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Floer Homology for Oriented 3-Manifolds

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Dedicated to Professor Akio Hattori on his sixtieth birthday

Contents

§1. Introduction

In [F], A. Floer introduced a new invariant for homology 3-spheres. In this paper we generalize his invariant to arbitrary closed and oriented 3-manifolds. In the case when the first homology group of the manifold is torsion free and nonzero, we also define invariants $I_k^s(M)$ for $s < 3$, which, in the case $s = 0$, is a generalization of Floer's one. The construction of this invariant is closely related also to the Donaldson's polynomial for closed 4-manifolds [D4]. The construction is based on the study of the moduli space of selfdual connections over $M \times \mathbf{R}$ and its compactification.

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In this section, we describe briefly the construction of our invariant. Throughout this paper, we let M be an oriented 3-manifold, σ a Riemannian metric on it. It induces the Hodge $*$-operator, $*_\sigma : \Lambda^k(M) \to \Lambda^{3-k}(M)$. We consider the trivial $SU(2)$ bundle over M. Let

$$\mathcal{A}(M) = \{d + a \mid a \in \Gamma(M, \Lambda^1 \otimes su(2))\}$$

be the set of all smooth connections of it. (In later sections, we work with Sobolev spaces but in this section we omit those details.) Put

$$\begin{aligned}
\widehat{\mathcal{G}}(M) &= \{g : M \to SU(2) \mid C^\infty\text{-maps}\}, \\
\mathcal{G}(M) &= \{g \in \widehat{\mathcal{G}}(M) \mid \deg g = 0\}, \\
\mathcal{B}(M) &= \mathcal{A}(M)/\widehat{\mathcal{G}}(M), \\
\widetilde{\mathcal{B}}(M) &= \mathcal{A}(M)/\mathcal{G}(M),
\end{aligned}$$

where $\mathcal{G}(M)$ acts on $\mathcal{A}(M)$ by

$$g^*(d + a) = d + g^{-1}dg + g^{-1}ag.$$

Following Taubes [T4] and Floer [F], we define a functional $\mathfrak{cs} : \widetilde{\mathcal{B}}(M) \to \mathbf{R}$ by

$$\mathfrak{cs}(a) = \int_M \operatorname{Tr}(\frac{1}{2}a \wedge da + \frac{1}{3}a \wedge a \wedge a) \tag{1.1}$$

(Here and hereafter, we shall write a in place of $d + a$.) It is well known that the right hand side is $\mathcal{G}(M)$-invariant. The gradient flow of this functional is described by

$$\frac{\partial a_t}{\partial t} = *_\sigma F^{a_t}. \tag{1.2}$$

The idea of Floer and Taubes is to use this gradient flow in order to define the $\infty/2$-dimensional homology group of $\mathcal{B}(M)$. It is not in general true that $\operatorname{grad}\mathfrak{cs}$ is a Morse-Smale flow, then in [T4], [F], they used a perturbation of it. In their case, where M is a homology sphere, the singular locus $\mathcal{SB}(M)$ and the set of critical points of the flow $\operatorname{grad}\mathfrak{cs}$ intersect at one point, the trivial connection. (Recall that the singular locus of $\mathcal{B}(M)$ is the set of reducible connections, and a critical point of the flow $\operatorname{grad}\mathfrak{cs}$ is a flat connection.) In our case the intersection is

$$\operatorname{Hom}(\pi_1(M), U(1))/\mathbf{Z}_2. \tag{1.3}$$

which is $b_1(M)$-dimensional. In §2, using the sum of the traces of the holonomy along the generators of $H_1(M;\mathbf{Z})$, we shall find a functional $f : \mathcal{B}(M) \to \mathbf{R}$, such that the equation

$$*_\sigma F^a - \operatorname{grad}_a f = 0 \tag{1.4}$$

has only a finite number of solutions, each of which is nondegenerate (see §2 for definition.) A connected component of elements of the set of elements $\mathcal{SB}(M)$, the reducible connections, satisfying (1.4) is identified to an element of

$$\operatorname{Hom}(\operatorname{Tor} H_1(M;\mathbf{Z}), U(1))/\mathbf{Z}_2. \tag{1.5.1}$$

And each connected component is identified to

$$\operatorname{Hom}\left(\frac{H_1(M;\mathbf{Z})}{\operatorname{Tor} H_1(M;\mathbf{Z})}, \mathbf{Z}_2\right) \tag{1.5.2}$$

or its quotient by $\mathbf{Z}_2$. Put

$$Fl = \{a \in \widetilde{\mathcal{B}}(M) \mid a \text{ satisfy } (1.4)\}, \tag{1.6.1}$$

$$Fl_0 = \{a \in Fl \mid a \text{ is irreducible}\}. \tag{1.6.2}$$

For $a, b \in Fl_0$, we set

$$\mathcal{M}(a,b) = \left\{ a_t \,\middle|\, \begin{array}{l} a_t : (-\infty,\infty) \to \widetilde{\mathcal{B}}(M), a_t \text{ satisfies } (1.7), \\ \lim_{t\to\infty} a_t = b, \lim_{t\to-\infty} a_t = a \end{array} \right\}.$$

(The precise definition is in §3.) Here

$$\frac{\partial a_t}{\partial t} = *_\sigma F^a - \operatorname{grad}_{a_t} f. \tag{1.7}$$

In a way similar to [F], we can find a map $\mu : Fl_0 \to \mathbf{Z}$ such that

$$\dim \mathcal{M}(a,b) = \mu(a) - \mu(b),$$

for $a, b \in Fl_0$ (§5.) We can also prove that $\mathcal{M}(a,b)$ is orientable (§6). Then, following Witten [W1] and Floer [F], we put

$$C_k^0 = \bigoplus_{\substack{a\in Fl_0 \\ \mu(a)=k}} \mathbf{Z}[a] \tag{1.8}$$

We define a boundary operator $\partial : C_k^0 \to C_{k-1}^0$ as follows. (Again our construction is the same as Floer's.) The action of $\mathbf{R}$ on $M \times \mathbf{R}$ induces a free action of $\mathbf{R}$ on $\mathcal{M}(a,b)$. We put, for $a \in Fl_0$, $\mu(a) = k$,

$$\partial([a]) = \sum_{\mu(b)=k-1} \langle \partial a, b \rangle [b],$$

where $\langle \partial a, b \rangle$ is the difference of the number of connected components of $\mathcal{M}(a,b)$ for which the direction of its orientation and the $\mathbf{R}$ action coincide and the number of connected components for which the orientation is the opposite direction to the $\mathbf{R}$-action. In a way similar to [F], we can prove $\partial\partial = 0$. Then we define

$$I_k^0(M) = \frac{\operatorname{Ker} \partial : C_k^0 \to C_{k-1}^0}{\operatorname{Im} \partial : C_{k+1}^0 \to C_k^0},$$

which, we shall prove, is an invariant of M. (In fact, we need to fix a basis of $H_1(M; \mathbf{Z})$.)

As is pointed out by Donaldson, Atiyah [A] and Witten [W2], Floer homology is closely related to the Donaldson polynomial [D4]. In fact, in the case when M is a homology sphere and is a boundary of a 4-manifold satisfying some additional assumptions, it is possible to define a relative Donaldson polynomial, which has a value in $I_k^0(M)$. But in the case when the first Betti number of M is positive, it seems that the above boundary operator is not enough for such a purpose. Then we construct other boundary operators. To motivate our construction we recall the definition of relative Donaldson polynomial very briefly. (Our description is not precise since it is anounced that the precise description will appear in [DFK].) Let X be a 4 manifold such that its boundary $\partial X = M$ is a homology sphere. Let $[\Sigma_1], \cdots, [\Sigma_\ell] \in H_2(X)$, $a \in Fl_0$. By $\mathcal{M}_k(X; a)$, we denote the set of all gauge classes of self dual connections ∇ with $c^2(\nabla) = k$, $\nabla|_{\partial X} = a$. Define a line bundle $\mathcal{L}_{\Sigma_i}$ on it by

$$\mathcal{L}_{\Sigma_i}(\nabla) = \bigwedge^{\text{top}} \left(\operatorname{Ker} \eth_{\nabla|_{\Sigma_i}} \right)^* \otimes \bigwedge^{\text{top}} \operatorname{Coker} \eth_{\nabla|_{\Sigma_i}},$$

where $\eth_{\nabla|_{\Sigma_i}}$ is a Dirac operator on Σ_i twisted by the restriction of ∇ to Σ_i. We put

$$Q_\ell([\Sigma_1], \cdots, [\Sigma_\ell])(a) = \int_{\mathcal{M}_k(X;a)} c^1(\mathcal{L}_{\Sigma_1}) \cup \cdots \cup c^1(\Sigma_\ell).$$

Here we choose k, ℓ so that $\dim \mathcal{M}_k(X, a) = 2\ell$. We regard $Q_\ell([\Sigma_1], \cdots, [\Sigma_\ell])$ as a cochain, an element of $\operatorname{Hom}(C_m, 0)$ with $m =$

$\mu(a)$. Under an appropriate assumtion this cochain is a cocycle and its cohomology class is an invariant of X.

In case $\partial X_1 = \partial X_2 = M$, $X = X_1 \amalg_M X_2$, $\Sigma_1 \cdots \Sigma_{\ell_1} \subset X_2$, $\Sigma'_1 \cdots \Sigma'_{\ell_2} \subset X_2$, one can prove, under appropriate assumption, that

$$\begin{aligned} Q_{\ell_1+\ell_2}&(\Sigma_1, \cdots, \Sigma_{\ell_1}, \Sigma'_1, \cdots, \Sigma'_{\ell_2}) \\ &= \langle Q_{\ell_1}(\Sigma_1, \cdots, \Sigma_{\ell_1}), Q_{\ell_2}(\Sigma'_1, \cdots, \Sigma'_{\ell_2})\rangle, \end{aligned} \tag{1.9}$$

where $\langle \quad , \quad \rangle$ is a coupling between Floer cohomologies of M and M^-, (M with opposite orientation). Note that in case $H_1 M = 0$, we have $H_2 X = H_2 X_1 \oplus H_2 X_2$.

Now we remove the assumption $H_1 M = 0$. Assume, for example $H_1 X_1 = H_1 X_2 = 0$. Then we have Mayer-Vietoris exact sequence:

$$H_2 X_1 \oplus H_2 X_2 \longrightarrow H_2 X \longrightarrow H_1 M \longrightarrow 0.$$

Fix a section $s : H_1 M \to H_2 X$. This is equivalent to choose, for each $[\gamma] \in H_1 M$, surfaces $\Sigma_{(i)}(\gamma) \subset X_i$ with $\partial \Sigma_i(\gamma) = \gamma$ such that $s([\gamma]) = [\Sigma_{(1)}(\gamma) \cup \Sigma_{(2)}(\gamma)] = [\Sigma(\gamma)]$. To generalize (1.9) one needs to calculate

$$Q_{\ell_1+\ell_2+\ell_3}(\Sigma_1, \cdots, \Sigma_{\ell_1}, \Sigma(\gamma_1), \cdots, \Sigma(\gamma_{\ell_3}), \Sigma'_1, \cdots, \Sigma'_{\ell_2}),$$

in terms of invariants of X_1,X_2. So it is natural to consider cochains such as

$$\begin{aligned} Q_{\ell+\ell'}&(\Sigma_{(1)}, \cdots, \Sigma_\ell, \Sigma_{(1)}(\gamma_1), \cdots, \Sigma_1(\gamma_{\ell'}))(a) \\ &= \int_{\mathcal{M}_k(X_1,a)} \mathcal{L}_{\Sigma_1} \cup \cdots \cup \mathcal{L}_{\Sigma_\ell} \cup \mathcal{L}_{\Sigma_{(1)}(\gamma_1)} \cup \cdots \cup \mathcal{L}_{\Sigma_{(1)}(\gamma_{\ell'})}. \end{aligned}$$

But one finds that this cochain is *not* a cocycle in general. Hence in our situation, the relative Donaldson polynomial should not take a value on usual Floer cohomology but a generalization of it. Our purpose is to find such a generalization.

We assume that $H_1(M;\mathbf{Z})$ is torsion free. Choose a set of closed loops $\{\gamma_1, \cdots, \gamma_d\}$ representing a basis of $H_1(M;\mathbf{Z})$. Put $\Sigma_i = \gamma_i \times \mathbf{R} \subset M \times \mathbf{R}$. Let $a_t \in \mathcal{M}(a,b)$, $a, b \in Fl_0$. It induces a connection of a trivial $SU(2)$ bundle over Σ_i. Let $\eth_{a_t}$ be the Dirac operator on Σ_i twisted by the connection. We may assume that $a(\gamma_i) \neq 1$ for each $a \in Fl_0$. It implies that $\eth_{a_t}$ is Fredholm. Put

$$\operatorname{Det} \eth_{a_t} = \bigwedge^{\text{top}} (\operatorname{Ker} \eth_{a_t})^* \otimes \bigwedge^{\text{top}} \operatorname{Coker} \eth_{a_t}.$$

By taking $(\mathrm{Det}\,\eth_{a_t})^{\otimes 2}$ and moving a_t on $\mathcal{M}(a,b)$, we obtain a complex line bundle on $\mathcal{M}(a,b)$, which is denoted by $\mathcal{L}_i^{(2)}$. (The reason why we have to take the square will be explained in §7.) Now, let $a, b \in Fl_0$ with $\mu(a) - \mu(b) = 2\ell + 1$. Put $\overline{\mathcal{M}}(a,b) = \mathcal{M}(a,b)/\mathbf{R}$. Then we can "define" the Chern number

$$\int_{\overline{\mathcal{M}}(a,b)} c^1(\mathcal{L}_{i_1}^{(2)}) \cup \cdots \cup c^1(\mathcal{L}_{i_\ell}^{(2)}) \in \mathbf{Z}.$$

This number is denoted by $\langle \partial_{i_1,\cdots,i_\ell} a, b\rangle$. (Since $\overline{\mathcal{M}}(a,b)$ has a boundary, the above number is, in fact, *not* well defined. This problem is discussed in §12.) We define $\partial_{i_1,\cdots,i_\ell} : C_k^0 \to C_{k-2\ell-1}^0$ by

$$\partial_{i_1,\cdots,i_\ell}([a]) = \sum_b \langle \partial_{i_1,\cdots,i_\ell} a, b\rangle [b].$$

Now we can state the main result of this paper. Let $\alpha \in \{1,\cdots,d\}^\ell / S_\ell$. (Here S_ℓ stands for the symmetric group.) We put $\partial_\alpha = \partial_{\alpha_1,\cdots,\alpha_\ell}$.

Theorem 1.10. *If $\sharp\alpha < 3$, and if $H_1(M;\mathbf{Z})$ is torsion free, then*

$$\sum_{\alpha^1 \cup \alpha^2 = \alpha} \partial_{\alpha^1}\partial_{\alpha^2} = 0.$$

Remark 1.11. In case when $\alpha = (1,1)$ the formula is:

$$\partial\partial_{1,1} + 2\partial_1\partial_1 + \partial_{1,1}\partial = 0.$$

Remark 1.12. For $\sharp\alpha > 2$ the formula is *not* correct. We discuss the reason in §12. There we also discuss why the formula may not be correct for $s > 0$ if $H_1(M;\mathbf{Z})$ has a torsion.

Now let $S^\ell H_1(M;\mathbf{Z})$ be the symmetric power. We put

$$C_k^s = \bigoplus_{\ell \le s} S^\ell H_1(M;\mathbf{Z}) \otimes C_{k+2\ell}^0.$$

Define $\partial_k^s : C_k^s \to C_{k-1}^s$ by

$$\partial_k^s(\gamma_\alpha \otimes [a]) = \sum_{\alpha^1 \cup \alpha^2 = \alpha} \gamma_{\alpha^1} \otimes \partial_{\alpha^2}[a],$$

where $\gamma_\alpha = \gamma_{\alpha_1} \otimes \cdots \otimes \gamma_{\alpha_\ell}$. Theorem 1.10 immediately implies

Corollary 1.13. *Suppose that $H_1(M;\mathbf{Z})$ is torsion free. For $s<3$ we have*

$$\partial_{k-1}^s \partial_k^s = 0.$$

We put

$$I_k^s(M) = \frac{\operatorname{Ker} \partial_k^s}{\operatorname{Im} \partial_{k-1}^s}.$$

Theorem 1.14. *Suppose that $H_1(M;\mathbf{Z})$ is torsion free. $I_k^s(M)$ does not depend on the choices of the metrics, γ_i's, etc, and is an invariant of M, equipped with a basis of $H_1(M;\mathbf{Z})$.*

By construction we have an exact sequence of complexes

$$0 \longrightarrow C_k^s \longrightarrow C_k^{s+1} \longrightarrow S^{s+1}(H_1(M;\mathbf{Z})) \otimes C_{k+2s+2}^0 \longrightarrow 0$$

It follows that:

Theorem 1.15. *Suppose that $H_1(M;\mathbf{Z})$ is torsion free. There exists a long exact sequence*

$$\longrightarrow I_k^s(M) \longrightarrow I_k^{s+1}(M) \longrightarrow S^{s+1}(H_1(M,\mathbf{Z})) \otimes I_{k+2s+2}^0(M) \longrightarrow$$

for $s=0$ or 1. The exact sequence is also an invariant of M.

The proof of these theorems is based on the detailed analysis of the end of the moduli space $\mathcal{M}(a,b)$. The results on it is in §7. In fact, we shall prove more general results than we need to construct our invariants. In the course, we develop various techniques, which might be useful in other situations.

Using our invariant $I_k^s(M)$, we can partially generalize the definition of relative Donaldson polynomial to the case when the boundary is not necessary a homology sphere. Those applications will appear elsewhere.

The organization of this paper is as follows.

In §2,3, we perturb the equation.

In §4, we review the sum formula for the index of the elliptic operators. We also discuss the sum formula of the family of indices.

This result is used in §5 to define the degree μ. In §5 we study also neighborhoods of various reducible connections.

In §6 we define the orientation of the moduli space. The fact that every oriented 3-manifolds bounds an oriented 4-manifold, is essentially used in the proof.

§§7–11 are devoted to the study of the end of moduli space $\mathcal{M}(a,b)$. The results of these sections are stated in §7.

In §8, we prove that the patching procedure of selfdual connections as in [T1] is possible in our situation, where various reducible connections must be dealt with.

In §9, we shall prove that the selfdual connections constructed in §8, contains all the connections in the end of the moduli space, except the concentrated ones. For this purpose, we establish a decay estimate such as in [FU].

Combining the results of §§8,9 we obtain a chart for a neighborhood of each point at infinity. In order to patch those charts, we introduce, in §10, the local action of the groups. This notion is a generalization of one introduced in [CG] to study the end of Riemannian manifolds. We use it to study the end of the moduli space.

The line bundle $\mathcal{L}_i^{(2)}$ is constructed and is extended to the boundary in §11. For this purpose we use the sum theorem for index bundles in §4 and the existence of the lift of the local action to the bundle.

Using the results of §§7–11, we define the boundary operator in §12 and prove Theorem 1.10. As is remarked before, the Chern number of the bundle $\mathcal{L}_i^{(2)}$ is not well defined. We shall prove in §12 that the boundary operator is well defined modulo isomorphism. In §12, we also discuss the case when $s = 3$ and describe why Theorem 1.10 does not hold in that case.

Finally we shall prove Theorems 1.14 and 1.15 in §13.

As the reader can find easily, this paper heavily depends on the brilliant ideas due to Donaldson, Floer, Taubes e.t.c. in their papers. Before this work is completed the author is informed (without the precise statement) that A. Floer generalized his invariant to homology $S^1 \times S^2$.

§2. Perturbation

Let L^p_ℓ be the Sobolev space of the sections, namely the set of sections L^p-norms of whose ℓ-th derivatives are finite. Put

$$\mathcal{A}^p_\ell(M) = \{d + a \mid a \in L^p_\ell(M, \wedge^1 \otimes su(2))\}$$
$$\mathcal{G}_\ell(M) = \text{the set of maps} : M \to SU(2) \text{ of } L^2_\ell\text{-class.}$$

$\mathcal{A}^2_\ell$ is denoted by $\mathcal{A}_\ell$. We choose sufficiently large ℓ and fix it throughout this paper. $\mathcal{G}_{\ell+1}$ acts on $\mathcal{A}_\ell$. (See [FU].) Put

$$\mathcal{B}_\ell(M) = \mathcal{A}_\ell(M)/\mathcal{G}_{\ell+1}(M).$$

Let $a \in \mathcal{A}_\ell(M)$. Then the set

$$\{u \in L^2_\ell(M, \wedge^1 \otimes su(2)) \mid d_a^* u = 0\} \tag{2.1}$$

is the orthonormal complement of $T_a\mathcal{G}_{\ell+1}a$ in $T_a\mathcal{A}_\ell(M)$. In the case when a is irreducible, the set (2.1) can be identified to $T_{[a]}\mathcal{B}_\ell(M)$. (See [FU].) We let the set (2.1) be denoted by $T_{[a]}\mathcal{B}_\ell(M)$ also in the case when a is reducible. In that case, $[a]$ is a singular point of $\mathcal{B}_\ell(M)$.

The purpose of this section is to perturb the functional $\mathfrak{cs}$ and the equation (1.2), so that (1.4) has only a finite number of solutions each of which is nondegenerate. We put

$$H_1'(M;\mathbf{Z}) = \frac{H_1(M;\mathbf{Z})}{\text{Torsion}}.$$

First we deal with singular points on

$$\mathrm{Hom}(H_1'(M;\mathbf{Z}), SU(2))/\text{conjugate} \subset \mathcal{B}_\ell(M).$$

Choose a set of loops $\{\ell_1^0, \cdots, \ell_d^0\}$ representing a basis of $H_1'(M;\mathbf{Z})$. Extend ℓ_i^0 to an embedding $\ell_i^0 : S^1 \times D^2 \to M$. Choose a nonnegative function u on D^2 with compact support such that

$$\int_{D^2} u(x)dx = 1.$$

For a loop $\ell : S^1 \to M$ and $a \in \mathcal{A}(M)$, let $h_\ell(a) \in SU(2)$ be the holonomy along ℓ. Define a functional f_0 on $\mathcal{B}_\ell(M)$ by

$$f_0(a) = \epsilon \sum_{i=1}^{d} \int \mathrm{Tr}\left(h_{\ell_i^0(\cdot,x)}(a)\right) u(x)dx, \tag{2.2}$$

where ϵ is a small positive number. Then by [F] 1b.1, $\mathrm{grad}_a f_o \in T_a\mathcal{B}_\ell(M)$ is well defined. Similarly we can define the hessian, $\mathrm{Hess}_a f_0 : T_{[a]}\mathcal{B}_\ell(M) \to T_{[a]}\mathcal{B}_\ell(M)$.

Here we examine the set, FR, of the flat reducible connections in $\mathcal{B}_\ell(M)$. The set of the conjugacy classes of the elements of $\mathrm{Hom}(\mathrm{Tor}\, H_1(M,\mathbf{Z}), U(1))$ has a one to one correspondence to $\pi_0(FR)$. For $\varphi \in \mathrm{Hom}(\mathrm{Tor}\, H_1(M,\mathbf{Z}), U(1))$, let FR_φ be the corresponding component. FR_φ is diffeomorphic to T^d if $\mathrm{Im}(\varphi) \not\subset \{\pm 1\}$, and is diffeomorphic to $T^d/\mathbf{Z}_2$ if $\mathrm{Im}(\varphi) \subset \{\pm 1\}$. Let $1 \in \mathrm{Hom}(\mathrm{Tor}\, H_1(M,\mathbf{Z}), \mathbf{Z})$ be the trivial representation.

Lemma 2.3. *There exists a neighborhood U of FR_1 such that, for sufficiently small ϵ, the set of elements of U satisfying*

$$*_\sigma F^a - \mathrm{grad}_a f_0 = 0 \tag{2.4}$$

is identified to $\mathrm{Hom}(H_1'(M,\mathbf{Z}),\mathbf{Z}_2) \simeq \{\pm 1\}^d$.

Proof. By identifying $FR_1 = \{(e^{i\theta_1},\cdots,e^{i\theta_d})\}/\mathbf{Z}_2$, we have

$$f_0(e^{i\theta_1},\cdots,e^{\theta_d}) = 2\epsilon \sum \cos\theta_i. \tag{2.5}$$

The lemma follows immediately.

Lemma 2.6. *Let* $a \in \mathrm{Hom}(H_1'(M,\mathbf{Z}),\mathbf{Z}_2)$. *Then* $\mathfrak{cs} - f_0$ *is nondegenerate at* a. *In other words*

$$*_\sigma d_a - \mathrm{Hess}_a f_0 : T_{[a]}\mathcal{B}_\ell(M) \to T_{[a]}\mathcal{B}_{\ell-1}(M)$$

is invertible.

Remark 2.7. $\mathrm{Hess}_a\, \mathfrak{cs} = *_\sigma d_a$. See [F],[T4].

Proof. We have

$$\mathrm{Ker}\, *_\sigma d_a \simeq H^1(M;\mathbf{R}) \otimes su(2) \simeq su(2)^d.$$

On this space $\mathrm{Hess}_a f_0$ is given by $-\epsilon \sum x_i^2$. Hence the lemma follows from the invertibility of the matrix

$$\begin{pmatrix} A+\epsilon E & \epsilon B \\ \epsilon C & \epsilon D \end{pmatrix}$$

for small ϵ and invertible A and D.

We take ϵ in (2.2) such that Lemma 2.6 holds and fix it.

Next we use a method similar to [D3] and [F]. Let $p_0 \in M$ and $v_0 \in T_{p_0}M$. Choose an embedding $I : D^2 \to M$, such that $I(0) = p_0$, and that $I_*(T_0D^2)$ is transversal to v_0. Let $\Gamma_1(p_0, I, v)$ be the set of smooth embeddings such that $\ell(1,0) = p_0$, $\frac{D\ell}{dt}(1,0) = v_0$, $\ell(0,x) = I(x)$. We put

$$\Gamma_m = \bigcup_{(p_0,v_0,I)} (\Gamma_1(p_0,v_0,I))^m.$$

Let $L_m = SU(2)^m/SU(2)$, where $SU(2)$ acts by conjugation. Define a map

$$\widetilde{\Phi}' : \mathcal{A}_\ell(M) \times \Gamma_m \to \mathrm{Map}(D^2, SU(2)^m)$$

by

$$\widetilde{\Phi}'(a,(\ell_1,\cdots,\ell_m))(x) = (h_{\ell_1(\cdot,x)}(a),\cdots,h_{\ell_m(\cdot,x)}(a)).$$

$\widetilde{\Phi}'$ induces a map

$$\Phi' : \mathcal{B}_\ell(M) \times \Gamma_m \to \mathrm{Map}(D^2, L_m).$$

Following [F], we choose $(\beta_i)_{i \in \mathbf{Z}_+}$ $(\beta_i > 0)$. and put

$$C^\beta(L_m, \mathbf{R}) = \{\psi \in C^\infty(L_m, \mathbf{R}) \mid \|\psi\|_\beta < \infty\},$$

where

$$\|\psi\|_\beta = \sum_{i=1}^{\infty} \beta_i \max_{x \in L_m} |D^i \psi(x)|.$$

Fix a function $u : D^2 \to [0, \infty)$ as before and define

$$\Phi : \mathcal{B}_\ell(M) \times \Gamma_m \times C^\beta(L_m, \mathbf{R}) \to \mathbf{R}$$

by

$$\Phi([a], (\ell_1, \cdots, \ell_m), \psi) = \int_{D^2} \psi(\Phi'([a], (\ell_1, \cdots, \ell_m)(x))u(x)dx.$$

For $v \in \Gamma_m \times C^\beta(L_m, \mathbf{R})$, we put $f_v([a]) = \Phi([a], v)$. For $\lambda = (\ell_1, \cdots, \ell_m) \in L_m$ and $\lambda' = (\ell'_1, \cdots, \ell'_{m'}) \in L_{m'}$, we say $\lambda \prec \lambda'$ if $\{\ell_1, \cdots, \ell_m\} \subset \{\ell'_1, \cdots, \ell'_{m'}\}$

Lemma 2.8. *There exists $\lambda_0 \in \Gamma_{m_0}$ and $\delta > 0$ such that for each $\lambda_0 \prec \lambda$, the set of $\psi \in C^\beta((L_m), \mathbf{R})$ satisfying the following conditions is of first category in $\{\psi \mid \|\psi\|_\beta < \delta\}$.*

(2.8.1) *The set $Fl(\psi)$ of the solution of*

$$*_\sigma F^a = \mathrm{grad}_a(f_0 + f_{(\lambda,\psi)}).$$

is finite.

(2.8.2) *For each $a \in Fl(\psi)$ the map*

$$*_\sigma d - \mathrm{Hess}_{[a]}(f_0 + f_{(\lambda,\psi)}) : T_a\mathcal{B}_\ell(M) \to T_a\mathcal{B}_{\ell-1}(M)$$

is invertible.

Proof. As is well known, (2.8.2) implies (2.8.1). Hence the problem is local on $\mathcal{B}_\ell(M)$. The argument in a neighborhood of irreducible connections is the same as [F] 2c.1. Then we study the neighborhood of the set of reducible connections. Precisely, we first take a perturbation so that (2.8.2) holds in a neighborhood of the set of the reducible connections, next we perturb again so that (2.8.1) and (2.8.2) holds, in the set of irreducible connections, as well.

Let $\varphi \in \mathrm{Hom}(\mathrm{Tor}\, H_1(M,\mathbf{Z}), SU(2))$. In the case when $\mathrm{Im}\,\varphi \subset \{\pm 1\}$, the proof of Lemma 2.6 works in a neighborhood of FR_{φ}. Then we assume that $\mathrm{Im}(\varphi) \not\subset \{\pm 1\}$. By the proof of Lemma 2.3, f_0 is a Morse function on Fl_{φ} and has exactly 2^d singular points on it. The same holds for $f_0 + f_{\lambda,\psi}$ if $\|\psi\|_{\beta}$ is small. Hence it suffices to work at a neighborhood of each singular point a_0. Choose a neighborhood U of a_0 with is of bounded L^2_{ℓ} norm.

Sublemma 2.9. *The set of ψ such that $*_{\sigma} d_a - \mathrm{Hess}_a(f_0 + f_{\lambda,\psi})$ is invertible for each $a \in U \cap Fl(\psi)$, is open.*

Proof. First we remark that the set

$$Fl(\psi) = \{[a] \in \mathcal{B}_{\ell}(M) \mid *_{\sigma} F^a = \mathrm{grad}_a(f_0 + f_{\lambda+\psi})\}$$

is independent of ℓ because the equation is elliptic modulo gauge transformation. Hence we can find a bounded subset L in $L^2_{\ell+2}(M, \wedge^1 \otimes su(2))$ such that if

(2.10.1) $\|\psi' - \psi\|_{\beta} < \delta$

(2.10.2) $[a] \in Fl(\psi)$

(2.10.3) $[a] \in U$

then $[a] = [a_0 + u]$ for some $u \in L$. Now, if the sublemma is false, then, there exists ψ,ψ_i and a_i such that

(2.11.1) $\lim_{i\to\infty} \|\psi_i - \psi\|_{\beta} = 0$,
(2.11.2) $[a_i] \in Fl(\psi_i)$,
(2.11.3) $[a_i] \in U$,
(2.11.4) $*_{\sigma} d_{a_i} - \mathrm{Hess}_{a_i}(f_0 + f_{\lambda,\psi_i})$ is not invertible,
(2.11.5) $*_{\sigma} d_a - \mathrm{Hess}_a(f_0 + f_{\lambda,\psi})$ is invertible for each $a \in Fl(\psi) \cap U$.

We can choose $u_i \in L$ such that $[a_0 + u_i] = [a_i]$. By Rellich's Theorem, we can find a subsequence such that u_i converges to u_{∞} in $L^2_{\ell+1}$. Hence by (2.11.1),(2.11.2) and (2.11.3), we have $[a_0 + u_{\infty}] = [a_{\infty}] \in U \cap Fl(\psi)$. Therefore $*_{\sigma} d_{a_{\infty}} - \mathrm{Hess}_{a_{\infty}}(f_0 + f_{\lambda,\psi})$ is invertible. On the other hand, we remark that the map

$$\begin{aligned} \mathcal{A}_{\ell+1}(M) \times L^2_{\ell}(M, \wedge^1 \otimes su(2)) &\to L^2_{\ell-1}(M, \wedge^1 \otimes su2)) \\ : (a,u) &\mapsto *_{\sigma} d_a u - \mathrm{Hess}_a(f_0 + f_{\lambda,\psi})u \end{aligned}$$

is continuous. (See [FU]). It follows that $*_{\sigma} d_{a_i} - \mathrm{Hess}_{a_i}(f_0 + f_{\lambda,\psi_i})$ is invertible for sufficiently large i. This contradicts (2.11.4). The proof of Sublemma 2.9 is now complete.

