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An Algebraic Character associated with the Poisson Brackets

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Dedicated to Professor Akio Hattori on his sixtieth birthday

§0. Introduction

Let N be a connected compact Kähler manifold, and $\mathrm{Aut}(N)$ the group of holomorphic automorphisms of N. Then if $c_1(N)_{\mathbb{R}} < 0$ or $c_1(N)_{\mathbb{R}} = 0$, the celebrated solution of Calabi's conjecture by Aubin [2] and Yau [19] asserts that N always admits an Einstein-Kähler metric. In the case $c_1(N)_{\mathbb{R}} > 0$, however, the existence problem is still open, and moreover a couple of obstructions to the existence are known. For instance, Futaki [7] introduced a complex Lie algebra homomorphism $F_N \colon H^0(N, \mathcal{O}(TN)) \to \mathbb{C}$ such that

(1) $F_N = 0$ if N admits an Einstein-Kähler metric;
(2) $F_N \neq 0$ for N in a fairly large family of compact Kähler manifolds (see also Koiso and Sakane [15]).

The purpose of this note is to give a systematic study of the obstruction F_N from a viewpoint of symplectic geometry. For instance, we relate it to the theorem of stationary phase of Duistermaat and Heckman [4], [5]. Another key to our approach is the following (cf. §6):

Theorem 0.1. *For any unipotent subgroup of* $\mathrm{Aut}(N)$*, the corresponding nilpotent Lie subalgebra of* $H^0(N, \mathcal{O}(TN))$ *sits in the kernel of* F_N*. Hence, if* $F_N \neq 0$*, then* N *admits a nontrivial biregular action of the algebraic group* $\mathbb{G}_m$ ($= \mathbb{C}^*$ *as a complex Lie group*).

Recall in particular that this theorem implies the identity

$$\psi(g) = |\det \phi(g)|^{\gamma} \qquad \text{for all } g \in \mathrm{Aut}(N), \tag{0.2}$$

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where ψ: $\mathrm{Aut}(N) \to \mathbb{R}_+$, ϕ: $\mathrm{Aut}(N) \to \mathrm{GL}_{\mathbb{C}}(V)$ and $\gamma \in \mathbb{Q}$ are just the same as in [10], so that γ is $2 \cdot (1+n)^{-1}$ or $(1+n)^{-1}$ according as the (complex) dimension n of N is even or odd. Moreover, taking the infinitesimal form of (0.2), we obtain the identity $F_N = \gamma\,(\det \circ \phi)_*$ on $H^0(N, \mathcal{O}(TN))$.

This note consists of rather independent seven sections including the first two introductory ones, and was written as an addendum to the preceding joint work [10] with A. Futaki. The author wishes to thank him and also Professor S. Kobayashi for valuable suggestions and encouragement.

§1. Notation and conventions

1.1. Throughout this note, we fix an n-dimensional complex connected manifold X. Let $|\mathcal{O}^*|^2$ be the multiplicative sheaf over X arising from the presheaf

$$U \to \{\, |f|^2;\ f \in H^0(U, \mathcal{O}^*)\,\}$$

with open subsets U of X. Then $H^0(U, |\mathcal{O}^*|^2) = \{\, \varphi \in C^\infty(U)_{\mathbb{R}}\,;\ \varphi > 0$ and $\partial\bar\partial \log \varphi = 0\,\}$, where $C^\infty(U)_{\mathbb{R}}$ denotes the set of all real-valued C^∞ functions on U. Let $\mathcal{Z}$ be the set of all real d-closed C^∞ (1,1)-forms on X, and $\mathcal{B}$ the space of all $\sqrt{-1}\partial\bar\partial\varphi$ with $\varphi \in C^\infty(X)_{\mathbb{R}}$. Put

$$H^{1,1}(X, \mathbb{R}) := \mathcal{Z}/\mathcal{B},$$

and by abuse of terminology, we say that $\omega, \omega' \in \mathcal{Z}$ are *cohomologous*, if $\omega - \omega' \in \mathcal{B}$. Note that the following isomorphism is more or less known (which I learned from Enoki and Tsunoda):

$$(1.1.1) \qquad H^{1,1}(X, \mathbb{R}) \cong H^1(X, \mathcal{O}^*/S^1)\,(= H^1(X, |\mathcal{O}^*|^2)).$$

