Advanced Studies in Pure Mathematics 23, 1994
Spectral and Scattering Theory and Applications
pp. 223–238

An $L^{q,r}$-Theory for Nonlinear Schrödinger Equations

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§1. Introduction

Consider the nonlinear Schrödinger equation:

$$\text{(NLS)} \qquad \partial_t u = i(\Delta u - F(u)), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^m,$$

where $F(u) = F \circ u$ is, for example, a Nemyckii operator defined by a function $F : \mathbb{C} \to \mathbb{C}$. There is an extensive literature on this problem, but it seems that all existing work assumes that either the initial value $\phi = u(0) = u(0, \cdot)$ or the limit $\phi_\pm = \lim_{t\to\pm\infty} e^{-it\Delta}u(t)$ is in L^2. The present paper is an attempt to solve (NLS) with the data in a larger class of functions.

As in most of the work on (NLS), we convert (NLS) into integral equations such as

$$\text{(INT)} \qquad u = \Phi u \equiv u_0 - iGF(u), \quad \text{or} \quad u = \Phi_\pm u \equiv u_\pm - iG_\pm F(u).$$

Here u_0 or $u_\pm$ is a *free wave* (solution of the free Schrödinger equation $\partial_t u = i\Delta u$), and G or $G_\pm$ is an integral operator defined by

$$(1.1) \qquad \begin{aligned} Gf(t) &= \int_0^t U(t-s)f(s)\,ds, \\ G_\pm f(t) &= \int_{\pm\infty}^t U(t-s)f(s)\,ds, \quad U(t) = e^{it\Delta}. \end{aligned}$$

The free term u_0 in (INT) is usually related to the initial value $u(0) = \phi$ by

$$(1.2) \qquad u_0 = \Gamma\phi, \quad \Gamma\phi(t) = U(t)\phi,$$

Received December 28, 1992.

but it is often convenient to take any *free wave* without regard to the initial value. The dual operator to Γ is formally given by

$$\Gamma^* f = \int_{-\infty}^{\infty} U(-s) f(s)\, ds. \tag{1.3}$$

We note that

$$G_1 \equiv G_- - G_+ = \Gamma\Gamma^*. \tag{1.4}$$

To deal with the different operators G, $G_\pm$ and G_1 simultaneously, it is convenient to consider operators of the general form

$$G_a f(t) = \int_{-\infty}^{\infty} a(t,s) U(t-s) f(s)\, ds, \tag{1.5}$$

where a is a measurable function such that $|a(t,s)| \leqslant 1$ (cf. Yajima [14]).

Our first task is to study the continuity properties of the operators Γ and G_a between wider classes of spaces than hitherto considered. Set $L^p = L^p(\mathbb{R}^m)$, $L^{q,r} = L^r(L^q) = L^r(\mathbb{R}; L^q)$. The following results are well known (see e.g. [7]). Γ is bounded on L^2 to $L^{q,r}$ if

$$1/q + 2/mr = 1/2, \quad 1/2 - 1/m < 1/q \leqslant 1/2. \tag{1.6}$$

G_a is bounded on $L^{s,t}$ to $L^{q,r}$ if either

$$1/q + 2/mr = 1/2 \quad \text{and} \quad 1/s + 2/mt = 1/2 + 2/m, \tag{1.7}$$

or

$$1/q + 1/s = 1 \quad \text{and} \quad 1/t - 1/r = 1 - (m/2)(1/s - 1/q), \tag{1.8}$$

with the parameters restricted by

$$1/2 - 1/m < 1/q \leqslant 1/2 \leqslant 1/s < 1/2 + 1/m \tag{1.9}$$

in either case. (Note that these results do not depend on a. This is obvious since they were deduced from the Sobolev inequalities using only absolute value estimates for the Green function of $U(t)$.)

We shall extend these results to wider ranges of the parameters.

Geometric notation. In order to describe various estimates in concise form, we find it convenient to use the geometric notation introduced in [7]. Slightly deviating from [7], we denote by $\square$ the closed unit square in $\mathbb{R}^2$, defined by $0 \leqslant x, y \leqslant 1$. Then we set $L(P) = L^{q,r}$ if

$P = (1/q, 1/r) \in \square$, and write $1/q = x(P)$, $1/r = y(P)$; $y(P)$ is sometimes called the *height* of P. The norm in $L(P)$ is denoted by $|||\ : L(P)|||$ or, more briefly, by $|||\ : P|||$. (If $y(P) = 0$, it is often convenient to replace $L(P) = L^{q,\infty}$ by $BC(L^q)$, where BC is the class of bounded and continuous functions. For simplicity, we do not use this modification in the present paper.)

The segment connecting $P, Q \in \square$ is denoted by $[PQ]$, $[PQ[$, $]PQ]$, or $]PQ[$, according as it is closed, open, etc. Sometimes we regard each $P \in \square$ also as a 2-vector (with origin $O = (0,0)$), so that $P + Q$ and kP $(k > 0)$ make sense as long as they are in $\square$.

The convenience of such notations will be seen from the following rules (see [7]).