Hence it suffices to show that the set of ψ for which

$$*_\sigma d_{a_0} - \mathrm{Hess}_{a_0}(f_0 + f_{(\lambda,\psi)})$$

is surjective, is dense. We can choose a loop ℓ_0 so that $\varphi(\ell_0) \notin \{\pm 1\}$ and assume $\{\ell_0\} \prec \lambda = (\ell_1, \cdots, \ell_m)$. Put

$$\widetilde{\Phi}'(a_0, \lambda)(0) = (g_1, \cdots, g_m).$$

We have

$$\{g \in SU(2) \mid g^{-1}(g_1, \cdots, g_m)g = (g_1. \cdots, g_m)\} \simeq U(1) \tag{2.12}$$

Hence $[g_1, \cdots, g_m]$ is contained in $U(1)^m/\mathbf{Z}_2 \subset SU(2)^m/SU(2)$ and is a regular point of $U(1)^m/\mathbf{Z}_2$. Put

$$\mathcal{B}_\ell^{\mathrm{red}}(M) = \{[a] \in \mathcal{B}_\ell^{\mathrm{red}}(M) \mid a \text{ is reducible.}\}$$

It follows from (2.12) that $[a_0]$ is a regular point of $\mathcal{B}_\ell^{\mathrm{red}}(M)$. Therefore, by a $U(1)$ analogue of [F] 2c.1, we may assume that

$$*_\sigma d_{a_0} - \mathrm{Hess}_{a_0}(f_0 + f_{\lambda,\psi}) : T_{[a_0]}(\mathcal{B}_\ell^{\mathrm{red}}(M)) \to T_{[a_0]}(\mathcal{B}_\ell^{\mathrm{red}}(M)) \tag{2.13}$$

is invertible. Put

$$K_\psi = \{u \in T_{[a_0]}\mathcal{B}_\ell(M) \mid *_\sigma d_{a_0} u - \mathrm{Hess}_{a_0}(f_0 + f_{\lambda,\psi})u = 0\}$$

By the invertibility of (2.13) we have

$$K_\psi \cap T_{[a_0]}\mathcal{B}_\ell^{\mathrm{red}}(M) = \{0\}. \tag{2.14}$$

The group

$$U(1) = \{g \in \mathcal{G}_\ell(M) \mid g^* a_0 = a_0\} \tag{2.15}$$

acts on K_ψ. By (2.14) and the finite dimensionality of K_ψ, we can identify $K_\psi \simeq \mathbf{C}^k$. Therefore by taking sufficiently large λ and m we may assume that

$$P : K_\psi \to T_{(g_1,\cdots,g_m)}SU(2)^m$$

is injective, where P is the differential at $[a_0]$ of the map : $[a] \mapsto \widetilde{\Psi}'(a, \lambda)(0) : \mathcal{A}_\ell(M) \to SU(2)^m$. By (2.8), $U(1)$ acts on $T_{(g_1,\cdots,g_m)}SU(2)^m$, which we can identify to $\mathbf{C}^m \oplus \mathbf{R}^m$. The map P

is $U(1)$ invariant. Hence we may assume that $P(K_\psi) \subset \mathbf{C}^m$. We define a function ψ' in a neighborhood of $(g_1, \cdots, g_m)$ by

$$\psi'(\exp_{(g_1,\cdots,g_m)}(z_1, \cdots, z_m, t_1, \cdots, t_m)) = -\sum |z_i|^2, \tag{2.16}$$

and extend it to a $SU(2)$ invariant function on $SU(2)^m$. We obtain a function on L_m, for which we use the same symbol. Now it is easy to see that

$$*_\sigma d_{a_0} - \mathrm{Hess}_{a_0}(f_0 + f_{\lambda,\psi+\epsilon\psi'})$$

is invertible for each sufficiently small ϵ. The proof of Lemma 2.7 is now completed.

Note that a linear function is used in [F] for the perturbation in a neighborhood of an irreducible connection. Here we use quadratic function to perturb the equation in a neighborhood of a reducible connection.

Remark 2.17. We choose the perturbation so that the zero eigenvalues of $*_\sigma d - \mathrm{Hess}_a(f_0 + f_{(\lambda,\mu)})$ is perturbed to positive one, if a is a reducible connection and if the corresponding eigenspace is identified to $\mathbf{C}^k$ with respect to the $U(1)$ action. The set of such connections is a subset of first category in an open set. This choice is used in the proof of Theorem 5.6. (See Remark 5.7.)

Now we put $f = f_0 + f_{\lambda,\psi}$ for generic ψ, and define Fl and Fl_0 by (1.6.1) and (1.6.2).

§3. Local structure of moduli space

Let $p : M \times \mathbf{R} \to M$ be the projection, $p^*(\wedge^i M)$ be the pull back of the vector bundles on $M \times \mathbf{R}$. Let δ be a number sufficiently close to 0. Choose a C^∞-map $\|\| : \mathbf{R} \to [0, \infty)$, such that $\|t\| = |t|$ outside a compact subset, put $e_\delta(t) = e^{\delta\|t\|}$. For a smooth section u of $p^*(\wedge^i M) \otimes su(2)$ with compact support, we put

$$\left(\|u\|^p_{\ell,\delta}\right)^p = \sum_{k\le\ell} \int_{M\times\mathbf{R}} e_\delta(t)|\nabla^k u|^p dxdt.$$

Let $L^p_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes p^*(\wedge^i M))$ be the completion with respect to this norm. We put

$$\mathcal{L}^i_{\ell,\delta} = L^2_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes p^*(\wedge^i M)).$$

Define $L^p_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes \wedge^i(M \times \mathbf{R}))$ in a similar way. Let $L^p_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes \wedge^2_{\pm}(M \times \mathbf{R}))$ be the subspace of $L^p_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes \wedge^2(M \times \mathbf{R}))$ consisting of the elements u satisfying $\widetilde{*}_\sigma u = \pm u$, respectively. Here and hereafter $\widetilde{*}_\sigma$ denotes the Hodge $*$ operator on $M \times \mathbf{R}$ with respect to the product metric $\sigma \oplus dt^2$. The Hodge operator on M induces $*_\sigma : p^*(\wedge^k M) \to p^*(\wedge^{3-k} M)$. We define isomorphisms

$$\begin{aligned}
&I^2_{\pm} : L^p_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes p^*(\wedge^1 M)) \to \\
&\qquad\qquad\qquad L^p_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes \wedge^2_{\pm}(M \times \mathbf{R})) \\
&I^1 : L^p_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes p^*(\wedge^0 M \oplus \wedge^1 M)) \to \\
&\qquad\qquad\qquad L^p_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes \wedge^1(M \times \mathbf{R})) \\
&I^0 : L^p_{\ell,\delta}(M \times \mathbf{R}, su(2)) \to L^p_{\ell,\delta}(M \times \mathbf{R}, su(2))
\end{aligned}$$

by

$$\begin{aligned}
I^2_{\pm}(\alpha) &= \alpha \pm (*_\sigma \alpha) \wedge dt \\
I^1(\varphi, \alpha) &= \varphi\, dt + \alpha \\
I^0 &= \text{identify}.
\end{aligned}$$

We put

$$\begin{aligned}
\Omega^0_{\ell,\delta} &= L^2_{\ell,\delta}(M \times \mathbf{R}, su(2)) \\
\Omega^1_{\ell,\delta} &= L^p_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes \wedge^1(M \times \mathbf{R})) \\
\Omega^2_{\ell,\delta} &= L^2_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes \wedge^2_-(M \times \mathbf{R}))
\end{aligned}$$

and identify $\mathcal{L}^0_{\ell,\delta} \simeq \Omega^0_{\ell,\delta}$, $\mathcal{L}^0_{\ell,\delta} \oplus \mathcal{L}^1_{\ell,\delta} \simeq \Omega^1_{\ell,\delta}$, $\mathcal{L}^1_{\ell,\delta} \simeq \Omega^2_{\ell,\delta}$, by I^i.

For $a, b \in Fl$, choose a connection $d + A^{a,b}$ of the trivial $SU(2)$ bundle on $M \times \mathbf{R}$ such that $A^{a,b} = b$ if $t > 1$ and that $A^{a,b} = a$ if $t < -1$. We put

$$\mathcal{A}_{\ell,\delta}(a, b) = \{d + A^{a,b} + \alpha \mid \alpha \in \Omega^1_{\ell,\delta}\}.$$

Clearly this space is independent of the choice of $A^{a,b}$. Hereafter we write A in place of $d + A$. Let $\mathcal{G}^0_{\ell,\delta}(M \times \mathbf{R})$ be the set of all locally L^2_ℓ map $g : M \times \mathbf{R} \to SU(2)$ such that there exists $\psi \in \mathcal{L}_{\ell,\delta}$ satisfying $\exp \psi = g$ outside a compact subset.

Lemma 3.1. *$\mathcal{G}^0_{\ell+1,\delta}(M \times \mathbf{R})$ acts on $\mathcal{A}_{\ell,\delta}(a, b)$ by*

$$g^* A = g^{-1} dg + g^{-1} A g.$$

The action is free if δ is positive or $a, b \in Fl_0$.

We omit the proof. (See [FU],[T3],[F].)
For $a \in \mathcal{A}_\ell(M)$, $A \in \mathcal{A}_{\ell,\delta}(a,b)$, we put

$$G_a = \{g \in \mathcal{G}_{\ell+1}(M) \mid g^* a = a\}$$
$$G_A = \{g : M \times \mathbf{R} \to G \mid g \text{ is a locally } L^2_{\ell+1} \text{ map satisfying } g^* A = A.\}$$

Remark 3.2. $G_A \subset G_a \cap G_b$.

Put

$$\mathcal{B}^{reg}_{\ell,\delta}(a,b) = \{[A] \mid A \in \mathcal{A}_{\ell,\delta}(a,b), G_A \neq \{\pm 1\}\}$$
$$T_{[A]}\mathcal{B}_{\ell,\delta}(a,b) = \{\alpha \in \Omega^1_{\ell,\delta} \mid e_\delta d_A^* e_\delta^{-1} \alpha = 0\}.$$

G_A acts on $\mathcal{B}_{\ell,\delta}(a,b)$ and $T_{[A]}\mathcal{B}_\ell(a,b)$.

Lemma 3.3. *The map $T_{[A]}B_{\ell,\delta}(a,b) \to \mathcal{B}_{\ell,\delta}(a,b) : \alpha \mapsto [A+\alpha]$, induces a G_A-invariant diffeomorphism from a neighborhood of* 0 *onto a neighborhood of A, if $a, b \in Fl_0$, or if $\delta > 0$.*

The proof is in [FU], [T3], [F].

Lemma 3.4. *$G_a \times G_b$ acts on $\mathcal{B}_{\ell,\delta}(a,b)$. The action is compatible with the diagonal inclusion : $G_A \to G_a \times G_b$.*

Proof. For each $g_1 \in G_a$ and $g_2 \in G_b$ choose a map $g : M \times \mathbf{R} \to SU(2)$ such that $g_t = g_1$ if $t < -1$ and that $g_t = g_2$ if $t > 1$. For $[A] \in \mathcal{B}_{\ell,\delta}(a,b)$ the element $g^* A$ is contained in $\mathcal{A}_{\ell,\delta}(a,b)$, and $[g^* A]$ depends only on $[A]$ and g_1, g_2. Clearly this induces a desired action.

Hereafter we put

$$g_1[A]g_2^{-1} = (g_1, g_2)[A]$$

for $A \in \mathcal{B}_{\ell,\delta}(a,b)$, $g_1 \in G_a$, $g_2 \in G_b$. Then G_a and G_b act from left and right on $\mathcal{B}_{\ell,\delta}(a,b)$, respectively.

Remark 3.5. The action is trivial if $\delta < 0$.

Now we consider a differential equation

$$F^A - \widetilde{*}_\sigma F^A - \operatorname{grad}_{a_t} f \wedge dt + *_\sigma \operatorname{grad}_{a_t} f = 0, \tag{3.6}$$

for $A \in \mathcal{A}_{\ell,\delta}(a,b)$. Here we put $A = I^1(a_t,\varphi)$. Let $\widehat{\mathcal{M}}_{\ell,\delta}(a,b)$ be the set of all solutions of (3.6) in $\mathcal{A}_{\ell,\delta}(a,b)$. Since $\mathrm{grad}_{g_t^* a_t} f = g_t^{-1}(\mathrm{grad}_{a_t} f) g_t$, it follows that

$$F^{g^*A} - \widetilde{*}_\sigma F^{g^*A} - \mathrm{grad}_{g_t^* a_t} f \wedge dt + *_\sigma \mathrm{grad}_{g_t^* a_t} f = \\ g^{-1}\left(F^A - \widetilde{*}_\sigma F^A - \mathrm{grad}_{a_t} f \wedge dt + *_\sigma \mathrm{grad}_{a_t} f\right) g.$$

Therefore $\widehat{M}_{\ell,\delta}(a,b)$ is $\mathcal{G}^0_{\ell+1,\delta}$ invariant. We put

$$\mathcal{M}_{\ell,\delta}(a,b) = \widehat{M}_{\ell,\delta}(a,b)/\mathcal{G}^0_{\ell+1,\delta}.$$

By a standard elliptic regularity estimate, $\mathcal{M}_{\ell,\delta}(a,b)$ is independent of ℓ. Then we omit ℓ and write $\mathcal{M}_\delta(a,b)$.

Here we remark that the set $G_a \backslash \mathcal{M}_\delta(a,b)/G_b$ is identified to the set $\mathcal{M}(a,b)$ in §1. In fact, the elements of the set $\mathcal{M}(a,b)$ have a one to one correspondence to the set of a_t's satisfying (1.7) and $\lim_{t\to-\infty} a_t = a$, $\lim_{t\to\infty}[a_t] = [b]$. Put $\lim_{t\to\infty} a_t = b'$. There exists g_∞ such that $g_\infty^* b' = b$. Choose g_t such that $\lim_{t\to-\infty} g_t = 1$, $\lim_{t\to\infty} g_t = g_\infty$. It is easy to see that $g^*(d+a_t) \in \mathcal{M}_\delta(a,b)$. This element depends only on $[a_t]$ and is independent of a_t. Conversely, if $A \in \widehat{\mathcal{M}}_\delta(a,b)$, we can find g such that g^*A has no dt factor. Let $(g^*A)(\cdot,t) = a_t$. Then $[a_t] \in \mathcal{M}(a,b)$.

Remark 3.7. It is *not* in general true that the set of loops joining $[a]$ and $[b]$ in $\mathcal{B}_\ell(M)$ has one to one correspondence to $\mathcal{B}_{\ell,\delta}(a,b)$. This is valid if the loop is contained in $\mathcal{B}_\ell(M) - \mathcal{SB}_\ell(M)$

For $A \in \mathcal{A}_\ell(a,b)$, we define $\mathcal{D}_A : \Omega^1_\ell \to \Omega^2_{\ell-1}$ by

$$\mathcal{D}_A\alpha = (d_A - \widetilde{*}_\sigma d_A)\alpha - \mathrm{Hess}_{a_t} f(u_t),$$

where $\alpha = I_1(u_t,\varphi)$, $d+A = d+a_t+\psi dt$. If we identify $\Omega^1_{\ell,\delta} \simeq \mathcal{L}^1_{\ell,\delta} \oplus \mathcal{L}^0_{\ell,\delta}$, $\Omega^2_{\ell-1,\delta} \simeq \mathcal{L}^1_{\ell-1,\delta}$, we have

$$\mathcal{D}_A(u,\varphi) = -\frac{\partial u}{\partial t} + (*_\sigma d_{a_t} - \mathrm{Hess}_{a_t} f - \psi_t \wedge)u + d_{a_t}\varphi. \tag{3.8}$$

Recall that $\mathcal{M}(a,b)$ is a C^∞-manifold in a neighborhood of $[A]$ if $\mathcal{D}_A$ is surjective.

Lemma 3.9. *There exists λ_0 and m_0 such that, for each $\lambda_0 \prec \lambda$, the set of $\psi \in C^\beta(L_m,\mathbf{R})$ satisfying the following is of first category in an open set. Let $a,b \in Fl$, $f = f_{\lambda,\psi}$.*

(3.9.1) *$\mathcal{M}_\delta(a,b)$ is a finite dimensional smooth manifold.*

(3.9.2) *For each* $[A] \in \mathcal{M}_\delta(a,b)$, $\mathcal{D}_A$ *is surjective.*

Proof. We write $\mathcal{M}_\delta^\psi(a,b)$, $\mathcal{D}_A^\psi$ while proving Lemma 3.9. In the set of irreducible connections, the proof of [F] 2c.2 works. Hence we study $\mathcal{M}_\delta^\psi(a,b)$ in the neighborhood of reducible connections. Put

$$\mathcal{B}_{\ell,\delta}^{\rm red}(a,b) = \{[A] \in \mathcal{B}_{\ell,\delta}(a,b) \mid G_A = U(1)\}$$
$$\mathcal{M}_\delta^{{\rm red},\psi}(a,b) = \mathcal{B}_{\ell,\delta}^{\rm red}(a,b) \cap \mathcal{M}_\delta^\psi(a,b)$$

Then by a $U(1)$ analogue of the argument by Floer [F] 2c.2, we may assume that $\mathcal{M}_\delta^{{\rm red},\psi}(a,b)$ is a C^∞-manifold, and, for each $[A] \in \mathcal{M}_{\ell,\delta}^{{\rm red},\psi}(a,b)$, the map

$$\mathcal{D}_A^{\rm red} : L^2_{\ell,\delta}(M \times \mathbf{R}, u(1) \otimes \wedge^1(M \times \mathbf{R})) \to L^2_{\ell-1,\delta}(M \times \mathbf{R}, u(1) \otimes \wedge^2_-(M \times \mathbf{R}))$$

is surjective. Let $[A] \in \mathcal{M}_\delta^{{\rm red},\psi}$. Choose a neighborhood U of $[A]$ in $\mathcal{B}_{\delta,\ell}^\psi(a,b)$, which is bounded in L^2_ℓ norm.

Sublemma 3.10. *The set of all* ψ' *such that* $\mathcal{D}_A^{\psi'}$ *is surjective for all* $A \in U \cap \mathcal{M}_\delta^{\psi'}(a,b)$, *is open.*

The proof is similar to one for Sublemma 2.9 and is omited.

Sublemma 3.11. *For each* $\epsilon > 0$ *and* ψ, *there exists* ψ' *and a neighborhood* U' *of* A, *such that* $\|\psi'\|_\beta < \epsilon$ *and that* $\mathcal{D}_{A'}^{\psi+\psi'}$ *is surjective for each* $[A'] \in U' \cap \mathcal{M}_\delta^{\psi+\psi'}(a,b)$.

Proof. By an argument similar to the proof of Sublemma 2.9, it suffices to find ψ' such that $\|\psi'\|_\beta < \epsilon$, and that $\mathcal{D}_A^{\psi+\psi'}$ is surjective. We put

$$Cok = \mathrm{Ker}\,(D_A^\psi)^* \subset \mathcal{L}^1_{\ell,\delta},$$
$$Ker = \{u \in \mathcal{L}^1_{\ell,\delta} \mid \mathcal{D}_A u = 0, d^*_{a_t} u_t = 0\}$$

The group $U(1) \simeq G_A$ acts on Ker and Cok. By the surjectivity of $\mathcal{D}_A^{\psi,{\rm red}}$, we have $Cok \simeq \mathbf{C}^k$ as $U(1)$ module. By the index calculation in §5, we can find a $U(1)$ invariant subspace K of Ker which is isomorphic to $\mathbf{C}^k$ as $U(1)$ module. (See Remark 5.7.) Choose an isomorphism $Q : Cok \to K$. For each t, let $K_t, Cok_t \subset T_{[a_t]}\mathcal{B}_\ell(M)$ be the projection of

K and Cok. By the unique continuation theorem ([Ar]), the projections $K \to K_t, Cok \to Cok_t$ are isomorphisms. Let $Q_t : Cok_t \to K_t$ be the projection of Q. We can choose sufficiently large m and λ such that the curve $t \mapsto \widetilde{\Psi}'(a_t, \lambda)(0) = a'_t$ is injective, and $P_t : T_{[a_t]}(\mathcal{B}_\ell(M)) \to T_{a'_t} SU(2)^m$ is injective on $K_t + Cok_t$ for each t. Since the action of $U(1)$ has no trivial component on Cok_t, it follows that $P_t(K_t + Cok_t)$ is transversal to the tangent vector of the curve a'_t. Hence we can find a function $\psi_0 \in C^\beta(L_m, \mathbf{R})$ such that

$$\left(\mathrm{Hess}_{a'_t}\, \psi_0\right)(P_t V, P_t W) = \langle Q_t V, W\rangle,$$

for each $V \in Cok_t$ and $W \in K_t$. It is easy to see that $\psi' = \psi + \delta\psi_0$ has the required property.

Lemma 3.9 follows easily from Sublemmas 3.10 and 3.11.

§4. Sum formula for index bundles

It seems that many parts of this section are well known to experts. But we include it here because of the lack of appropriate reference and because we need a part of the proof in §11. However we omit the detail of the proof since the results are essentially known. First we shall work in the following situation.

Situation 4.1. Let X^{n+1} be an oriented complete Riemannian manifold, E, F be vector bundles on it, K a compact subset. Suppose that $X - K$ is isometric to the direct product $M \times (0, \infty)$. Let V be a vector bundle on M and $\Psi_E : E \to p^*V$, and $\Psi_F : F \to p^*V$ be isomorphisms of vector bundles. (Here $p : M \times (0, \infty) \to M$ is the projection.) Let $\mathcal{D}^0 : \Gamma(V) \to \Gamma(V)$ and $\mathcal{D} : \Gamma(E) \to \Gamma(F)$ be elliptic operators of first order. Suppose that $\mathcal{D}^0$ is selfadjoint. Assume that M is decomposed to $M_+ \amalg M_-$ such that

$$\mathcal{D} = \Psi_F^{-1}(\pm\frac{\partial}{\partial t} + \mathcal{D}^0)\Psi_E$$

respectively on $M_\pm \times (0, \infty)$. Let $\{\lambda_i | i \in \mathbf{Z}\}$ be the set of all eigenvalues of $\mathcal{D}^0$. Put $\lambda_0 = \min_{i \in \mathbf{Z}} \lambda_i^2$.

Theorem 4.2. *Suppose $\lambda_0 > 0$. Then $\mathcal{D}$ is Fredholm. Moreover, for $\lambda < \lambda_0$, there exists a finite dimensional subspace L_λ of $L^2(E)$, such that*

(4.2.1) *If $u \in L_\lambda^\perp$ then $|\mathcal{D}u| > \sqrt{\lambda}|u|$.. Here $L_\lambda^\perp$ is a orthonormal complement of L_λ*

(4.2.2) *L_λ is generated by the vectors v satisfying $\mathcal{D}^*\mathcal{D}v = \lambda' v$ with $\lambda' \le \lambda$.*

We omit the proof. See [LM],[T3]. Theorem 4.2 implies that

$$\operatorname{Index}\mathcal{D} = \dim\operatorname{Ker}\mathcal{D} - \dim\operatorname{Ker}\mathcal{D}^*$$

is well defined.

Situation 4.3. Let $X_i, M_i, E_i, F_i, V_i, \mathcal{D}_i, \mathcal{D}_i^0$ be as in Situation 4.1. We assume that there are unions of connected components, say $M_{1,+}^0$ and M_{2-}^0, of $M_{1,+}$ and $M_{2,-}$ respectively, and an orientation reversing diffeomorphism from $M_{1,+}^0$ to $M_{2,-}^0$, by which we can identity $V_1, \mathcal{D}_1^0$ and $V_2, \mathcal{D}_1^0$. We patch $X_1 - M_{1,+}^0 \times (T,\infty)$ and $X_2 - M_{2,-}^0 \times (T,\infty)$ by the diffeomorphism $M_{1,+}^0 \times \{T\} \to M_{2,-}^0 \times \{T\}$ to obtain $X(T)$. (Figure 1)

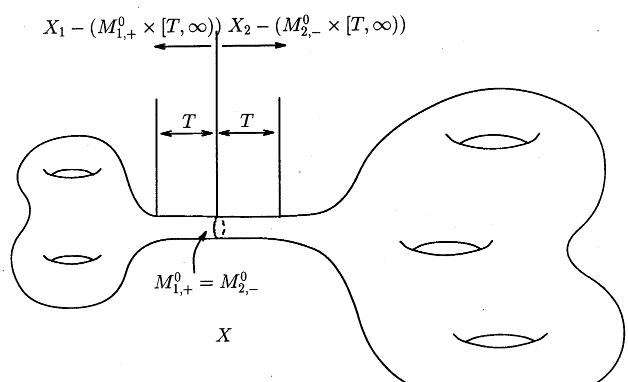

Figure 1.

Let $E(T)$ (resp. $F(T)$) be a vector bundle on $X(T)$ obtained by patching E_1 and E_2 (resp. F_i) by $\Psi_{E_2}^{-1}\Psi_{E_1}$ (resp. $\Psi_{F_2}^{-1}\Psi_{F_1}$). Define an operator $\mathcal{D} : \Gamma(E(T)) \to \Gamma(F(T))$ by

$$\mathcal{D} = \begin{cases} \mathcal{D}_1 & \text{on} \quad X_1 \\ \mathcal{D}_2 & \text{on} \quad X_2 \end{cases}$$

Theorem 4.4. *If $\lambda_0 > 0$ then we have*

$$\mathrm{Index}\,\mathcal{D} = \mathrm{Index}\,\mathcal{D}_1 + \mathrm{Index}\,\mathcal{D}_2.$$

Proof. Let $0 < \lambda < \lambda_0$. We may assume that λ is not an eigenvalue of $\mathcal{D}^*\mathcal{D}$ or $\mathcal{D}_i^*\mathcal{D}_i$. Let $L_\lambda \subset L^2(E)$ be the vector space generated by the vectors v such that $\mathcal{D}^*\mathcal{D}v = \lambda' v$ with $\lambda' < \lambda$. Define $L_\lambda^* \subset L^2(F)$, L_λ^i, L_λ^{i*} in the same way. Note that an embedding $X_1 - M_{1,+}^0 \times [T, \infty) \to X$ can be extended to an embedding $X_1 - M_{1,+}^0 \times [2T, \infty)$. Let $M_{1,+}^0 \times [0, 2T] \to X$ be its restriction. Put $d(t) = \min(|t|, |2T - t|)$.

Lemma 4.5. *If $u \in L_\lambda$ then*

$$|\nabla^k \varphi|(I(x,t)) < C_k e^{-\sqrt{\lambda_0 - \lambda}\,d(t)} \|u\|_{L^2}.$$

Proof. We may assume $\mathcal{D}^*\mathcal{D}u = \lambda' u$, $\lambda' < \lambda$. Let $\varphi_1, \cdots$ be the eigenvectors of $\mathcal{D}_0^*\mathcal{D}_0$. We put

$$u(I(x,t)) = \sum_{i=1}^{\infty} u_i(t)\varphi_i(x).$$

Since

$$\mathcal{D}^*\mathcal{D} = -\frac{\partial^2}{\partial t^2} + (\mathcal{D}^0)^2,$$

we have

$$-\frac{d^2 u_i}{dt^2} + \lambda_i^2 u_i = \lambda' u_i.$$

It follows that

$$|u_i(t)| \le C e^{-\sqrt{\lambda_0 - \lambda'}\,d(t)} \max\{|u_i(0)|, |u_i(T)|\},$$

from which the lemma follows by the standard estimates for elliptic operators.

Let $\chi : [-1, 1] \to [0, 1]$ be a nondecreasing C^∞ function such that

$$\chi(t) = \begin{cases} 0 & \text{if} \quad t < -1 \\ 1 & \text{if} \quad t > 1. \end{cases}$$

We define $P_i' : L_\lambda \to \Gamma_c(X_i, E_i)$ as follows. (Here Γ_c stands for the set of smooth sections with compact support.)

$$\begin{cases} (P_1'u)(x,t) = (1 - \chi(\frac{t-T}{T}))u(x,t) & \text{if} \quad (x,t) \in M_{1,+}^0 \times [0, 2T] \\ (P_1'u)(x,t) = 0 & \text{if} \quad (x,t) \in M_{1,+}^0 \times [2T, \infty) \\ (P_1'u)(z) = u(z) & \text{if} \quad z \notin M_{1,+}^0 \times [0, \infty) \end{cases}$$

$$\begin{cases} (P_2'u)(x,t) = \chi(\frac{t-T}{T})u(x,t) & \text{if} \quad (x,t) \in M^0_{2,-} \times [0,2T] \\ (P_1'u)(x,t) = 0 & \text{if} \quad (x,t) \in M^0_{2,-} \times [2T,\infty) \\ (P_1'u)(z) = u(z) & \text{if} \quad z \notin M^0_{2,-} \times [0,\infty) \end{cases}$$

Let $P_i(u)$ be the orthonormal projection of $P_i'(u)$ to L^i_λ. Put $P_\lambda = (P_1, P_2) : L_\lambda \to L^1_\lambda \oplus L^2_\lambda$. Then using Lemma 4.5 we can prove that P_λ is an isomorphism for large T. Similarly we can construct an isomorphism $P^*_\lambda : L^*_\lambda \to L^{1*}_\lambda \oplus L^{2*}_\lambda$. On the other hand, $\mathcal{D}$ defines an isomorphism: $L_\lambda \cap (\operatorname{Ker}\mathcal{D})^\perp \to L^*_\lambda \cap (\operatorname{Ker}\mathcal{D}^*)^\perp$. Therefore

$$\operatorname{Index}\mathcal{D} = \dim L_\lambda - \dim L^*_\lambda.$$

Similarly, we have

$$\operatorname{Index}\mathcal{D}_i = \dim L^i_\lambda - \dim L^{i*}_\lambda.$$

The theorem follows immediately. (Recall that $\operatorname{Index}\mathcal{D}^T$ does not depend on T.)

Remark 4.6. By the same method, we can prove that, if $\mathcal{D}_0$ is invertible, then the $Ce^{-\sqrt{\lambda_0-\lambda}T/C}$-neighborhood of the set

$$\{\text{eigenvalues of } \mathcal{D}^{T*}\mathcal{D}^T \text{ smaller than } \lambda_0\}$$

contains the set

$$\begin{aligned} &\{\text{eigenvalues of } \mathcal{D}_1\mathcal{D}_1^* \text{ smaller than } \lambda_0\} \\ &\cup \{\text{eigenvalues of } \mathcal{D}_2\mathcal{D}_2^* \text{ smaller than } \lambda_0\}. \end{aligned}$$

Also the $Ce^{-\sqrt{\lambda_0-\lambda}T/C}$-neighborhood of the later set contains the former set.

Moreover we can prove the following:

Corollary 4.7. *In Situation* 4.1, *let M^0_+, M^0_- be unions of components of M_+, M_-, respectively. Suppose that M^0_+, together with $\mathcal{D}_0$,V on it, is diffeomorphic to M^0_-. Construct $X(T), E(T), F(T), \mathcal{D}^T$, e.t.c. as before.* (*Figure* 2) *Then we have*

$$\operatorname{Index}\mathcal{D}^T = \operatorname{Index}\mathcal{D}.$$

In §6 and §11, we need also a family version of Theorem 4.4.

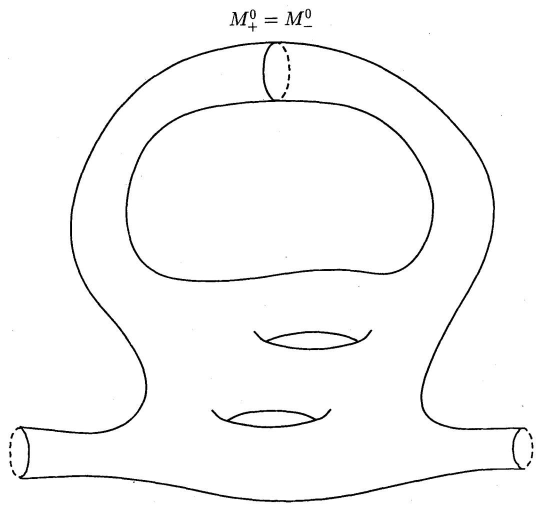

Figure 2.

Situation 4.8. Let Y be a manifold, $p_i : W_i \to Y$, $q : Z \to Y$ be fibre bundles. Let $\widetilde{E}_i, \widetilde{F}_i :\to W_i$, $\widetilde{V} \to Z$ be vector bundles and $\widetilde{\mathcal{D}}_i : \Gamma(\widetilde{E}_i) \to \Gamma(\widetilde{F}_i)$, $\widetilde{\mathcal{D}}^0 : \Gamma(\widetilde{V}) \to \Gamma(\widetilde{V})$ be families of elliptic operators. Suppose that $p_i^{-1}(y) = X_i(y)$, $q^{-1}(y) = M(y)$, $\widetilde{E}_i\Big|_{X_i(y)} = E_i(y)$, $F_i(y)$, $V(y)$, $\mathcal{D}_i(y)$, $\mathcal{D}^0(y)$ are as in Situation 4.3, for each $y \in Y$. As before, we can construct, $W(T) \to Y$, $\widetilde{E}(T), \widetilde{F}(T) \to W(T)$, $\mathcal{D}(T) : \Gamma(\widetilde{E}(T)) \to \Gamma(\widetilde{F}(T))$. As in [AS], the index bundles

$$\operatorname{Index} \mathcal{D}_i, \operatorname{Index} \mathcal{D}^T \in K(Y),$$

are well defined if $\mathcal{D}^0(y)$ is invertible.