By introducing somewhat new objects such as $\mathcal{L}_\zeta$ down below, we shall here give a differential geometric treatment of this isomorphism. Let ζ be an element of $H^1(X, |\mathcal{O}^*|^2)$ represented by a Čech 1-cocycle $\{\zeta_{ij}\}$ with respect to a sufficiently fine Stein cover $X = \cup_{i \in I} U_i$. We then have the corresponding $\mathbb{R}$-line bundle $\mathcal{L}_\zeta$ over X such that the restriction $\mathcal{L}_{\zeta|U_i}$ of $\mathcal{L}_\zeta$ over each U_i is identified with $U_i \times \mathbb{R}$ by

$$U_i \times \mathbb{R} \cong \mathcal{L}_{\zeta|U_i}, \qquad (x, s) \leftrightarrow s \cdot \mathbf{e}_i,$$

where $\mathbf{e}_i$ is a local C^∞ base for $\mathcal{L}_\zeta$ over U_i satisfying $\mathbf{e}_i(x) = \zeta_{ij}(x)\,\mathbf{e}_j(x)$, $x \in U_i \cap U_j$. Let $\mathcal{L}_\zeta^*$ be the dual $\mathbb{R}$-line bundle over X. Then a C^∞ section

$\mathbf{h}$ of $\mathcal{L}_\zeta^*$ over X is called a *norm* for $\mathcal{L}_\zeta$ if $\mathbf{h}_i := < \mathbf{h}, \mathbf{e}_i >$ is positive on U_i for each $i \in I$. Note that any norm $\mathbf{h}$ for $\mathcal{L}_\zeta$ is locally written as $\mathbf{h}_i \mathbf{e}_i^*$ on U_i, and the local data $\{\mathbf{h}_i\}_{i \in I}$ are characterized by the property $\mathbf{h}_i = \zeta_{ij} \cdot \mathbf{h}_j$. We now define the first Chern form $c_1(\mathcal{L}_\zeta, \mathbf{h})$ for $\mathcal{L}_\zeta$ with respect to $\mathbf{h}$ by

$$c_1(\mathcal{L}_\zeta, \mathbf{h}) := \frac{\sqrt{-1}}{2\pi} \bar{\partial}\partial \log \mathbf{h}_i.$$

Then, given ζ, the (1,1)-form $c_1(\mathcal{L}_\zeta, \mathbf{h})$ is easily shown to define a common cohomology class in $H^{1,1}(X, \mathbb{R})$ (denoted by $c_1(\mathcal{L}_\zeta)$) for all $\mathbf{h}$. Conversely, for any real d-closed C^∞ (1,1)-form ω on X, we can write

$$\omega_{|U_i} = \frac{\sqrt{-1}}{2\pi} \bar{\partial}\partial \log \mathbf{h}_i, \qquad i \in I,$$

for some $\mathbf{h}_i \in C^\infty(U_i)_{\mathbb{R}}$ with $\mathbf{h}_i > 0$. Then by setting $\zeta(\omega)_{ij} := \mathbf{h}_i / \mathbf{h}_j$, we have an element $\zeta(\omega) = \{\zeta(\omega)_{ij}\}$ of $H^1(X, |\mathcal{O}^*|^2)$, depending only on ω, such that $\{\mathbf{h}_i\}_{i \in I}$ form a norm $\mathbf{h}$ for the $\mathbb{R}$-line bundle $\mathcal{L}_{\zeta(\omega)}$ with

$$\omega = c_1(\mathcal{L}_{\zeta(\omega)}, \mathbf{h}). \tag{1.1.2}$$

Moreover, $\zeta(\omega_1) = \zeta(\omega_2)$ whenever ω_1 and ω_2 are cohomologous. Hence, denoting by $[\omega]$ the cohomology class in $H^{1,1}(X, \mathbb{R})$ represented by ω, we have the inverse

$$H^{1,1}(X, \mathbb{R}) \to H^1(X, |\mathcal{O}^*|^2), \qquad [\omega] \mapsto \zeta(\omega) \tag{1.1.3}$$

of the mapping: $H^1(X, |\mathcal{O}^*|^2) \ni \zeta \mapsto c_1(\mathcal{L}_\zeta) \in H^{1,1}(X, \mathbb{R})$. This then gives the isomorphism (1.1.1).

