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# Rational Solutions of the Ernst Equation

**Kiyokazu Nagatomo**

**Abstract.**

We find infinitely many rational solutions of the Ernst equation in general relativity. These are constructed by solving a formal version of Hilbert's homogeneous problem and are expressed in terms of determinants of Toeplitz type matrices.

## §1. Introduction

In this note we consider a family of rational solutions of the Ernst equation

$$
\left\{
\begin{array}{l}
f\nabla^2 f - (\partial_z f)^2 - (\partial_\rho f)^2 + (\partial_z e)^2 + (\partial_\rho e)^2 = 0,\\
f\nabla^2 e - 2(\partial_z f \partial_z e + \partial_\rho f \partial_\rho e) = 0,
\end{array}
\right.
\tag{1}
$$

where $\nabla^2$ is the 3-dimensional Laplace operator acting on axially symmetric functions ; $\nabla^2 = \partial_\rho^2 + (1/\rho)\partial_\rho + \partial_z^2$. In the previous paper [1] we have discussed the following initial value problem for Equation (1) with an initial value at $\rho = 0$:

$$
\begin{array}{cc}
f(z,\rho)\mid_{\rho=0} = f(z), & e(z,\rho)\mid_{\rho=0} = e(z)\\
f(z) \in \mathbf{R}[[z]]^\times, & e(z) \in \mathbf{R}[[z]]
\end{array}
\tag{2}
$$

where $\mathbf{R}[[z]]$ denotes the set of the formal power series in $z$ and $\mathbf{R}[[z]]^\times$ is the set of invertible elements in $\mathbf{R}[[z]]$. We have proven that the above initial value problem is uniquely soluble in the category $\mathbf{R}[[z,\rho]]$ and have found several special solutions which are rational with respect to the variables $z$ and $\rho$. The aim of this note is to clarify the reason why these solutions are rational. Recall that the initial values of all these solutions have the following algebraic properties.

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**Rationality.** *The initial values $f(z)$ and $e(z)$ are rational functions of $z$ of the form*

$$f(z) = \frac{1}{a(z)}, \qquad e(z) = \frac{b(z)}{a(z)} \tag{3}$$

*where $a(z)$ and $b(z)$ are polynomials of $z$ such that $a(0) \neq 0$ and $a|1+b^2$.*

We will show that this rationality is not accidental, that is, any solution whose initial value has the above property is a rational function.

**Theorem.** *Let $a$ and $b$ be polynomials of $z$ such that $a(0) \neq 0$ and $a|1+b^2$. Then the solution of the Ernst equation with the initial value $f(z,\rho)\,|_{\rho=0} = 1/a(z)$, $e((z,\rho)\,|_{\rho=0} = b(z)/a(z)$ is a rational function of $z$ and $\rho$.*

We prove this theorem by solving a formal Hilbert's homogeneous problem associated with our initial value problem. In this proof we determine the explicit form of the solution from the prescribed initial value; if the initial value has the property (3), then corresponding solution is expressed in terms of determinants of Toeplitz type matrices whose components are polynomials of the variables $z$ and $\rho$. Hence the rationality is immediately proven.

To reduce our problem to Hilbert's homogeneous problem , we need another expression of the Ernst equation. It is well known [2] that the Ernst equation is equivalent to the following 2nd order differential equation for a $2 \times 2$ matrix $\tau$ with supplementary conditions:

$$\partial_z(\rho \partial_z \tau \cdot \tau^{-1}) + \partial_\rho(\rho \partial_\rho \tau \cdot \tau^{-1}) = 0 \tag{4a}$$

$$\det(\tau) = 1, \qquad {}^t\tau = \tau \tag{4b}$$

This equivalence is given by

$$\tau = \begin{bmatrix} -\frac{f^2+e^2}{f} & \frac{e}{f} \\ \frac{e}{f} & -\frac{1}{f} \end{bmatrix}. \tag{5}$$

In this expression, Condition (3) is replaced by Condition (4b) and "every component of $\tau(z)$ is a polynomial of $z$". Hence it suffices to prove that if an initial value $\tau(z)$ satisfies the conditions mentioned above, then the corresponding solution of Equation (4) surely exists and every component of this is a rational function of the variables $z$ and $\rho$.

## §2. Linear problem and Hilbert's homogeneous problem

Let $\mathcal{C}$ be the set of all $2 \times 2$ matrices with components in $\mathbf{R}$ (real number). Hereafter, $\mathcal{C}[[z]]$ and $\mathcal{C}[[z, \rho]]$ denote respectively the set of all formal power series in $z$ and $(z, \rho)$ with coefficients in $\mathcal{C}$.