Theorem 4.9. *Suppose $\mathcal{D}^0(y)$ is invertible for each y, then we have*

$$\operatorname{Index} \mathcal{D}_1 + \operatorname{Index} \mathcal{D}_2 = \operatorname{Index} \mathcal{D}^T,$$

in $K(Y)$.

Theorem 4.9 follows from the proof of Theorem 4.4, since P_λ and L_λ, e.t.c. there depend smoothly on operators.

Remark 4.10. The results of this section hold in the case when, for example, in Situation 4.1 the operator $\mathcal{D}$ is not exactly equal to $\Psi_F^{-1}(\pm\frac{\partial}{\partial t} + \mathcal{D}^0)\Psi_E$, but the difference is estimated by $Ce^{-|t|/C}$. (See [T3].)

§5. Dimension of moduli space

We put $\overline{\mathcal{M}}_\delta(a,b) = G_a\backslash\mathcal{M}_\delta(a,b)/G_b$. Recall that the action of $G_a \times G_b$ is trivial if $\delta < 0$. We can prove that $\overline{M}_\delta(a,b)$ is independent of δ. Hence we write $\overline{\mathcal{M}}(a,b)$.

Theorem 5.1. *There exists a map $\mu : Fl \to \mathbf{Z}$ such that $\mu(1) = 0$ and that*

$$\dim \overline{\mathcal{M}}(a,b) = \mu(a) - \mu(b) - \dim G_a, \tag{5.1}$$

except the component containing no irreducible connection.

Proof. First we assume that $a, b \in Fl_0$. In this case $\dim \overline{\mathcal{M}}(a,b) = \dim \mathcal{M}_\delta(a,b)$. We can use the perturbed Atiyah-Hitchin-Singer complex

$$\Omega^0_{\ell+1,0} \xrightarrow{d_A} \Omega^1_{\ell,0} \xrightarrow{\mathcal{D}_A} \Omega^2_{\ell-1,0}. \tag{5.2}$$

(definitions of operators and spaces are in §3), to calculate the dimension as

$$\dim \mathcal{M}_\delta(a,b) = \dim \frac{\operatorname{Ker}\mathcal{D}_A}{\operatorname{Im} d_A}.$$

Since $a \in Fl_0$, it follows that d_A is injective. By Lemma 3.9, $\mathcal{D}_A$ is surjective. Hence $\dim \mathcal{M}_\delta(a,b)$ is equal to the index of the complex (5.2). We put

$$(\mathcal{D}_A, d_A^*) : \Omega^1_{\ell,0} \to \Omega^2_{\ell,0} \oplus \Omega^0_{\ell-1,0}.$$

Then we have:

$$\dim \mathcal{M}_\delta(a,b) = \operatorname{Index}(\mathcal{D}_A, d_A^*).$$

We identify Ω^1_ℓ and $\Omega^2_\ell \oplus \Omega^0_\ell$ to $\mathcal{L}^1_{\ell,\delta} \oplus \mathcal{L}^0_{\ell,\delta}$ as in §3. For $a \in \mathcal{A}_\ell(M)$, define

$$D_a : L^2_\ell(M, (\wedge^1 \oplus \wedge^2) \otimes su(2)) \to L^2_\ell(M, (\wedge^1 \oplus \wedge^2) \otimes su(2))$$

by

$$D_a(u,\varphi) = (*_\sigma d_a u - \operatorname{Hess}_a u + d_a\varphi, d_a^* u).$$

Then when $t \to \infty$ the operator $(\mathcal{D}_A, d_A^*)$ is asymptotic to $-\frac{\partial}{\partial t} + D_b$ and when $t \to -\infty$ it is asymptotic to $-\frac{\partial}{\partial t} + D_a$. Since $a, b \in Fl_0$ it follows that

$$d_a : L^2(M, su(2)) \to L^2(M, \wedge^1 \otimes su(2))$$

is injective. Hence by (2.8.2), D_a and D_b are invertible. Therefore by Theorem 4.3, $(\mathcal{D}_A, d_A^*)$ is Fredholm for each $A \in \mathcal{B}_{\ell,\delta}(a,b)$. Since $\mathcal{B}_{\ell,\delta}(a,b)$ is connected, it follows that its index is independent of A. Therefore, we can use Theorem 4.4 to show

$$\mathrm{Index}(\mathcal{D}_C, d_C^*) = \mathrm{Index}(\mathcal{D}_A, d_A^*) + \mathrm{Index}(\mathcal{D}_B, d_B^*),$$

for $A \in \mathcal{M}_\delta(a,b)$, $B \in \mathcal{M}_\delta(b,c)$, $C \in \mathcal{M}_\delta(a,c)$, $a, b, c \in Fl_0$. In the case when b is reduced, way we can prove

$$\begin{aligned}\mathrm{Index}(\mathcal{D}_C, e_\delta d_C^* e_\delta^{-1}) = &\,\mathrm{Index}(\mathcal{D}_A, e_\delta d_A^* e_\delta^{-1}) + \mathrm{Index}(\mathcal{D}_B, e_\delta d_B^* e_\delta^{-1}) \\ &- \dim G_b,\end{aligned}$$

in a similar way, for $\delta > 0$. Therefore the theorem follows by putting

$$\mu(a) = \mathrm{Index}(\mathcal{D}_A, e_\delta d_A^* e_\delta^{-1}) - 3,$$

for an element $[A] \in \mathcal{B}_{\ell,\delta}(1,a)$.

Next we study the neighborhood of a reducible connection $A \in \mathcal{M}_\delta(a,b)$. There are two cases:

Case I. $\dim G_a = \dim G_b = 3$, $G_A = U(1)$.
Case II. $\dim G_a = \dim G_b = 1$, $G_A = U(1)$.

In case I, there exists $\varphi : \mathrm{Tor}\, H_1(M, \mathbf{Z}) \to \{\pm 1\} \subset U(1)$ such that $a, b \in RF_\varphi$. (See §2.) Then we can renumber the loops $\ell_1^0, \cdots, \ell_d^0$, which we choose at the beginning of §2, such that

$$\begin{aligned} a(\ell_i^0) = 1 &\Longleftrightarrow i \le p \\ b(\ell_i^0) = 1 &\Longleftrightarrow i \le p + k. \end{aligned}$$

(At this point, it is not yet clear that $k > 0$.)

Replacing the element b by a gauge equivalent one, we may assume that there exists $a_t \in \mathcal{A}_\ell(M)$ such that $d + A = d + a_t$. (Namely A has no dt component.) The group $U(1) = G_A$ acts on the complex $(\mathcal{D}_A, e_\delta d_A^* e_\delta^{-1})$. It follows that its index is a $U(1)$ module.

Lemma 5.3.

$$\operatorname{Index}(\mathcal{D}_A, e_\delta d_A^* e_\delta^{-1}) \simeq \begin{cases} \mathbf{C}^{k+1} \oplus \mathbf{R}^{k+1} & \textit{if} \quad \delta > 0 \\ \mathbf{C}^{k-1} \oplus \mathbf{R}^{k-1} & \textit{if} \quad \delta < 0. \end{cases}$$

Proof. We replace the complex $(\mathcal{D}_A, d_A^*)$ by $(\mathcal{D}_{A,1}+\epsilon, d_A^*+\epsilon)$, where

$$\mathcal{D}_{A,1}(u,\varphi) = -\frac{\partial u}{\partial t} + *_\sigma d_{a_t} u + d_{a_t}\varphi.$$

Put

$$\operatorname{Index}(\mathcal{D}_{A,1} + \epsilon, d_A^* + \epsilon) = \mathbf{C}^{k_1} \oplus \mathbf{R}^{k_2}.$$

The trivial $su(2)$ bundle together with (nontrivial) connection $d + a_t$ on $M \times \mathbf{R}$ splits into a real line bundle $\mathcal{L}^{\mathbf{R}}$ and a complex line bundle $\mathcal{L}^{\mathbf{C}}$, since $d + a_t$ is reducible. Note that the image of holonomy representation of a and b is contained in $\{\pm 1\}$, the center of $SU(2)$. Therefore the line bundles together with their connections, have canonical trivializations on their ends. Hence we can apply Corollary 4.7 to obtain bundles $\overline{\mathcal{L}}^{\mathbf{R}}$ and $\overline{\mathcal{L}}^{\mathbf{C}}$ on $M \times S^1$ such that

$$\begin{aligned} k_1 &= \dim_{\mathbf{C}} \operatorname{Index}\left((P_- d_A, d_A^*) \otimes \overline{\mathcal{L}}^{\mathbf{C}}\right) \\ k_2 &= \dim_{\mathbf{R}} \operatorname{Index}\left((P_- d_A, d_A^*) \otimes \overline{\mathcal{L}}^{\mathbf{R}}\right). \end{aligned}$$

Here

$$\overline{\mathcal{L}}^{\mathbf{C}} \xrightarrow{d_A} \wedge^1(M \times S^1) \otimes \overline{\mathcal{L}}^{\mathbf{C}} \xrightarrow{P_- d_A} \wedge^2_-(M \times S^1) \otimes \overline{\mathcal{L}}^{\mathbf{C}},$$

and similarly for $\overline{\mathcal{L}}^{\mathbf{R}}$. Therefore, as in Atiyah-Hichin-Singer [AHS], we have

$$\begin{aligned} k_1 &= \int_{M \times S^1} \left(2 + \frac{p^1(M \times S^1)}{3}\right)\left(1 + c^1(\overline{\mathcal{L}}^{\mathbf{C}}) + \frac{c^1(\overline{\mathcal{L}}^{\mathbf{C}}) \wedge c^1(\overline{\mathcal{L}}^{\mathbf{C}})}{2}\right) \\ &= 0, \end{aligned}$$

since

$$c^1(\overline{\mathcal{L}}^{\mathbf{C}}) = \sum_{i=p+1}^{p+k} [\ell_i^0] \cup [S^1].$$

Similarly $k_2 = 0$.

Next we compare the index of $(\mathcal{D}_{A,1} + \epsilon, d_A^* + \epsilon)$ to one of $(\mathcal{D}_A, e_\delta d_A^* e_\delta^{-1})$. For this purpose, we use the notion of spectral flow due to Atiyah-Patodi-Singer [APS]. Put

$$D_{a_t,1}(u,\varphi) = (*_\sigma d_{a_t} u + d_{a_t}\varphi, d_{a_t}^*\varphi).$$

The spectral flow of the operator $D_{a_t,1} + \epsilon$ gives the index of $(\mathcal{D}_A + \epsilon, d_A^* + \epsilon)$. The operator $D_{a,1}$ has zero as eigenvalue. The eigenspace is identified to $(\mathbf{C} \oplus \mathbf{R})^{d+1} \simeq (H_0(M;\mathbf{R}) \oplus H_1(M;\mathbf{R})) \otimes su(2)$. Replacing $\mathcal{D}_A$ by $\mathcal{D}_A + \epsilon$ is equivalent to push these eigenvalues a bit to positive direction. Next we examine the effect of the perturbation. We put

$$D_{a_t,2}(u,\varphi) = (*_\delta d_{a_t} u - \mathrm{Hess}_{a_t} f(u) + d_{a_t}\varphi, d_{a_t}^*\varphi).$$

We take the basis $(z_1, \cdots, z_d, t_1, \cdots, t_d)$ of $H_1(M;\mathbf{R}) \otimes su(2)$ such that z_i and t_i correspond to ℓ_i^0. Then, by (2.5) and our choice of a and b, replacement of $D_{a_t,1}$ by $D_{a_t,2}$ is equivalent to push the zero eigenvalues corresponding $z_1, \cdots, z_p$ and $t_1, \cdots, t_p$ a bit to positive direction and the others to negative direction while $t \to -\infty$, and to push the zero eigenvalue corresponding to $z_1, \cdots, z_{p+k}$ and $t_1, \cdots, t_{p+k}$ a bit to positive direction and the others to negative direction while $t \to \infty$. It follows from $k_1 = k_2 = 0$ that the index of the spectral flow $D_{a_t,2}$ is $\mathbf{C}^k \oplus \mathbf{R}^k$.

Finally we examine the effect replacing D_{a_t} by $(\mathcal{D}_{a_t}, e_\delta d_A^* e_\delta^{-1})$. If $\delta > 0$, this is equivalent to push the zero eigenvalues in $H_0(M;\mathbf{R}) \otimes su(2)$ to positive direction while $t \to \infty$ and push them to negative direction while $t \to -\infty$. If $\delta < 0$ this is equivalent to the perturbation to the opposite direction. Lemma 5.3 follows.

Lemma 5.3 implies $k > 0$. Using Lemma 5.3, we have a description of the moduli space in a neighborhood of reducible connections. First let $k = 1$, $\delta > 0$. The group $SU(2) \times SU(2) \times \mathbf{R}$ acts on $\mathcal{M}_\delta(a,b)$. Here $SU(2) \times SU(2) \simeq G_a \times G_b$ acts on $\mathcal{M}_\delta(a,b)$ by Lemma 3.4, and the action of $\mathbf{R}$ is induced by its action on $M \times \mathbf{R}$. Since $G_A = U(1)$ there exists an embedding

$$\frac{SU(2) \times SU(2)}{U(1)} \times \mathbf{R} \to \mathcal{M}_\delta(a,b).$$

By Lemma 5.3, this map is a diffeomorphism onto a connected component containing $[A]$. It follows that all the connections on this component is reducible. In the case $k \geq 2$ we can use a similar argument. Summing up we obtain

Theorem 5.4. *Suppose* $\dim G_a = \dim G_b = 3$, $\dim G_A = 1$, $[A] \in \mathcal{M}_\delta(a,b)$, $\delta > 0$. *Then* $\mu(a) = 3k + \mu(b)$ *for some* $k \leq d$ *and that there exists a diffeomorphism from*

$$\frac{SU(2) \times \mathbf{C}^{k-1} \times SU(2)}{U(1)} \times \mathbf{R}^k$$

onto a neighborhood of the $G_a \times G_b \times \mathbf{R}$ *orbit of* $[A]$. *The diffeomorphism is compatible with* $G_a \times G_b \times \mathbf{R} \simeq SU(2) \times SU(2) \times \mathbf{R}$ *action.*

Remark 5.5. In case $k = 1$ the formula (5.1) does not hold for this component. This is similar to the fact that the virtual dimension of the trivial connection on S^3 is -3. In case $k > 1$ the neighborhood of $[A]$ in $\overline{\mathcal{M}}(a,b)$ is diffeomorphic to the product of $C\mathbf{C}P^{k-1} \times \mathbf{R}^k$. Here C means the cone. (Compare [D1].)

By a similar but simpler argument we can examine the case when $G_0 = U(1)$ and obtain:

Theorem 5.6. *Let* $G_a = G_b = G_A = U(1)$, $A \in \mathcal{M}_\delta(a,b)$ *and* $\delta > 0$. *Then* $\mu(a) = \mu(b) + k$ *for some* $k \leq d$. *All the connections contained in the connected component of* $\mathcal{M}_\delta(a,b)$ *containing* $[A]$ *are reducible.*

Remark 5.7. We used the above index calculation in the proof of Sublemma 3.10. The fact we used there is that the $\mathbf{C}$-part of the index is always of nonnegative dimension.

If we use different perturbation from one we gave in §§2,3, (for example if we change the sign in Formula (2.16) from point to point) then the above fact is no longer true. As the consequence, Lemma 3.9 does not necessary hold in that case, and we have an obstruction in second homology of Atiyah-Hitchin-Singer complex.

Finally we remark:

Lemma 5.8. *Let* $[a], [b] \in Fl$, $b = g^*a$, *where* $g : M \to SU(2)$ *and* $\deg g = k$. *Then,*

$$\mu(b) = 8k + \mu(a).$$

For the proof see [F].

§6. Orientation of moduli space

Lemma 6.1. $\mathcal{M}_\delta(a,b)$ *is orientable.*

Proof. Let $\mathcal{DET}(a,b) = \mathcal{DET}(\mathcal{D}_A, e_\delta d_A^* e_\delta^{-1})$ be the determinant bundle of the Atiyah-Hitchin-Singer complex (5.2). We can extend $\mathcal{DET}(a,b)$ to a real line bundle on $\mathcal{B}_{\ell,\delta}(a,b)$. On $\mathcal{M}_\delta(a,b)$, the bundle $\mathcal{DET}(a,b)$ is isomorphic to the bundle of $\dim \mathcal{M}_\delta(a,b)$-forms. Hence it suffices to show :

Lemma 6.2. *The bundle $\mathcal{DET}(a,b)$ on $\mathcal{B}_{\ell,\delta}(a,b)$ is trivial.*

Proof. Since $\mathcal{M}_{\ell,\delta}(a,b)$ is not simply connected, the argument in [D1],[F], can not be applied directly to our situation. Instead we shall proceed as follows. Since 3-dimensional oriented cobordism group is trivial, we can find oriented manifolds $\overline{X}_\pm$ such that $\partial \overline{X}_+ = M$, $\partial \overline{X}_- = M^-$, where M^- is the manifold M with opposite orientation. Let W be a closed oriented 4-manifold obtained by patching X_+ and X_- along M. Take trivial $SU(2)$ bundles on them. Let $\mathcal{A}_\ell(W)$ be the set of all L^2_ℓ connection on W, and $\mathcal{G}_\ell(W)$ be the group of transformations. We put $\mathcal{B}_\ell(W) = \mathcal{A}_\ell(W)/\mathcal{G}_{\ell+1}(W)$. Put a metric on $X_\pm = \overline{X}_\pm - \partial \overline{X}_\pm$, such that $X_\pm - K_\pm$ is isometric to $M \times (0,\infty)$ for some compact subset $K_\pm$. Let e_δ be a function on $X_\pm$ such that $e_\delta(x,t) = e^{-\delta\|t\|}$ outside $K_\pm$. For $a \in Fl$ choose a connection $d + A^a$ on $X_\pm$ such that $A^a = a$ outside $K_\pm$. Put

$$L^2_{\ell,\delta}(X_\pm, \wedge^1 \otimes su(2)) = \left\{ u \;\middle|\; \begin{array}{l} u \text{ is a locally } L^2_\ell \text{ section} \\ \quad \text{of } \wedge^1 \otimes su(2) \\ \displaystyle\sum_{k=0}^{\ell} \int_{X_\pm} e_\delta |\nabla^k u| < \infty \end{array} \right\}$$

$$\mathcal{A}_{\ell,\delta}(X_\pm, a) = \{ d + A^a + u \mid u \in L^2_{\ell,\delta}(X_\pm, \wedge^1 \otimes su(2)) \}.$$

Define $\mathcal{G}^0_{\ell,\delta}$ as in §2. Put

$$\mathcal{B}_{\ell,\delta}(X_\pm, a) = \mathcal{A}_{\ell,\delta}(X_\pm, a)/\mathcal{G}^0_{\ell+1,\delta}(X_\pm).$$

Let $\mathcal{DET}_\pm(a)$ be the determinant bundle of Atiyah-Hitchin-Singer complex on $\mathcal{B}_{\ell,\delta}(X_\pm, a)$. First we shall prove that $\mathcal{DET}_\pm(a)$ is trivial. For simplicity, we assume that $a \in Fl_0$. It suffices to show that $\mathcal{DET}_\pm(a)$ is trivial on each compact subset $L_\pm$ of $\mathcal{B}_{\ell,\delta}(X_\pm, a)$. We define a map Pat : $L_+ \times L_- \to \mathcal{B}_{\ell,\delta}(W)$ as follows. Define a Riemannian manifold $X(T)$ by patching X_+ and X_- along M as in Situation 4.3. Then $M \times [0, 2T]$ is embedded in $X(T)$. Choose a C^∞ function $\chi : [-1,1] \to [0,1]$ by

$$\chi(t) = \begin{cases} 0 & \text{if } \; t < -1 \\ 1 & \text{if } \; t > 1. \end{cases}$$

For $[d+A] \in L_+, [d+B] \in L_-$ define $\mathrm{Pat}([A],[B])$ by

$$\begin{cases} \mathrm{Pat}([A],[B])(z) = A(z) \quad \text{if} \quad z \in X_+ - M \times (0,\infty) \\ \mathrm{Pat}([A],[B])(x,t) = \left(1 - \chi\left(\frac{t-T}{T}\right)\right) A(x,t) + \chi\left(\frac{t-T}{T}\right) B(x,t) \\ \mathrm{Pat}([A],[B])(z) = B(z) \quad \text{if} \quad z \in X_- - M \times (0,\infty) \end{cases}$$

Let $\mathcal{DET}_{X(T)} \to \mathcal{B}_\ell(X(T))$ be the determinant bundle of the Atiyah-Hitchin-Singer complex on $X(T)$. By Theorem 4.9, we have

$$\mathrm{Pat}^*(\mathcal{DET}_X(T)) \simeq \mathcal{DET}_+(a) \otimes \mathcal{DET}_-(a).$$

For sufficiently large T. By [D3], $\mathcal{DET}_{X(T)}$ is trivial. It follows that $\mathcal{DET}_\pm(a)$ is trivial.

Next, Let $L \subset \mathcal{B}_{\ell,\delta}(a,b)$, $L' \subset \mathcal{B}_{\ell,\delta}(X^+,a)$ be compact subsets. In a similar way, we define a map $\mathrm{Pat} : L \times L' \to \mathcal{B}_{\ell,\delta}(X^+,b)$. By Theorem 4.9, we have

$$\mathrm{Pat}^*(\mathcal{DET}_+(b)) \simeq \mathcal{DET}(a,b) \otimes \mathcal{DET}_+(a).$$

Therefore the trivializations of $\mathcal{DET}_+(a)$ and $\mathcal{DET}_+(b)$ induces a trivialization of $\mathcal{DET}(a,b)$, if $a,b \in Fl_0$. The case when a and/or b are reducible can be proved in a similar way, by using a perturbation of the complex around the boundaries. The proof of Lemma 6.2 is now complete.

§7. Partial compactification of moduli space

Let $\mathcal{M}'_\delta(a,b)$, $\overline{\mathcal{M}}'(a,b)$ be the quotients of $\mathcal{M}_\delta(a,b)$ and $\overline{\mathcal{M}}(a,b)$ by the $\mathbf{R}$-action. The proof of the theorems in §1 is based on the following Theorems 7.1 and 7.3 on the structure of the ends of $\overline{\mathcal{M}}'(a,b)$. Hereafter we fix sufficiently small positive number δ and write $\mathcal{M}(a,b)$ e.t.c. in place of $\mathcal{M}_\delta(a,b)$.

Theorem 7.1. *For $a,b \in Fl$, let $C\overline{\mathcal{M}}'(a,b)$ be the disjoint union of*

$$\overline{\mathcal{M}}'(a,c_0) \times \prod_{i=0}^{k-1} \overline{\mathcal{M}}'(c_i,c_{i+1}) \times \overline{\mathcal{M}}'(c_k,b),$$

for $c_0,\cdots,c_k \in Fl$, with $\mu(a) > \mu(c_0) > \cdots > \mu(c_k) > \mu(b)$. Put $m = \dim \overline{\mathcal{M}}'(a,b)$.

Then we can define a smooth structure on $C\overline{\mathcal{M}}'(a,b)$ such that the following holds.

(7.1.1) *If*

$$x \in \overline{\mathcal{M}}'(a,c_0) \times \prod_{i=0}^{k-1} \overline{\mathcal{M}}'(c_i,c_{i+1}) \times \overline{\mathcal{M}}'(c_k,b),$$

with $G_{c_i} = \{\pm 1\}$. Then a neighborhood of x in $C\overline{\mathcal{M}}'(a,b)$ is diffeomorphic to $[0,\infty)^{k+1} \times \mathbf{R}^{m-k-1}$.
(7.1.2) *If $x = ([A],[B]) \in \overline{\mathcal{M}}'(a,c) \times \overline{\mathcal{M}}'(c,b)$, with $G_c = U(1), G_A = G_B = \{\pm 1\}$. Then a neighborhood of x is diffeomorphic to $\mathbf{R}^m$.*
(7.1.3) *If $x = ([A],[B]) \in \overline{\mathcal{M}}'(a,c) \times \overline{\mathcal{M}}'(c,b)$, with $G_c = SU(2)$, $G_A = G_B = \{\pm 1\}$. Then a neighborhood of x is diffeomorphic to*

$$\frac{\mathbf{C}^2}{\mathbf{Z}_2} \times \mathbf{R}^{m-4}.$$

(7.1.4) *If $x = (A,B,C) \in \overline{\mathcal{M}}'(a,c_1) \times \overline{\mathcal{M}}'(c_1,c_2) \times \overline{\mathcal{M}}'(c_2,b)$, with $G_{c_1} = G_{c_2} = SU(2)$, $G_B = U(1)$, $G_A = G_C = \{\pm 1\}$, $3k = \mu(c_1) - \mu(c_2)$. Then a neighborhood of x is diffeomorphic to*

$$\left(\left(\frac{SO(3) \times \mathbf{C}^{k-1} \times SO(3)}{U(1)} \times (0,\infty]^2\right) / \sim\right) \times \mathbf{R}^{m-2k-5},$$

where $\sim$ is defined by

$$([g_1,z,g_2],(\infty,t)) \sim [g_1 g, z, g_2], (\infty,t))$$
$$([g_1,z,g_2],(t,\infty)) \sim [g_1, z, g g_2], (t,\infty)).$$

(7.1.5) *If $x = ([A],[B],[C]) \in \overline{\mathcal{M}}'(a,c_1) \times \overline{\mathcal{M}}'(c_1,c_2) \times \overline{\mathcal{M}}'(c_2,b)$, with $G_{c_1} = G_{c_2} = G_B = U(1)$, $G_A = G_C = \{\pm 1\}$. Then a neighborhood of x is diffeomorphic to $\mathbf{R}^m$.*
(7.1.6) *Let $\Lambda \in \mathbf{R}_+$. Then the set*

$$\overline{\mathcal{M}}'(a,b;\Lambda) = \{[A] \in \overline{\mathcal{M}}'(a,b) \mid \sup |F^A| < \Lambda\}$$

is relatively compact in $C\overline{\mathcal{M}}'(a,b)$.
(7.1.7) *The orientations of $\overline{\mathcal{M}}'(c_i,c_{i+1})$ are compatible in $C\overline{\mathcal{M}}'(a,b)$.*

Remark 7.2. (7.1.1) ... (7.1.5) above do not cover all the possible cases. The general case is the combination of them and the reader can easily supply it.

Next we construct the bundles in §1. Choose a set of loops $\{\gamma_1, \cdots, \gamma_d\}$ representing a basis of $H_1'(M;\mathbf{Z})$. Put $\Sigma_i = \gamma_i \times \mathbf{R} \subset M \times \mathbf{R}$. The surface Σ_i has a canonical spin structure. For $A \in \mathcal{A}_{\ell,\delta}(a,b)$, we let

$$\eth_A^i : \Gamma_c(\Sigma_i, su(2) \otimes \mathbf{C}) \to \Gamma_c(\Sigma_i, su(2) \otimes \mathbf{C})$$

be the Dirac operator twisted by the connection A. For each $a, b \in Fl$, $\eth_A^i + \epsilon$ is a Fredholm operator. (We add ϵ since $\eth_A^i$ is not Fredholm when a or b is reducible.) Then we obtain a complex line bundle

$$\mathcal{L}_i(a,b) \to \mathcal{B}_{\ell,\delta}(a,b)$$

by

$$\mathcal{L}_i(a,b)|_{[A]} = \bigwedge^{\text{top}} \left(\operatorname{Ker}(\eth_A^i + \epsilon)\right)^* \otimes \bigwedge^{\text{top}} \operatorname{Coker}(\eth_A^i + \epsilon).$$

(Note the action of $\mathcal{G}_{\ell,\delta}$ is free on $\mathcal{A}_{\ell,\delta}(a,b)$). The action of $G_a \times G_b$ on $\mathcal{B}_{\ell,\delta}(a,b)$ is lifted to this line bundle. The group $\{\pm 1\}$ acts trivially on $\mathcal{B}_{\ell,\delta}(a,b)$. The lift of the action of $\{\pm 1\}$ to $\mathcal{L}_i(a,b)$ is not necessary trivial. (Compare [D2], where the similar action is trivial because the numerical index of the Dirac operator on a *closed* surface is zero.) Then we consider the tensor product $\mathcal{L}_i(a,b) \otimes \mathcal{L}_i(a,b)$. It induces a complex line bundle $\overline{\mathcal{L}}_i^{(2)}(a,b)$ on $\overline{\mathcal{M}}'_*(a,b)$, the set of irreducible connections in $\overline{\mathcal{M}}'(a,b)$. (If we want to "define" the first Chern class $c^1(\mathcal{L}_i(a,b))$ itself, we have to invert 2.)

Theorem 7.3. *Collection of line bundles*

$$\mathcal{L}_i^{(2)}(a,c_0) \otimes \cdots \otimes \mathcal{L}_i^{(2)}(c_k,b) \to \overline{\mathcal{M}}'_*(a,c_0) \times \prod_{i=0}^{k-1} \overline{\mathcal{M}}'_*(c_i,c_{i+1}) \times \overline{\mathcal{M}}'_*(c_k,b),$$

can be patched together to give a complex line bundle on $\mathcal{C}\overline{\mathcal{M}}'_*(a,b)$.

Here and hereafter $\mathcal{M}_*$ stands for the set of irreducible connections. We can not extend the line bundle to the neighborhood of the connections described in Theorems 5.4 and 5.6. This is the reason why Theorem 1.10 does not hold for $s > 2$ when $H_1(M;\mathbf{Z})$ is torsion free and $s > 0$ when $H^1(M;\mathbf{Z})$ has a torsion. (We shall explain this point a bit more detail in §12.)

The proofs of Theorems 7.1 and 7.3 occupy §§7–11. We include the analysis of the structure of moduli space and the line bundle on it in the neighborhood of the connection described in Theorems 5.4 and 5.6, though the author does not know how to use it to deduce a topological information. In order to explain the outline of the proofs of Theorems 7.1 and 7.3, we introduce the following notion. (Compare Donaldson [D2].)

Definition 7.4. Let $K_0 \subset \overline{\mathcal{M}}'(a, c_0), \cdots, K_k \subset \overline{\mathcal{M}}'(c_k, b)$ be compact subsets and $\epsilon, T, C > 0$. We say that $[A] \in \overline{\mathcal{M}}'(a, b)$ is a *standard model of type* $(K_0, \cdots, K_k, T, \epsilon, C)$, if there exist $[A_i] \in K_i$, $S_{i+1} > T + S_i$, and $[A'] = [A]$, with the following property.

Let $I_i : M \times [-T, T] \to M \times \mathbf{R}$ be the embedding defined by $I_i(x, t) = (x, t + S_i)$. Then we have

(7.4.1) $$\|I_i^*(A') - A_i\|_{C^\ell}(x, t) < \epsilon,$$

(7.4.2) $$|A' - c_i|_{C^\ell}(x, t) < C \exp\{-\min\{|S_i + T/2 - t|, |S_{i+1} - T/2 - t|\}/C\},$$

if $t \in [S_i + T/2, S_{i+1} - T/2]$.

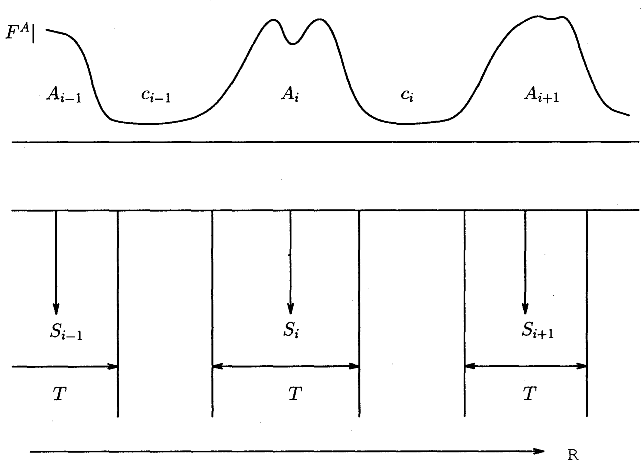

Figure 3.

The proof of Theorem 7.1 is based on the following two Theorems 7.5 and 7.6.

Theorem 7.5. *There exists C such that, for each $T, \Lambda, \epsilon > 0$, we can find a compact subset $K_{a,b}$ of $\overline{\mathcal{M}}(a, b)$ for each $a, b \in Fl$, with the* *following property. If* $[A] \in \overline{\mathcal{M}}(a,b)$, $\sup|F^A| < \Lambda$, *and if* $[A] \notin K_{a,b}$, *then there exist* $c_0, \cdots, c_k \in Fl$ *such that* $[A]$ *is a standard model of type* $(K_{a,c_0}, \cdots, K_{c_k,b}, T, \epsilon, C)$.