1.2. By a *log-harmonic* $\mathbb{R}$-line bundle over X, we mean a C^∞ $\mathbb{R}$-line bundle $\mathcal{L}$ over X written in the form $\mathcal{L} = \mathcal{L}_\zeta$ for some $\zeta \in H^1(X, |\mathcal{O}^*|^2)$. Now, let $p\colon \mathcal{L} \to X$, $p'\colon \mathcal{L}' \to X$ be arbitrary log-harmonic $\mathbb{R}$-line bundles over X. Then by abuse of terminology, a diffeomorphism $g\colon \mathcal{L} \to \mathcal{L}'$ is called *log-harmonic*, if the following conditions are satisfied:

(1) There exists a holomorphic automorphism $\tilde{g}$ of X such that the identity $\tilde{g} \circ p = p' \circ g$ holds.
(2) For each $x \in X$, the restriction $g_{|p^{-1}(x)}\colon p^{-1}(x) \to p^{-1}(\tilde{g}x)$ is an $\mathbb{R}$-linear isomorphism.
(3) $g(\mathbf{e}_i)/\mathbf{e}'_j \in H^0(\tilde{g}(U_i) \cap U_j, |\mathcal{O}^*|^2)$, for all $i, j \in I$, where $\{\mathbf{e}_i\}$ (resp. $\{\mathbf{e}'_i\}$) are the local bases for $\mathcal{L}$ (resp. $\mathcal{L}'$) as defined in 1.1.

Furthermore, $\mathcal{L}$ and $\mathcal{L}'$ are said to be *equivalent* (denoted by $\mathcal{L} \sim \mathcal{L}'$), if there exists a log-harmonic diffeomorphism $g\colon \mathcal{L} \to \mathcal{L}'$ such that the corresponding automorphism $\tilde{g}$ of X is id_X. By setting

$$\mathrm{Pic}_{\mathbb{R}}(X) := \{\text{all log-harmonic } \mathbb{R}\text{-line bundles over } X\}/\sim,$$

we have (see (1.1.1), (1.1.2), (1.1.3) above):

$$H^{1,1}(X,\mathbb{R}) \cong H^1(X,|\mathcal{O}^*|^2) \cong \mathrm{Pic}_{\mathbb{R}}(X), \qquad [\omega] \leftrightarrow \zeta(\omega) \leftrightarrow [\mathcal{L}_{\zeta(\omega)}],$$

where $[\mathcal{L}_{\zeta(\omega)}] \in \mathrm{Pic}_{\mathbb{R}}(X)$ is the class represented by $\mathcal{L}_{\zeta(\omega)}$. For a log-harmonic line bundle $\mathcal{L}$ over X, let $\mathcal{Z}_{\mathcal{L}}$ denote the set of all real d-closed C^∞ (1,1)-forms on X in the cohomology class $c_1(\mathcal{L})$. We then set

$$\begin{aligned} \mathbf{H}_{\mathcal{L}} &: \ \text{the set of all norms for } \mathcal{L},\\ \mathcal{S}_{\mathcal{L}} &:= \{\, \omega \in \mathcal{Z}_{\mathcal{L}};\ \omega \text{ is nowhere degenerate}\,\},\\ \mathcal{ES}_{\mathcal{L}} &:= \{\, \omega \in \mathcal{S}_{\mathcal{L}};\ \sqrt{-1}\bar{\partial}\partial\log(\omega^n) = r\omega \text{ for some } r \in \mathbb{R}\},\\ \mathcal{EK}_{\mathcal{L}} &:= \{\, \omega \in \mathcal{ES}_{\mathcal{L}};\ \omega \text{ is a Kähler form}\,\} \end{aligned}$$

where elements of $\mathcal{ES}_{\mathcal{L}}$ (resp. $\mathcal{EK}_{\mathcal{L}}$) are called *Einstein symplectic* (resp. *Einstein-Kähler*) *forms* on X. Note that, in view of (1.1.2), the mapping

$$\mathbf{H}_{\mathcal{L}} \to \mathcal{Z}_{\mathcal{L}}, \qquad \mathbf{h} \mapsto c_1(\mathcal{L},\mathbf{h}),$$

is surjective. Moreover, for elements $\mathbf{h}_1, \mathbf{h}_2$ in $\mathbf{H}_{\mathcal{L}}$, the identity $c_1(\mathcal{L},\mathbf{h}_1) = c_1(\mathcal{L},\mathbf{h}_2)$ holds if and only if $\mathbf{h}_1/\mathbf{h}_2 \in H^0(X,|\mathcal{O}^*|^2)$. Hence, whenever X is compact, $\mathbf{h}$ is uniquely determined by $c_1(\mathcal{L},\mathbf{h})$ up to constant multiple. Finally, a positive real C^∞ (n,n)-form Ω on X is called an *Einstein volume form* if $(\sqrt{-1}\bar{\partial}\partial\log\Omega)^n = r\Omega$ for some $r \in \mathbb{R}$. We put

$$\tilde{\mathcal{E}} : \ \text{the set of all Einstein volume forms on } X.$$

Obviously, $\tilde{\mathcal{E}}$ is nonempty if there exists a log-harmonic line bundle $\mathcal{L}$ over X with $\mathcal{ES}_{\mathcal{L}} \neq \phi$.