(1.10a) $\quad L(P)^* = L(P') \quad \text{if} \quad P + P' = (1,1), \quad y(P) > 0,$

(1.10b)

$$|||fg : P + Q||| \leqslant |||f : P|||\,|||g : Q|||, \quad |||f^k : kP||| = |||f : P|||^k, \quad k > 0,$$

(1.10c) $\quad L(P) \cap L(Q) \subset L(R) \subset L(P) + L(Q) \quad \text{for} \quad R \in [PQ].$

We introduce some special points in $\square$:

$$B = (1/2, 0), \quad C = (1/2 - 1/m, 1/2) \quad (C = (0, 1/4) \text{ if } m = 1),$$
$$E = (1/2 - 1/m, 1), \quad F = (1/2 - 1/m, 0)$$
$$(E = (0, 1/2),\ F = (0,0) \text{ if } m = 1),$$
$$B' = (1/2, 1), \quad C' = (1/2 + 1/m, 1/2) \quad (C' = (1, 3/4) \text{ if } m = 1),$$
$$E' = (1/2 + 1/m, 0), \quad F' = (1/2 + 1/m, 1)$$
$$(E' = (1, 1/2),\ F' = (1,1) \text{ if } m = 1).$$

We further introduce the triangles $T = \triangle(BEF)$ and $T' = \triangle(B'E'F')$; these are assumed to be open except that B and B' are included. Note that $[BC[\in T$, $[B'C'[\in T'$.

With these notations, the known results (1.6)–(1.9) can be stated as follows.

(i) Γ is bounded on L^2 to $L(P)$ for any $P \in [BC[$.

(ii) G_a is bounded on $L(\bar{P})$ to $L(P)$ if either

(iia) $P \in [BC[$ and $\bar{P} \in [B'C'[$, or

(iib) $P \in T$ and $\bar{P} \in T'$ with

$$x(P) + x(\bar{P}) = 1, \quad x(\bar{P}) + 2y(\bar{P})/m - x(P) - 2y(P)/m = 2/m.$$

§2. The operator G_a

In this section we generalize the estimates (ii) for G_a given in Section 1, using the geometric notation throughout. It is convenient to introduce the linear functional

$$\pi(P) = x + 2y/m \quad \text{for} \quad P = (x, y) \in \square. \tag{2.1}$$

Theorem 2.1. *G_a is bounded on $L(\bar{P})$ to $L(P)$ if $P \in T$, $\bar{P} \in T'$ with $\pi(\bar{P}) - \pi(P) = 2/m$.*

Remark. Theorem 2.1 can be improved by admitting certain points P on $[BF[$ and $\bar{P}$ on $[B'F'[$. The improvement requires deeper results, and will be given in next section.

Theorem 2.1 may be expressed in still another way. The set of $P \in \mathbb{R}^2$ with $\pi(P) = \text{const}$ is a straight line with slope $-m/2$; such a line [or a segment on it] will be called a π-line [or π-segment]. $[BC[$ and $[B'C'[$ are π-segments. T is composed of a one-parameter family of π-segments l (such as $[BC[$), and likewise T' by a family of segments $\bar{l}$ of π-segments (such as $[B'C'[$). The constant value of $\pi(P)$ for $P \in l$ will be denoted by $\pi(l)$, and similarly for $\bar{l}$. The possible values of $\pi(l)$ range over $(1/2 - 1/m, 1/2 + 1/m)$ ($(0,1)$ if $m = 1$), and those of $\pi(\bar{l})$ over $(1/2 + 1/m, 1/2 + 3/m)$ ($(2,3)$ if $m = 1$); these intervals do not overlap. l will be said to be *conjugate* to $\bar{l}$, and vice versa, if $\pi(\bar{l}) - \pi(l) = 2/m$. For each l, there is a conjugate $\bar{l}$, and vice versa. In particular, $[BC[$ and $[B'C'[$ are conjugate. It is easy to see that a conjugate pair l, $\bar{l}$ have equal length, while the upper end of l and the lower end of $\bar{l}$ have equal height.

Theorem 2.1 is equivalent to saying that given any conjugate pair l, $\bar{l}$, G_a is bounded on $L(\bar{P})$ to $L(P)$ for any $P \in l$ and any $\bar{P} \in \bar{l}$.

It is obvious how Theorem 2.1 generalizes the known results (iia) and (iib) (see Section 1). In (iia), P and $\bar{P}$ were restricted on a particular conjugate pair $[BC[$, $[B'C'[$. In (iib), P may be on any l and $\bar{P}$ on any $\bar{l}$ if l, $\bar{l}$ are conjugate, but they had to correspond to each other one to one due to the condition $x(P) + x(\bar{P}) = 1$. Theorem 2.1 unites these two cases by eliminating the restrictions.

Theorem 2.1 will be proved by interpolating between these special cases using the following lemma.

Interpolation Lemma. *Assume that none of P, $\bar{P}$, Q, $\bar{Q}$ has height zero. If a linear operator maps $L(\bar{P})$ into $L(P)$ and $L(\bar{Q})$ into*

$L(Q)$ (continuously), then it maps $L((1-\theta)\bar{P}+\theta\bar{Q})$ into $L((1-\theta)P+\theta Q)$, where $0<\theta<1$.

This lemma follows directly from Bergh-Löfström [1;Theorem 5.1.2], which shows that $(L(P), L(Q))_{[\theta]} = L((1-\theta)P+\theta Q)$ with equal norm.

To prove Theorem 2.1, we may assume that $y(P), y(\bar{P}) > 0$, since the only case to the contrary is $P = B$, $\bar{P} \in [B'C'[$, for which the result is known by (iia). We begin the proof by invoking the map $\bar{P} \to P$ involved in (iib); it is defined by $x(P) + x(\bar{P}) = 1$ and $\pi(\bar{P}) - \pi(P) = 2/m$, and can be extended to an *affine* map Λ of $\mathrm{cl}(T')$ onto $\mathrm{cl}(T)$ (cl denotes the closure). Λ sends B' into B, E' into F, and F' into E. The known special case (iib) shows that G_a is bounded on $L(\bar{P})$ to $L(P)$ if $P = \Lambda(\bar{P})$, *provided that $P \in T$, $\bar{P} \in T'$.*

Now take any pair $P \in T$, $\bar{P} \in T'$ with $\pi(\bar{P}) - \pi(P) = 2/m$. We have to show that G_a maps $L(\bar{P})$ to $L(P)$. First take the case that $\bar{P}$ is above $[B'C'[$, which implies that P is above $[BC[$. Take a point $\bar{Q} \in T'$ sufficiently close to F' that the prolongation of $[\bar{Q}\bar{P}]$ meets $[B'C'[$, say at $\bar{R}$. Let Q be the image of $\bar{Q}$ under Λ, so that Q is close to E. Prolong $[QP]$ until it meets $[BC[$, say at R (this is possible if Q is sufficiently close to E, which is guaranteed if $\bar{Q}$ is close enough to F').