Theorem 7.6. *For each compact set* $K_0 \subset \overline{\mathcal{M}}'(a,c_0), \cdots, K_k \subset \overline{\mathcal{M}}'(c_k,b)$ *and* C, *there exist* $\epsilon = \epsilon(K_0,\cdots,K_k,C)$ *and* $T = T(K_0,\cdots,K_k,C)$, *such that the set of elements of* $\mathcal{M}'(a,b)$ *which is a standard form of type* $(K_0,\cdots,K_k,\epsilon,T,C)$ *is parametrized by*

$$\widetilde{K}_0 \times_{G_{c_0}} \widetilde{K}_1 \times_{G_{c_1}} \cdots \times_{G_{c_k}} \widetilde{K}_k \times (T,\infty)^{k+1}.$$

Here $\widetilde{K}_i \subset \mathcal{M}'(c_{i-1},c_i)$ *is the lift of* K_i.

Here $\widetilde{K}_0 \times_{G_0} \widetilde{K}_1$ is the quotient of $\widetilde{K}_0 \times \widetilde{K}_1$ by the action $g([A],[B]) = ([A]g^{-1}, g[B])$ of G_0. The proof of Theorem 7.6 is in §8. For the proof of Theorem 7.1, we need a bit more complicated version of Theorem 7.5.

Theorem 7.5′. *For each* $\Lambda > 0$ *we can find* $K_{a,b} \subset \overline{\mathcal{M}}(a,b)$ *and* C_k *such that the conclusion of Theorem* 7.5 *holds for*

$$\begin{aligned}\epsilon_k &= \epsilon(K_{a,c_0},\cdots,K_{c_k,b},C_k),\\ T_k &= T(K_{a,c_0},\cdots,K_{c_k,b},C_k)\end{aligned}$$

where $\epsilon(\cdots)$, $T(\cdots)$, *and* $C(\cdots)$ *are as in Theorem* 7.6.

The proof of Theorem 7.5′ is in §9. Now we are ready to explain the outline of the proof of Theorem 7.1. Let $a,b \in Fl_0$. Choose $K_{c,c'}$ for $\mu(a) \ge \mu(c) \ge \mu(c') \ge \mu(b)$, as in Theorem 7.5′. For $\mathfrak{c} = (c_0,\cdots,c_k)$, Let $\epsilon(\mathfrak{c})$ and $T(\mathfrak{c})$ be the number in Theorem 7.6. Define an equivalence relation $\sim$ on

$$\widetilde{K}_{a,c_0} \times \cdots \times \widetilde{K}_{c_k,b} \times (T(\mathfrak{c}),\infty]^{k+1}$$

by

$$\begin{cases} (x_0,\cdots,x_{k+1},t_0,\cdots,t_{k+1}) \sim (x_0,\cdots,x_ig,g^{-1}x_{i+1},\cdots,t_{k+1}) \\ \qquad\qquad \text{for each } t_0,\cdots,t_{k+1} \\ (x_0,\cdots,x_{k+1},t_0,\cdots,t_{k+1}) \sim (x_0,\cdots,x_ig,x_{i+1},\cdots,t_{k+1}) \\ \qquad\qquad \text{if} \quad t_i = \infty. \end{cases}$$

Put

$$\begin{aligned}
\widetilde{X}(\mathfrak{c}) &= \frac{\widetilde{K}_{a,c_0} \times \cdots \times \widetilde{K}_{c_k,b} \times (T(\mathfrak{c}),\infty]^{k+1}}{\sim},\\
X(\mathfrak{c}) &= G_a \backslash \widetilde{X}(\mathfrak{c})/G_b,\\
\widetilde{X}^{\circ}(\mathfrak{c}) &= \frac{\widetilde{K}_{a,c_0} \times \cdots \times \widetilde{K}_{c_k,b} \times (T(\mathfrak{c}),\infty)^{k+1}}{\sim},\\
\overset{\circ}{X}(\mathfrak{c}) &= G_a \backslash \widetilde{X}^{\circ}(\mathfrak{c})/G_b.
\end{aligned}$$

By Theorem 7.6, we have a diffeomorphism

$$\Phi_{\mathfrak{c}} : \overset{\circ}{X}(\mathfrak{c}) \to \overline{\mathcal{M}}'(a,b).$$

to its image. If $\mathfrak{c}' \subset \mathfrak{c}$, we have, by Theorem 7.6,

$$\Phi_{\mathfrak{c},\mathfrak{c}'} : X(\mathfrak{c}) \to G_a \backslash \mathcal{M}'(a,c'_0) \times_{G_{c'_0}} \cdots \times_{G_{c'_{k'}}} \mathcal{M}'(c'_{k'},b)/G_b \times [T,\infty]^{k'+1}.$$

We put

$$U(\mathfrak{c},\mathfrak{c}') = \{z \in X(\mathfrak{c}) \mid \Phi_{\mathfrak{c},\mathfrak{c}'}(z) \in \overset{\circ}{X}(\mathfrak{c}')\}.$$

If $\Phi_{\mathfrak{c}'}\Phi_{\mathfrak{c},\mathfrak{c}'} = \Phi_{\mathfrak{c}}$ is true, then we are able to use these maps to define the smooth structure on $\mathcal{CM}'(a,b)$. But the above equality does not exactly hold but holds modulo some small difference. Hence we have to perturb them. The argument needed for it is in §10, where we define the notion of local action and construct it on the end of $\mathcal{M}'(a,b)$. To extend line bundle we use an argument similar to the proof of the theorems in §4 and a lift of the local action to the line bundle.

§8. Taubes construction

We prove Theorem 7.6 in this section. Theorem 7.6 corresponds Donaldson [D2] §4. There Donaldson used the "alternating method". His method might work in our situation, where we have to deal with various types of reducible connections. But, since the organization needed for alternating method is a bit complicated, we use here more direct argument. (Maybe this is one Donaldson suggested in [D2] p 302.)

For simplicity of notation, we shall prove a special (but the most difficult) case. Let $a, c_1, c_2, b \in Fl$ such that $G_a = G_b = \{\pm 1\}$, $G_{c_1} = G_{c_2} = SU(2)$, $\mu(c_1) = \mu(c_2) + 3$, and $\widetilde{K} \subset \mathcal{M}'(c_1,c_2)$ be a component consisting of reducible connections. (We have, by Theorem 5.4,

$$\widetilde{K} \simeq \frac{SU(2) \times SU(2)}{U(1)}.\,)$$

Let $K_1 \subset \overline{\mathcal{M}}'(a,c_1)$, $K_2 \subset \overline{\mathcal{M}}'(c_2,b)$ be compact subsets and $\widetilde{K}_1 \subset \mathcal{M}'(a,c_1)$, $\widetilde{K}_2 \subset \mathcal{M}'(c_2,b)$ be their lifts. We shall construct a diffeomorphism $\Phi_{K,K_1,K_2} : \widetilde{K}_1 \times_{G_{c_1}} \widetilde{K} \times_{G_{c_2}} \widetilde{K}_2 \times [T,\infty)^2 \times \mathbf{R} \to \mathcal{M}(a,b)$, whose image contains all standard model of type $(K_1, K, K_2, T, \epsilon, C)$.

Choose a finite open covering

$$\begin{aligned} U_1^1 \cup \cdots \cup U_N^1 \supseteq K_1 \\ U_1^2 \cup \cdots \cup U_N^2 \supseteq K_2, \end{aligned}$$

and sections $\overline{s}_j^i : U_j^i \to \widetilde{K}_i$. Let $s_j^1 : U_j^1 \to \mathcal{A}_{\ell,\delta}(a,c_1)$, $s_j^2 : U_j^2 \to \mathcal{A}_{\ell,\delta}(c_2,b)$ be their lifts. Choose also an open covering

$$V_1 \cup \cdots \cup V_N = SU(2),$$

such that V_k is contractible. We have maps

$$\begin{aligned} J_k^1 &: V_k \times \mathbf{R} \to SU(2) \\ J_k^2 &: V_k \times \mathbf{R} \to SU(2) \end{aligned}$$

such that

$$\begin{cases} J_k^1(g,t) = 1 & \text{if} \quad t < -1 \\ J_k^1(g,t) = g & \text{if} \quad t > 0 \end{cases}$$
$$\begin{cases} J_k^2(g,t) = 1 & \text{if} \quad t > 1 \\ J_k^2(g,t) = g & \text{if} \quad t < 0. \end{cases}$$

Let $d + a_t^0 \in \mathcal{A}_{\ell,\delta}(c_1,c_2)$ be a representative of $G_{c_1}\backslash\widetilde{K}/G_{c_2} =$ one point. Choose a nonincreasing smooth function $\chi : \mathbf{R} \to [0,1]$ such that

$$\chi(t) = \begin{cases} 1 & \text{if} \quad t < 0 \\ 0 & \text{if} \quad t > 1. \end{cases}$$

Now, we define a map

$$\widetilde{\Phi}'_{j_1,j_2,k_1,k_2} : U_{j_1}^1 \times V_{k_1} \times V_{k_2} \times U_{j_2}^2 \times [T,\infty)^2 \times \mathbf{R} \to \mathcal{A}_{\ell,\delta}(a,b),$$

as follows. Let $A_i = s^i_{j_i}([A_i])$, $S_i \in [T,\infty)$, $S \in \mathbf{R}$, $g_i \in V_{k_i}$. Then

$$
\begin{aligned}
&\widetilde{\Phi}'_{j_1,j_2,k_1,k_2}([A_1], g_1, g_2, [A_2], S_1, S_2, S)\\
&\quad = (J_{k_1}(g_1,\cdot)^* A_1)(x, t-S) \qquad \text{for} \quad t < S + S_1/3\\
&\quad = \chi\left(\frac{t-S-S_1/3}{S_1/3}\right) g_1^* A_1(x, t-S)\\
&\qquad\quad + \left(1-\chi\left(\frac{t-S-S_1/3}{S_1/3}\right)\right) a^0_{t-S-S_1}\\
&\qquad\quad \text{for} \quad t \in [S+S_1/3, S+2S_1/3]\\
&\quad = a^0_{t-S-S_1} \qquad \text{for} \quad t \in [S+2S_1/3, S+S_1+S_2/3]\\
&\quad = \chi\left(\frac{t-S-S_1-S_2/3}{S_2/3}\right) a^0_{t-S_1-S}\\
&\qquad\quad + \left(1-\chi\left(\frac{t-S-S_1-S_2/3}{S_2/3}\right)\right) g_2^* A_2(x, t-S-S_1-S_2)\\
&\qquad\quad \text{for} \quad t \in [S+S_1+S_2/3, S+S_1+2S_2/3]\\
&\quad = (J^2_{k_2}(g_2,\cdot)^* A_2)(s, t-S-S_1-S_2) \qquad \text{for} \quad t > S+S_1+2S_2/3.
\end{aligned}
$$

Here $J^i_k(g,\cdot)$ is regarded as a map $M \times \mathbf{R} \to SU(2)$ and a gauge transformation.

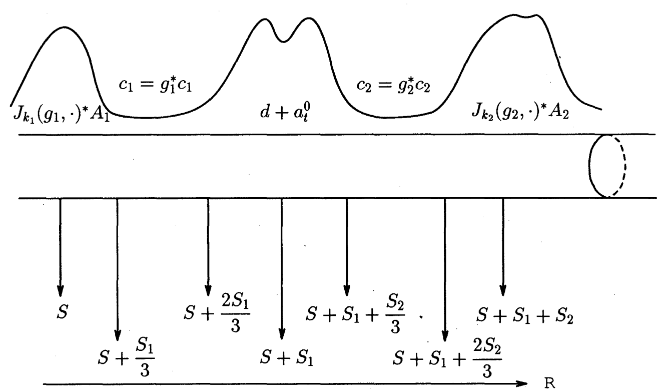

Figure 4

We remark that, by the compactness of K_1, we have a constant C such that

$$
(8.1) \qquad \begin{cases} |(d+A_1) - (d+a)| < Ce^{t/C}, \\ |(d+A_1) - (d+c_1)| < Ce^{-t/C}, \end{cases}
$$

for $A_1 \in K_1$. (Compare the decay estimate in next section.) A similar estimate holds for K_2 and K. Using (8.1) we can prove the following:

Lemma 8.2. *If*

$$[A_1] \in U^1_{j_1} \cap U^1_{j'_1},$$
$$[A_2] \in U^1_{j_2} \cap U^2_{j'_2},$$
$$g_1 \in V_{k_1} \cap V'_{k_2},$$
$$g_2 \in V_{k_2} \cap V'_{k_2},$$

then there exists a gauge transformation $\widehat{g}$, such that

$$\widehat{g}^* \widetilde{\Phi}'_{j_1,j_2,k_1,k_2}([A_1], g_1, g_2, [A_2], S_1, S_2, S)(t, x) = \widetilde{\Phi}'_{j'_1,j'_2,k'_1,k'_2}([A_1], g_1, g_2, [A_2], S_1, S_2, S)(t, x),$$

if $t \notin [S + S_1/3, S + 2S_1/3] \cup [S + S_1 + S_2/3, S + S_1 + 2S_2/3]$, *and*

$$|\widehat{g}^* \widetilde{\Phi}'_{j_1,j_2,k_1,k_2} - \widetilde{\Phi}'_{j'_1,j'_2,k'_1,k'_2}| < e(S_1, S_2).$$

Here and hereafter, we put

$$e(S_1, S_2) = C \exp(-\min\{S_1, S_2\}/C).$$

Choose an embedding $U(1) \subset SU(2)$ such that a^0_t is invariant by the image. By Lemma 8.2 and the construction, we can apply the partition of unity associated to the coverings $\{U^1_j\}$ and $\{U^2_j\}$ to prove the following:

Lemma 8.3. *There exists*

$$\widetilde{\Phi}''_{j_1,j_2,k_1,k_2} : U^1_{j_1} \times V_{k_1} \times V_{k_2} \times U^2_{j_2} \times [T, \infty)^2 \times \mathbf{R} \to \mathcal{A}_{\ell,\delta}(a, b),$$

such that

(8.3.1) $$|\widetilde{\Phi}''_{j_1,j_2,k_1,k_2} - \widetilde{\Phi}'_{j_1,j_2,k_1,k_2}| < e(S_1, S_2),$$

(8.3.2) *the maps $\widetilde{\Phi}''_{j_1,j_2,k_1,k_2}$ can be patched together to give a map*

$$\Phi'_{K_1,K,K_2} : \widetilde{K}_1 \times_{SU(2)} \frac{SU(2) \times SU(2)}{U(1)} \times_{SU(2)} \widetilde{K}_2 \times [T, \infty)^2 \times \mathbf{R} \to \mathcal{B}_{\ell,\delta}(a, b).$$

By (8.1) we have:

Lemma 8.4. *Let* $[A] \in \operatorname{Im} \Phi'_{K_1,K,K_2}$ *then*

$$|F^A + \widetilde{*}_\sigma F^A - \operatorname{grad}_{a_t} f \wedge dt - *_\sigma \operatorname{grad}_{a_t} f|_{L^2_\ell} < e(S_1, S_2).$$

We put

$$|u|_{\ell,S_1,S_2,S} = |u|_{L^2_\ell(M\times\mathbf{R})} + |u|_{L^1_\ell(M\times S,S+S_1+S_2)}.$$

Then we have also

Lemma 8.4′. *Let* $[A] \in \operatorname{Im} \Phi'_{K_1,K,K_2}$ *then*

$$|F^A + \widetilde{*}_\sigma F^A - \operatorname{grad}_{a_t} f \wedge dt - *_\sigma \operatorname{grad}_{a_t} f|_{\ell,S_1,S_2,S} < e(S_1, S_2).$$

We shall apply Taubes' method as in [FU], to deform Φ'_{K_1,K,K_2} to a map to $\mathcal{M}(a,b)$. For this purpose, the following estimate is essential.

Lemma 8.5. *There exists* $\lambda > 0$ *independent of* S_i *such that if* $A \in \operatorname{Im} \Phi'_{K_1,K,K_2}$, $u \in \Omega^2_\ell$ *we have*

$$|\mathcal{D}_A \mathcal{D}_A^* u|_{L^2_{\ell-2}} > \lambda |u|_{L^2_\ell}.$$

This lemma is an immediate consequence of Lemma 3.9 and Remark 4.6. Furthermore since $a \to \operatorname{grad}_a f$ is a C^2 map with respect to the L^2_ℓ norm for large ℓ, it follows that

$$\operatorname{grad}_{a_t+u_t} f = \operatorname{grad}_{a_t} f + (\operatorname{Hess}_{a_t} f)(u_t) + E(a,u)$$

with

$$|E(a,u)|_{L^2_\ell} \le C|u|^2_\ell$$
$$|E(a,u)|_{\ell,S_1,S_2,S} \le C|u|^2_{\ell,S_1,S_2,S}.$$

Hence we can apply the argument of [FU] pp.132–139, and obtain

Lemma 8.6. *There exists* T_0, *and* $\widetilde{\Phi}_{j_1,j_2,k_1,k_2} : U^1_{j_1} \times V_{k_1} \times V_{k_2} \times U^2_{j_2} \times [T_0,\infty) \times \mathbf{R} \to \widehat{M}(a,b)$ *such that*

(8.6.1) $\widetilde{\Phi}_{j_1,j_2,k_1,k_2}$ *can be patched together to give a map*

$$\Phi_{K_1,K,K_2} : \widetilde{K}_1 \times_{SU(2)} \frac{SU(2)\times SU(2)}{U(1)} \times_{SU(2)} \widetilde{K}_2 \times [T,\infty)^2 \times \mathbf{R}$$
$$\to \mathcal{M}(a,b).$$

$$\left|\widetilde{\Phi}''_{j_1,j_2,k_1,k_2} - \widetilde{\Phi}_{j_1,j_2,k_1,k_2}\right|_{C^1,\ell,S_1,S_2,S} < e(S_1,S_2). \tag{8.6.2}$$

The definition of the norm in (8.6.2) is as follows. $U^1_{j_1} \times V_{k_1} \times V_{k_2} \times U^2_{j_2} \times [T_0,\infty) \times \mathbf{R}$ has a natural Riemannian metric. We define a norm on $\mathcal{A}_{\ell,\delta}(a,b)$ by using (ℓ, S_1, S_2, S)-norm. Then the norm in (8.6.2) is the C^1-norm with respect to this metric and norm.

Note that the linear equation solved in [FU] pp.132–139 is gauge invariant. (8.6.1) follows from this fact.

We shall prove that the map Φ_{K_1,K,K_2} is an immersion, surjective to the set of standard model, and that injective.

Let $g_1, g_2 \in V_{k_1}, V_{k_2}$, and $\Pi \subset T_{(g_1,g_2)}(V_{k_1}, V_{k_2})$ be an orthonormal complement of $T_{(g_1,g_2)}(U(1)\cdot(g_1,g_2))$.

Lemma 8.7. *There exists C independent of S_1, S_2 such that, for each $v \in \Pi$ we have:*

$$\left|\Phi'_{j_1,j_2,k_1,k_2*}(v)\right|_{\ell,S_1S_2,S} \geq C|v|,$$

for sufficiently large S_i. Here we choose $[A_i] \in U^i_{j_i}$, S_i,S and regard

$$\Pi \subset T_{([A_1],g_1,g_2,[A_2],S_1,S_2,S)}(U^1_{j_1} \times V_{k_1} \times V_{k_2} \times U^2_{j_2} \times [T,\infty)^2 \times \mathbf{R}).$$

Remark 8.8. The lemma does not hold if we replace the $||_{\ell,S_1,S_2,S}$-norm by L^2_ℓ-norm, since c_1 and c_2 are reducible.

Proof. For simplicity, we put $g_1 = g_2 = 1$. Set

$$A = \widetilde{\Phi}'_{j_i,j_2,k_1,k_2}([A_1],1,1,[A_2],S_1,S_2,S)$$
$$v = (\overline{v}_1,\overline{v}_2) \in su(2) \oplus su(2).$$

Define $v_i : \mathbf{R} \to su(2)$, by

$$v_i(t) = \frac{d}{ds} J^i_{k_i}(1+s\overline{v}_i,t)\Big|_{s=0}.$$

Then by definition

$$\widetilde{\Phi}'_{j_1,j_2,k_1,k_2*}(v_1,v_2) = \begin{cases} (d^{A_1}v_1)(x,t-S) \\ \qquad \text{for} \quad t<S \\ (d^{A_2}v_2)(x,t-S_1-S_2-S) \\ \qquad \text{for} \quad t > S+S_1+S_2 \\ 0 \qquad \text{otherwise.} \end{cases} \tag{8.9}$$

Let the differential form in the above formula be denoted by w. Lemma 8.7 is a consequence of the following:

Lemma 8.10. *There exists C such that*

$$\left|w - d^A u\right|_{\ell,S_1,S_2,S} > C(|v_1| + |v_2|)$$

for each $u \in \Omega^0_{\ell+1}$ and sufficiently large S_i.

(In the statement we omit δ, since a and b are irreducible.)

Proof. We prove by construction. Then we assume that we have $\overline{v}_i^n \in su(2)$ with $|\overline{v}_i^n| = 1$, and $S_i^n \to \infty$, $[A_i^n]$,u^n such that

$$\lim_{n\to\infty} \left|w^n - d^{A_i^n} u^n\right|_{\ell,S_1^n,S_2^n,S} = 0.$$

Since $[A_i^n]$ and $\overline{v}_i^n$ move on compact sets, we may assume that they are independent of n. Hence we have

$$\begin{aligned} &S_i^n \to \infty \\ &\left|w^n - d^{A^n} u^n\right|_{\ell,S_1^n,S_2^n,S} \to 0. \end{aligned}$$

Here w^n is as in (8.9) with $S_i = S_i^n$, and

$$A^n = \widetilde{\Phi}'_{j_1,j_2,k_1,k_2}([A_1], 1, 1, [A_2], S_1^n, S_2^n, S).$$

(Since everything is invariant by the $\mathbf{R}$ action, we may assume that S is independent of n.) By construction, there exists α independent of n such that

(8.11)
$$\begin{aligned} &|d + A^n - d|_{C^{\ell'}} < Ce^{-\beta_1(t)/C} \quad \text{if} \quad t \in S+\alpha, [S + S_1^n - \alpha] \\ &|d + A^n - d|_{C^{\ell'}} < Ce^{-\beta_2(t)/C} \quad \text{if} \quad t \in [S + S_1^n + \alpha, S + S_1^n + S_2^n - \alpha], \end{aligned}$$

where

$$\begin{aligned} \beta_1(t) &= d(t, \partial[S+\alpha, S + S_1^n - \alpha]) \\ \beta_2(t) &= d(t, \partial[S + S_1^n + \alpha, S + S_1^n + S_2^n - \alpha]). \end{aligned}$$

Hence, by (8.9), we have, for each $\alpha' > \alpha$, that

$$|du^n|_{L^1_\ell(S+\alpha'.S+S_1^n+S_2^n-\alpha')} < \epsilon_n + Ce^{-\alpha'/C},$$

where $\epsilon_n \to 0$. Therefore there exists $s_1^n, s_2^n \in su(2)$ such that

$$\begin{aligned}
&|u^n - s_1^n|_{C^{\ell'}}(x,t) < C\epsilon_n + Ce^{-\beta_1(t)/C} \\
&\qquad \text{if} \quad t \in [S+\alpha', S+S_1^n-\alpha'] \\
&|u^n - s_2^n|_{C^{\ell'}}(x,t) < C\epsilon_n + Ce^{-\beta_2(t)/C} \\
&\qquad \text{if} \quad t \in [S+S_1^n+\alpha', S+S_1^n+S_2^n-\alpha'].
\end{aligned}$$

(This is the step we can not work with L^2 norm.)

Then patching u with s_1^n and s_2^n, we have $u_1^n, u_2^n, u_3^n \in L^2_{\ell+1}(M \times \mathbf{R}, su(2))$ such that

$$|d^{A_1}(v_1 - u_1^n)|_{C^{\ell'}} < C\epsilon_n \tag{8.12.1}$$

$$|d^{A_2}(v_2 - u_2^n)|_{C^{\ell'}} < C\epsilon_n \tag{8.12.2}$$

$$|d^{a_t^0} u_3|_{C^{\ell'}} < C\epsilon_n \tag{8.12.3}$$

$$|u_1^n(t,x) - s_1^n|_{C^{\ell'}} < Ce^{-t/C} \tag{8.12.4}$$

$$|u_2^n(t,x) - s_2^n|_{C^{\ell'}} < Ce^{t/C} \tag{8.12.5}$$

$$|u_3^n(t,x) - s_2^n|_{C^{\ell'}} < Ce^{-t/C} \tag{8.12.6}$$

$$|u_3^n(t,x) - s_1^n|_{C^{\ell'}} < Ce^{t/C} \tag{8.12.7}$$

(u_1^n,u_2^n, and u_3^n are constructed from the restrictions of u^n to $(-\infty, S+S_1^n/3]$, $[S+S_1^n+2S_2^n/3, \infty)$, $[S+2S_1^n/3, S+S_1^n+S_2^n/3]$, respectively.)

We may assume that $\lim s_1^n = s_1$ and $\lim s_2^n = s_2$. Therefore, by (8.12.3), (8.12.6),(8.12.7) and the fact $G_{a_t^0} = U(1)$ imply that $s_1 = s_2 \in u(1) \subset su(2)$. ($u(1)$ is a Lie algebra of $G_{a_y^0} = U(1)$.) Hence, using the fact that $(\overline{v}_1, \overline{v}_2)$ is perpendicular to $u(1) \subset su(2) \oplus su(2)$, we can find t_0 such that

$$|v_1 - u_1^n|(x, t_0) > C \tag{8.13}$$

or

$$|v_2 - u_2^n|(x, -t_0) > C,$$

for some C independent of n. Suppose, for example (8.13) holds. By scaling, we can find $(u^n)'$ such that

$$\begin{aligned}
&\infty > C_2 > |(u^n)'|(x,t_0) > C_1 > 0 \\
&\left|d^{A_1}(u^n)'\right|_{C^{\ell}} < \epsilon_n \to 0.
\end{aligned}$$

Therefore, by taking a subsequence, $(u^n)'$ converges to u' such that $d^{A_1}u' = 0$, with respect to the compact uniform topology. This contradicts the irreducibility of A_1. The proof of Lemma 8.10 is now complete.

An estimate similar to Lemma 8.7 for TK_i direction and $[T,\infty)^2 \times \mathbf{R}$ direction is easier. Then, combined with (8.6.2), they imply:

Lemma 8.14. *If V is a tangent vector of*

$$\widetilde{K}_1 \times_{SU(2)} \frac{SU(2) \times SU(2)}{U(1)} \times_{SU(2)} \widetilde{K}_2 \times [T,\infty)^2 \times \mathbf{R},$$

at $([A_1], g_1, g_2, [A_2], S_1, S_2, S)$, *then we have*

$$|\Phi_{K_1,K,K_2,*}(V)|_{\ell,S_1,S_2,S} > C|V|.$$

Lemma 8.14 implies that Φ_{K_1,K,K_2} is of maximal rank.

Remark 8.15. By Hölder's inequality, we have

$$||_{\ell,S_1,S_2,S} < C(S_1 + S_2)||_{L^2}.$$

Hence, Lemma 8.14 implies

$$|\Phi_{K_1,K,K_2*}(v)|_{L^2} > \frac{C|v|}{S_1 + S_2}.$$

It seems that this reflects the fact that the sectional curvature K of $\mathcal{M}(a,b)$ at $\Phi(A_1, g_1, g_2, A_2, S_1, S_2, S)$ is estimated as $|K| < C(S_1+S_2)^2$.

Lemma 8.16. *For each C, there exist T,S,ϵ, such that if $[A]$ is a standard model of type $(K_1, K, K_2, T, \epsilon, C)$, then*

$$[A] \in \Phi_{K_1,K,K_2}(\widetilde{K}_1 \times_{G_{c_1}} \widetilde{K} \times_{G_{c_2}} \widetilde{K}_2 \times [S,\infty)^2 \times \mathbf{R}).$$

Proof. The definition of the standard model implies that there exist $[A_1]$,$[A_2]$,g_1,g_2,S_1,S_2,S such that

$$|\widetilde{\Phi}'_{i_1,i_2,k_1,k_2}([A_1], g_1, g_2, [A_2], S_1 S_2, S) - A|_{L^2_\ell} < e(S_1, S_2).$$

Here A is a representative of A, and $A_j \in U_{i_j}$, $g_j \in V_{i_j}$. Let $\ell : [0,1] \to \mathcal{A}_{\ell,\delta}(a,b)$ be the straight line connecting them. The length of ℓ is smaller than $e(S_1, S_2)$. By [FU] pp.132–139, we can deform this path to a path ℓ' in $\widehat{\mathcal{M}}(a,b)$ connecting $\widetilde{\Phi}_{i_1,i_2,k_1,k_2}([A_1], g_1, g_2, [A_2], S_1, S_2, S)$ and A. The length of ℓ' is also estimated by $e(S_1,S_2)$. By using Lemma 8.14, we can lift this path to $\widetilde{\ell}: [0,1] \to \widetilde{K}_1 \times_{G_{c_1}} \widetilde{K} \times_{G_{c_2}} \widetilde{K}_2 \times [T,\infty)^2 \times \mathbf{R}$ such that $\widetilde{\ell}(0) = ([A_1], g_1, g_2, [A_2], S_1, S_2, S)$. Therefore

$$\Phi_{K_1,K,K_2}(\widetilde{\ell}(1)) = [A],$$

as required.

Finally we shall prove that Φ_{K_1,K,K_2} is injective.

Lemma 8.17. *If*

$$\Phi_{K_1,K,K_2}([A_1], g_1, g_2, [A_2], S_1, S_2, S) = \Phi_{K_1,K,K_2}([A'_1], g'_1, g'_2, [A'_2], S'_1, S'_2, S')$$

then

$$\begin{aligned} &|A_i - A'_i|_{\ell,S_1,S_2,S} < e(S_1,S_2) \\ &|S_i - S'_i| < e(S_1,S_2) \\ &|S - S'| < e(S_1,S_2), \end{aligned}$$

and there exists $h \in SU(2)$ such that

$$|hg_i - g'_i| < e(S_1,S_2).$$

Proof. The proof is similar to the proof of Lemma 8.7. Suppose $A_j \in U_{i_j}$, $A'_j \in U_{i'_j}$, $g_j \in V_{k_j}$, $g \in V_{k'_j}$. The proof of the statement on S_i and S is easy, then we assume that $S_i = S'_i$, $S = S'$, for simplicity. By assumption, there exists a gauge transformation $\widehat{g} : M \times \mathbf{R} \to SU(2)$ such that

$$\widehat{g}^* \widetilde{\Phi}_{i_1,i_2,k_1,k_2}([A_1], g_1, g_2, [A_2], S_1, S_2, S) = \widetilde{\Phi}_{i'_1,i'_2,k'_1,k'_2}([A'_1], g'_1, g'_2, [A'_2], S_1, S_2, S).$$

Then

$$\begin{aligned} &|\widehat{g}^* \widetilde{\Phi}'_{i_1,i_2,k_1,k_2}([A_1], g_1, g_2, [A_2], S_1, S_2, S) - \\ &\qquad \widetilde{\Phi}'_{i'_1,i'_2,k'_1,k'_2}([A'_1], g'_1, g'_2, [A'_2], S_1, S_2, S)|_{\ell,S_1,S_2,S} < e(S_1,S_2). \end{aligned}$$

Therefore, we have

$$|d\widehat{g}|_{C^\ell} < \begin{cases} Ce^{-\beta_1(t)/C} & \text{if} \quad t \in [S+\alpha, S+S_1-\alpha] \\ Ce^{-\beta_2(t)/C} & \text{if} \quad t \in [S+S_1+\alpha, S+S_1+S_2-\alpha]. \end{cases}$$

Here β_i is as in (8.11). Hence we have $g_i^0 \in SU(2)$ such that

$$|\widehat{g} - g_1^0| < Ce^{-\beta_1(t)/C} \quad \text{if} \quad t \in [S+\alpha, S+S_1-\alpha]$$
$$|\widehat{g} - g_2^0| < Ce^{-\beta_2(t)/C} \quad \text{if} \quad t \in [S+S_1+\alpha, S+S_1+S_2-\alpha].$$

Hence as in the proof of Lemma 8.10, we obtain $\widehat{g}_i : M \times \mathbf{R} \to SU(2)$, $i = 1, 2, 3$, such that

$$(8.18.1) \qquad |(\widehat{g}_1 J^1_{k_1}(g_1, \cdot))^* A_1 - J^1_{k_1'}(g_1', \cdot)^* A_1'|_{L^2_\ell} < e(S_1, S_2)$$

$$(8.18.2) \qquad |(\widehat{g}_2 J^2_{k_2}(g_2, \cdot))^* A_2 - J^2_{k_2'}(g_2', \cdot)^* A_2'|_{L^2_\ell} < e(S_1, S_2)$$

$$(8.18.3) \qquad |\widehat{g}_3^* a_t^0 - a_t^0|_{L^2_\ell} < e(S_1, S_2)$$

and

$$(8.18.4) \qquad |\widehat{g}_1(x,t) - g_1^0|_{C^\ell} < Ce^{-t/C}$$

$$(8.18.5) \qquad |\widehat{g}_2(x,t) - g_2^0|_{C^\ell} < Ce^{t/C}$$

$$(8.18.6) \qquad |\widehat{g}_3(x,t) - g_2^0|_{C^\ell} < Ce^{-t/C}$$

$$(8.18.7) \qquad |\widehat{g}_3(x,t) - g_1^0|_{C^\ell} < Ce^{t/C}$$

(8.18.3),(8.18.6),(8.18.7) and $G_{a_t^0} = U(1)$ implies that we have $h \in U(1)$ such that

$$|g_i^0 - h| < e(S_1, S_2).$$

Hence (8.18.1),(8.18.2),(8.18.4),(8.18.5) and the irreducibility of A_i, A_i' imply

$$|g_i' - hg_i| < e(S_1, S_2)$$
$$|A_i - A_i'|_{L^2_\ell} < e(S_1, S_2).$$

The proof of Lemma 8.17 is now complete.