1.3. Let $X = \cup_{i\in I} U_i$ be a sufficiently fine Stein cover, and L a holomorphic line bundle over X with transition functions θ_{ij} $(i,j \in I)$. To this L, we can naturally associate the Čech cohomology class $\{\theta_{ij}\} \in H^1(X,\mathcal{O}^*)$. Put $\zeta := \{|\theta_{ij}|^2\} \in H^1(X,|\mathcal{O}^*|^2)$, and denote by $L_{\mathbb{R}}$ the corresponding $\mathbb{R}$-line bundle $\mathcal{L}_\zeta$ over X. Let $\mathcal{H}_L$ be the set of all C^∞ Hermitian (fibre) metrics of L over X, and for each $h \in \mathcal{H}_L$, let $c_1(L,h)$ be the first Chern form for L with respect to h. We then have the map

$$\mathrm{ord}_{\mathbb{R}}\colon L \to L_{\mathbb{R}}, \qquad \ell \mapsto \mathrm{ord}_{\mathbb{R}}(\ell) := \ell\cdot\bar{\ell}$$

such that, for each $h \in \mathcal{H}_L$, there exists a unique norm (denoted by $h_{\mathbb{R}}$) for $L_{\mathbb{R}}$ satisfying the following conditions:

(1) $h(\ell, \ell) = h_{\mathbb{R}}(\mathrm{ord}_{\mathbb{R}}(\ell))\,(= h_{\mathbb{R}}(\ell \cdot \bar{\ell})), \qquad \ell \in L.$

(2) The mapping $\mathcal{H}_L \ni h \mapsto h_{\mathbb{R}} \in \mathbf{H}_{L_{\mathbb{R}}}$ is a bijection.

(3) $c_1(L, h) = c_1(L_{\mathbb{R}}, h_{\mathbb{R}})$.

If $L = K_X^{-1}$, then we have a natural identification of $\mathcal{H}_{\mathcal{L}}$ with the space of volume forms on X. Moreover, in this case, the cohomology class $c_1((K_X^{-1})_{\mathbb{R}}) \in H^{1,1}(X, \mathbb{R})$ will be denoted simply by $c_1(X)_{\mathbb{R}}$.

1.4. From now on, until the end of this note, we fix an arbitrary log-harmonic $\mathbb{R}$-line bundle $\mathcal{L}$ over X with $\mathcal{S}_{\mathcal{L}} \neq \phi$. Consider moreover a complex Lie subgroup G of the group $\mathrm{Aut}(X)$ of holomorphic automorphisms of X such that the natural G-action on X lifts to a quasi-holomorphic G-action on $\mathcal{L}$, where an action of G on $\mathcal{L}$ is said to be *quasi-holomorphic*, if the following conditions are satisfied:

(1) Each element g of G induces a log-harmonic diffeomorphism of the $\mathbb{R}$-line bundle $\mathcal{L}$ (cf. 1.2).

(2) Let $\{\mathbf{e}_i\,; i \in I\}$ be the local bases for $\mathcal{L}$ as defined in 1.1. Then for each $i, j \in I$, the functions $g(\mathbf{e}_i)/\mathbf{e}_j$ ($g \in G$) are, wherever defined, written in the form $|w_{ij;g}|^2$ for some holomorphic functions $w_{ij;g}$ ($g \in G$) depending holomorphically on g.

In this note, we fix such a lifting once for all, and look at the left G-action

$$G \times \mathbf{H}_{\mathcal{L}} \to \mathbf{H}_{\mathcal{L}}, \qquad (g, \mathbf{h}) \mapsto g \cdot \mathbf{h} := (g^{-1})^* \mathbf{h},$$

where $((g^{-1})^*\mathbf{h})(\ell) := \mathbf{h}(g^{-1} \cdot \ell)$ for all $\ell \in \mathcal{L}$. Let $\mathfrak{g}$ be the complex Lie subalgebra of $H^0(X, \mathcal{O}(TX))$ associated with G in $\mathrm{Aut}(X)$. For each $\mathcal{Y} \in \mathfrak{g}$, we define the corresponding real vector field $\mathcal{Y}_{\mathbb{R}}$ on X by