Let us consider a (formal) initial value problem

$$\begin{gathered} \partial_z(\rho\partial_z\tau \cdot \tau^{-1}) + \partial_\rho(\rho\partial_\rho\tau \cdot \tau^{-1}) = 0 \\ \tau(z, \rho)\,|_{\rho=0} = \tau(z) \end{gathered} \tag{6}$$

where $\tau(z, \rho) \in \mathcal{C}[[z, \rho]]^\times$ and $\tau(z) \in \mathcal{C}[[z]]^\times$. In [1] we have shown that the initial value problem (6) is uniquely soluble and that if an initial value $\tau(z)$ satisfies Condition (4b), then this property is preserved for any value of $\rho$.

The key to analyze the above initial value problem is that Equation (4a) implicitly involves infinitely many conservation laws. The collection of these conservation laws is simply expressed by using "wave function" as follows. Let $P = \partial_z\tau \cdot \tau^{-1}$ and $Q = \partial_\rho\tau \cdot \tau^{-1}$ and introduce a new variable $\lambda$ (so-called spectral parameter). A solution of the following system of linear differential equations is called a wave function:

$$D_1 W = PW, \qquad D_2 W = QW \tag{7}$$

where $D_1 = \partial_z - \lambda\rho\partial_\rho + 2\lambda^2\partial_\lambda$ and $D_2 = \lambda\rho\partial_z + \partial_\rho$. Here $\lambda$ is regarded as a formal variable. However, in analytic category, $\lambda$ is to be considered as a homogeneous coordinates of the Riemann sphere $\mathbb{P}^1(\mathbf{C})$ and the above linear system admits many kinds of solutions corresponding to the specification of the variable $\lambda$. There are two important solutions, one is analytic at $\lambda = \infty$ and the other is analytic at $\lambda = 0$; we use notations $W$ and $V$ respectively. We can also define these two class of the solutions in our formal category. We first give fundamental properties of $W$.

**Lemma 1.** *Let $\tau(z, \rho)$ be a solution of the Ernst equation. Then there exists uniquely a solution of Equation* (7) *of the form*

$$W = 1_2 + \sum_{j=1}^{\infty} w_j(z, \rho)\lambda^{-j}, \qquad w_j(z, \rho) \in \mathcal{C}[[z, \rho]]. \tag{8}$$

*The value at $\rho = 0$ of this unique solution is evaluated by*

$$W(z, \rho, \lambda)\,|_{\rho=0} = \tau(z) \cdot [\tau(z + 1/2\lambda)]^{-1}.$$

*Furthermore if $\tau(z)^{-1}$ is a polynomial of $z$ with degree $m$, then $W$ is a polynomial of $\lambda^{-1}$ with degree at most $m$.*

*Proof.* The first two statements have been proven in [1] (Proposition 2.1). We have to prove the last statement. Using the expression of $W$ (see Equation (8)), Equation (7) is equivalent to the following infinite series of the differential equations

$$\text{(9-j)} \qquad \begin{aligned} \rho\partial_\rho w_j + 2jw_j &= \partial_z w_{j-1} - Pw_{j-1}, \\ \rho\partial_z w_j &= -\partial_\rho w_{j-1} + Qw_{j-1}. \end{aligned}$$

Eliminating $w_{j+1}$ by using Equation (9-j+1), we have

(10-j)

$$\begin{aligned} \rho\partial_\rho^2 w_j + &(2j+1-\rho Q)\partial_\rho w_j \\ &= -\rho\partial_z^2 w_j + \rho P\partial_z w_j + (\rho\partial_z P + \rho\partial_\rho Q + 2j+1)w_j. \end{aligned}$$

We first note that $w_j(z,0) = 0$ for $j \geqq m+1$. Hence it is sufficient to prove that any solution of Equation (10-j) such that $w_j(z,0) = 0$ is trivial. To do this, differentiate $r$ times both sides of Equation (10-j) by $\rho$ and set $\rho = 0$. Then we find

$$\begin{aligned} (r+2j+1)c_j^{r+1}(z) = (2j+1)&c_j^r(z) - r\partial_z^2 c_j^{r-1}(z) \\ &+ r\sum_{k=0}^{r-1}\binom{r-1}{k}\partial_\rho^k Q\,|_{\rho=0} c_j^{r-k}(z) \\ &+ r\sum_{k=0}^{r-1}\binom{r-1}{k}\partial_\rho^k P\,|_{\rho=0}\partial_z c_j^{r-1-k}(z) \\ &+ r\sum_{k=0}^{r-1}\binom{r-1}{k}(\partial_\rho^k Q + \partial_\rho^k\partial_z P)\,|_{\rho=0} c_j^{r-1-k}(z). \end{aligned}$$

where we set

$$c_j^r(z) = \partial_\rho^r w_j(z,\rho)\,|_{\rho=0}.$$

Since $c_j^0(z) = 0$ for any $j \geqq m+1$, we have $c_j^r(z) = 0, \quad r \geqq 0, \quad j \geqq m+1$. Q.E.D.