G_a maps $L(\bar{Q})$ to $L(Q)$ by (iib), because $Q = \Lambda(\bar{Q})$. G_a maps $L(\bar{R})$ into $L(R)$ by (iia), because $R \in [BC[$ and $\bar{R} \in [B'C'[$. According to Interpolation Lemma, therefore, the theorem will follow if we show that P divides $[QR]$ at the same ratio as $\bar{P}$ does $[\bar{Q}\bar{R}]$.

This is a simple geometric problem. Indeed, let θ be such that $\bar{P} = (1-\theta)\bar{Q}+\theta\bar{R}$. Since π is linear, we have $\pi(\bar{P}) = (1-\theta)\pi(\bar{Q})+\theta\pi(\bar{R})$. On the other hand, $\pi(\bar{R}) = \pi(R) + 2/m$, $\pi(\bar{Q}) = \pi(Q) + 2/m$, and $\pi(\bar{P}) = \pi(P) + 2/m$, by conjugacy. Hence $\pi(P) = \pi((1-\theta)Q + \theta R)$. But π is injective on $[QR]$, which has slope different from $-m/2$. It follows that $P = (1-\theta)Q + \theta R$, as required.

The case that $\bar{P}$ is below $[B'C'[$ follows from this by duality, or one may repeat the above arguments with $\bar{Q}$ close to E'. This completes the proof of Theorem 2.1.

§3. The operators Γ and Γ^*

According to the known result (i) (see Section 1), Γ is bounded on L^2 to $L(P)$ if $P \in [BC[$. In this section, we generalize this result to some other domain spaces, and deduce corresponding results for the dual operator Γ^*. We begin by noting that certain $L(P)$'s are never realized by Γ.

Lemma 3.1. *If $P \in \Box$, $P \neq B$, is on or to the right of $[BE]$ (i.e. $x(P) + y(P)/m \geqslant 1/2$), there is no nontrivial $\phi \in \mathcal{S}'$ such that $\Gamma\phi \in L(P)$. (Note that $[BE]$ has slope $-m$, twice the slope of π-lines.)*

This is an immediate consequence of the following lemma (due to Strauss [10] for $q \geqslant 2$), which limits the decay rate of a free wave.

Decay Lemma. *For any nontrivial $\phi \in \mathcal{S}'$ and $1 \leqslant q \leqslant \infty$, one has*

$$\|U(t)\phi\|_q \geqslant K\langle t\rangle^{m(1/q-1/2)}, \quad t \in \mathbb{R}, \quad \langle t\rangle = (1+t^2)^{1/2},$$

where $K > 0$ is a constant depending on ϕ. (Set $\|\psi\|_q = +\infty$ if $\psi \notin L^q$.)

Proof. Let $u = \Gamma\phi$, $v = \Gamma\psi$, with $0 \neq \phi \in \mathcal{S}'$, $\psi \in \mathcal{S}$. Then $\langle u(t), v(t)\rangle = \langle \phi, \psi\rangle \equiv K$, hence $|K| \leqslant \|u(t)\|_q \|v(t)\|_{q'}$. If we choose a special function $\psi(x) = \exp[-(x-a)^2/4s]$, $s > 0$, a direct computation gives $\|v(t)\|_{q'} = c\langle t\rangle^{m(1/q'-1/2)}$. Hence $\|u(t)\|_q \geqslant c|K|\langle t\rangle^{m(1/q-1/2)}$. This proves the required result if we can show that $K \neq 0$ for some choice of a and s. But $K = 0$ for all a and s would imply that $e^{-s\Delta}\phi = 0$ for $s > 0$, as is seen from Green's formula. On passing to the limit $s \to 0$, this gives $\phi = 0$, a contradiction.

We now prove that Γ maps certain L^p's into certain $L(P)$'s. To this end we introduce further special points

$$D = ((m-2)/2(m-1), m/2(m-1)) \in [BE[,$$
$$(D = E = (0, 1/2) \ \text{ if } \ m = 1),$$
$$D' = (m/2(m-1), (m-2)/2(m-1)) \in [B'E'[,$$
$$(D' = E' = (1, 1/2) \ \text{ if } \ m = 1).$$

(Note that O, C, D are colinear.) We set $\hat{T} = \triangle(BCD) \subset T$, which is supposed to include the side $]CD[$ (except for $m = 2$) but no other boundary points. Similarly we define $\hat{T}' = \triangle(B'C'D') \subset T'$.

Theorem 3.2. *Let $1/2 < 1/p < m/2(m-1)$ ($1/2 < 1/p \leqslant 1$ if $m = 1$). Then Γ is bounded on L^p to $L(P)$ for any $P \in \hat{T}$ with $\pi(P) = 1/p$. Γ^* is bounded on $L(\bar{P})$ to $L^{p'}$ for any $\bar{P} \in \hat{T}'$ with $\pi(\bar{P}) = 1/p' + 2/m$.*

Corollary 3.3. *If $(2m+2)/(m+2) < p \leqslant 2$, Γ is bounded on L^p to $L^q(\mathbb{R} \times \mathbb{R}^m)$ for $q = (m+2)p/m$.*

Remark. Corollary 3.3 generalizes the well known result of Strichartz [12]. The restriction on p comes from the fact that the line

$\pi(x,y) = 1/p$ must meet the diagonal $x = y$ *inside* $\hat{T}$. The lower limit of the possible values of q is $2 + 2/m$, and corresponds to the maximal decay.