Lemma 8.19. *For sufficiently large T, the map Φ_{K_1,K,K_2} is injective.*

Proof. Let A_i, A_i', g_i, g_i', S_i, S_i', S, S' be as in the proof of Lemma 8.17. Replacing g_i by hg_i, we may assume that $|g_i - g_i'| < e(S_1, S_2)$. Hence we can find a path $\ell : [0,1] \to \widetilde{K}_1 \times_{G_{c_1}} \widetilde{K} \times_{G_{c_2}} \widetilde{K}_2 \times [T, \infty)^2 \times \mathbf{R}$ connecting $([A_1], g_1, g_2, [A_2], S_1, S_2, S)$ and $(A_1', g_1', g_2', A_2', S_1', S_2', S')$. The length of ℓ is smaller than $e(S_1, S_2)$. We may assume that A_j and A_j' are in the same $U^i_{j_i}$, and that g_j and g_j' are in the same V_{k_j}. Therefore the map

$$\overline{\ell} = \widetilde{\Phi}_{U^1_{j_1}, U^2_{j_2}, V_{j_1}, V_{j_2}} \circ \ell : [0,1] \to \widehat{\mathcal{M}}(a,b)$$

is well defined. Note $\overline{\ell}(0) = \overline{\ell}(1)$ and the length of $\overline{\ell}$ with respect to the $||_{\ell,S_1,S_2}$-norm is smaller than $e(S_1,S_2)$. Hence we can find $H : D^2 \to \mathcal{A}_{\ell,\delta}(a,b)$ such that $H|_{\partial D^2} = \overline{\ell}$. By [FU] pp.132–139, we can deform H to $H' : D^2 \to \widehat{\mathcal{M}}_{\ell,\delta}(a,b)$ such that $H = H'$ on ∂D^2. Since the diameter of $H'(D^2)$ is smaller than $e(S_1,S_2)$, we can lift H' to $\widetilde{K}_1 \times_{G_{c_1}} \widetilde{K} \times_{G_{c_2}} \widetilde{K}_2 \times [T,\infty)^2 \times \mathbf{R}$, by Lemma 8.14. We conclude $\ell(0) = \ell(1)$. The proof of Lemma 8.19 is complete.

Thus, we have proved that the set of the standard model of type $(K_1, K, K_0, T, \epsilon, C)$ in $\mathcal{M}'(a,b)$ is parametrized by

$$\widetilde{K}_1 \times_{SU(2)} \frac{SU(2)\times SU(2)}{U(1)} \times_{SU(2)} \widetilde{K}_2.$$

We divide it by $G_a \times G_b = \{\pm 1\} \times \{\pm 1\}$ and obtain

$$\widetilde{K}_1 \times_{SU(2)} \frac{SO(3)\times SO(3)}{U(1)} \times_{SU(2)} \widetilde{K}_2.$$

This proves Theorem 7.6, in our case. The proof of the general case is the same, but the notations will be more complicated.

Remark 8.20. It seems that the proofs of Lemmas 8.17 and 8.19 reflect the fact that the injectivity radius of $\overline{\mathcal{M}}'(a,b)$ at $\Phi_{K_1,K,K_2}([A_1], g_1, g_2, [A_2], S_1, S_2, S)$ is larger than $C(\frac{1}{|S_1|+|S_2|})$.

§9. Decay estimate

In this section we shall prove Theorem 7.5$'$. This theorem corresponds to [FU] §9. There Weitzenbeck formula was used for the proof. We can not use it here because, in our case, M is not S^3 and because we perturbed the equation.

Lemma 9.1. *There exist ϵ, λ and C independent of T such that if $d + a_t$ is a $su(2)$ connection on $M \times [-T,T]$ without dt component, $c \in Fl$ and if*

$$|a_t - c|_{L^2_\ell} < \epsilon \tag{9.2.1}$$

$$\frac{\partial a_t}{dt} = *_\sigma F^{a_t} - \mathrm{grad}_{a_t} f \tag{9.2.2}$$

$$d_c^* a_0 = 0, \tag{9.2.3}$$

then we have

$$|a_t - c|_{L^2_\ell} \le Ce^{-\lambda\beta_T(t)}. \tag{9.3}$$

Here $\beta_T(t) = \inf\{T-t, T+t\}$.

Proof. We put $u(t) = a_t - c$. We have

$$\begin{aligned} *_\sigma F^{c+u(t)} - \operatorname{grad}_c f \\ &= *_\sigma d_c u(t) - \operatorname{Hess}_c f\,(u(t)) + E(u(t)), \end{aligned}$$

with

$$|E(u(t))|_{L^2_\ell} \le C|u(t)|^2_{L^2_\ell}, \tag{9.4}$$

for sufficiently large ℓ. Decompose $u(t) = \alpha(t) + \beta(t)$ with

$$\begin{cases} d_c^*\alpha(t) = 0 \\ \beta(t) \in \operatorname{Im} d_c \end{cases}$$

Then we have

$$|\alpha(t)|_{L^2_\ell} < C\epsilon, \quad |\beta|_{L^2_\ell} < C\epsilon, \tag{9.5.1}$$

$$\frac{\partial\alpha(t)}{\partial t} = *_\sigma d_c\alpha(t) - \operatorname{Hess}_c f(\alpha(t)) + E_1(\alpha(t),\beta(t)) \tag{9.5.2}$$

$$\frac{\partial\beta(t)}{\partial t} = E_2(\alpha(t),\beta(t)), \tag{9.5.3}$$

with

$$|E_i(\alpha(t),\beta(t))|_{L^2_\ell} < C\left(|\alpha(t)|_{L^2_\ell} + |\beta(t)|_{L^2_\ell}\right)^2. \tag{9.6}$$

We decompose

$$\alpha(t) = \alpha_+(t) + \alpha_-(t),$$

where α_+, α_- belong to the spaces spanned by positive and negative eigenspaces of $*_\sigma d_c - \operatorname{Hess}_c f$, respectively. (Note that by Lemma 2.8, zero is not an eigenvalue of $*_\sigma d_c - \operatorname{Hess}_c f$.) We put $g_\pm(t) = |\alpha_\pm(t)|_{L^2}$, $h(t) = |\beta(t)|_{L^2}$. By (9.2.2) and (9.4), we have

$$|E_1(\alpha(t),\beta(t))|_{L^\infty} < C(g_+(t) + g_-(t) + h(t))^2.$$

Therefore, we have

$$\frac{dg_+}{dt} \geq \lambda g_+ - C_0(g_- + h)^2, \tag{9.7.1}$$

$$\frac{dg_-}{dt} \leq -\lambda g_- + C_0(g_+ + h)^2, \tag{9.7.2}$$

$$\left|\frac{dh}{dt}\right| \leq C_0(g_+ + g_- + h)^2. \tag{9.7.3}$$

Hence, by elliptic regularity, it suffices to show the following:

Sublemma 9.8. *There exists a constant C and ϵ depending only on C_0 and λ and is independent of T such that if g_+,g_- and h be non-negative functions satisfying* (9.7.1)–(9.7.3) *and*

$$|g_\pm(t)| < \epsilon, |h(t)| < \epsilon, \tag{9.7.4}$$

$$h(0) = 0, \tag{9.7.5}$$

then

$$|g_\pm(t)|, |h(t)| < Ce^{-\lambda\beta_T(t)}. \tag{9.9}$$

Proof. First we replace the assumption (9.7.5) by $|h(0)| < \delta$, and prove

$$|g_\pm(t)|, |h(t)| < C(e^{-\lambda\beta_T(t)} + \delta).$$

when $\delta^2 T < \mu_0$, $\epsilon T < \mu_0$ for some μ_0 depending only on C_0 and λ. For this purpose we prove

$$|h| < C_0(\epsilon^n + \epsilon e^{-\lambda\beta_T(t)} + \delta) \tag{9.10.2n}$$

$$|g_\pm| < C_0(\epsilon^n + \epsilon e^{-\lambda\beta_T(t)} + \delta) \tag{9.11.2n.$\pm$}$$

by an induction on n. (Here n is a half integer.) Assume (9.10.2n). Let $t_0 \in [-T, T]$. We put

$$\widehat{g}_+(t) = e^{-\lambda(t-t_0)} g_+(t).$$

Then, by (9.7.1),(9.7.4),(9.10.2n), and (9.11.2$n-1$,$\pm$), we have:

$$\begin{aligned}\epsilon e^{-\lambda(T-t_0)} &\geq \widehat{g}_+(T)\\ &\geq g_+(t_0) - \int_{t_0}^{T} C_0^3 e^{-\lambda(t-t_0)}(\epsilon^{n-1/2} + \epsilon e^{-\lambda\beta_T(t)} + \delta)^2 dt.\end{aligned}$$

(9.11.2n,+) follows. For the proof of (9.11.2n,−), we use $\widehat{g}_- = e^{\lambda(t-t_0)}g_-(t)$ in a similar way.

It is easy to see that (9.10.2n) and (9.11.2n) imply (9.10.2n+1).

For general T, we proceed as follows. Apply the first step to $T_0 = \mu_0/\epsilon$, and $\delta = 0$. We have $h(3T_0/4) < C_0 e^{-T_0\lambda/4}$. Then we apply the first step to $g_\pm(t - 3T_0/4)$, $h(t - 3T_0/4)$ and $T = T_0$. We obtain

$$\sup_{0<t<4T_0/3} |g_\pm(t)| < C_0 e^{-5T_0\lambda/12}$$

$$\sup_{0<t<4T_0/3} |h(t)| < C_0 e^{-5T_0\lambda/12},$$

if $3T_0/2 < T$. And similarly for $-4T_0/3 < t < 0$. Hence we can apply the first step to $T = 4T_0/3$. Iterating this, we obtain the desired result. The proof of Lemma 9.1 is now complete.

Lemma 9.12. *For each δ, C, there exists ϵ such that if $a \in \mathcal{A}_\ell(M)$,*

$$| *_\sigma F^a - \mathrm{grad}_a f|_{L^2_\ell} < \epsilon$$

$$|a|_{L^2_\ell} < C,$$

then there exists $c \in Fl$ and $g \in \mathcal{G}_{\ell+1}$ such that

$$|g^* a - c|_{L^2_\ell} < \delta.$$

Proof. If not, there exists $a_i \in \mathcal{A}_\ell(M)$ and $\delta > 0$, such that

$$\lim_{i\to\infty} | *_\sigma F^{a_i} - \mathrm{grad}_{a_i} f|_{L^2_\ell} = 0, \tag{9.13.1}$$

$$|a_i|_{L^2_\ell} < C, \tag{9.13.2}$$

$$|g_i^* a_i - c|_{L^2_\ell} > \delta \tag{9.13.3}$$

for each i, $g_i \in G_{\ell+1}$, and $c \in Fl$. (9.13.2) implies that, by taking a subsequence, a_i converges to an element a_∞ of $\mathcal{A}_{\ell-1,\delta}(a, b)$. Then, (9.13.1) implies that

$$| *_\sigma F^{a_\infty} - \mathrm{grad}_{a_\infty} f|_{L^2_\ell} = 0.$$

Hence there exists $g_i \in \mathcal{G}_{\ell+1}(M)$ and $c \in Fl$ such that $g_i^* a_i$ converges to c in $\mathcal{A}_{\ell-1}(M)$. By replacing g_i if necessary, we may assume that

$$d_c^*(g_i^* a_i - c) = 0. \tag{9.14}$$

(See FU.) By (9.13.1) we have

(9.15) $$\lim_{i\to\infty} | *_\sigma F^{g_i^* a_i} - \mathrm{grad}_{g_i^* a_i} f|_{L^2_\ell} = 0.$$

By (9.14),(9.15), $\lim |g_i^* a_i - c|_{L^2_{\ell-1}} = 0$, and an elliptic estimate, we have

(9.16) $$\lim_{i\to\infty} |g_i^* a_i - c|_{L^2_\ell} = 0.$$

(9.16) contradicts (9.13.3).

Using this lemma, we can improve Lemma 9.1 as follows.

Lemma 9.17. *There exists T_0,ϵ,λ, and C, such that if $d + a_t$ be a $su(2)$-connection on $M \times [-T, T]$ without dt component, and if*

(9.18.1) $$T > T_0$$

(9.18.2) $$\frac{\partial a_t}{\partial t} = *_\sigma F^{a_t} - \mathrm{grad}_{a_t} f$$

(9.18.3) $$\left|\frac{\partial a_t}{\partial t}\right|_{L^2_\ell} < \epsilon,$$

then there exists $c \in Fl$ and $g \in \mathcal{G}_{\ell+1}(M)$ such that

(9.19) $$|g^* a_t - c|_{L^2_\ell} < Ce^{-\lambda\beta_T(t)}.$$

Here g is regarded as a gauge transformation on $M \times \mathbf{R}$ independent of the $\mathbf{R}$ factor. The constants C,ϵ,λ are independent of T.

Proof. Let ϵ_0 be the number determined in Lemma 9.1, and S be a sufficiently large positive number determined later. Put $\delta = \epsilon_0/2S$. Then we obtain ϵ by Lemma 9.12. We may assume that $\epsilon < \delta$. By Lemma 9.12, we obtain $c \in Fl$. Replacing a_t by gauge transformation independent of t, we may assume that

(9.20.1) $$|a_0 - c|_{L^2_\ell} < \delta$$

(9.20.2) $$d_c^*(a_0 - c) = 0.$$

By (9.20.1),(9.18.3), and $2S\epsilon < \epsilon_0$, we can apply Lemma 9.1 to $M \times [-S, S]$, and obtain

$$|a_t - c|_{L^2_\ell} < Ce^{-\lambda\beta_S(t)}.$$

Hence by taking S sufficiently large, we have

(9.21.1) $$|a_{3S/4} - c|_{L^2_\ell} < \epsilon_0/K$$

(9.21.2) $$|a_{-3S/4} - c|_{L^2_\ell} < \epsilon_0/K.$$

Here K is a sufficiently large positive number determined later. Therefore there exists $g \in \mathcal{G}_{\ell+1}(M)$ such that

$$\begin{aligned}
&|g-1|_{L^2_\ell} < C\epsilon_0/K \\
&d_c^*(g^* a_{3S/4} - c) = 0 \\
&|g^* a_{3S/4} - c|_{L^2_\ell} < C\epsilon_0/K.
\end{aligned}$$

Here C depends only on M. Hence we can apply Lemma 9.1 to $g^* a_{t+3S/4}$, on $M \times [-S, S]$. By choosing S sufficiently large, we obtain

$$|g^* a_t - c|_{L^2_\ell} < C\epsilon_0/K,$$

for $t \in [0, 4S/3]$, provided $3S/2 < T$. By taking K sufficiently large, we have

$$|a_t - c|_{L^2_\ell} < \delta,$$

for $t \in [0, 4S/3]$. By using (9.21.2) we have the same estimate for $t \in [-3S/4, 0]$. Hence we can apply Lemma 9.1 to $M \times [-4S/3, 4S/3]$ if $3S/2 < T$. Repeating this we obtain the lemma.

Lemma 9.22. *There exists $\theta > 0$ such that, if $[A] \in \mathcal{M}_\delta(a,b)$ with $\mu(a) \neq \mu(b)$, and if $g^*A = d + a_t$, where $d + a_t$ is a connection without dt factor, then we have*

$$\int_{M\times\mathbf{R}} \left|\frac{\partial a_t}{dt}\right|^2 dxdt > \theta.$$

Proof. By [F] p122, the integral in the lemma is independent of A but depends only on a and b. Hence the lemma follows from (2.8.1).

Proof of Theorem 7.5$'$. Fix $a, b \in Fl$. Put $k_0 = \mu(a) - \mu(b)$. We shall prove that, for each $\mu(a) \geq \mu(c) \geq \mu(c') \geq \mu(b)$ there exists $K_{c,c'}$, such that the conclusion of Theorem 7.5 holds for

$$\begin{aligned}
\epsilon &= \frac{\epsilon(K_{a,c_0}, \cdots, K_{c_k,b})}{2^k} \\
T &= \frac{T(K_{a,c_0}, \cdots, K_{c_k,b})}{2^k}.
\end{aligned}$$

The proof is by induction on k. The first step is obvious, since $\overline{\mathcal{M}}'(c,c')$ is a finite set if $\mu(c) = \mu(c') + 1$. Hence it is enough to show the last step of the induction. We assume that the last step is false. Then we have $A_i \in \overline{\mathcal{M}}'(a,b)$, such that

$$\sup |F^{A_i}| < \Lambda, \tag{9.23.1}$$

$$[A_i] \text{ is unbounded in } \overline{\mathcal{M}}'(a,b), \tag{9.23.2}$$

$$\text{non of } A_i \text{ is a standard model.} \tag{9.23.3}$$

Let g_i be a gauge transform such that $g_i^* A_i = d + a_t^i$ has no dt component. We have

$$\frac{da_t^i}{dt} = *_\sigma F^{a_t^i} - \mathrm{grad}_{a_t^i} f.$$

If

$$\left| \frac{\partial a_t^i}{dt} \right|_{L^2_\ell} < \epsilon,$$

were true for each t, then Lemma 9.17 would imply that $a_t^i = c$ for some $c \in Fl$. It would follow that $a = b$. This is a contradiction. Hence there exists t_i^1 such that

$$\left| \frac{\partial a_{t_i^1}^i}{dt} \right|_{L^2_\ell} > \epsilon.$$

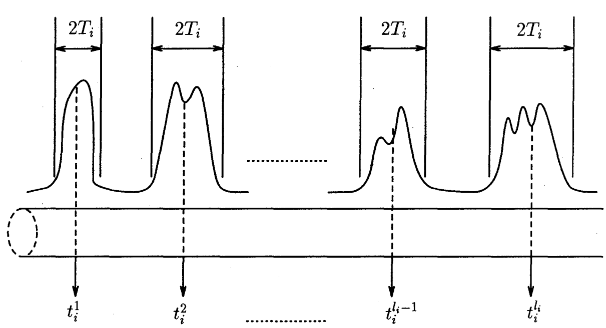

Figure 5.

Lemma 9.24. *There exists L independent of i, and there exist T_i,*

$t_i^1, \ldots, t_i^{\ell_i}$, *such that*

$$\ell_i < L, \tag{9.24.1}$$

$$\lim T_i = \infty, \tag{9.24.2}$$

$$\left|\frac{\partial a_t^i}{\partial t}\right|_{L^2_\ell} < \epsilon \quad \text{if } |t - t_i^j| > T_i \text{ for each } i, \tag{9.24.3}$$

$$|t_i^j - t_i^{j'}| > T_i \quad \text{if } j \neq j'. \tag{9.24.4}$$

Proof. The existence of the upperbound L of ℓ_i independent of i is the essential part of the statement. Hence, if Lemma 9.24 is not true, then, by taking a subsequence, we may assume that there exist $t_i^1, \cdots, t_i^{\ell_i} \in \mathbf{R}$, T_i such that (9.24.2),(9.24.4) and

$$\lim \ell_i = \infty \tag{9.24.5}$$

$$\left|\frac{\partial a^i_{t_i^j}}{dt}\right|_{L^2_\ell} > \epsilon \tag{9.24.6}$$

hold. By $|a_t| < \Lambda$, and by Uhlenbeck's theorem [FU] p117, we can find $g_i^j \in \mathcal{G}_{\ell+1}(M)$ such that a subsequence of the connection

$$t \mapsto g_i^{j*} a^i_{t-t_i^j},$$

converges to an element $d + a^\infty_{j,t}$ of $\mathcal{M}(c_j, c_j')$, for fixed j, (in C^2 topology on any compact set.) Here $c_j, c_{j'} \in Fl$. By (7.24.6), we have $c_j \neq c_{j'}$. Hence by Lemma 9.22

$$\int_{M\times\mathbf{R}} \left|\frac{\partial a^\infty_{j,t}}{dt}\right|^2 dt > \theta,$$

for each j. Therefore, Fatou's lemma implies

$$\begin{aligned}
&\lim_{i\to\infty} \int \left|\frac{\partial a_t^i}{\partial t}\right|^2 dt \\
&\geq \sum_{j=1}^{\infty} \int_{M\times\mathbf{R}} \left|\frac{\partial a^\infty_{j,t}}{dt} dt\right|^2 dt \\
&= \infty.
\end{aligned}$$

This contradicts the fact that

$$\int \left| \frac{\partial a_t^i}{dt} \right|^2 dt$$

is independent of i but depends only on a and b. The proof of the lemma is complete.

By Lemma 9.24 and $|F^{a_i}| < \Lambda$, we can take a subsequence such that the following holds : $\ell_i = \ell$ is independent of i : let $\widehat{a}_t^{i,j} = a^i_{t-t_i^j}$: there exists $g_{i,j}$ such that $\lim_{i\to\infty} g_{i,j}^* \widehat{a}_t^{i,j}$ converges to an element $a_i^{\infty,j}$ of $\mathcal{M}(c'_j, c''_j)$ uniformly on every compact set, for some c'_j, c''_j. If $\ell = 1$, we can easily prove that A_i is bounded in $\mathcal{M}'(a,b)$. This contradicts (9.23.2). On the other hand, by induction hypothesis, $\widehat{a}_t^{\infty,j}$ is either an element of $K_{c'_j,c''_j}$, or a standard model. Therefore, using Lemma 9.17 and (9.24.3), we can prove that A_i is a standard model for large i. This contradicts (9.23.3). The proof of Theorem 7.5′ is now complete.

§10. Local action on the end of moduli space

Using the results in §§8,9, we obtain charts $\Phi_{\mathfrak{c}} : X(\mathfrak{c}) \to \overline{\mathcal{M}}'(a,b)$ for each $\mathfrak{c}$. As we pointed out in §7 these charts are not compatible. Then we have to perturb them. Also, in order to extend bundles $\mathcal{L}_i^{(2)}$ to the boundary, we have to examine its behaviour on the image of each chart. For these purposes, it is useful to use the notion, local action of groups, which is a generalization of one introduced by Cheeger-Gromov [CG]. They used the local action to study the end of Riemannian manifolds with bounded curvature. In their case, a special kind of local action, F-structure, (that is the local action of Torus,) arises, and the direction of the orbits is the collapsed one. In our case, the curvature is not bounded from above. (It might be bounded from below.) Hence the group acting on the end is not necessary Abelian. (The group $SU(2)$ arises as well.) However the end is also collapsed and the collapsed direction is homogeneous. (For example, in the case we studied in §8, the collapsed direction is parametrized by $SO(3) \times SO(3)/S^1$.

Before stating our result we shall discuss examples. First consider the case, when $G_a = G_b = \{\pm 1\}$, $G_c = G_{c'} = U(1)$, $\mu(a) > \mu(c) > \mu(c') > \mu(b)$. Choose a compact subset $K_{c,c'}$ of $\overline{\mathcal{M}}'(c,c')$, consisting of irreducible connections. Then, by Theorem 7.6, the intersection of $\overline{\mathcal{M}}'(a,b)$ and a neighborhood of $K_{a,c} \times K_{c,c'} \times K_{c',b}$ in $\mathcal{C}\overline{\mathcal{M}}'(a,b)$ is diffeomorphic to

$$G_a \backslash \widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,c'} \times_{G_{c'}} \widetilde{K}_{c',b}/G_b \times (T,\infty)^2.$$

On this set we can define an action of $U(1) \times U(1) = G_c \times G_{c'}$ by

$$(h,h')([x,y,z],t,s) = ([xh,y,h'z],t,s).$$

Note that $\widetilde{K}_{a,c} \to K_{a,c}$ is a principal $U(1)$ bundle, hence $U(1)$ acts on $\widetilde{K}_{a,c}$. As in §7, we have a map

$$\begin{aligned}
\Phi_{(c,c'),(c')} &: G_a \backslash \widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,c'} \times_{G_{c'}} \widetilde{K}_{c',b}/G_b \times (T,\infty)^2 \\
&\to G_a \backslash \mathcal{M}'(a,c') \times_{G_{c'}} \widetilde{K}_{c',b}/G_b \times (T,\infty) \\
\Phi_{(c,c'),(c)} &: G_a \backslash \widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,c'} \times_{G_{c'}} \widetilde{K}_{c',b}/G_b \times (T,\infty)^2 \\
&\to G_a \backslash \mathcal{M}'(a,c) \times_{G_c} \widetilde{K}_{c,b}/G_b \times (T,\infty)
\end{aligned}$$

Let Z_2, Z_1 be inverse images of $G_a \backslash \widetilde{K}_{a,c'} \times_{G_{c'}} \widetilde{K}_{c',b}/G_b \times (T,\infty)$ and $G_a \backslash \widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,b}/G_b \times (T,\infty)$ respectively. (See Figure 6.) $G_a \backslash \widetilde{K}_{a,c'} \times_{G_{c'}} \widetilde{K}_{c',b}/G_b \times (T,\infty)$ has a $U(1)$ action. This action is identified to the action on the second factor of $U(1) \times U(1)$ on Z_2. Similarly the $U(1)$ action of $G_a \backslash \widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,b}/G_b \times (T,\infty)$ is identified to the action of the first factor of $U(1) \times U(1)$ on Z_1. This is exactly the situation of T-structure defined in [CG].

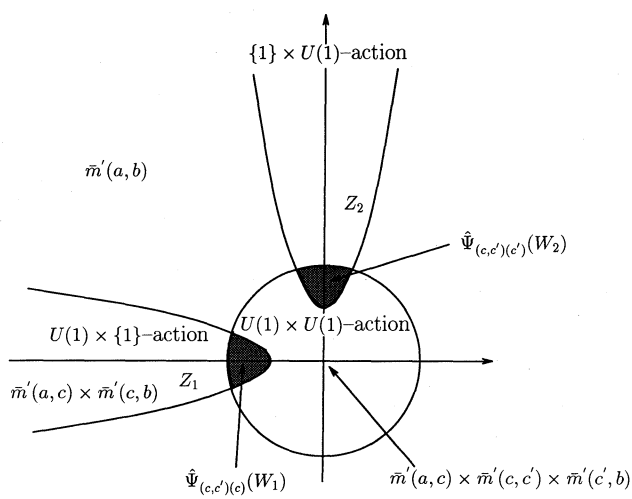

Figure 6

Next, consider the case, $G_a = G_b = \{\pm 1\}$, $G_c = SU(2)$. A neighborhood of $K_{a,c} \times K_{c,b}$ in $\mathcal{C}\overline{\mathcal{M}}'(a,b)$ is diffeomorphic to

$$G_a \backslash \widetilde{K}_{a,c} \times_{SU(2)} \widetilde{K}_{c,b}/G_b \times (T, \infty).$$

On this set $SU(2)$ does not has a global action, but has a local action in the following sense. Consider the principal $SU(2)$ bundle : $\widetilde{K}_{a,c} \to \widetilde{K}_{a,c}/SU(2)$. Let $SU(2)$ act on itself by conjugation, and $P' \to \widetilde{K}_{a,c}/SU(2)$ be the associated bundle. P' has a structure of Lie group bundle. P' induces a bundle $P \to \widetilde{K}_{a,c}/G_c \times G_c \backslash \widetilde{K}_{c,b}$. P has a fibrewise action to

$$\widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,b} \to \widetilde{K}_{a,c}/G_c \times G_c \backslash \widetilde{K}_{c,b},$$

induced from the fibrewise action of P' to $\widetilde{K}_{a,b}$ from left. (Note $SU(2)$ act globally on $\widetilde{K}_{a.b}$ from right.) This fibrewise action defines a local action. If $\mu(c) > \mu(c') > \mu(b)$, the local action of $G_c = SU(2)$ can be made to be compatible with the local action of $G_c \times G_{c'}$.

Note that this action is *not* an action of a sheaf of groups in the sense of [CG], because the fibre bundle $P \to \widetilde{K}_{a,c}/G_c \times G_c \backslash \widetilde{K}_{c,b}$ is not flat, in general.

Take a principal bundle $\widetilde{K}_{c,b} \to SU(2) \backslash \widetilde{K}_{c,b}$ and construct a Lie group bundle $Q \to \widetilde{K}_{a,c}/G_c \times G_c \backslash \widetilde{K}_{c,b}$ in a similar way. Q has also a fibrewise action on

$$G_a \backslash \widetilde{K}_{a,c} \times_{SU(2)} \widetilde{K}_{c,b}/G_b \times (T, \infty).$$

This action does not coincide to the action of P. But they have the same orbits. By convention, we use only the action of P.

Definition 10.1. Let X be a C^∞ manifold. A local action on X is a collection $(U_i, G_i, \varphi_{i,j})$ such that

(10.1.1) U_i is an open covering of X.
(10.1.2) $\cdot : G_i \times U_i \to U_i$ is a smooth action of a Lie group G_i on U_i.
(10.1.3) $U_i \cap U_j$ is G_i and G_j invariant.
(10.1.4) Let $Em(G_i, G_j)$ be the set of all injective homomorphisms. For $i < j$, there exists a smooth map $\varphi_{i,j} : \frac{U_i \cap U_j}{G_i} \to Em(G_i, G_j)$ such that

$$g(x) = \varphi_{i,j}([x])(g)(x)$$

holds for each $x \in U_i \cap U_j$, $g \in G_i$.

Example 10.2. Let $X \to N$ be a principal G bundle. (G acts on X from right.) Let $P = X \times_{ad} G$. P is a Lie group bundle and has a fibrewise left action on X. This gives a local action on X.

Example 10.3. Let $\widetilde{X}^o(c)$ be as in §7. There exists a fibration

$$\widetilde{X}^o(c) \to G_a \backslash \widetilde{K}_{a,c_0}/G_{c_0} \times \cdots \times G_{c_k} \backslash \widetilde{K}_{c_k,b}/G_b \times (T(c),\infty)^{k+1}$$

the fibre of which is $G_a \times G_{c_0} \times \cdots \times G_{c_k} \times G_b$. We have a Lie group bundle

$$P \to G_a \backslash \widetilde{K}_{a,c_0}/G_{c_0} \times \cdots \times G_{c_k} \backslash \widetilde{K}_{c_k,b}/G_b \times (T(c),\infty)^{k+1}$$

whose fibre is $G_a \times G_{c_0} \times \cdots \times G_{c_k} \times G_b$. The bundle P has a fibrewise action to $\widetilde{X}^o(c)$. This gives a local action on $\widetilde{X}^o(c)$.

Theorem 10.4. *There exist a local action on $\overline{\mathcal{M}}'(a,b)$ and maps*

$$\Psi_{\mathfrak{c}} : \overset{\circ}{X}(\mathfrak{c}) \to \overline{\mathcal{M}}'(a,b),$$
$$\Psi_{\mathfrak{c},\mathfrak{c}'} : U(\mathfrak{c},\mathfrak{c}') \to X(\mathfrak{c}),$$

such that

(10.4.1) *The restriction by Ψ_c of the local action on $\overset{\circ}{X}(c)$ of the local action coincides to one in Example* 10.3.

