $B_1(x,y)$, $B_2(x,y)$ and $C_1(x,y)$ are C^1-functions whose first order derivatives satisfy the condition of Definition* 3. *We assume $A_1C_1 - (B_1+B_2)^2/4 > 0$.*

Moreover, let $D(x,y)$ and $E(x,y)$ are functions which satisfy the condition of Definition 3. Then the equation

$$(C_1\xi_x + B_1\xi_y)_x + (B_2\xi_x + A_1\xi_y)_y + D\xi = E$$

has a C^2-solution in some neighborhood of $x = y = 0$.

From the assumed regularity of A, B and C, it is easy to see that the conditions of this theorem are fulfilled by putting $A_1 = \tilde{A},\ B_1 = B_2 = \tilde{B},\ C_1 = \tilde{C}$. Thus we obtain the desired ξ. Set

$$v(t,\theta) = \int_{\text{fixed point}}^{(t,\theta)} (Bu_t + At^{-1}u_\theta)dt - (Ctu_t + Bu_\theta)d\theta.$$

Then a straightforward calculation shows that the pair of $u = -\log t + \xi$ and v form a set of isothermal coordinates. The estimates for their derivatives up to second order are derived from the C^2 property of ξ with respect to t and θ.

What is left is to show $\int^\infty \varphi(u)du = \infty$, $\varphi(u)$ being $\operatorname{ess\,inf}_v \partial\tau/\partial u$. This calculation is somewhat involved, but eventually we are led to the conclusion that there exist constants $K > 0$ and $r_1 \geq r_0$ for which

$$\tau_u \geq K\rho^2\rho' \quad (r \geq r_1)$$

holds and that the contour $\{u = \text{const.}\}$ lies between the circles of radii r and $r + h(r;m)$, m being some constant not depending on r. We know that $\varphi(u)$ is the infimum of τ_u on the contour $\{u = \text{const.}\}$ while $\varphi(r;m)$ is the infimum of τ_u in the region between those circles. This fact establishes the lemma.

Example. Consider $\rho(r)$ which has the form $\rho(r) = \rho_0(r) - \rho_0'(r)\big(1 - k(r)\big)\sin r$ where $\rho_0(r)$ is a positive function having absolutely continuous derivative and $k(r)$ is an absolutely continuous function. We assume (i) $\rho_0(r) \to \infty$ (ii) $0 \leq \rho'(r) \leq 1$ (iii) $0 < k(r) \leq 1$ (iv) $k(r)^{-1}k'(r) \to 0$ (v) $\rho_0'(r)k(r)$ is nonincreasing (vi) $\rho_0'(r)k(r)\exp\big(\int_{r_0}^r [\rho_0(s)+1]^{-1}ds\big) \to \infty$ (vii) $\int_{r_0}^\infty \rho_0(r)\rho_0'(r)k(r)dr = \infty$ (viii) $\rho'(r)^{-1}\rho_0''(r)k(r)^{-1} \to 0$. Then we can show that $\rho(r)$ satisfies all the conditions. If we choose $\rho_0(r) = r^\alpha$ $(0 < \alpha \leq 1)$ or $\rho_0(r) = \log r$ then it fulfills (i)(ii)(iii). It also satisfies (vi)(vii) and (viii) if we choose a nondecreasing $k(r)$. In particular, by setting $k(r) = 1$, $\rho(r) = r^\alpha$ and $\rho(r) = \log r$ themselves meet the conditions.

Example. The following example shows how fast A, B, C should tend to their limits. Let $\rho(r) = r$ and put $a = 1 - r^{-\alpha}\cos\theta$, $b = r^{-\alpha}\sin\theta$

and $c = 1+r^{-\alpha}\cos\theta$ where $\alpha > 2$. Then $t = r^{-1}$ and $g = \sqrt{1-r^{-2\alpha}}$. The crucial terms are $t^{-3}C_r$ and $t^{-2}B_\theta$. But they are close to $-\alpha t^{\alpha-2}\cos\theta$ and $-t^{\alpha-2}\cos\theta$ respectively. Therefore they fit the conditions.

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Department of Mathematics
Meiji University
1-1-1 Higashimita, Tama-ku
Kawasaki-shi, 214
Japan