Proof of Theorem 3.2. The following is an adaptation of a method used by Giga [4] for the heat operator $e^{-t\Delta}$. First fix q such that

$$(3.1) \qquad 1/2 - 1/m < 1/q < 1/2 \quad (0 \leqslant 1/q < 1/2 \text{ if } m = 1).$$

Let $Q \in [BC[$ with $x(Q) = 1/q$, so that $\pi(Q) = 1/2$. The special case (i) (Section 1) shows that Γ maps L^2 (continuously) into $L(Q)$. On the other hand, $\phi \in L^{q'}$ implies that $\|U(t)\phi\|_q \leqslant c|t|^{-m(1/2-1/q)}\|\phi\|_{q'}$. Thus Γ maps $L^{q'}$ into $L_*(R)$, where $R = (1/q, m(1/2 - 1/q)) \in [BE[$, hence $\pi(R) = 1/q'$, and where L_* denotes the *weak* L-space with respect to the time variable. Since Q and R are on the same vertical line $x = 1/q$, it follows from Marcinkiewitz's interpolation theorem that if

$$(3.2) \qquad 1/2 < 1/p < 1/q',$$

then Γ maps L^p into $L(P)$ with

$$(3.3) \qquad x(P) = 1/q \quad \text{and} \quad \pi(P) = 1/p,$$

provided that

$$(3.4) \qquad y(P) \leqslant 1/p.$$

We now change the viewpoint and vary q, with $p < 2$ fixed. Then (3.3) shows that P moves on a π-segment with $x(P) = 1/q$, restricted by $1/2 - 1/m < x(P) < 1/p'$, due to (3.1) and (3.2). This proves the theorem for $m \leqslant 2$, since (3.4) is automatically satisfied. If $m \geqslant 3$, (3.4) introduces a new restriction; combined with (3.3), it requires that $y(P) \leqslant \pi(P) = x(P) + 2y(P)/m$, hence $x(P)/y(P) \geqslant (m-2)/m$. This means that P must be below the ray extending $[OD[$. Thus P must belong to $\hat{T}$. Summing up, we have proved Theorem 3.2.

If $p > 2$, Theorem 3.2 is not true. However, there is an analogous result with L^p replaced by a certain subspace. As is well known, the Fourier transform $\mathcal{F}$ on $\mathbb{R}^m$ maps $L^{p'}$ into L^p. We shall denote its image by $\tilde{L}^p$, and make it into a normed space with the norm $\|\phi\|_p^\sim = \|\mathcal{F}^{-1}\phi\|_{p'}$. Obviously $\tilde{L}^p$ is a Banach space, isometrically isomorphic with $L^{p'}$.

Theorem 3.4. *Let $2 \leqslant p \leqslant \infty$. The map Γ is bounded on $\tilde{L}^p$ to $L(P)$ if P is in the triangle $\Delta(OBC)$ with $\pi(P) = 1/p$. The triangle is assumed to exclude $]OC[$ but otherwise closed.*

Corollary 3.5. *If $2 \leqslant p \leqslant \infty$, Γ is bounded on $\tilde{L}^p$ to $L^q(\mathbb{R} \times \mathbb{R}^m)$ for $q = (m+2)p/m$.*

Proof of Theorem 3.4. In view of the definition of $\tilde{L}^p$, Theorem 3.4 is equivalent to saying that $\Gamma \circ \mathcal{F}$ maps $L^{p'}$ into $L(P)$ if P is as stated in the theorem. This is true for $p' = 2 = p$ by (i). Moreover, $\Gamma \circ \mathcal{F}$ maps L^1 into $BC(L^\infty)$. Indeed, $\psi \in L^1$ implies $U(t)\mathcal{F}\psi = \mathcal{F}\omega(t)$, where $\omega(t)(\xi) = \exp(-it\xi^2)\psi(\xi)$, so that $\omega \in BC(L^1)$, hence $\Gamma\mathcal{F}\psi = \mathcal{F}\omega \in BC(L^\infty)$. The assertion then follows by another application of the interpolation theorem [1;Theorem 5.1.2] to the pair $BC(L^\infty) \subset L(O)$ and $L(P)$, with P varying on $[BC[$.

Unfortunately, the range of the P's in Theorems 3.2, 3.4 does not cover the basic triangle T. But this does not mean that the region left out cannot be realized. In fact it is easy to see that $\Gamma\phi \in L(P)$ for all $P \in \square$ to the left of $[BE[$, if ϕ is a sufficiently nice function. Actually we are not so much interested in P outside the triangle $T = \Delta(BEF)$. Thus the following theorem gives a convenient criterion; here Σ denotes the Ginibre-Velo class $H^1 \cap L^2_1$, where L^2_1 is the weighted L^2-space $\langle x \rangle^{-1} L^2$, $\langle x \rangle = (1 + |x|^2)^{1/2}$.