(10.4.2) $\Psi_{\mathfrak{c}'}\Psi_{\mathfrak{c},\mathfrak{c}'} = \Psi_{\mathfrak{c}}$. *(The subset $U(\mathfrak{c},\mathfrak{c}') \subset X(\mathfrak{c})$ is as in* §7.*)*

Theorem 7.1 follows immediately from Theorem 10.4. We have also

$$|\Phi_{\mathfrak{c}} - \Psi_{\mathfrak{c}}|(z) < e(S_1,\cdots,S_k). \tag{10.5}$$

Here $\Phi_{\mathfrak{c}}$ is the map constructed in §8, $z = ([A_1,\cdots,A_k],S_1,\cdots,S_k)$ and

$$e(S_1,\cdots,S_k) = \sum Ce^{-S_i/C}.$$

To prove Theorem 10.4, we modify the maps $\Psi_{\mathfrak{c}}$ inductively on $\mathfrak{c}$. First we take $\mathfrak{c}$ which is maximal with respect to the inclusion and put $\Psi_{\mathfrak{c}} = \Phi_{\mathfrak{c}}$. We do not change $\Psi_{\mathfrak{c}'}$ while modifying $\Phi_{\mathfrak{c}}$ with $\mathfrak{c}' \supset \mathfrak{c}$. For simplicity of the notation, we discuss one step of modifications. We consider the following case. Let $\mu(a) < \mu(c) < \mu(c') < \mu(b)$, with $G_a = \{\pm 1\}$, $G_c = G_{c'} = G_b = U(1)$, and consider the component of $K_{c,c'}$ consisting of irreducible connections. Suppose, by induction hypothesis, we have

$$\begin{aligned}\Psi_{(c,c')} &: \widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,c'} \times_{G_{c'}} \widetilde{K}_{c',b} \times (T,\infty)^2 \\ &\to \mathcal{M}'(a,b) \\ \Psi_{(c,c'),(c)} &: \widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,c'} \times_{G_{c'}} \widetilde{K}_{c',b} \times (T,\infty)^2 \\ &\to \widetilde{\mathcal{M}}'(a,c) \times_{G_c} \mathcal{M}'(c,b) \times (T,\infty) \\ \widehat{\Psi}_{(c,c'),(c')} &: \widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,c'} \times_{G_{c'}} \widetilde{K}_{c',b} \times (T,\infty)^2 \\ &\to \widetilde{M}'(a,c') \times_{G_{c'}} \mathcal{M}'(c',b) \times (T,\infty),\end{aligned}$$

and a local action on the image of $\Psi_{(c,c')}$. We shall define $\Psi_{(c)}$ and $\Psi_{(c')}$ such that

$$\Psi_{(c)}\Psi_{(c,c'),(c)} = \Psi_{(c,c')}$$

on

$$W_1 = \Psi^{-1}_{(c,c'),(c)}(\widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,b} \times [T,\infty)),$$

and

$$\Psi_{(c')}\Psi_{(c,c'),(c')} = \Psi_{(c,c')}$$

on

$$W_2 = \Psi^{-1}_{(c,c'),(c')}(\widetilde{K}_{a,c'} \times_{G_{c'}} \widetilde{K}_{c',b} \times [T,\infty)).$$

(See Figure 6.) By induction hypothesis, $\Psi_{(c,c'),(c)}$ and $\Psi_{(c,c'),(c')}$ preserves $G_c \times G_b$ and $G_{c'} \times G_b$ actions respectively. (In this case, those actions are defined globally since the groups are abelian.) The maps $\Psi_{(c)}$ and $\Psi_{(c')}$ we shall construct must be G_b invariant. Once we obtain such maps $\Psi_{(c)}$ and $\Psi_{(c')}$ we can define a local action on their images by pushing out one by those maps. These local actions can be patched together with one on the image of $\Psi_{(c,c')}$ by the $G_c \times G_b$ and $G_{c'} \times G_b$ invariance of the maps $\Psi_{(c,c'),(c)}$ and $\Psi_{(c,c'),(c')}$.

We begin the construction of Ψ_c. We choose an open coverings U_j^1, U_j^2, U_j^3, U_j^4, of $\widetilde{K}_{a,c}/G_c$, $K_{c,c'}$, $G_{c'}\backslash\widetilde{K}_{c',b}/G_b$, $\widetilde{K}_{a,c'}/G_{c'}$, respectively. Let V_k be an open covering of $U(1)$. Take maps J_k^1 and J_k^2 as in §8. Choose sections $s_j^1 : U_j^1 \to \mathcal{A}_\ell(a,c)$ and s_j^2,s_j^3.s_j^4. As in §8, define a map

$$\widetilde{\Phi}'_{j_1,j_2,j_3,k_1,k_2} : U^1_{j_1} \times U^2_{j_2} \times U^3_{j_3} \times V_{k_1} \times V_{k_2} \times (T,\infty) \times \mathbf{R} \to \mathcal{A}_{\ell,\delta}(a,b)$$

by

$$\widetilde{\Phi}'_{j_1,j_2,j_3,k_1,k_2}([A_1],[A_2],[A_3],g_1,g_2,S_1,S_2,S)$$

$$\begin{cases} = (J_{k_1}(g_1,\cdot)^*A_1)(x,t-S) \quad \text{for} \quad t < S + S_1/3 \\ = \chi\left(\dfrac{t-S-S_1/3}{S_1/3}\right) g_1^* A_1(x,t-S) \\ \qquad + \left(1-\chi\left(\dfrac{t-S-S_1/3}{S_1/3}\right)\right) A_2(t-S-S_1) \\ \qquad\qquad \text{for} \quad t \in [S+S_1/3, S+2S_1/3] \\ = A_2(t-S-S_1) \quad \text{for} \quad t \in [S+2S_1/3, S+S_1+S_2/3] \\ = \chi\left(\dfrac{t-S-S_1-S_2/3}{S_2/3}\right) A_2(t-S_1-S) \\ \qquad + \left(1-\chi\left(\dfrac{t-S-S_1-S_2/3}{S_2/3}\right)\right) g_2^* A_3(x,t-S-S_1-S_2) \\ \qquad\qquad \text{for} \quad t \in [S+S_1+S_2/3, S+S_1+2S_2/3] \\ = (J_{k_2}^2(g_2,\cdot)^*A_3)(s,t-S-S_1-S_2) \quad \text{for} \quad t > S+S_1+2S_2/3. \end{cases}$$

By perturbing this map as in §8, we obtain a map

$$\widetilde{\Phi}_{j_1,j_2,j_3,k_1,k_2} : U^1_{j_1} \times U^2_{j_2} \times U^3_{j_3} \times V_{k_1} \times V_{k_2} \times (T,\infty) \times \mathbf{R} \to \widehat{\mathcal{M}}_{\ell,\delta}(a,b)$$

which is a lift of the map $\Phi_{(c,c')}$ of Theorem 7.6. By construction in §8, we have

$$|\widetilde{\Phi}'_{j_1,j_2,j_3,k_1,k_2} - \widetilde{\Phi}_{j_1,j_2,j_3,k_1,k_2}| < e(S_1,S_2).$$

Similarly we have

$$\begin{aligned} \widetilde{\Phi}'^{(1)}_{j_1,j_2,k_1} &: U^1_{j_1} \times U^2_{j_2} \times V_{k_1} \times (T,\infty) \times \mathbf{R} \to \mathcal{A}_\ell(a,c') \\ \widetilde{\Phi}^{(1)}_{j_1,j_2,k_1} &: U^1_{j_1} \times U^2_{j_2} \times V_{k_1} \times (T,\infty) \times \mathbf{R} \to \widehat{\mathcal{M}}_\ell(a,c'), \end{aligned}$$

such that $\widetilde{\Phi}^{(1)}_{j_1,j_2,k_1}$ is a lift of

$$\Phi_{(c)} : G_a \backslash \widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,c'}/G_{c'} \times (T,\infty) \times \mathbf{R} \to \overline{\mathcal{M}}(a,c').$$

Here $\widetilde{\Phi}^{(1)\prime}_{j_1,j_2,k_1}$ is obtained by a similar patching procedure as $\widetilde{\Phi}'_{j_1,j_2,j_3,k_1,k_2}$, and that

$$|\widetilde{\Phi}'^{(1)}_{j_1,j_2,k_1} - \widetilde{\Phi}^{(1)}_{j_1,j_2,k_1}| < e(S_1).$$

We may assume that for each j_1,j_2 with

$$G_a \backslash \widetilde{U}^1_{j_1} \times_{G_c} \widetilde{U}^2_{j_2} \times_{G_{c'}} \widetilde{K}_{c',b} \times (T,\infty)^2 \subset W_1,$$

there exists $j = j(j_1, j_2, k_1)$ such that

$$\operatorname{Im} \widetilde{\Phi}_{j_1,j_2,k_1} \subset U_j^4.$$

We have maps

$$\begin{aligned}
\widetilde{\Phi}'^{(2)}_{j,j_3,k_2} &: U_j^4 \times U_{j_3}^3 \times V_{k_2} \times (T,\infty) \times \mathbf{R} \to \mathcal{A}_\ell(a,b) \\
\widetilde{\Phi}^{(2)}_{j,j_3,k_1} &: U_j^4 \times U_{j_3}^3 \times V_{k_2} \times (T,\infty) \times \mathbf{R} \to \widehat{\mathcal{M}}_\ell(a,b)
\end{aligned}$$

such that $\widetilde{\Phi}^{(2)}_{j,j_3,k_2}$ is a lift of

$$\Phi_{(c')} : \widetilde{K}_{a,c'} \times_{G_{c'}} \widetilde{K}_{c',b}/G_b \times (T,\infty) \times \mathbf{R} \to \overline{\mathcal{M}}(a,b),$$

Here $\widetilde{\Phi}^{(2)\prime}_{j,j_3,k_2}$ is obtained by a similar patching procedure as $\widetilde{\Phi}'_{j_1,j_2,j_3,k_1,k_2}$, and that

$$|\widetilde{\Phi}'^{(2)}_{j,j_3,k_2} - \widetilde{\Phi}^{(2)}_{j,j_3,k_2}| < e(S_2).$$

By construction, we can choose lifts s_j^1 e.t.c. so that

$$\begin{aligned}
&\widetilde{\Phi}'^{(2)}_{j(j_1,j_2,k_1),j_3,k_2}\left(\widetilde{\Phi}'^{(1)}_{j_1,j_2,k_1}([A_1],[A_2],g_1,S_1,S),[A_3],g_2,S_2,S'\right) \\
&\quad = \widetilde{\Phi}'_{j_1,j_2,j_3,k_1,k_2}([A_1],[A_2],[A_3],g_1,g_2,S_1,S_2,S'').
\end{aligned}$$

(Here S'' is determined by S,S',S_1 and S_2.) It follows that

$$|\Phi_{(c')}\Phi_{(c,c'),(c')} - \Phi_{(c,c')}| < e(S_1,S_2).$$

Using induction hypothesis (10.5), we obtain

$$|\Phi_{(c')}\Psi_{(c,c'),(c')} - \Psi_{(c,c')}| < e(S_1,S_2).$$

Let $\widetilde{\Psi}_{j_1,j_2,j_3,k_1,k_2}$ and $\widetilde{\Psi}^{(1)}_{j_1,j_2,k_1}$ be the lifts of $\Psi_{c,c'}$ and $\Psi_{(c,c')(c')}$, respectively. Then we have

$$\begin{aligned}
&|\widetilde{\Phi}^{(2)}_{j(j_1,j_2,k_1),j_3,k_2}\left(\widetilde{\Psi}^{(1)}_{j_1,j_2,k_1}([A_1],[A_2],g_1,S_1,S),[A_3],g_2,S_2,S'\right) \\
&\quad - \widetilde{\Psi}_{j_1,j_2,j_3,k_1,k_2}([A_1],[A_2],[A_3],g_1,g_2,S_1,S_2,S'')| < e(S_1,S_2).
\end{aligned}$$

Therefore we can define

$$\widetilde{\Xi}'_{j_1,j_2,j_3,k_1,k_2} : U_{j_1} \times U_{j_2} \times U_{j_3} \times V_{k_1} \times V_{k_2} \times (T,\infty)^2 \times [0,1] \to \mathcal{A}_\ell(a,b)/\mathbf{R}$$

by

$$\widetilde{\Xi}'_{j_1,j_2,j_3,k_1,k_2}([A_1],[A_2],[A_3],g_1,g_2,S_1,S_2,s) =$$
$$(1-s)\cdot\widetilde{\Phi}^{(2)}_{j(j_1,j_2,k_1),j_3,k_2}\left(\widetilde{\Psi}^{(1)}_{j_1,j_2,k_1}([A_1],[A_2],g_1,S_1,S),[A_3],g_2,S_2,S'\right)$$
$$+\,s\cdot\widetilde{\Psi}_{j_1,j_2,j_3,k_1,k_2}([A_1],[A_2],[A_3],g_1,g_2,S_1,S_2,S'').$$

Since gauge transformation is an affine map (namely $g^*(sA+(1-s)B) = sg^*A+(1-s)g^*B$ holds for each connections A, B and gauge transformation g), it follows from an argument similar to the proof of Lemma 8.3 that we can perturb $\widetilde{\Xi}'_{j_1,j_2,j_3,k_1,k_2}$ so that it defines a map Ξ' : $W_1\times[0,1]\to\mathcal{B}_\ell(a,b)$, which is G_b invariant. Using Taubes' method as in §8, we can perturb this map and obtain $\Xi : W_1\times[0,1]\to\mathcal{M}'_\ell(a,b)$. This map Ξ is an isotopy between $\Psi_{(c,c')}$ and $\Phi_{(c')}\Psi_{(c,c'),(c')}$. Take a small open neighborhood W_1' of W_1 in

$$\mathcal{M}(a,c)\times_{G_c}\mathcal{M}(c,c')\times_{G_{c'}}\mathcal{M}(c',b)\times(T,\infty)^2.$$

Ξ can be extend to W_1'. Let $\varphi : W_1'\to[0,1]$ be a G_b-invariant function such that

$$\begin{cases}\varphi(x)=0 & \text{if}\quad x\in\partial W_1', \qquad\qquad \text{and if}\quad \Psi_{(c,c'),(c')}(x)\in X_{(c)}\\ \varphi(x)=1 & \text{if}\quad x\in W_1.\end{cases}$$

(See Figure 7.) Define $\Psi_{(c')}$ on $\Psi_{(c,c'),(c')}(W_1')$ by

$$\Psi_{(c')}(\Psi_{(c,c'),(c')}(x)) = \Xi(x,\varphi(x)).$$

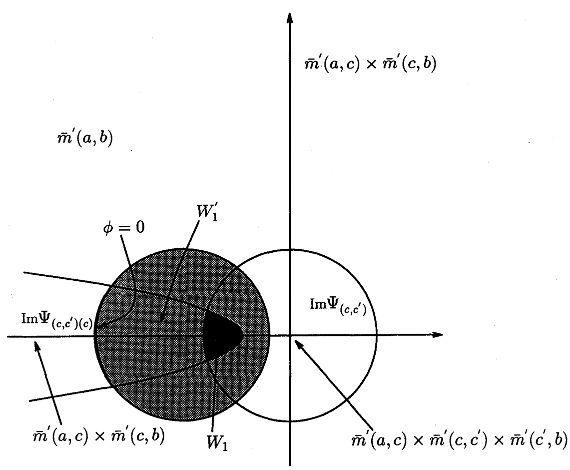

Figure 7.

Since

$$\Xi(x,0) = \Phi_{(c')}\Psi_{(c,c'),(c')}(x),$$

we can extend $\Psi_{(c')}$, by putting $\Psi_{(c')} = \Phi_{(c')}$ outside $\Psi_{(c,c'),(c')}(W_1')$. Since

$$\Xi(x,1) = \Psi_{(c,c')}(x),$$

we have $\Psi_{(c')}\Psi_{(c,c'),(c')} = \Psi_{(c,c')}$, on W_1. The inequality (10.5) holds by construction. Using Lemma 8.14, we can prove that $\Psi_{(c')}$ is a diffeomorphism to its image. Thus the patching argument for the proof of Theorem 10.2 is completed in our case. The proof of general case is the same, but the notation will be more complicated.

Remark 10.6. If we can establish rigorously what we suggested in Remarks 8.15 and 8.20 we might be able to prove Theorem 10.2 using the center of mass technique in Riemannian geometry. (See [GK].) But the direct argument we gave above might be simpler.

§11. Extension of the line bundle to the boundary

In this section, we shall prove Theorem 7.3. First we consider the case when none of c_i are reducible. We put

$$\mathcal{C}_1\overline{\mathcal{M}}'(a,b) = \bigcup_{c_0,\cdots,c_k,G_{c_i}=\{\pm1\}} \overline{\mathcal{M}}'(a,c_0)\times\prod_{i=0}^{k-1}\overline{\mathcal{M}}'(c_i,c_{i+1})\times\overline{\mathcal{M}}'(c_k,b).$$

Lemma 11.1. *Let* $\mathfrak{c} = (c_0,\cdots,c_k)$, $\mu(a) > \mu(c_0) > \cdots > \mu(c_k) > \mu(b)$, $G_{c_i} = \{\pm1\}$, *and*

$$\Psi_{\mathfrak{c}} : K_{a,c_0}\times\prod K_{c_i,c_{i+1}}\times K_{c_k,b}\times(T,\infty)^k \to \overline{\mathcal{M}}'(a,b)$$

be the map given in §10. *Then there exists an isomorphism of line bundles*

$$\varphi^i_{\mathfrak{c}} : \Psi^*_{\mathfrak{c}}\mathcal{L}^{(2)}_i(a,b) \to \mathcal{L}^{(2)}_i(a,c_0)\otimes\cdots\otimes\mathcal{L}^{(2)}_i(c_k,b).$$

This lemma follows from Theorem 4.9 and the construction of $\Psi_{\mathfrak{c}}$. Hereafter we write

$$\mathcal{L}^{(2)}_i(\mathfrak{c}) = \mathcal{L}^{(2)}_i(a,c_0)\otimes\cdots\otimes\mathcal{L}^{(2)}_i(c_k,b).$$

Similarly, for $\mathfrak{c}'\subset\mathfrak{c}$, we have an isomorphism

$$\varphi^i_{\mathfrak{c},\mathfrak{c}'} : \Psi^*_{\mathfrak{c},\mathfrak{c}'}\mathcal{L}^{(2)}_i(\mathfrak{c}') \to \mathcal{L}^{(2)}_i(\mathfrak{c}).$$

Lemma 11.2. *On*

$$K_{a,c_0} \times \prod K_{c_i,c_{i+1}} \times K_{c_k,b} \times \{(S_1, \cdots, S_k)\}$$

we have

$$\|\varphi^i_{\mathfrak{c},\mathfrak{c}'} \circ \varphi^i_{\mathfrak{c}'} - \varphi^i_{\mathfrak{c}}\| < e(S_1, \cdots, S_k).$$

This lemma follows from the construction of $\varphi^i_{\mathfrak{c}}$. By Lemma 11.2, we can perturb $\varphi^i_{\mathfrak{c}}$, $\varphi^i_{\mathfrak{c},\mathfrak{c}'}$ such that

$$\varphi^i_{\mathfrak{c},\mathfrak{c}'} \circ \varphi^i_{\mathfrak{c}'} = \varphi^i_{\mathfrak{c}}.$$

Using these isomorphisms, we can patch the bundles $\mathcal{L}^{(2)}_i(\mathfrak{c})$ and obtain a line bundle over $\mathcal{C}_1\overline{\mathcal{M}}'(a,b)$.

Next we consider the case when some c_i are reducible. The following three results are used for this purpose.

Theorem 11.3. *The local action on $\overline{\mathcal{M}}'(a,b)$ constructed in* §10, *can be lifted to $\mathcal{L}^{(2)}_i(a,b)$.*

Hence, for each $\mathfrak{c}$, the line bundle $\Psi^*_{\mathfrak{c}}\mathcal{L}^{(2)}_i(a,b)$ on $\widetilde{K}_{a,c_0} \times_{G_{c_0}} \cdots \times_{G_{c_k}} \widetilde{K}_{c_k,b} \times (T,\infty)^k$ has a local $G_a \times G_{c_0} \times \cdots \times G_{c_k} \times G_b$ action. Therefore we obtain a bundle $\overline{\Psi^*_{\mathfrak{c}}\mathcal{L}^{(2)}_i(a,b)}$ on

$$K^*_{a,c_0} \times \prod K^*_{c_i,c_{i+1}} \times K^*_{c_k,b} \times (T,\infty)^k.$$

Here $K^*_{c_i,c_{i+1}}$ denotes the set of reducible connections. As before we put

$$\mathcal{L}^{(2)}_i(\mathfrak{c}) = \mathcal{L}^{(2)}_i(a,c_0) \otimes \cdots \otimes \mathcal{L}^{(2)}_i(c_k,b),$$

which is a line bundle on

$$K^*_{a,c_0} \times \prod K^*_{c_i,c_{i+1}} \times K^*_{c_k,b} \times (T,\infty)^k.$$

Lemma 11.4. *There exist isomorphisms*

$$\varphi^i_{\mathfrak{c}} : \overline{\Psi^*_{\mathfrak{c}}\mathcal{L}^{(2)}_i(a,b)} \to \mathcal{L}^{(2)}_i(\mathfrak{c})$$
$$\varphi^i_{\mathfrak{c},\mathfrak{c}'} : \overline{\Psi^*_{\mathfrak{c},\mathfrak{c}'}\mathcal{L}^{(2)}_i(\mathfrak{c}')} \to \mathcal{L}^{(2)}_i(\mathfrak{c}).$$

Lemma 11.5. *On*

$$K^*_{a,c_0} \times \prod K^*_{c_i,c_{i+1}} \times K^*_{c_k,b} \times \{(S_1, \cdots, S_k)\}$$

we have

$$\|\varphi^i_{\mathfrak{c},\mathfrak{c}'} \circ \varphi^i_{\mathfrak{c}'} - \varphi^i_{\mathfrak{c}}\| < e(S_1, \cdots, S_k).$$

Using these results, we can prove Theorem 7.3 in a way similar to the case when none of c_i are reducible. The proof of Lemmas 11.4 and 11.5 are similar to one of Lemma 11.1 and 11.2 respectively. In the rest of this section, we prove Theorem 11.3.

First we lift the action on the image $\Psi_{\mathfrak{c}}(\widetilde{X}^\circ(\mathfrak{c})) \subset \mathcal{M}'(a,b)$. We are studying the determinant bundle of the operator $\eth^i_A + \epsilon$ defined on $\Sigma_i \simeq S^1 \times \mathbf{R} \subset M \times \mathbf{R}$. On their ends, these operators are asymptotic to $\frac{\partial}{\partial t} + \eth^i_a + \epsilon$, for some $a \in Fl$. Here the operator $\eth^i_a$ is defined on S^1. We choose λ_0 such that the first eigenvalue of $(\eth^i_a + \epsilon)^*(\eth^i_a + \epsilon)$ is larger than λ_0 for each a.

For simplicity, we shall consider the case where $\mathfrak{c} = (c)$, $G_c \neq \{\pm 1\}$. In this case, $\Psi_{\mathfrak{c}}$ is a perturbation of the map Φ defined below. (See §8.)

Choose an open covering

$$\begin{aligned}
&U^1_1 \cup \cdots \cup U^1_N \supseteq K_{a,c},\\
&U^2_1 \cup \cdots \cup U^2_N \supseteq K_{c,b},\\
&V_1 \cup \cdots \cup V_N = G_c,
\end{aligned}$$

and sections $s^1_j : U^1_j \to \mathcal{A}_{\ell,.\delta}(a,c)$, $s^2_j : U^2_j \to \mathcal{A}_{\ell,\delta}(c,b)$. Let $J_k : V_k \times \mathbf{R} \to G_c$ be a map such that

$$J_k(g,t) = \begin{cases} 1 & \text{if } t < -1 \\ g & \text{if } t > 0 \end{cases}.$$

Then the map

$$\widetilde{\Phi}'_{j_1,j_2,k} : U^1_{j_1} \times V_k \times U^2_{j_2} \times [T,\infty) \times \mathbf{R} \to \mathcal{A}_{\ell,\delta}(a,b)$$

is defined by

$$\begin{aligned}
&\widetilde{\Phi}'_{j_1,j_2,k}([A_1], g, [A_2], S', S)\\
&= \begin{cases} (J_k(g,\cdot)^* A_1)(x, t-S) & \text{if } t < S + S'/3. \\ \chi\left(\dfrac{t-S-S'/3}{S'/3}\right) g^* A_1(x,t-S) & \\ \qquad + \left(1-\chi\left(\dfrac{t-S-S'/3}{S'}\right)\right) A_2(t-S-S') & \\ \qquad\qquad \text{if } S+S'/3 < t < S+2S'/3 & \\ A_2(t-S-S') & \text{if } t > S + 2S'/3. \end{cases}
\end{aligned}$$

Here χ is the cut function in §8. The maps $\widetilde{\Phi}'_{j_1,j_2,k}$ induce a map Φ : $\widetilde{X}^\circ((c)) \to \mathcal{B}_{\ell,\delta}(a,c)$. They satisfy

$$\|(\Psi_{(c)} - \Phi)([A_1], g, [A_2], S', S)\|_{L^2_\ell} < Ce^{-S'/C}.$$

Therefore, there exists an isomorphism $\Psi^*_{(c)}\mathcal{L}^{(2)}_i(a,b) \to \Phi^*\overline{\mathcal{L}}^{(2)}_i(a,b)$. We shall lift the local action of G_c on $\widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,b}$, to a local action on $\Phi^*\mathcal{L}^{(2)}_i(a,c)$.

Replacing $U^1_{j_1}$ and $U^2_{j_2}$ by a smaller one if necessary, we can find positive numbers $\lambda_{j_1,j_2} < \lambda_0$, such that the following holds.

(11.6.1) If $[a_t] \in U^1_{j_1}$ then λ_{j_1,j_2} is not an eigenvalue of $(\eth_{a_t}+\epsilon)^*(\eth_{a_t}+\epsilon)$ on Σ_i.
(11.6.2) If $[a_t] \in U^2_{j_2}$ then λ_{j_1,j_2} is not an eigenvalue of $(\eth_{a_t}+\epsilon)^*(\eth_{1,t}+\epsilon)$ on Σ_i.

Then, by Remark 4.6, λ_{j_1,j_2} is not an eigenvalue of $(\eth_A+\epsilon)^*(\eth_A+\epsilon)$ on Σ_i, if

$$[A] \in \Phi(U^1_{j_1} \times G_c \times U^2_{j_2} \times (T,\infty) \times \mathbf{R})$$

for sufficiently large T. Let $[A_1] \in U^1_{j_1}$, $[A_2] \in U^2_{j_2}$, $g \in V_k \subset G_c$, and $A = \widetilde{\Phi}'_{j_1,k,j_2}([A_1], g, [A_2], S', S)$, we put

$$L(A_1, g, A_2, S', S) = \bigoplus_{\lambda<\lambda_{j_1,j_2}} \{u \mid (\eth_A+\epsilon)^*(\eth_A+\epsilon)u = \lambda u\},$$

$$L'(A_1, g, A_2, S', S) = \bigoplus_{\lambda<\lambda_{j_1,j_2}} \{u \mid (\eth_A+\epsilon)(\eth_A+\epsilon)^*u = \lambda u\},$$

$$L = \bigcup_{A_1,g,A_2,S',S} L(A_1, g, A_2, S', S),$$

$$L' = \bigcup_{A_1,g,A_2,S',S} L'(A_1, g, A_2, S', S).$$

By (11.6.1) and (11.6.2), the dimensions of L and L' are constant. By definition,

$$\Phi^*(\mathcal{L}^{(2)}_i(a,b))|_{([A_1],g,[A_2],S',S)}$$
$$\simeq \left(\bigwedge^{\text{top}}(L(A_1,g,A_2,S',S))^* \otimes \bigwedge^{\text{top}} L'(A_1,g,A_2,S',S)\right)^{\otimes 2}.$$

Lemma 11.7. *Let $t \in [S+S'/3, S+2S'/3]$, $u \in L(A_1, g, A_2, S', S)$. Then*

$$|u(x,t)|_{C^\ell} < Ce^{-\sqrt{\lambda_0 - \lambda_{j_1,j_2}}\beta(t)}\|u\|_{L^2}$$

Here $\beta(t) = d(t, \partial[S + S'/3, S + 2S'/3])$

The proof of the lemma is similar to one of Lemma 4.5.

For $u \in L(A_1, g, A_2, S', S)$, $g, h \in G_c$ with $g, hg \in V_k$, we put

$$I_1(h)(u)(t,x) = \begin{cases} J_k(hg, t-S)J_k(g, t-S)^{-1}u(x,t) & \text{if } t < S + S'/3. \\ \chi\left(\dfrac{t-S-S'/3}{S'/3}\right) hu(x,t) + \left(1 - \chi\left(\dfrac{t-S-S'/3}{S'/3}\right)\right) u(x,t) & \\ \qquad \text{if } S + S'/3 < t < S + 2S'/3, & \\ u(x,t) \qquad \text{if } t > S + 2S'/3. & \end{cases}$$

Let $I_2(h)(u)$ is the orthonormal projection of $I_1(h)(u)$ to $L(A_1, hg.A_2, S', S)$. Lemma 11.7 implies:

Lemma 11.8.

$$\|I_2(h)(u) - I_1(h)(u)\|_{L^2} < Ce^{-S'/C}\|u\|_{L^2}.$$

Lemma 11.9. *If $g \in V_k$, $hg \in V_k$ and $h'hg \in V_k$, then*

$$\|I_2(h'h)(u) - I_2(h')I_2(h)(u)\|_{L^2} < Ce^{-S'/C}\|u\|_{L^2}.$$

Next we extend I_2 to I_5 which is defined also for h such that $g \in V_k$ and $hg \notin V_k$. Note that $G_c = U(1)$ or $= SU(2)$. Hence, in fact, we need only two charts V_1 and V_2 to cover G_c. (This fact is not essential for the proof but we use it to simplify the notation.) Choose $g_0 \in V_1 \cap V_2$. For $g \in V_1$, $hg \in V_2$, we take h_1 and h_2 such that $h_1 g = g_0$ and $h_2 h_1 = h$. Then, for $h \in L(A_1, g, A_2, S', S)$, the element $I_2(h_1)(u) \in L(A_1, g_0, A_2, S', S)$ is well defined. We put

$$I_3(h)(u) = I_2(h_2)I_2(h_1)(u).$$

Since $h_2(h_1 g), h_1 g \in V_2$, it follows that $I_2(h_2)$ in the above formula is well defined. Choose $\chi : G_c \to [0,1]$ such that

$$\chi(g) = \begin{cases} 1 & \text{if } g \in V_1 - (V_1 \cap V_2). \\ 0 & \text{if } g \in V_2 - (V_1 \cap V_2). \end{cases}$$

Put

$$I_4^1(u) = \begin{cases} I_2(h)(u) & \text{if } hg \in V_1 - (V_1 \cap V_2), \\ \chi(hg)I_2(h)(u) + (1 - \chi(hg))I_3(h)(u) & \\ \qquad \text{if } hg \in V_1 \cap V_2, & \\ I_3(h)(u) & \text{if } hg \in V_2 - (V_1 \cap V_2). \end{cases}$$

In the case when $g \in V_2$, we define $I_4^2(h)$ in a similar way. Finally we put, for $u \in L(A_1, g, A_2, S', S)$

$$I_5(h)(u) = \begin{cases} I_4^1(h)(u) & \text{if } g \in V_1 - (V_1 \cap V_2), \\ \chi(g)I_4^1(h)(u) + (1 - \chi(g))I_4^2(h)(u) & \\ \qquad \text{if } g \in V_1 \cap V_2, & \\ I_4^2(h)(u) & \text{if } g \in V_2 - (V_1 \cap V_2). \end{cases}$$

Then I_5 is defined for every h and g and depends smoothly on them. By perturbing I_5 a bit we obtain $I_6(h)$ which is a linear isometry

$$L(A_1, g, A_2, S', S) \to L(A_1, hg, A_2, S', S).$$

By construction, we have

$$\|I_6(h'h)(u) - I_6(h')I_6(h)(u)\|_{L^2} < Ce^{-S'/C}\|u\|_{L^2}. \tag{11.10}$$

Next we use the center of mass technique, to perturb I_6 and obtain I satisfying $I(h)I(h') = I(hh')$. Namely we use the following:

Lemma 11.11. *For each compact Lie group G and $n, \epsilon > 0$, there exists $\delta_n(G, \epsilon) > 0$, such that the following holds.*

Let $\pi : L \to X$ be a hermitian vector bundle of rank n, G act on X, and $\varphi : G \times L \to L$ be a map. Suppose

(11.12.1) $\quad \pi(\varphi(g, v)) = g(\pi(v))$,

(11.12.2) $\quad \varphi$ *is a linear isometry on each fibre,*

(11.12.3) $\quad |\varphi(g_1, g_2, v) - \varphi(g_1(\varphi(g_2, v))| < \delta_n(G, \epsilon)$.

Then, there exists a lift of the action of G to L, such that

$$|\varphi(g, v) - g \cdot v| < \epsilon.$$

In the case when X is a point, Lemma 11.11 means that an almost homomorphism $G \to U(n)$ is approximated by a homomorphism. This case is proved in [GKR]. The proof of Lemma 11.11 is identical to that case and hence is omitted. (See also [BK] p138.) Note that $\delta_n(G,\epsilon)$ in the lemma is independent of X.

Now, using Lemma 11.11, we can perturb I_6 to obtain a lift I of the local action on $U^1_{j_1} \times G_c \times U^2_{j_2} \times (T,\infty) \times \mathbf{R}$ to the vector bundle $L(A_1, g, A_2, S', S)$ on it. In a similar way, we can lift the action to $L'(A_1, g, A_2, S', S)$. Hence we obtain a lift of the action to the restriction of $\Phi^*\overline{\mathcal{L}}^{(2)}_{a,b}$ to $\widetilde{U}^1_{j_1} \times_{G_c} \widetilde{U}^2_{j_2} \times (T,\infty) \times \mathbf{R} = U^1_{j_1} \times G_c \times U^2_{j_2} \times (T,\infty) \times \mathbf{R}$. (Here $\widetilde{U}^1_{j_1}$ and $\widetilde{U}^2_{j_2}$ are the inverse images of $U^1_{j_1}$ and $U^2_{j_2}$ in $\widetilde{K}_{a,c}$ and $\widetilde{K}_{c,b}$, respectively.) We denote the lift by I_{j_1,j_2}. By construction, we have, on $(\widetilde{U}^1_{j_1} \times_{G_c} \widetilde{U}^2_{j_2}) \cap (\widetilde{U}^1_{j_1'} \times_{G_c} \widetilde{U}^2_{j_2'}) \times (T,\infty) \times \mathbf{R}$,

$$d(I_{j_1,j_2}(h), I_{j_1',j_2'}(h)) < Ce^{-T/C}.$$

Hence using a partition of unity, we can patch them as an almost action. Therefore, using Lemma 11.11, we obtain a lift of the local action to $\Phi^*\mathcal{L}^{(2)}_i(a,b)$.