Secondly we consider another important solution, the formal version of a locally analytic solution at $\lambda = 0$.

**Lemma 2.** *Let $\tau(z,\rho)$ be a solution of the Ernst equation. Then there exists uniquely a solution of Equation* (7) *of the form*

$$V = \sum_{j=0}^{\infty} v_j(z,\rho)\lambda^j, \qquad v_j(z,\rho) \in \mathcal{C}[[z,\rho]].$$

*which satisfies* $V(z,\rho,\lambda)\,|_{\rho=0} = \tau(z)$.

*Proof.* Substituting the above expression of $V$ into Equation (7), we have

$$\text{(11-j)} \qquad \partial_z v_j - \rho\partial_\rho v_{j-1} + 2(j-1)v_{j-1} = Pv_j,$$

$$\text{(12-j)} \qquad \partial_\rho v_j + \rho\partial_z v_{j-1} = Qv_j.$$

where we set $v_{-1} = 0$. We now solve Equation (12-j) with an initial value $v_0(z,0) = \tau(z)$ and $v_j(z,0) = 0, \quad j \geqq 1$. We can easily find a unique solution of Equation (12-j);

$$v_0 = \tau(z,\rho), \qquad v_j(z,\rho) = -\tau(z,\rho)\int_0^\rho r[\tau(z,r)]^{-1}\partial_z v_{j-1}(z,r)\,dr.$$

Then we show that $v_j(z,\rho)$ defined by the above equation also satisfies Equation (11-j). Clearly $v_0$ satisfies Equation (11-0). Assume Equation (11-j) is satisfied for $v_j$. Eliminating $v_{j-1}$ by using Equations (11-j) and (12-j), we have

$$\rho(\partial_z^2 v_j + \partial_\rho^2 v_j) - (2j-1)\partial_\rho v_j + (2j-1)Qv_j - \rho P\partial_z v_j - \rho Q\partial_\rho v_j = 0.$$

Substituting this into

$$\partial_z v_{j+1} = -\partial_z\tau\int_0^\rho r\tau^{-1}\partial_z v_j\,dr - \tau\int_0^\rho r\partial_z(\tau^{-1})\partial_z v_j\,dr - \tau\int_0^\rho r\tau^{-1}\partial_z^2 v_j\,dr,$$

we have the desired result. Q.E.D.

By using $W$ and $V$ we can define a kind of transition function $u(z,\rho,\lambda)$ by

$$u(z,\rho,\lambda) = W^{-1}\cdot V.$$

Noticing the recursive definition of $v_j$ in the proof of Lemma 2, we can easily show that $v_j(z,\rho) = \rho^{2j}\widehat{v}_j, \widehat{v}_j \in \mathcal{C}[[z,\rho]]$, and hence $u(z,\rho,\lambda)$ is well defined as an element of $\mathcal{C}[[z,\rho,\lambda,\lambda^{-1}]]$. Clearly $u(z,\rho,\lambda)$ satisfies the linear differential equation $D_1 u = D_2 u = 0$. By virtue of this equation and our choice of an initial value for $V$, we can prove an important relation

$$\text{(13)} \qquad u(z,\rho,\lambda) = \tau(-\rho^2\lambda/2 + z + 1/2\lambda)$$

as follows. First we note $u(z,0,\lambda) = \tau(z+1/2\lambda)$. The right hand side of (13) clearly satisfies $D_1 u = D_2 u = 0$. It is sufficient to prove uniqueness of the solution of the equation $D_1 u = D_2 u = 0$ under the prescribed initial value. Express $u$ as $u = \sum_{j\in\mathbb{Z}} u_j(z,\rho)\lambda^j$. Similarly as the proof of Lemma 2 we have

$$(14)\qquad \begin{aligned} \partial_\rho u_{j+1} &= -\rho\partial_z u_j, \\ \partial_z u_{j+1} &= \rho\partial_\rho u_j - 2ju_j \end{aligned}$$

and hence

$$\rho(\partial_z^2 + \partial_\rho^2)u_j + (1-2j)\partial_\rho u_j = 0.$$

Therefore $u_j(z,\rho), j \leqq 0$ is uniquely determined from an initial value $u_j(z,0)$. For $j > 0$, $u_j(z,\rho)$ is uniquely determined by Equation (14) and an initial value.