Theorem 3.6. *For any $P \in T \cup [BF[$, Γ is bounded on Σ to $L(P)$. For any $\bar{P} \in T' \cup [B'F'[$, Γ^* is bounded on $L(\bar{P})$ to Σ^*.*

Proof. $\phi \in \Sigma$ implies that $\phi \in L^{q'}$ for $1/2 \leqslant 1/q' < 1/2 + 1/m$ and that $\phi \in H^1$. Hence $\|U(t)\phi\|_q \leqslant K\langle t \rangle^{-m(1/2-1/q)}$ (maximal decay) for $1/2 - 1/m < q \leqslant 1/2$, which implies that $\Gamma\phi \in L^{q,r}$ for $0 \leqslant 1/r < m(1/2 - 1/q)$. Thus $\Gamma\phi \in L(P)$ for any $P \in T \cup [BF[$. The second part of the theorem follows by duality.

Finally we prove the promised improvement of Theorem 2.1. For this we need another set of special points. Let

$$H = ((m-2)/2(m-1), 0), \quad H' = (m/2(m-1), 1)$$
$$(H = (0,0),\ H' = (1,1) \text{ if } m = 1).$$

Theorem 2.1 (improved). *Let $P \in T \cup [BH[$ and $\bar{P} \in T' \cup [B'H'[$ with $\pi(\bar{P}) - \pi(P) = 2/m$. Then G_a is bounded on $L(\bar{P})$ to $L(P)$. (H and H' are introduced to avoid empty statement.)*

Proof. It suffices to consider the case $P \in [BH[$ or $\bar{P} \in [B'H'[$. In the first case, let $P = (1/q, 0) \in [BH[$ and set $g = G_a f$, $f \in L(\bar{P})$. Then

$$
\begin{aligned}
g(t) &= \int a(t,s)U(t-s)f(s)\,ds \\
&= \int a(t,s+t)U(-s)f(s+t)\,ds = \Gamma^*(a_t f_t),
\end{aligned}
\tag{3.5}
$$

where

$$
a_t(s) = a(t, s+t), \quad f_t(s) = f(s+t). \tag{3.6}
$$

But Γ^* is bounded on $L(\bar{P})$ to L^q by Theorem 3.2, since $\pi(\bar{P}) - 1/q = 2/m$. Hence $\|g(t)\|_q \leqslant c|||a_t f_t : \bar{P}||| \leqslant c|||f_t : \bar{P}||| = c|||f : \bar{P}|||$. This shows that G_a is bounded on $L(\bar{P})$ to $L^{q,\infty} = L(P)$. The case $\bar{P} \in [B'H'[$ then follows by duality.

§4. Further estimates

1. *Free waves.* By a free wave in general we mean a solution $u \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}^m)$ of the free Schrödinger equation $\partial_t u - i\Delta u = 0$. Such u may be identified with a function $u \in C^\infty(\mathbb{R}; \mathcal{S}')$, where $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^m)$ (see Schwartz [8]). Equivalently, we may write $u = \Gamma\phi$, where $\phi = u(0) \in \mathcal{S}'$. In fact $\{U(t)\}$ forms a C^∞-group on $\mathcal{S}'$. Thus $\Gamma\phi$ is a general form of the free wave if we allow all $\phi \in \mathcal{S}'$. It is also well known that $U(t)$ forms a strongly continuous group on Σ (for Σ see Section 3). Since Σ is a Hilbert space, it follows by duality that $U(t)$ also forms a strongly continuous group on Σ^*. However, these groups are not uniformly bounded.

2. *Free waves in $L(P)$.* We denote by $\underline{L}(P)$ the set of free waves belonging to $L(P)$. It is easy to see that $\underline{L}(P)$ is a closed linear manifold in $L(P)$. Lemma 3.1 shows that $\underline{L}(P) = \{0\}$ if P is on or to the right of $[BE]$; otherwise $\underline{L}(P)$ is a rather large space, as is seen from Theorem 3.6.

Lemma 4.1. *Let $P \in T$. If $u \in \underline{L}(P)$, then $u \in \dot{C}(\mathbb{R}; \Sigma^*)$. ($\dot{C}$ denotes the class of continuous functions that tend to zero as $t \to \pm\infty$.)*

Proof. $u \in \underline{L}(P)$ implies that $u(s) \in L^q$ for almost all s, where $1/q = x(P)$. But $L^q \subset \Sigma^*$, since $\Sigma \subset L^{q'}$ by $1/2 - 1/m < 1/q \leqslant 1/2$. Since $u(t) = U(t-s)u(s)$, it follows that $u \in C(\mathbb{R}; \Sigma^*)$.

To analyze the behavior of $u(t)$ for large t, let $\psi \in \Sigma$ and $v(t) = U(t)\psi \in \Sigma$. We shall estimate $\langle u(t), \psi\rangle$.

$$
|\langle u(t), \psi\rangle| = |\langle u(t+s), v(s)\rangle| \leqslant \|u(t+s)\|_q \|v(s)\|_{q'}.
$$

But $\|\omega\|_{q'} \leqslant \|\langle x\rangle\omega\|_2 \|\langle x\rangle^{-1}\|_\sigma$ for any $\omega \in L^{q'}$, where $1/\sigma = 1/q' - 1/2 = 1/2 - 1/q < 1/m$ (see above) so that $\|\langle x\rangle^{-1}\|_\sigma = c < \infty$. Thus

$$\|v(s)\|_{q'} \leqslant c\|\langle x\rangle v(s)\|_2 = c\|\langle x\rangle U(s)\psi\|_2 = c\|U(s)\langle x + 2is\partial\rangle\psi\|_2$$
$$= c\|\langle x + 2is\partial\rangle\psi\|_2 \leqslant c\langle s\rangle\|\psi\|_\Sigma.$$

(Here we have used the operator calculus involving $x\cdot$ and $U(s)$ (see e.g. Ginibre-Velo [5]).) Thus we obtain

$$|\langle u(t), \psi\rangle| \leqslant c\langle s\rangle\|u(t+s)\|_q\|\psi\|_\Sigma.$$

We integrate this inequality in s, after multiplying with a weight function $\kappa(s) \geqslant 0$ with L^1-norm one, with a bounded support including $s = 0$. Since $\|u(\cdot)\|_q$ has finite L^r-norm $|||u : P|||$, where $1/r = y(P)$, it follows that $|\langle u(t), \psi\rangle| \leqslant c|||\kappa u_t : P|||\|\psi\|_\Sigma$, where $u_t(s) = u(t+s)$. Since this is true for any $\psi \in \Sigma$, we conclude that

$$u(t) \in \Sigma^* \quad \text{with} \quad \|u(t)\|_{\Sigma^*} \leqslant c|||\kappa u_t : P|||.$$

Since $|||u : P|||$ is finite, the right member tends to zero as $t \to \pm\infty$ if $y(P) > 0$.