In order to lift the local action on $\mathcal{M}(a,b)$, we have to patch those lifts we constructed above. By construction, they are compatible modulo a difference estimated by $e(S_1,\cdots,S_k)$ on $\cdots \times \{(S_1,\cdots,S_k)\} \times \mathbf{R}$. Hence we can apply a similar patching procedure as above. The proof of Theorem 11.3 is now complete.

§12. Boundary operators

In this section, we define the boundary operators

$$\partial : C^0_k \to C^0_{k-1}$$
$$\partial_\gamma : C^0_k \to C^0_{k-3}$$
$$\partial_{\gamma_1,\gamma_2} : C^0_k \to C^0_{k-5}.$$

The definition of ∂ is the same as Floer's. Let $a, b \in Fl$, with $\mu(a) = \mu(b) + 1$. Then, $\overline{\mathcal{M}}'(a,b)$ consists of finitely many points each of which is given an orientation $+$ or $-$. We let $\langle \partial a, b\rangle$ be the number of the points with $+$ orientation minus the number of points with $-$ orientation. Put

$$\partial[a] = \sum \langle \partial a, b\rangle [b].$$

Next we define ∂_γ. For a closed loop γ on M we obtain a line bundles $\mathcal{L}^{(2)}_\gamma(c,c')$, over $\overline{\mathcal{M}}'(c,c')$. We choose sections $s_\gamma(c,c')$ to $\mathcal{L}^{(2)}_\gamma(c,c')$, such that the following holds.

(12.1.1) For each $a, b \in Fl$, the collection of the sections

$$s_\gamma(a, c_0) \otimes \cdots \otimes s_\gamma(c_k, b)$$

to

$$\mathcal{L}^{(2)}_\gamma(a, c_0) \otimes \cdots \otimes \mathcal{L}^{(2)}_\gamma(c_k, b)$$

can be patched together to give a smooth section on $\mathcal{C}\overline{\mathcal{M}}'(a, b)$. (We use the symbol $s_\gamma(a, b)$ also for this extension.)
(12.1.2) The zeros of $s_\gamma(c, c')$ are transversal and transversal to each other.

Since we restrict ourselves to the case when $s < 3$ if $H_1(M; \mathbf{Z})$ is torsion free, and when $s = 0$ otherwise, then we need only to study the case when $\mu(a) < \mu(b) + 8$, $H_1(M; \mathbf{Z})$ is torsion free and a and b are irreducible. In this case, if $\mu(a) \geq \mu(c) \geq \mu(c') \geq \mu(b)$, and if $\mathcal{M}(a, c) \neq \emptyset$, $\mathcal{M}(c', b) \neq \emptyset$, then $\mathcal{M}(c, c')$ does not contain a reducible connection. Also in our case, Lemma 5.8 implies that bubbling off of instanton does not happen. Hence (7.1.6) implies that the set $\mathcal{C}\overline{\mathcal{M}}'(a, b)$ is compact. The later fact is not really necessary for the argument. (We can discuss as in Donaldson [D4], in case when a and b are irreducible.) However the former point is essential. We discuss it at the end of this section.

Now, let $\mu(a) = \mu(b) + 3$. Set

$$\Sigma_\gamma(a, b) = \left\{ x \in \mathcal{C}\overline{\mathcal{M}}'(a, b) | s_\gamma(a, b)(x) = 0 \right\}.$$

Dimension counting, the compactness of $\mathcal{C}\overline{\mathcal{M}}'(a, b)$ and the transversality (12.1.2) imply

$$\Sigma_\gamma(a, b) \cap \partial \mathcal{C}\overline{\mathcal{M}}'(a, b) = \emptyset$$
$$\sharp \Sigma_\gamma(a, b) < \infty.$$

The orientation of $\overline{\mathcal{M}}'(a, b)$ induces an orientation of each point of Σ_i. We define $\langle \partial_\gamma a, b \rangle$ by

$$\langle \partial_\gamma a, b \rangle = \sharp \Sigma_\gamma.$$

Here and hereafter $\sharp$ stands for the number of points with $+$ orientation minus the number of points with $-$ as orientation. We set

$$\partial_\gamma [a] = \sum_b \langle \partial_\gamma a, b \rangle [b].$$

For $\mu(b) = \mu(a) + 5$, and loops γ_1 and γ_2, we put

$$\Sigma_{\gamma_1,\gamma_2}(a,b) = \{x \in \mathcal{C}\overline{\mathcal{M}}'(a,b) | s_{\gamma_1}(a,b)(x) = s_{\gamma_2}(a,b)(x) = 0.\},$$

and define

$$\begin{aligned} \langle \partial_{\gamma_1,\gamma_2} a, b\rangle &= \sharp \Sigma_{\gamma_1,\gamma_2}(a,b) \\ \partial_{\gamma_1,\gamma_2}[a] &= \sum_b \langle \partial_{\gamma_1,\gamma_2} a, b\rangle [b]. \end{aligned}$$

Now we prove Theorem 1.10. For simplicity, we discuss the case $\alpha = \{\gamma\}$, and prove $\partial_\gamma \partial + \partial\partial_\gamma = 0$. Let $a, b \in Fl$ with $\mu(a) = \mu(b) + 4$. The line bundle $\mathcal{L}^{(2)}_\gamma(a,b) \to \overline{\mathcal{M}}'(a,b)$ can be extended to $\mathcal{C}\overline{\mathcal{M}}'(a,b)$ by Theorem 7.3. Since $\dim \overline{\mathcal{M}}'(a,b) = 3$, the set

$$\Sigma_\gamma(a,b) = \{x \in \mathcal{C}\overline{\mathcal{M}}'(a,b) | s_\gamma(a,b)(x) = 0\}$$

is one dimensional oriented manifold. And

$$\partial \Sigma_\gamma(a,b) = \Sigma_\gamma(a,b) \cap \partial \overline{\mathcal{M}}'(a,b).$$

By transversality and dimension counting we have

$$\begin{aligned} \partial \Sigma_\gamma(a,b) = \{(x,y) \in \overline{\mathcal{M}}'(a,b) \times \overline{\mathcal{M}}'(c,b) | \\ s_\gamma(a,c)(x) \cdot s_\gamma(c,b)(y) = 0, c \text{ is irreducible.}\}. \\ = \coprod_{\mu(c)=\mu(b)+1} \Sigma_\gamma(a,b) \times \overline{\mathcal{M}}'(c,b) \cup \\ \coprod_{\mu(c')=\mu(b)+2} \overline{\mathcal{M}}'(a,c') \times \Sigma_\gamma(c',b). \end{aligned}$$

The orientations are also compatible. Therefore we have

$$\sum_c \langle \partial_\gamma a, c\rangle \langle \partial c, b\rangle + \sum_{c'} \langle \partial a, c'\rangle \langle \partial_\gamma c', b\rangle = 0.$$

Hence $\partial_\gamma \partial + \partial \partial_\gamma = 0$, as required.

The proof of $\partial_{\gamma_1,\gamma_2}\partial + \partial_{\gamma_1}\partial_{\gamma_2} + \partial_{\gamma_2}\partial_{\gamma_1} + \partial\partial_{\gamma_1,\gamma_2} = 0$ is similar.

Now put

$$C^s_k = \bigoplus_{\ell \le s} S^\ell H_1(M, \mathbf{Z}) \otimes C^0_{k-2\ell},$$

and define $\widehat{\partial} : C_k^s \to C_{k-1}^s$, by

$$\widehat{\partial}(\gamma_\alpha \otimes [a]) = \sum_{\alpha^1 \cup \alpha^2 = \alpha} \gamma_{\alpha^1} \otimes \partial_{\alpha^2}[a].$$

(Here we fix a basis $\gamma_1, \cdots, \gamma_d$ of the first homology group and put

$$\partial_\alpha = \sum_{j_1, \cdots, j_\ell} \prod_i C_{i,j_i} \partial_{\gamma_{j_1} \cdots \gamma_{j_\ell}}$$

if $\alpha = (\sum_{j_1} C_{1,j_1}[\gamma_{j_1}], \cdots, \sum_{j_\ell} C_{\ell,j_\ell}[\gamma_{j_\ell}])$. Later, in Lemma 12.10, we shall prove that ∂_γ are additive with respect to γ.) Theorem 1.10 implies $\widehat{\partial}\widehat{\partial} = 0$.

As we pointed out in §1, the boundary operator $\widehat{\partial}$ itself *does* depend on the choice of the sections $s_\gamma(c, c')$, because the spaces $\mathcal{C}\overline{\mathcal{M}}'(c, c')$ have boundaries. Next we prove that the chain complex $(C_\cdot^s, \widehat{\partial})$ is independent of the choice of the section.

Theorem 12.2. *Suppose $H_1(M; \mathbf{Z})$ is torsion free and $s < 3$. Let $s_\gamma(a, b)$ and $s'_\gamma(a, b)$ are the sections satisfying* (12.1.1) *and* (12.1.2). *Let $(C^s, \widehat{\partial})$ and $(C^s, \widehat{\partial}')$ be the corresponding chain complexes. Then there exist maps $\psi, \varphi : C^s \to C^s$ such that*

(12.2.1) $$\widehat{\partial}'\varphi = \varphi\widehat{\partial}$$

(12.2.2) $$\widehat{\partial}\psi = \psi\widehat{\partial}'$$

(12.2.3) $$\varphi\psi = \psi\varphi = \textit{identity}.$$

Proof. For each loop γ and $c, c' \in Fl$, we choose a section $\widetilde{s}_\gamma(c, c')$ to $\mathcal{L}_\gamma^{(2)}(c, c') \times [0, 1] \to \overline{\mathcal{M}}'(c, c') \times [0, 1]$ such that

(12.3.1) $$\widetilde{s}_\gamma(c, c')(x, 0) = s_\gamma(c, c')(x)$$
$$\widetilde{s}_\gamma(c, c')(x, 1) = s'_\gamma(c, c')(x)$$

(12.3.2) For each $a, b \in Fl$, the collections of sections

$$\widetilde{s}_\gamma(a, c_0) \otimes \cdots \otimes \widetilde{s}_\gamma(c_k, b)$$

can be patched together to give a smooth section on $\mathcal{C}\overline{\mathcal{M}}'(a, b) \times [0, 1]$.
(12.3.3) The zeros of $\widetilde{s}_{\gamma_i}$ are transversal and are transversal to each other.

Now, let $\mu(a) = \mu(b) + 3$, and put

$$\widetilde{\Sigma}_\gamma(a,b) = \{(x,t) \in C\overline{\mathcal{M}}'(a,b) \times [0,1] | \widetilde{s}_\gamma(a,b)(x,t) = 0\}.$$

Then $\dim \widetilde{\Sigma}_\gamma(a,b) = 1$. Note that (12.3.2) implies that

$$\widetilde{\Sigma}_\gamma(a,b) \cap (\overline{\mathcal{M}}'(a,c) \times \overline{\mathcal{M}}'(c,b) \times [0,1]) \neq \emptyset$$

only if c is irreducible and $\mu(c) = \mu(b) + 1$ or 2. Therefore

(12.4)

$$\begin{aligned} &\partial\widetilde{\Sigma}_\gamma(a,b) = \\ &\quad \{(x,0)|\widetilde{s}_\gamma(a,b)(x,0) = 0\} \cup \{(x,1)|\widetilde{s}_\gamma(a,b)(x,1) = 0\} \cup \\ &\quad \coprod_c \{(x_1,x_2,t)|\widetilde{s}_\gamma(c,b)(x_1,t) \cdot \widetilde{s}_\gamma(a,c)(x_2,t) = 0\}. \end{aligned}$$

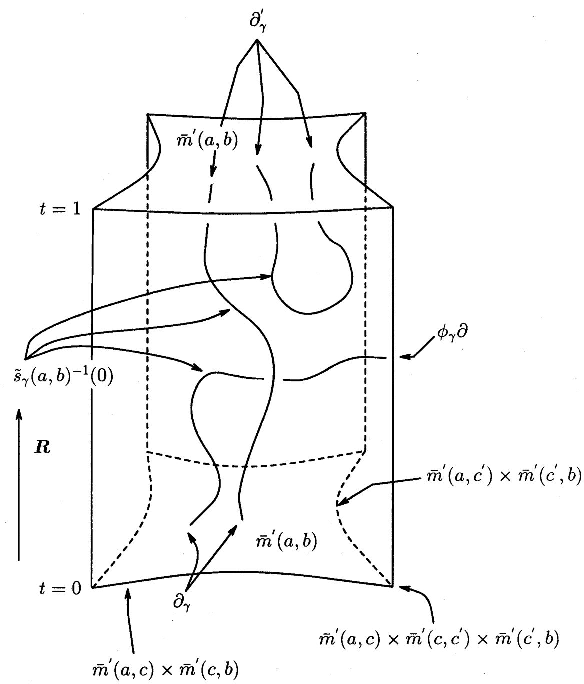

Figure 8.

For each $a, c \in Fl$, with $\mu(a) = \mu(c) + 2$, we put

$$\langle \varphi_\gamma a, c \rangle = \sharp\{(x,t) \in \overline{\mathcal{M}}'(a,c) \times [0,1] | \widetilde{s}_\gamma(x,t) = 0\}.$$

Note the set in the right hand side is a finite set, by (12.3.3) and dimension counting. Define $\varphi_\gamma : C_k^0 \to C_{k-2}^0$ by

$$\varphi_\gamma[a] = \sum \langle \varphi_\gamma a, c \rangle [c].$$

Then (12.4) implies

$$\partial_\gamma - \partial'_\gamma + \partial \varphi_\gamma - \varphi_\gamma \partial = 0. \tag{12.5}$$

Now define $\varphi, \psi : C^1 \to C^1$ by

$$\begin{aligned} \varphi(1 \otimes [a]) &= 1 \otimes [a] \\ \varphi(\gamma \otimes [a]) &= \gamma \otimes [a] + 1 \otimes \varphi_\gamma[a], \\ \psi(1 \otimes [a]) &= 1 \otimes [a] \\ \psi(\gamma \otimes [a]) &= \gamma \otimes [a] - 1 \otimes \varphi_\gamma[a]. \end{aligned}$$

Then using (12.5), it is easy to verify (12.2.1),(12.2.2), and (12.2.3).

Next we consider the case $s = 2$. Let $\mu(a) = \mu(b) + 5$. Put

$$\widetilde{\Sigma}_{\gamma_1,\gamma_2}(a,b) = \{(x,t) \in \mathcal{C}\overline{\mathcal{M}}'(a,b) \times [0,1] | \widetilde{s}_{\gamma_1}(x,t) = \widetilde{s}_{\gamma_2}(x,t) = 0\}.$$

We have

(12.6)

$$
\begin{aligned}
&\partial\widetilde{\Sigma}_{\gamma_1,\gamma_2}(a,b) = \\
&\quad \{(x,0)|s_{\gamma_1}(a,b)(x) = s_{\gamma_2}(a,b)(x) = 0\} \\
&\quad \cup \{(x,1)|s'_{\gamma_1}(a,b)(x) = s'_{\gamma_2}(a,b)(x) = 0\} \\
&\quad \cup \coprod_{\mu(c_1)=\mu(b)+1} \{(x,y,t) \in \overline{\mathcal{M}}'(a,c_1) \times \overline{\mathcal{M}}'(c_1,b) \times [0,1]| \\
&\qquad\qquad\qquad \widetilde{s}_{\gamma_1}(a,c_1)(x,t) = \widetilde{s}_{\gamma_2}(a,c_1)(x,t) = 0\}, \\
&\quad \cup \coprod_{\mu(c_4)=\mu(b)+4} \{(x,y,t) \in \overline{\mathcal{M}}'(a,c_4) \times \overline{\mathcal{M}}'(c_4,b) \times [0,1]| \\
&\qquad\qquad\qquad \widetilde{s}_{\gamma_1}(c_4,b)(x,t) = \widetilde{s}_{\gamma_2}(c_4,b)(y,t) = 0\},
\end{aligned}
$$

$$
\cup \coprod_{\mu(c_2)=\mu(b)+2} \left\{ (x,y,t) \;\middle|\; \begin{array}{c} (x,y,t) \in \overline{\mathcal{M}}'(a,c_2) \times \overline{\mathcal{M}}'(c_2,b) \times [0,1] \\ \widetilde{s}_{\gamma_1}(a,c_2)(x,t) = 0 = \widetilde{s}_{\gamma_2}(c_2,b)(y,t) \\ \text{or} \\ \widetilde{s}_{\gamma_1}(c_2,b)(x,t) = 0 = \widetilde{s}_{\gamma_2}(a,c_2)(y,t) \end{array} \right\}
$$

$$
\cup \coprod_{\mu(c_3)=\mu(b)+3} \left\{ (x,y,t) \;\middle|\; \begin{array}{c} (x,y,t) \in \overline{\mathcal{M}}'(a,c_3) \times \overline{\mathcal{M}}'(c_3,b) \times [0,1] \\ \widetilde{s}_{\gamma_1}(a,c_3)(x,t) = 0 = \widetilde{s}_{\gamma_2}(c_3,b)(y,t) \\ \text{or} \\ \widetilde{s}_{\gamma_1}(c_3,b)(x,t) = 0 = \widetilde{s}_{\gamma_2}(a,c_3)(y,t) \end{array} \right\}.
$$

Let $\Lambda_0, \Lambda_5, \Lambda_1, \Lambda_4, \Lambda_2, \Lambda_3$ be the sets in the above formula, respectively. We have

$$\sharp\Lambda_0 = \langle \partial_{\gamma_1,\gamma_2} a, b\rangle, \tag{12.7.1}$$

$$\sharp\Lambda_5 = -\langle \partial'_{\gamma_1,\gamma_2} a, b\rangle. \tag{12.7.2}$$

For $a, c \in Fl$ with $\mu(a) = \mu(c) + 4$, we put

$$\langle \varphi_{\gamma_1,\gamma_2} a, c\rangle = \sharp\{(x,t) \in \overline{\mathcal{M}}'(a,c) \times [0,1] | \widetilde{s}_{\gamma_1}(x,t) = \widetilde{s}_{\gamma_2}(x,t) = 0\}.$$

Then we have

$$\sharp\Lambda_1 = \sum_{c_1} \langle \varphi_{\gamma_1,\gamma_2} a, c_1\rangle \langle \partial c_1, b\rangle, \tag{12.7.3}$$

$$\sharp\Lambda_4 = -\sum_{c_4} \langle \partial a, c_4\rangle \langle \varphi_{\gamma_1,\gamma_2} c_4, b\rangle. \tag{12.7.4}$$

To examine $\sharp\Lambda_2$ and $\sharp\Lambda_3$, we remark that the sections $\widetilde{s}_\gamma(c,c')$ can be defined by an induction on $\mu(c)-\mu(c')$. Then, we can assume the following conditions (12.8). For $c,c'\in Fl$ with $\mu(c)=\mu(c')+2$, we put

$$T(c,c')=\sup\{t|\exists x\ \ (x,t)\in\widetilde{\Sigma}_\gamma(c,c')\},$$
$$S(c,c')=\inf\{t|\exists x\ \ (x,t)\in\widetilde{\Sigma}_\gamma(c,c')\}.$$

(12.8.1) If $\mu(c)=\mu(c')+3=\mu(c'')+5$, and if $t>T(c',c'')$ then

$$\widetilde{s}'_\gamma(c,c')(x,t)=\widetilde{s}'_\gamma(c,c')(x,1)$$

(12.8.2) If $\mu(c)=\mu(c')+2=\mu(c'')+5$, and if $t<S(c,c')$, then

$$\widetilde{s}'_\gamma(c',c'')(x,t)=\widetilde{s}'_\gamma(c',c'')(x,0)$$

Using (12.8.1), we can prove:

$$\begin{aligned}\Lambda_2=&\coprod_{c_2}\{x\in\overline{\mathcal{M}}'(a,c_2)|s'_{\gamma_1}(x)=0\}\times\\&\qquad\{(y,t)\in\overline{\mathcal{M}}'(c_2,b)\times[0,1]|\widetilde{s}_{\gamma_2}(y,t)=0\}\\&\cup\coprod_{c_2}\{x\in\overline{\mathcal{M}}'(a,c_2)|s'_{\gamma_2}(x)=0\}\times\\&\qquad\{(y,t)\in\overline{\mathcal{M}}'(c_2,b)\times[0,1]|\widetilde{s}_{\gamma_1}(y,t)=0\}.\end{aligned}$$

Therefore

(12.9.1) $$\sharp\Lambda_2=-\sum_{c_2}\langle\partial'_{\gamma_1}a,c_2\rangle\langle\varphi_{\gamma_2}c_2,b\rangle-\sum_{c_2}\langle\partial'_{\gamma_2}a,c_2\rangle\langle\varphi_{\gamma_1}c_2,b\rangle.$$

Similarly, using (12.8.2), we can prove:

(12.9.2) $$\sharp\Lambda_3=\sum_{c_3}\langle\varphi_{\gamma_1}a,c_3\rangle\langle\partial_{\gamma_2}c_3,b\rangle+\sum_{c_3}\langle\varphi_{\gamma_2}a,c_3\rangle\langle\partial_{\gamma_1}c_3,b\rangle.$$

By (12.6.1),(12.7),(12.9), we have
(12.10)
$$\partial_{\gamma_1,\gamma_2}+\varphi_{\gamma_1}\partial_{\gamma_2}+\varphi_{\gamma_1}\partial_{\gamma_2}+\varphi_{\gamma_1,\gamma_2}\partial=\partial'_{\gamma_1,\gamma_2}+\partial'_{\gamma_1}\varphi_{\gamma_2}+\partial'_{\gamma_2}\varphi_{\gamma_1}+\partial'\varphi_{\gamma_1,\gamma_2}.$$

Now we put

$$\begin{aligned}\varphi(\gamma_1\gamma_2\otimes[a])&=\gamma_1\gamma_2\otimes[a]+\gamma_1\otimes\varphi_{\gamma_2}[a]+\gamma_2\otimes\varphi_{\gamma_1}[a]+1\otimes\varphi_{\gamma_1,\gamma_2}[a]\\\psi(\gamma_1\gamma_2\otimes[a])&=\gamma_1\gamma_2\otimes[a]-\gamma_1\otimes\varphi_{\gamma_2}[a]-\gamma_2\otimes\varphi_{\gamma_1}[a]\\&\quad-1\otimes(\varphi_{\gamma_1,\gamma_2}+\varphi_{\gamma_1}\varphi_{\gamma_2}+\varphi_{\gamma_2}\varphi_{\gamma_1})[a].\end{aligned}$$

Formulas (12.2.1),(12.2.2),(12.2.3) follow immediately from (12.5) and (12.10). The proof of Theorem 12.2 is now complete.

Next we shall prove the following:

Lemma 12.11. *Let $\gamma_1, \gamma_2, \gamma, \gamma'$ be closed loops on M with $[\gamma_1] + [\gamma_2] = [\gamma]$ in $H_1(M; \mathbf{Z})$. Then we can find collections of sections $s_{\gamma_1}(c,c')$,$s_{\gamma_2}(c,c')$, $s_\gamma(c,c')$,$s_{\gamma'}(c,c')$ with* (12.1.1), (12.1.2) *such that the corresponding boundary operators satisfy*

$$\partial_{\gamma_1} + \partial_{\gamma_2} = \partial_\gamma \tag{12.11.1}$$

$$\partial_{\gamma_1,\gamma'} + \partial_{\gamma_2,\gamma'} = \partial_{\gamma,\gamma'}. \tag{12.11.2}$$

Proof. Let $\mu(a) = \mu(b) + 3$. Consider $C\overline{\mathcal{M}}(a,b)$. (We do not divide it by the $\mathbf{R}$ action.) Let Σ be a surface on $M \times \mathbf{R}$ which is asymptotic to $(\gamma_1 \cup \gamma_2) \times \mathbf{R}$ as $t \to -\infty$, and to $\gamma \times \mathbf{R}$ as $t \to \infty$. Using the Dirac operator on Σ, we can define a line bundle $\mathcal{L}^{(2)}_\Sigma(a,b)$ on $C\overline{\mathcal{M}}(a,b) = C\overline{\mathcal{M}}'(a,b) \times \mathbf{R}$. We put

$$CC\overline{\mathcal{M}}(a,b) = C\overline{\mathcal{M}}(a,b) \times [-\infty, \infty].$$

By construction and Theorem 4.9, the bundles $\mathcal{L}^{(2)}_\Sigma(a,b)$ on $C\overline{\mathcal{M}}(a,b)$, and $\mathcal{L}^{(2)}_{\gamma_1}(a,b) \otimes \mathcal{L}^{(2)}_{\gamma_2}(a,b)$ on $C\overline{\mathcal{M}}'(a,b) \times \{-\infty\}$, and $\mathcal{L}^{(2)}_{\gamma}(a,b)$ on $C\overline{\mathcal{M}}'(a,b) \times \{\infty\}$ can be patched together to give a line bundle over $CC\overline{\mathcal{M}}(a,b)$. We extend the sections $s_{\gamma_1}(a,b) \otimes s_{\gamma_2}(a,b)$ and $s_\gamma(a,b)$ to a section on $CC\overline{\mathcal{M}}(a,b)$. Then, by an argument similar to the proof of Theorem 12.2, we can find φ_γ such that

$$\partial_\gamma - (\partial_{\gamma_1} + \partial_{\gamma_2}) = \partial\varphi_\gamma - \varphi_\gamma\partial.$$

Using this map φ_γ, we can modify the section s_γ such that (12.11.1) is satisfied. The proof of (12.11.2) is similar.

Finally, we discuss what happens when $s \geq 1$ in case $H_1(M; \mathbf{Z})$ has a torsion, and when $s \geq 3$ in case $H_1(M; \mathbf{Z})$ is torsion free.

Suppose first that $H_1(M; \mathbf{Z})$ has a torsion, and $\mu(a) = \mu(b) + 5$. In this case, there may be reducible connections c and c' such that $G_c = G_{c'} = U(1)$ and that $\mu(c) = \mu(c') + 1 = \mu(b) + 2$. Then

$$\dim \overline{\mathcal{M}}'(a,c) = \dim \overline{\mathcal{M}}'(c,c') = \dim \overline{\mathcal{M}}'(c',b) = 0.$$

The set $\overline{\mathcal{M}}'(c,c')$ may have a 0 dimensional orbit $\overline{\mathcal{M}}'_{red}(c,c')$ which consists only of reducible connections. (See Theorem 5.6.) A neighborhood of each point of $\overline{\mathcal{M}}'(a,c) \times \overline{\mathcal{M}}'_{red}(c,c') \times \overline{\mathcal{M}}'(c',b)$, in $\mathcal{C}\overline{\mathcal{M}}'(a,b)$ is identified to $(0,\infty] \times (0,\infty] \times U(1)/\sim$, where $(t,s,g_1) \sim (t,s,g_2)$ if and only if $t = \infty$ or $s = \infty$. Here $\{\infty\} \times (0,\infty) \times U(1)/\sim$ and $(0,\infty) \times \{\infty\} \times U(1)/\sim$ are identified to $\overline{\mathcal{M}}'(a,c) \times \overline{\mathcal{M}}'(c,b)$ and $\overline{\mathcal{M}}'(a,c') \times \overline{\mathcal{M}}'(c',b)$ respectively. The bundle $\mathcal{L}^{(2)}_{\gamma}(a,b)$ is extended outside $\infty \times \infty \times U(1)/\sim = \mathit{point}$. The neighborhood of this point is a cone of S^2. (It may be more natural to regard that this S^2 has two singular points.)

Using the basis $[\ell_i]$ of $H'_1(M;\mathbf{Z})$, chosen at the beginning of §2, we can find ℓ_{i_0} such that

$$c(\ell_i) = c'(\ell_i) \qquad \text{if } i \neq i_0. \tag{12.12.1}$$

$$c(\ell_{i_0}) = 1, \quad c'(\ell_{i_0}) = -1. \tag{12.12.2}$$

In this case we can prove that the restriction of the line bundle $\mathcal{L}^{(2)}_{\ell_{i_0}}(a,b)$ to this S^2 is nontrivial. (Its chern number is ± 1.) (See the proof of Lemma 12.13 below.) Then the formula

$$\partial_\gamma \partial + \partial_\gamma \partial = 0$$

does not hold in general.

Next suppose that $H_1(M;\mathbf{Z})$ is torsion free. Let c and c' be reducible connections such that $G_c = G_{c'} = SU(2)$, $A \in \overline{\mathcal{M}}'(c,c')$, $G_A = U(1)$, $\mu(c) = \mu(c') + 3$. Then, if $a, b \in Fl$ and if $\mathcal{M}(a,c) \neq \emptyset, \mathcal{M}(c',b) \neq \emptyset$, then $\mu(a) \geq \mu(c) + 4, \mu(b) \leq \mu(c') - 1$. Hence, the first case we are to examine is the case when $\mu(a) = \mu(b) + 8 = \mu(c') + 7 = \mu(c) + 4$. In this case,

$$\dim \overline{\mathcal{M}}'(a,c) = \dim \overline{\mathcal{M}}'_{red}(c,c') = \dim \overline{\mathcal{M}}'(c',b) = 0.$$

Here $\overline{\mathcal{M}}'_{red}(c,c')$ is the component of $[A]$, which consists of one point. By Theorem 7.1 a neighborhood of each point of

$$\overline{\mathcal{M}}'(a,c) \times \overline{\mathcal{M}}'_{red}(c,c') \times \overline{\mathcal{M}}'(c',b)$$

in $\mathcal{C}\overline{\mathcal{M}}'(a,b)$ is

$$\left(\frac{SO(3) \times SO(3)}{U(1)} \times (0,\infty]^2\right) / \sim$$

where $\sim$ is as in (7.1.4). In other words, it is a cone of $\mathbf{C}P^3/\mathbf{Z}_2 = X$. (See the proof of Lemma 12.13.) Here $\mathbf{Z}_2$ acts by

$$\tau[z_0, z_1, z_2, z_3] = [z_0, z_1, -z_2, -z_3].$$

The fixed points set of this action has two components. The fixed points correspond to the singular points of X. Those singular locus are identified to

$$\begin{aligned}&\left(\frac{SO(3) \times SO(3)}{U(1)} \times \{\infty\} \times (0, \infty)\right) / \sim \\ &\subset \overline{\mathcal{M}}'(a, c) \times \overline{\mathcal{M}}'(c, b),\end{aligned}$$

and

$$\begin{aligned}&\left(\frac{SO(3) \times SO(3)}{U(1)} \times (0, \infty) \times \{\infty\}\right) / \sim \\ &\subset \overline{\mathcal{M}}'(a, c') \times \overline{\mathcal{M}}(c', b),\end{aligned}$$

respectively. We can find ℓ_{i_0} such that (12.12.1) and (12.12.2) are satisfied.

Lemma 12.13.

$$\int_X c^1(\mathcal{L}^{(2)}_{\ell_{i_0}}(a, b))^3 = \pm 4.$$

Proof. Let a_t^0 be a representative of $\overline{\mathcal{M}}'(c, c') = point$, (used in §8.) On $\ell_{i_0} \times \mathbf{R}$, a_t^0 converges to the trivial connection as t goes to $-\infty$, and, as t goes to ∞, it converges to a flat connection -1 whose holonomy, $\rho_{-1} : \mathbf{Z} = \pi_1(S^1) \to SU(2)$ is given by $\rho_{-1}(1) = -1$.

Sublemma 12.14.

$$\operatorname{Index}(\eth_{a_t^0} + \epsilon) = -1.$$

Proof. We put $S^1 = \mathbf{R}/2\pi\mathbf{Z}$. Let x be the coordinate of S^1. We have

$$\eth_{trivial} = \frac{\partial}{\partial t} + i\frac{\partial}{\partial x}.$$

We can perturb a_t^0 so that it is a connection with holonomy

$$\begin{pmatrix} e^{\pi i t} & 0 \\ 0 & e^{-\pi i t} \end{pmatrix}.$$

(a_0^0 is a trivial connection and $a_1^0 = -1$.) Then the spectral flow corresponding to the operator $\eth_{a_t^0} + \epsilon$ is as in Figure 9. (Here we take $\epsilon > 0$.)