**Proposition 3.**

$$(15)\qquad W^{-1}\cdot V = \tau(-\rho^2\lambda/2 + z + 1/2\lambda)$$

*If $\tau(z)$ and $\tau(z)^{-1}$ are polynomials of $z$, then both $W$ and $V$ are polynomials of $\lambda^{-1}$ and $\lambda$ respectively with coefficients in $\mathcal{C}[[z,\rho]]$.*

*Proof.* Since $\tau(z)^{-1}$ is a polynomial of $z$, Lemma 1 asserts that $W$ is a polynomial of $\lambda^{-1}$ with coefficients in $\mathcal{C}[[z,\rho]]$. On the other hand, the relation $V = W\tau(-\rho^2\lambda/2 + z + 1/2\lambda)$ shows that $V$ is a polynomial of $\lambda$, since $W$ involves no positive power of $\lambda$. Q.E.D.

In the proof of Lemma 2 we have shown $v_0 = \tau$. Hence by using Equation (15) we get

$$(16)\qquad \tau(z,\rho) = \sum_{j=0}^{\infty} w_j(z,\rho)\chi_j(z,\rho)$$

where we set $\tau(-\rho^2\lambda/2 + z + 1/2\lambda) = \sum_{j\in\mathbb{Z}} \chi_j(z,\rho)\lambda^j$. Here we remark that

$$\chi_j(z,\rho) = \rho^{2j}\widehat{\chi}_j(z,\rho), \quad \widehat{\chi}_j \in \mathcal{C}[[z,\rho]],$$

hence the right hand side of Equation (16) is well defined as an element of $\mathcal{C}[[z,\rho]]$.

## §3. Construction of rational solutions

In this section we assume that the initial value $\tau(z)$ is a polynomial of $z$ with degree $m$ and $\det\tau(z) = 1$. In this case $\tau(z)^{-1}$ is also a polynomial of $z$ with degree $m$. From Proposition 3, both $W$ and $V$ are polynomials of $\lambda^{-1}$ and $\lambda$ respectively with degree at most $m$. Substituting the expression $W = 1_2 + \sum_{j=1}^{m} w_j(z,\rho)\lambda^{-j}$ into Equation (15), since $V$ involves no negative power of $\lambda$ we have

$$\chi_k + \sum_{j=1}^{m} w_j \chi_{j+k} = 0, \qquad -1 \leqq k \leqq -m. \tag{17}$$

The important point is that $\chi_j$ is determined exactly from the initial value only. Hence if we can seek $W_j, 1 \leqq j \leqq m$ from Equation (17), by virtue of Equation (16) we have an expression of a solution of the Ernst equation in terms of the initial value. We now introduce the following three matricies $X = (w_1, w_2, \ldots, w_m)$, $A = (\chi_{j+k})_{-1 \leqq k \leqq -m, 1 \leqq j \leqq m}$, $b = {}^t(-\chi_{-1}, \ldots, -\chi_{-m})$. Then Equation (17) is simply expressed as $XA = b$. Since

$$A\,|_{z=\rho=0} = \begin{bmatrix} \tau(0) & & & \\ & \cdot & & * \\ & & \cdot & \\ & 0 & & \cdot \\ & & & \tau(0) \end{bmatrix},$$

the matrix $A$ is invertible in a neighborhood of $(z,\rho) = (0,0)$. Further every entry of $A^{-1}$ is a rational function of the variables $z$ and $\rho$.

**Proposition 4.** *Let $\tau(z) \in C[[z,\rho]]$ be a polynomial of $z$ with degree $m$ such that* $\det \tau(z) = 1$. *Then the unique solution of the initial value problem (6) is given by*

$$\tau(z,\rho) = \sum_{j=0}^{m} w_j(z,\rho)\chi_j(z,\rho)$$

*where $\tau(-\rho^2\lambda/2 + z + 1/2\lambda) = \sum_{j\in\mathbb{Z}} \chi_j(z,\rho)\lambda^j$ and $w_j(z,\rho)$ is a unique solution of the linear equations (17).*

Using the above proposition and equivalence of the Ernst equation and Equation (4) we have

**Theorem.** *Let $a$ and $b$ be polynomials of $z$ such that $a(0) \neq 0$ and $a|1+b^2$. Then the solution of the Ernst equation with the initial value $f(z,0) = 1/a(z)$, $e(z,0) = b(z)/a(z)$ is a rational function of $z$ and $\rho$.*

*Proof.* Let us define an initial value by

$$\tau(z)=\begin{bmatrix} -\frac{1+b^2}{a} & b \\ b & -a \end{bmatrix}.$$

Then the theorem is immediately derived by Proposition 4. Q.E.D.

## References

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*Department of Mathematics*
*Osaka University*
*Toyonaka, Osaka 560*
*Japan*