This argument does not work if $y(P) = 0$. But $y(P) = 0$ occurs only if $P = B$, in which case $u(t) \in L^2$ for almost all t, hence $u \in L(Q)$ for every $Q \in [BC[$ by (i) (Section 1). Choosing any such Q with $y(Q) > 0$, we see that the required result holds also for $P = B$.

Remark. Given $u \in \underline{L}(P)$ with $P \in T$, how can one characterize $\phi = u(0)$, or $u(t)$ in general? Unfortunately we have no answer to this question, beyond the fact that $u(t) \in \Sigma^*$.

3. *The range of* G_a. In Section 2 we proved that G_a is bounded on $L(\bar{P})$ to $L(P)$ for certain P and $\bar{P}$. Since G_a is an integral operator, it is expected that the functions produced by G_a are continuous in some sense or other, unless the function a is ill-behaved.

Lemma 4.2. *Suppose that a has the property that for each $t \in \mathbb{R}$, $t_n \to t$ implies $a(t_n, s + t_n) \to a(t, s + t)$ for almost every $s \in \mathbb{R}$. (This condition is satisfied for $G_a = G$, $G_\pm$.) If $f \in L(\bar{P})$ with $\bar{P} \in T'$, then $G_a f \in \dot{C}(\mathbb{R}; \Sigma^*)$.*

Proof. Let $g = G_a f$ where $f \in L(\bar{P})$, $\bar{P} \in T'$. Then we have the relations (3.5–6). Since Γ^* maps $L(\bar{P})$ continuously into Σ^* (see Theorem 3.6), we have $g(t) \in \Sigma^*$, with $\|g(t)\|_{\Sigma^*} \leqslant c|||f : \bar{P}|||$.

Next we prove that $g(t) \in \Sigma^*$ is continuous in t. To this end we compute

$$g(\tau) - g(t) = \Gamma^*(a_\tau f_\tau - a_t f_t) = \Gamma^*[a_\tau(f_\tau - f_t) + (a_\tau - a_t)f_t].$$

It suffices to show that the expression in [ ] tends to zero in $L(\bar{P})$ as $\tau \to t$ along any sequence t_n. This is true of $a_\tau(f_\tau - f_t)$, since translation is continuous on $L(\bar{P})$. The same is true of $(a_\tau - a_t)f_t$ by dominated convergence, since by hypothesis $a(t_n, s + t_n) \to a(t, s + t)$ as $t_n \to t$, for almost all s. This proves the continuity of $g(t)$.

It remains to show that $g(t) \to 0$ in Σ^* as $t \to \pm\infty$. To this end we take any $\epsilon > 0$ and write $f = f' + f''$, where f' is supported on $(-\infty, \tau)$ and f'' on (τ, ∞), with τ sufficiently large that $|||f'' : \bar{P}||| < \epsilon$. Set $g' = G_a f'$, $g'' = G_a f''$. It follows from the preceding results that both $g'(t)$ and $g''(t)$ are continuous and bounded in Σ^*, with $\|g''(t)\|_{\Sigma^*} \leqslant c\epsilon$. On the other hand $g'(t)$ coincides with a free wave for $t > \tau$. Thus Lemma 4.1 shows that $g'(t)$ tends in Σ^* to zero as $t \to \infty$. Since ϵ may be arbitrarily small, we have shown that $g(t) \to 0$ as $t \to \infty$. Similarly we can prove the same result for $t \to -\infty$.

Lemma 4.3. *Suppose that for each $t \in \mathbb{R}$, $t_n \to t$ implies $a(t_n, s) \to a(t, s)$ for almost all s. (This condition is met for $G_a = G$, $G_\pm$.) Let $h(t) = U(-t)(G_a f)(t)$, where $f \in L(\bar{P})$ with $\bar{P} \in T'$. Then $h \in BC(\mathbb{R}; \Sigma^*)$. If, in particular, $G_a = G_+$ $[G_-]$, then $h(t) \to 0$ in Σ^* as $t \to \infty$ $[-\infty]$.*

Proof. We have

$$h(t) = \int_{-\infty}^{\infty} a(t, s) U(-s) f(s)\, ds = \Gamma^* q_t, \quad q_t(s) = a(t, s) f(s).$$

Since $|||q_t : \bar{P}||| \leqslant |||f : \bar{P}|||$, the result follows as in the proof of Lemma 4.2, except that h need not tend to zero as $t \to \pm\infty$. (In fact h is constant if $a \equiv 1$.)

If $G_a = G_+$, then $a(t, s) = 0$ for $s < t$, so that $q_t \to 0$ in $L(\bar{P})$ as $t \to \infty$. Hence $h(t) \to 0$ in Σ^* as $t \to \infty$. G_- can be handled in the same way.