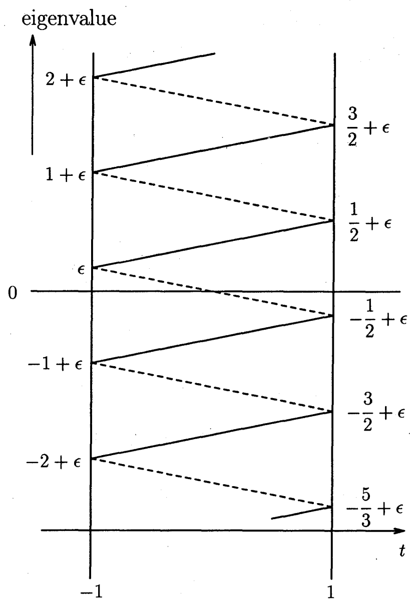

Figure 9.

The sublemma follows.

Remark 12.15. In our case, the half spin bundle $\otimes \mathbf{C}^2$ together with connection a_t^0 splits to the direct sum of two complex line bundles. The dotted lines in Figure 9 correspond to the second factor and the others to the first factor.

The group $U(1) = I_{a_t^0}$ acts on the eigenspaces, and the index in Sublemma 12.14 can be regarded as an element of the representation ring $R(U(1)) \sim \mathbf{Z}[t, t^{-1}]$. Here t be the representation corresponding to $z \mapsto z$ and t^{-1} to $z \to z^{-1}$, where we identify $U(1) = \{z||z| = 1\}$. By Figure 9, The index is equal to $-t^{-1}$.

If we choose $\epsilon < 0$ then the index is t.

Now we consider the map $\pi : SU(2) \times SU(2) \to \mathcal{M}'(c, c')$ constructed in Theorem 5.4. Let $\mathcal{L}_i(c, c')$ be the line bundle defined in §7. (We have not yet divided it by $G_c \times G_{c'}$.) $\pi^* \mathcal{L}_i(c, c')$ is trivial.

On $SU(2) \times SU(2)$, the group $U(1) = I_{a_t^0}$ acts by

$$h(g_1, g_2) = (g_1 h, h^{-1} g_2).$$

This action lifts to $\pi^*(\mathcal{L}_i(c, c'))$. The quotient is identified to the restriction of $\mathcal{L}_i(c, c')$ to the image of π, which is diffeomorphic to $SU(2) \times SU(2)/U(1)$. By Sublemma 12.14 and Remark 12.15, the action of $U(1)$ on $\pi^*(\mathcal{L}_i(c, c'))$ is given by

$$h((g_1, g_2), v) = ((g_1 h^{-1}, h g_2), hv), \tag{12.16}$$

(in both cases $\epsilon > 0$ and $\epsilon < 0$.)

We put

$$\widehat{X} = \frac{SU(2) \times SU(2) \times [0,1]}{\sim},$$

where

$$(g_1, g_2, 0) = (g_1', g_2, 0),$$
$$(g_1, g_2, 1) = (g_1, g_2', 1).$$

$\widehat{X}$ is diffeomorphic to S^7. By Theorem 7.1,

$$X = \frac{\widehat{X}}{U(1) \times \mathbf{Z}_2}.$$

Here $h \in U(1)$ and $\tau = -1 \in \mathbf{Z}_2$ acts on $\widehat{X}$ by

$$h([g_1, g_2, t]) = [g_1 h, h^{-1} g_2, t],$$
$$\tau([g_1, g_2, t]) = [-g_1, g_2, t].$$

Hence $\widehat{X}/U(1) \simeq \mathbf{C}P^3$. By (12.16), the bundle $\mathcal{L}_i(a, b)$ on $\widehat{X}/U(1) \subset \mathcal{CM}'(a, b)$ is isomorphic to the canonical bundle on $\mathbf{C}P^3$. Hence, its Chern class is equal to the generator, u. Therefore,

$$\int_X c^1\left(\mathcal{L}_i^{(2)}(a, b)\right)^3 = \int_{\mathbf{C}P^3} (2u)^3/2 = 4.$$

The proof of Lemma 12.13 is now complete.

Using Lemma 12.13, we can discuss as in the proof of Theorem 1.10, to show

$$\sum_{\alpha_1 \cup \alpha_2 = \alpha} \partial_{\gamma_{\alpha_1}} \partial_{\gamma_{\alpha_2}} = 4 \sum_{c, c'} \sharp \overline{\mathcal{M}}'(a, c) \cdot \sharp \overline{\mathcal{M}}'(c', b),$$

in the case when $\alpha = (\ell_{i_0}, \ell_{i_0}, \ell_{i_0})$.

It might be possible to define an invariant mod 4 using the above formula. But the author does not try to do it here, because he suspects if it is a correct way.

From the above observation, it seems that we need to examine the reducible connections more seriously when we generalize the invariant for larger s.

§13. Independence of the metrics and the perturbations

The proof of Theorem 1.14 is based on an argument similar to one in §§7–12 and [F]. Let σ_1, σ_2 be two metrics on M and f_1,f_2 be two perturbations as in §§2,3. Let Fl_1 and Fl_2 be the set of solutions of

$$*_{\sigma_1} F^a - \mathrm{grad}_a f_1 = 0,$$

and

$$*_{\sigma_2} F^a - \mathrm{grad}_a f_2 = 0,$$

respectively. Let $(C^s_{(1)}, \partial^1)$ and $(C^s_{(2)}, \partial^2)$ be corresponding complexes constructed in §12. Choose a family of metrics g_t such that

$$\sigma_t = \sigma_1 \quad \text{for } t < -1. \tag{13.1.1}$$

$$\sigma_t = \sigma_2 \quad \text{for } t > 1. \tag{13.1.2}$$

Choose χ such that

$$\begin{aligned} \chi(t) &= 1 \quad \text{for } t > 1, \\ \chi(t) &= 0 \quad \text{for } t < 0. \end{aligned}$$

Let σ_t be the metric $\sigma_t \oplus dt^2$ on $M \times \mathbf{R}$. We consider the equation

$$\begin{aligned} F^A - \widetilde{*}_{\sigma_t} F^A - &\chi(-t) \left(\mathrm{grad}_{a_t} f_1 \wedge dt - *_{\sigma_t} \mathrm{grad}_{a_t} f_1\right) \\ &- \chi(t) \left(\mathrm{grad}_{a_t} f_2 \wedge dt - *_{\sigma_t} \mathrm{grad}\, f_2\right) = 0, \end{aligned} \tag{13.2}$$

for $A \in \mathcal{A}_{\ell,\delta}(a, b)$. (Compare (3.6).) Here $a \in Fl_1$ and $b \in Fl_2$. The linearization of (13.2) is given by

$$0 = \mathcal{D}_A(u, \varphi) =$$
$$-\frac{\partial u}{\partial t} + (*_{\sigma_t} d_{a_t} - \psi_t - \chi(-t)\, \mathrm{Hess}_{a_t} f_1 - \chi(t)\, \mathrm{Hess}_{a_t} f_2) \wedge u + d_{a_t} \varphi$$

Here u, φ e.t.c are the same as in (3.8). Let $\mathcal{D}^1_A$ and $\mathcal{D}^2_A$ be the operators in (3.8) for $\sigma = \sigma_1 \oplus dt^2, \sigma_2 \oplus dt^2$ and $f = f_1, f_2$, respectively.

Lemma 13.3. *If $A \in \mathcal{A}_{\ell,\gamma}(a,b)$ with $a \in Fl_1$ $b \in Fl_2$, then*

$$\dim \operatorname{Coker} \mathcal{D}_A < \infty.$$

Proof. If not we have (u_i, φ_i) such that

$$\begin{aligned} &\mathcal{D}_A^*(u_i, \varphi_i) = 0, \\ &< (u_i, \varphi_i), (u_j, \varphi_j) >= \delta_{i,j}. \end{aligned}$$

Then, by elliptic regularity, we have $|t_i| \to \infty$ such that

$$|(u_i(x_0, t_i), \varphi_i(x_0, t_i))| > C_0 > 0.$$

We may assume that $t_i \to \infty$. Put $u_i'(t,x) = u_i(t - t_i, x)$, $\varphi_i'(t,x) = \varphi_i(t - t_i, x)$. By taking a subsequence we may assume that (u_i', φ_i') converges to $(\widehat{u}, \widehat{\varphi})$ with respect to the C^∞ topology on each compact set. Then we have

$$\begin{aligned} &\mathcal{D}_b^{(2)*}(\widehat{u}, \widehat{\varphi}) = 0 \\ &(\widehat{u}, \widehat{\varphi}) \neq 0. \end{aligned}$$

This contradicts (2.6).

Using Lemma 13.3, we can apply the argument of [D3] to obtain a perturbation $Q(\cdot)$, such that the linearized operator $\mathcal{D}_A'$ of

$$\begin{aligned} (13.4) \quad F^A - \widetilde{*}_{\sigma_t} F^A &- \chi(-t)(\operatorname{grad}_{a_t} f_1 \wedge dt - *_{\sigma_t} \operatorname{grad}_{a_t} f_1) \\ &- \chi(t)(\operatorname{grad}_{a_t} f_2 \wedge dt - *_{\sigma_t} \operatorname{grad}_{a_t} f_2) + Q(A) = 0. \end{aligned}$$

is surjective. Here $Q(A)$ depends only on a restriction of A to $M \times [-1,1]$ and its support is also contained in it. Let $\overline{\mathcal{M}}(a,b)$ be the set of solutions of (13.4) divided by gauge transformations. Let $\overline{\mathcal{M}}'_{(1)}(a,b)$ and $\overline{\mathcal{M}}'_{(2)}(a,b)$ be the set of solutions of (3.6) for $\sigma = \sigma_1$, $f = f_1$ and $\sigma = \sigma_2$, $f = f_2$, divided by the gauge transformations and $\mathbf{R}$ action, respectively.

Theorem 13.5. *For $a \in Fl_1$ and $b \in Fl_2$, let $C\overline{\mathcal{M}}(a,b)$ be the* *disjoint union of*

$$\overline{\mathcal{M}}(a,b),$$

$$\overline{\mathcal{M}}(a,c_0) \times \prod_{i=0}^{k-1} \overline{\mathcal{M}}'_{(2)}(c_i,c_{i+1}) \times \overline{\mathcal{M}}'_{(2)}(c_k,b),$$

$$\overline{\mathcal{M}}'_{(1)}(a,c_0) \times \prod_{i=0}^{k-1} \overline{\mathcal{M}}'_{(1)}(c_i,c_{i+1}) \times \overline{\mathcal{M}}(c_k,b),$$

$$\overline{\mathcal{M}}'_{(1)}(a,c_0) \times \prod_{i=1}^{k_0-1} \overline{\mathcal{M}}'_{(1)}(c_i,c_{i+1}) \times \overline{\mathcal{M}}(c_{k_0},c_{k_0+1})$$

$$\times \prod_{i=k_0+1}^{k-1} \overline{\mathcal{M}}'_{(2)}(c_i,c_{i+1}) \times \overline{M}'_{(2)}(c_k,b).$$

Then $C\overline{\mathcal{M}}(a,b)$ *has a smooth structure with properties similar to* (7.1.1) -(7.1.7).

The proof is similar to the proof of Theorem 7.1 and is omitted.

We remark here the reason why we need to fix a basis of $H_1'(M;\mathbf{Z})$. Let μ_1,μ_2 be the maps defined in Theorem 5.1 for metrics σ_1,σ_2 and let f_1 and f_2 be functions we used in sections 2 and 3. If we use the same basis of $H_1'(M;\mathbf{Z})$ (or more precisely $H_1'(M;\mathbf{Z})\otimes\mathbf{Z}_2$), then we have $\mu_1(c) = \mu_2(c)$ for each reducible connection c. This fact is essential for the argument of the rest of this section. In fact, suppose, for example, there exists reducible c such that

$$\mu_1(c) = \mu_2(c) - 10.$$

Then for some $a \in Fl_1$,$b \in Fl_2$ with $\mu_1(a) = \mu_2(b)+1$, the space $\overline{\mathcal{M}}(a,b)$ may have an end described by

$$\overline{\mathcal{M}}'_{(1)}(a,c) \times \overline{\mathcal{M}}(c,c) \times \overline{\mathcal{M}}'_{(2)}(c,b).$$

And $\mu_1(a) - \mu_1(c)$ can be greater than 7. Therefore, in the compactification of $\overline{\mathcal{M}}'_{(1)}$ the end we discussed at the end of §12 can appear. These ends can cause serious problem for the argument of the well definedness. The point is that the virtual dimension of $\overline{\mathcal{M}}(a,b)$ is -10 but we can not find perturbation to make it empty

The author has no explicit example which shows that our invariant does depend on the choice of the basis of $H_1(M;\mathbf{Z})$. But it seems quite unlikely that it is independent.

We return to the proof of invariance. For $\gamma \simeq S^1 \subset M$, we define bundles

$$\mathcal{L}^{(2)}_{\gamma,1}(a,a') \quad \text{on } \overline{\mathcal{M}}_{(1)*}(a,a'),$$
$$\mathcal{L}^{(2)}_{\gamma,2}(b,b') \quad \text{on } \overline{\mathcal{M}}_{(2)*}(b,b'),$$
$$\mathcal{L}^{(2)}_{\gamma}(a,b) \quad \text{on } \overline{\mathcal{M}}(a,b).$$

Theorem 13.6. *The tensor products of $\mathcal{L}^{(2)}_{\gamma,1}$,$\mathcal{L}^{(2)}_{\gamma,2}$, and $\mathcal{L}^{(2)}_{\gamma}$ can be patched together to give a line bundle on $C\overline{\mathcal{M}}_*(a,b)$.*

The proof is the same as the proof of Theorem 7.3.

Now we define $\varphi : (C^s_{(1)}, \partial^1) \to (C^s_{(2)}, \partial^2)$. We put

$$< \varphi_\emptyset(a), b >= \sharp\overline{\mathcal{M}}(a,b)$$

if $\mu(a) = \mu(b)$. (Here $\sharp$ is the same as in §12.) Set

$$\varphi[a] = \sum_b < \varphi_\emptyset a, b > [b].$$

This defines the map $\varphi : C^0_{(1)} \to C^0_{(2)}$.

Next we fix sections $s_\gamma(a,b)$, $s_{\gamma,1}(a,a')$, $s_{\gamma,2}(b,b')$ to $\mathcal{L}^{(2)}_{\gamma}(a,b)$, $\mathcal{L}^{(2)}_{\gamma,1}(a,a')$, $\mathcal{L}^{(2)}_{\gamma,2}(b,b')$ such that (12.1.2) holds and that they can be patched together to give a section of the line bundle obtained in Theorem 13.6. Now, for $\mu(a) = \mu(b) + 2$, we put

$$< \varphi_\gamma a, b >= \sharp\{x \in \overline{\mathcal{M}}(a,b) | s_\gamma(x) = 0.\}.$$

For $\mu(a) = \mu(b) + 4$, we put

$$< \varphi_{\gamma_1,\gamma_2} a, b >= \sharp\{x \in \overline{\mathcal{M}}(a,b) | s_{\gamma_1}(x) = s_{\gamma_2}(x) = 0\}.$$

Set

$$\varphi_\gamma[a] = \sum_b < \varphi_\gamma a, b > [b],$$
$$\varphi_{\gamma_1,\gamma_2}[a] = \sum_b < \varphi_{\gamma_1,\gamma_2} a, b > [b].$$

Lemma 13.7. *If $|\alpha| < 3$, then*

$$\sum_{\alpha_1 \cup \alpha_2 = \alpha} \partial^2_{\alpha_1} \varphi_{\alpha_2} = \sum_{\alpha_1 \cup \alpha_2 = \alpha} \varphi_{\alpha_1} \partial^1_{\alpha_2}.$$

(If $|\alpha| > 0$ we assume that $H_1(M;\mathbf{Z})$ is torsion free.)

The proof is the same as the proof of Theorem 1.10 in §12. Put

$$\varphi(\gamma_\alpha \otimes a) = \sum_{\alpha_1 \cup \alpha_2 = \alpha} \gamma_{\alpha_1} \otimes \gamma_{\alpha_2} a.$$

Lemma 13.7 implies that $\varphi : (C^s_{(1)}, \partial^1) \to (C^s_{(2)}, \partial^2)$ is a chain map.

Lemma 13.8. *The chain map φ modulo chain homotopy is independent to the choice of the homotopy σ_t of the metrics and the perturbation Q in* (13.4).

Proof. Let σ^1_t,σ^2_t,Q_1,Q_2 be the homotopies and perturbations and φ_1, φ_2 be corresponding chain maps. Choose homotopies σ^u_t and Q_u among them. Let $\overline{\mathcal{M}}'_u(a,b)$ be the set of solutions of (13.4) for $\sigma_t = \sigma^u_t$ and $Q = Q_u$. Let $\mathcal{C}\overline{\mathcal{M}}'_u(a,b)$ be the disjoint union of

$$\overline{\mathcal{M}}_u(a,b)$$

$$\overline{\mathcal{M}}_u(a,c_0) \times \prod_{i=0}^{k-1} \overline{\mathcal{M}}'_{(2)}(c_i,c_{i+1}) \times \overline{\mathcal{M}}'_{(2)}(c_k,b),$$

$$\overline{\mathcal{M}}'_{(1)}(a,c_0) \times \prod_{i=0}^{k-1} \overline{\mathcal{M}}'_{(1)}(c_i,c_{i+1}) \times \times \overline{\mathcal{M}}_u(c_k,b),$$

$$\overline{\mathcal{M}}'_{(1)}(a,c_0) \times \prod_{i=0}^{k_0-1} \overline{\mathcal{M}}'_{(1)}(c_i,c_{i+1}) \times \overline{\mathcal{M}}_u(c_{k_0},c_{k_0+1})$$
$$\times \prod_{i=k_0+1}^{k-1} \overline{\mathcal{M}}'_{(2)}(c_i,c_{i+1}) \times \overline{\mathcal{M}}'_{(2)}(c_k,b).$$

(Here we do *not* assume that $\mu(a) > \mu(c_0) > \cdots > \mu(c_k) > \mu(b)$.) (Note that $\mathcal{M}_{(1)}(a,b) \neq \mathcal{M}_1(a,b)$.)

Put

$$\mathcal{H}\overline{\mathcal{M}}(a,b) = \bigcup_u \overline{\mathcal{M}}_u(a,b) \times \{u\},$$

$$\mathcal{CH}\overline{\mathcal{M}}(a,b) = \bigcup_u \mathcal{C}\overline{\mathcal{M}}_u(a,b) \times \{u\}.$$

Theorem 13.9. *We can take σ^u_t and Q_u such that $\mathcal{CH}\overline{\mathcal{M}}(a,b)$ has a smooth structure which has properties similar to* (7.1.1)–(7.1.7).

The proof of Theorem 13.9 is a bit more difficult than that of Theorem 7.1. The reason is that we can *not* assume that the operator $\mathcal{D}_A^{(u)}$ obtained by linearizing (13.4) is surjective for every u, (even if we choose σ_t^u and Q_u to be generic.) Then we have to use the Kuranishi map as in [T2], [D2]. For simplicity we prove the case $\mu(a) = \mu(b)$. Here $a \in Fl_1, b \in Fl_2$. Then $\dim \mathcal{HM}'(a,b) = 1$. In this case, Theorem 13.9 follows immediately from the following two lemmas.

Lemma 13.10. *Suppose that the sequence $(A_i, u_i) \in \mathcal{H}\overline{\mathcal{M}}(a,b)$ is unbounded. Then, by taking a subsequence if necessary, there exist either $c \in Fl_1$, t_i, $B \in \overline{\mathcal{M}}_u(a,c)$, $C \in \overline{\mathcal{M}}'_{(2)}(c,b)$ with $\mu(c) = \mu(a) + 1$ or $c' \in Fl_2$, t'_i, $B' \in \overline{\mathcal{M}}_{(1)}(a,c')$, $C' \in \overline{\mathcal{M}}_u(c',b)$ with $\mu(c') = \mu(a) - 1$ such that the Conditions* (13.10.1)–(13.10.3) *or* (13.10.1) –(13.10.3)$'$ *below hold.*

(13.10.1) $\quad u_i \to u$

(13.10.2) $\quad |A_i(x,t) - B(x,t)| \to 0$

(13.10.3) $\quad |A_i(x,t-t_i) - C(x,t)| \to 0$

(13.10.2)$'$ $\quad |A_i(x,t+t_i) - B'(x,t)| \to 0$

(13.10.3)$'$ $\quad |A_i(x,t) - C'(x,t)| \to 0.$

(See Figure 10.) Note that $\overline{\mathcal{M}}_u(a,c) = \emptyset = \overline{\mathcal{M}}_u(c',b)$ for generic u. (The virtual dimension of them is -1.) But "1-parameter family of -1-dimensional spaces is a finite set". Hence by a generic choice of σ_t^u and Q_u there exist a finite number of u's, for which $\overline{\mathcal{M}}_u(a,c)$ or $\overline{\mathcal{M}}_u(c',b)$ is nonempty.

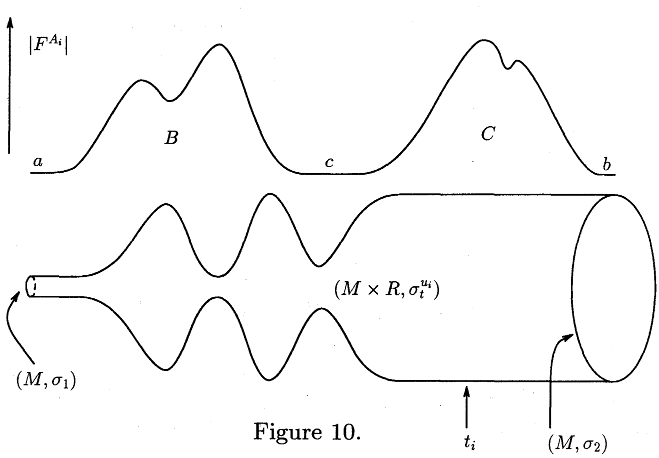

Figure 10.

Lemma 13.11. *Let $B \in \overline{\mathcal{M}}_u(a,c)$, $C \in \overline{\mathcal{M}}'_{(2)}(c,b)$. Then there exist $u(v) : (0,\infty) \to 0,1$, $A(v) \in \overline{\mathcal{M}}_{u(v)}(a,b)$ and $t(v), t'(v) \in \mathbf{R}$, such that*

$$\lim_{v \to \infty} u(v) = u \tag{13.11.1}$$

$$\lim_{v \to \infty} |A(v)(x, t - t(v)) - B(x,t)| = 0 \tag{13.11.2}$$

$$\lim_{v \to \infty} |A(v)(x, t + t'(v)) - C(x,t)| = 0. \tag{13.11.3}$$

Moreover, if A_i satisfies (13.10.1) - (13.10.3) *then $[A_i] = [A(v_i)]$ for large i. A similar statement holds for c'.*

The proof of Lemma 13.10 is similar to the proof in §9 and is omitted. Before proving Lemma 13.11 we complete the proof of Lemma 13.8 in the case when $s = 0$.

In this case, Theorem 13.9 implies

$$\begin{aligned} &\partial \mathcal{H}\overline{\mathcal{M}}(a,b) - \overline{\mathcal{M}}_1(a,b) - \overline{\mathcal{M}}_2(a,b) \\ &\qquad = \bigcup_{u,c} \overline{\mathcal{M}}_u(a,c) \times \overline{\mathcal{M}}'_{(2)}(c,b) \cup \bigcup_{u,c'} \overline{\mathcal{M}}'_{(1)}(a,c') \times \overline{\mathcal{M}}_u(c',b). \end{aligned}$$

We put

$$< \Phi a, c >= \sum_u \sharp \overline{\mathcal{M}}_u(a,c)$$

$$< \Phi c', b >= \sum_u \sharp \overline{\mathcal{M}}_u(c',b),$$

and

$$\Phi[a] = \sum_c < \Phi a, c > [c]$$

$$\Phi[c'] = \sum_b < \Phi c', b > [b].$$

Then we have

$$\varphi_1 - \varphi_2 = \partial \Phi - \Phi \partial.$$

Here φ_1 and φ_2 are the chain maps constructed using σ_t^1, Q_1 and σ_t^2, Q_2, respectively. This proves Lemma 13.8 when $s = 0$. The case when $s > 0$ can be proved by combining the methods of §§7 - 12 and Theorem 13.9. (In fact, the case $s > 0$ is simpler, because we do not have to use Kuranishi map in that case.)

Proof of Lemma 13.11. Let $\mathcal{D}^u_A$ be the operator obtained by linearizing the equation (13.4) for $\sigma_t = \sigma^u_t$ and $Q = Q_u$. By the generic choice of σ^u_t and Q_u we have $\dim \operatorname{Coker} \mathcal{D}^u_B = 1$. We consider the set X of the connections which is a standard form of type $(\{B\}, \{C\}, \epsilon, T)$. By Remark 4.6, there exists a positive number λ_0, such that, if $A \in X$ and if $|u - u'| < \epsilon$, then, there is exactly one eigenvalue of $\mathcal{D}^{u'}_A \mathcal{D}^{u'*}_A$ smaller than λ_0. Let Π_I be the orthonormal projection to this eigenspace, (which is isomorphic to $\mathbf{R}$). Put $\Pi_{II} = identify - \Pi_I$. For $A \in \mathcal{A}(a,b)$, $u' \in [0,1]$ we consider the equation

(13.12)
$$\begin{aligned}\Pi_{II}(F^A &- \widetilde{*}_{\sigma^{u'}_t} F^A - \chi_{u'}(-t)(\mathrm{grad}_{a_t} f_1 \wedge dt - *_{\sigma^{u'}_t} \mathrm{grad}_{a_t} f_1) \\ &- \chi_{u'}(t)(\mathrm{grad}_{a_t} f_2 \wedge dt - *_{\sigma^{u'}_t} \mathrm{grad}_{a_t} f_2) + Q_{u'}(A)) = 0.\end{aligned}$$

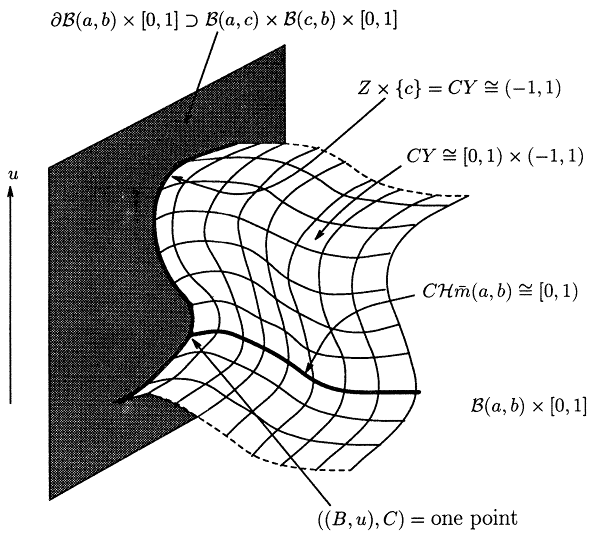

Figure 11.

The set of solutions of (13.12) divided by gauge transformations consists a 2-dimensional family Y. Let Z be the set of solutions of (13.12) for $A \in \mathcal{A}(a,c)$ and $u' \in [0,1]$. ($\dim Z = 1$.) Then, using the method of the proof of Theorem 7.1, we can compactify Y by adding $Z \times \{C\}$. Put $\mathcal{C}Y = Y \cup (Z \times \{C\})$. A neighborhood of $((B,u),C)$ in $\mathcal{C}Y$ is identified to $[0,1) \times (0,1)$, where $\{0\} \times (0,1) \subset Z \times \{C\}$. (See Figure 11.) For

(A, u'), we put

$$f(A,u') = \Pi_I(F^A - \widetilde{*}_{\sigma_t^{u'}} F^A - \chi_{u'}(-t)(\mathrm{grad}_{a_t} f_1 \wedge dt - *_{\sigma_t^{u'}} \mathrm{grad}_{a_t} f_1) \\ - \chi_{u'}(t)(\mathrm{grad}_{a_t} f_2 \wedge dt - *_{\sigma_t^{u'}} \mathrm{grad}_{a_t} f_2) + Q_{u'}(A)).$$

We identify the image of Π_I to $\mathbf{R}$ and regard f as a function. Using the decay estimate in §9 we can extend the function f to a *smooth* function on $\mathcal{C}Y$. The set of zero's of f is identified to a neighborhood of $((B,u),C)$ in $\mathcal{CH}\overline{\mathcal{M}}(a,b)$. We consider the restriction of f to $\{0\} \times (0,1) \subset Z$. If we choose g_t^u and Q_u generic, we may assume that the derivative of this restriction is nonzero at $((B,u),C) \in \{0\} \times (0,1)$. It follows from implicit function theorem that the zero of f in $\mathcal{C}Y$ is diffeomorphic to $[0,1)$ where $0 \in [0,1)$ corresponds to $((B,u),C)$. Lemma 13.11 follows immediately.

The proof of Lemma 13.8 is now complete.

Next we take another metric σ_3 and another perturbation f_3. Choose homotopies $\sigma_t^{1,2}$ and $\sigma_t^{2,3}$ from σ_1 to σ_2 and from σ_2 to σ_3. Choose also perturbations $Q_{1,2}$ and $Q_{2,3}$. Let $\varphi_{1,2}$ and $\varphi_{2,3}$ be the chain maps obtained by them, respectively.

Lemma 13.12. *We can find homotopy of metric $\sigma_t^{1,3}$ from σ_1 to σ_3 and a perturbation $Q_{1,3}$ such that the chain map $\varphi_{1,3} : C^s_{(1)} \to C^s_{(3)}$ satisfies*

$$\varphi_{3,2}\varphi_{1,2} = \varphi_{1,3}.$$

Proof. We put

$$\sigma_t^s = \chi(-t-s)\sigma_{t+2s}^{1,2} + \chi(t-s)\sigma_{t-2s}^{2,3}.$$

We shift the perturbation $Q_{1,2}$ by $2s$ to the negative direction and shift $Q_{2,3}$ by $2s$ to the positive direction. Let $Q_{1,3}^s$ be the sum of them. We consider the equation

(13.13)
$$F^A - \widetilde{*}_{\sigma_t^s} F^A - \chi(-t-s)(\mathrm{grad}_{a_t} f_1 \wedge dt - *_{\sigma_t^s} \mathrm{grad}_{a_t} f_1) \\ - \chi(t+s)\chi(s-t)(\mathrm{grad}_{a_t} f_2 \wedge dt - *_{\sigma_t^s} \mathrm{grad}_{a_t} f_2) \\ - \chi(t-s)(\mathrm{grad}_{a_t} f_3 \wedge dt - *_{\sigma_t^s} \mathrm{grad}_{a_t} f_3) + Q_{1,3}(A) = 0$$

Let $\overline{\mathcal{M}}(s;a,e)$ be the set of solutions of (13.13) divided by gauge transformations. Let $\overline{\mathcal{M}}_{1,2}(a,b)$ and $\overline{\mathcal{M}}_{2,3}(b,e)$ be the moduli spaces used in the definitions of $\varphi_{1,2}$ and $\varphi_{2,3}$ respectively. (Here $a \in Fl_1$, $b \in Fl_2$, $e \in Fl_3$.)

By using Remark 4.6, we can prove that the linearized equation for (13.13) is surjective for sufficiently large s. Consider the disjoint union of

$$\mathcal{C}\overline{\mathcal{M}}(s; a, e) \times \{s\} \qquad s \in [s_0, \infty)$$

and

$$\prod_{i=-1}^{k_0-1} \overline{\mathcal{M}}'_{(1)}(c_i, c_{i+1}) \times \overline{\mathcal{M}}_{1,2}(c_{k_0}, c_{k_0+1})$$
$$\times \prod_{i=k_0+1}^{k_1-1} \overline{\mathcal{M}}'_{(2)}(c_i, c_{i+1}) \times \overline{\mathcal{M}}_{2,3}(c_{k_1}, c_{k_1+1})$$
$$\times \prod_{i=k_1+1}^{k_2} \overline{\mathcal{M}}'_{(3)}(c_i, c_{i+1}) \times \{\infty\}.$$

(Here we put $a = c_{-1}$, $e = c_{k_2+1}$.) The later one is a compactification of $\cup_b \overline{\mathcal{M}}_{1,2}(a, b) \times \overline{\mathcal{M}}_{2,3}(b, e)$. Let $\mathcal{CC}\overline{\mathcal{M}}(a, e)$ be the union. Using this moduli space, the proof of the lemma goes in a way similar to the argument of §§7 - 13.

Now we are in the position to complete the proof of Theorem 1.14. Suppose $\sigma_1 = \sigma_3$, in Lemma 13.12. Then we can take a trivial homotopy $\sigma^{1,3} = \sigma_1$ and $Q_{1,3} = 0$. In this case, it is easy to see that the corresponding chain map is the identity map. Therefore by Lemma 13.12 and Lemma 13.8, $\varphi_{2,3}\varphi_{1,2}$ is chain homotopic to identity. (In this case $\varphi_{2,3} = \varphi_{2,1}$.) Thus the chain map $\varphi_{1,2}$ we constructed gives an isomorphisms on the homology groups. Also the isomorphism is canonical because of Lemma 13.8. The proof of Theorem 1.14 is now complete. The proof of the independence of the exact sequence 1.15 is similar.

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