§5. A miniature scattering theory for NLS

In this section we shall construct a scattering theory for small solutions of (NLS), assuming, for simplicity, that

(5.1) $\quad |F'(\zeta)| \leqslant M'|\zeta|^{k-1}, \quad F(0) = 0,$ where $k > 1$ is a constant.

This implies that $|F(\zeta)| \leqslant M|\zeta|^k$ with some M; we may set $M' = M$.

Our solution u will belong to $L(P)$, where $P \in T$ is a *k-point*, by which we mean that P and kP form a conjugate pair (see Section 2). Obviously $y(P) > 0$ for a k-point P.

If P is a k-point, then $kP \in T'$ and $(k-1)\pi(P) = \pi(kP) - \pi(P) = 2/m$, hence

$$\pi(P) = 2/(k-1)m. \tag{5.2}$$

Thus $\pi(P)$ is determined by k only and decreases with increasing k. Moreover, since $P \in T$ implies $1/2 - 1/m < \pi(P) < 1/2 + 1/m$, it follows from (5.2) that $1 + 4/(m+2) < k < 1 + 4/(m-2)$. But this is not sufficient; we have

Lemma 5.1. *In order that there exist a k-point, it is necessary and sufficient that*

$$[m + 2 + (m^2 + 12m + 4)^{1/2}]/2m < k < 1 + 4/(m-2). \tag{5.3}$$

The right member should read ∞ *if* $m \leqslant 2$.

Remark. Lemma 5.1 wil be proved below. (5.3) is a familiar condition that recurs in various situations for NLS, see e.g. [2, 3, 11, 13]. It is of some interest that it occurs here as a simple geometric condition. Under condition (5.3), a typical k-point is given by

$$P = (1/(k+1), 1/(k-1) - m/2(k+1)). \tag{5.4}$$

Of course any points sufficiently close to P on the π-line through P are k-points.

In what follows we have to do with free waves that are *asymptotic* to solutions u of (NLS). In general we say that two functions $u, v \in C(\mathbb{R}; \mathcal{S}')$ are asymptotic to each other at ∞, and write "$u \sim v$ at ∞", if $U(-t)(u(t) - v(t)) \to 0$ as $t \to \infty$. Similarly we define "$u \sim v$ at $-\infty$". Obviously the relation $u \sim v$ is invariant under simultaneous translation of u, v in t. We also note that given u, there is at most one *free wave* v such that $u \sim v$ at ∞, and similarly at $-\infty$. This follows from the fact that $U(-t)v(t) = v(0)$ for a free wave v.

Theorem 5.2. *Let P be a k-point, and $u \in L(P)$ a solution of* (NLS). *Then there are unique free waves $u_\pm \in \underline{L}(P)$ that are asymptotic to u at $\pm\infty$. The maps $u \mapsto u_\pm$ are continuous and injective from $L(P)$ to $\underline{L}(P)$, and in fact uniformly continuous on bounded sets in $L(P)$.*

Proof. Uniqueness of $u_\pm$ is obvious from the remark above. We shall construct u_+ (u_- can be similarly handled). Set $w = -iG_+F(u) \in$

$L(P)$, which exists because $F(u) \in L(kP)$ (by (1.10b)) and P, kP are conjugate. Then $(\partial_t - i\Delta)w = -iF(u)$. Since $(\partial_t - i\Delta)u = -iF(u)$, we have $(\partial_t - i\Delta)(u-w) = 0$, so that $u_+ \equiv u - w \in \underline{L}(P)$, and we can write $u = u_+ - iG_+F(u)$. That $u \sim u_+$ at ∞ follows from Lemma 4.3. The map $u \mapsto u_+$ is uniformly continuous on bounded sets since $u \mapsto F(u)$ from $L(P)$ to $L(kP)$ and $F(u) \mapsto w = G_+F(u)$ from $L(kP)$ to $L(P)$ have the same property (see Theorem 2.1).

The proof that $u \mapsto u_+$ is injective is more complicated. Suppose that there is another solution $v \in L(P)$ of (NLS). Then we have as above $v = v_+ - iG_+F(v)$, where $v_+ \in \underline{L}(P)$ and $v \sim v_+$ at ∞. we claim that if $v_+ = u_+$ then $v = u$. Indeed $v_+ = u_+$ implies

$$u - v = -iG_+(F(u) - F(v)) \tag{5.5}$$

on subtraction. We divide $(-\infty, \infty)$ into a finite number of subintervals $I_0 = (-\infty, T_1)$, $I_1 = (T_1, T_2)$, $\ldots$, $I_n = (T_n, \infty)$, and set $u_j = \chi_j u$, $v_j = \chi_j v$, where χ_j is the characteristic function of I_j. Since $|||u : P|||$ and $|||v : P|||$ are finite, for any $\epsilon > 0$ we can choose n and the I_j so that $|||u_j : P|||^{k-1} + |||v_j : P|||^{k-1} \leqslant \epsilon$.

Let us compute $u_j - v_j$ by multiplying (5.5) with χ_j. Since G_+ is of Volterra type, with integration on (t, ∞), there is no contribution from the parts u_i, v_i with $i \leqslant j$. Since G_+ is bounded on $L(kP)$ to $L(P)$ and since

$$|F(u_i) - F(v_i)| \leqslant cM|u_i - v_i|(|u_i|^{k-1} + |v_i|^{k-1}),$$

we obtain (cf. [7] for this computation)

$$\begin{aligned} |||u_j - v_j : P||| &\leqslant c\sum_{i=j}^{n} |||F(u_i) - F(v_i) : kP||| \\ &\leqslant cM\sum_{i=j}^{n} |||u_i - v_i : P|||(|||u_i : P|||^{k-1} + |||v_i : P|||^{k-1}) \\ &\leqslant cM\epsilon\sum_{i=j}^{n} |||u_j - v_j : P|||. \end{aligned} \tag{5.6}$$

Now assume that ϵ is chosen so small that $cM\epsilon < 1$. If we set $j = n$ in (5.6), we obtain $|||u_n - v_n : P||| \leqslant cM\epsilon|||u_n - v_n : P|||$, hence $u_n = v_n$. On setting $j = n-1$, then, we have $|||u_{n-1} - v_{n-1} : P||| \leqslant cM\epsilon|||u_{n-1} - v_{n-1} : P|||$, hence $u_{n-1} = v_{n-1}$. Proceding in the same way, we obtain $u_j = v_j$ for $j = 0, 1, \ldots, n$, hence $u = v$.

We now construct a scattering theory for small solutions in $L(P)$.

Theorem 5.3. *Let P be a k-point. Then there exist balls $B_\pm$ in $\underline{L}(P)$ and a ball B in $L(P)$, with center O and positive radii, with the following properties.*

(a) *If $u_- \in B_-$,* (NLS) *has a unique global solution $u \in B$ such that $u \sim u_-$ at $-\infty$.*

(b) *There is a unique free wave $u_+ \in \underline{L}(P)$ such that $u \sim u_+$ at ∞.*

(c) *The scattering operator $S : u_+ = Su_-$ is well defined and is continuous and injective on B_- to $\underline{L}(P)$.*

(d) *The range of S covers B_+.*

(e) *All u and $u_\pm$ belong to $\dot{C}(\mathbb{R}; \Sigma^*)$.*

Remark. Our scattering operator S acts on space-time functions, and differs from the conventional ones, which act on space functions. Our viewpoint is in conformity with the idea of Segal (see e.g. [9]).

Proof. To construct the solution u, we solve the integral equation $u = \Phi_-(u) \equiv u_- - iG_-F(u)$ by a routine method (such as was used in [6,7]; see Section 1 for $G_\pm$). Indeed, given $v \in L(P)$, we have $F(v) \in L(kP)$, with $|||F(v) : kP||| \leqslant M|||v : P|||^k$. Since P and kP are conjugate, we obtain $|||\Phi_-(v) : P||| \leqslant |||u_- : P||| + cM|||v : P|||^k$ by Theorem 2.1. It follows that Φ_- sends a certain ball B of $L(P)$ into itself if $|||u_- : P|||$ is sufficiently small. An analogous estimate using the Lipschitz continuity of F shows that Φ_- is a contraction on B. Thus Φ_- has a unique fixed point u in B, which is a (weak) solution of (NLS). Lemma 4.3 then shows that $u \sim u_-$ at $-\infty$.

Since we are using the contraction theorem, the uniqueness of u in B is obvious. Moreover, the continuity of the map $u_- \mapsto u$ follows easily.

The existence of u_+, hence of S too, follows from Theorem 5.2. Since the map $u \mapsto u_+$ is injective and uniformly continuous on bounded sets, the same is true of S. Property (e) follows from Lemmas 4.1–2.

Finally we note that the role of u_- and u_+ may be reversed to construct the inverse operator $S^{-1} : u_- = S^{-1}u_+$ for sufficiently small $u_+ \in \underline{L}(P)$. Since $|||u_- : P||| \leqslant \text{const}|||u_+ : P|||$ for sufficiently small $|||u_+ : P|||$ (due to the uniform continuity proved above), we have $S^{-1}B_+ \subset B_-$ if B_+ is sufficiently small. This shows that the range of S covers B_+.

Proof of Lemma 5.1. We recall some properties of the generic conjugate pair $l, \bar{l}$. l and $\bar{l}$ are parallel and have the same length; the upper end Q of l is on the vertical side $]EF[$ of T, the lower end $\bar{Q}$ of $\bar{l}$ is on the vertical side $]E'F'[$ of T', and $Q, \bar{Q}$ have the same height, which we denote by h. Let R denote the lower end of l, and $\bar{R}$ the upper end of $\bar{l}$.

Obviously a k-point $P \in l$ exists *with some* $k > 1$ if and only if there is a ray OX from the origin O that meets both l and $\bar{l}$; in this case

$k = \pi(\bar{l})/\pi(l)$, since l and $\bar{l}$ are parallel, so that k does not depend on the exact position of the ray.

If $h \leqslant 1/2$ so that l is on or below $[BC[$, R is on the bottom side $[BF[$ of T. Thus the ray $O\bar{P}$ meets l if $\bar{P} \in \bar{l}$ is sufficiently low, hence k-points exist on l for some k. If we let $h \to 0$, so that l shrinks to the point $F = (1/2 - 1/m, 0)$, and $\bar{l}$ to $E' = (1/2 + 1/m, 0)$, the ratio $k = \pi(\bar{l})/\pi(l)$ approaches $(1/2 + 1/m)/(1/2 - 1/m) = (m + 2)/(m - 2)$. If $h = 1/2$, then $l = [BC[$, $\bar{l} = [B'C'[$, and $k = 1 + 4/m$.

The case that l is above $[BC[$ is more complicated. In this case R is on the hypotenuse BE of T and $\bar{R}$ is on the upper side $[B'F'[$ of T'. If h is not too large, the ray OR is still below the ray $O\bar{R}$, so that there is a ray OX that meets both l and $\bar{l}$. If h is increased, this ceases to be the case eventually. The critical value of h can be determined by the condition that the two rays OR and $O\bar{R}$ coincide. An elementary algebra gives the value of h, then of k, which turns out to be the value on the left side of (5.3). Since k decreases with increasing h, we have proved the lamma.

Acknowledgement. The writer is indebted to A. Jensen, G. Ponce, and W. Strauss for various critical and informative comments.

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