$$\mathcal{Y}_{\mathbb{R}} := \mathcal{Y} + \overline{\mathcal{Y}}.$$

Let J be the complex structure of X, and put $\mathfrak{g}_{real} := \{\mathcal{Y}_{\mathbb{R}}; \mathcal{Y} \in \mathfrak{g}\}$. Then by sending $\mathcal{Y} \in \mathfrak{g}$ to $\mathcal{Y}_{\mathbb{R}} \in \mathfrak{g}_{real}$, we have the complex Lie algebra isomorphism $(\mathfrak{g}, \sqrt{-1}) \cong (\mathfrak{g}_{real}, J)$ with $\mathcal{Y} = \frac{1}{2}(\mathcal{Y}_{\mathbb{R}} - \sqrt{-1}J \cdot \mathcal{Y}_{\mathbb{R}})$. Now for each $(\mathcal{V}, \mathbf{h}) \in \mathfrak{g}_{real} \times \mathbf{H}_{\mathcal{L}}$, we define a C^∞ section $\mathcal{V}\mathbf{h}$ for $\mathcal{L}^*$ by

$$\mathcal{V}\mathbf{h} := \frac{\partial}{\partial t}\bigg|_{t=0} (\exp(t\mathcal{V}))^* \mathbf{h} \;\Big(= \frac{\partial}{\partial t}\bigg|_{t=0} (\exp(-t\mathcal{V})) \cdot \mathbf{h}\Big).$$

Denote by $\mathcal{L} \otimes_{\mathbb{R}} \mathbb{C}$ the complex line bundle on X obtained as the complexification of $\mathcal{L}$. Note then that

$$\mathcal{Y}\mathbf{h} := \frac{1}{2}(\mathcal{Y}_{\mathbb{R}} - \sqrt{-1}J \cdot \mathcal{Y}_{\mathbb{R}})\mathbf{h}, \tag{1.4.1}$$

is a global C^∞ section for $\mathcal{L}\otimes_{\mathbb{R}}\mathbb{C}$. If X is compact, we can further define a $\mathbb{C}$-linear map $T_{\mathcal{L},\mathbf{h}}\colon \mathfrak{g} \to \mathbb{C}$ by

$$(1.4.2) \qquad T_{\mathcal{L},\mathbf{h}}(\mathcal{Y}) := \int_X \mathbf{h}^{-1}(\mathcal{Y}\mathbf{h})\, c_1(\mathcal{L},\mathbf{h})^n, \qquad \mathcal{Y} \in \mathfrak{g}.$$

Then by (1.4.1), the corresponding real and imaginary parts are written in the form

$$\mathrm{Re}(T_{\mathcal{L},\mathbf{h}}(\mathcal{Y})) = \frac{1}{2}\int_X \mathbf{h}^{-1}(\mathcal{Y}_{\mathbb{R}}\mathbf{h})\, c_1(\mathcal{L},\mathbf{h})^n,$$
$$\mathrm{Im}(T_{\mathcal{L},\mathbf{h}}(\mathcal{Y})) = -\frac{\sqrt{-1}}{2}\int_X \mathbf{h}^{-1}((J\cdot\mathcal{Y}_{\mathbb{R}})\mathbf{h})\, c_1(\mathcal{L},\mathbf{h})^n.$$

Now, just by the same argument as in Donaldson [3; Proposition 6] (see also [17; Appendix A]), both $\mathrm{Re}(T_{\mathcal{L},\mathbf{h}})$ and $\mathrm{Im}(T_{\mathcal{L},\mathbf{h}})$ (and therefore $T_{\mathcal{L},\mathbf{h}}$) are independent of the choice of $\mathbf{h}$ in $\mathbf{H}_{\mathcal{L}}$. (Hence, $T_{\mathcal{L},\mathbf{h}}$ is often denoted by $T_{\mathcal{L}}$.) We later study this independence from quite different viewpoints (cf. §5).

1.5. In this note, by N, we always denote a compact complex connected manifold with a holomorphic line bundle L over it. Here in 1.5, we further set $X = N$, and assume that $\mathcal{L}$ is *quantized* by L, i.e.,

(1) $\mathcal{L} = L_{\mathbb{R}}$, and
(2) the natural G-action on X lifts to a holomorphic bundle action on L in such a way that the mapping $\mathrm{ord}_{\mathbb{R}}\colon L \to \mathcal{L}$ is G-equivariant.

Then by (3) of 1.3, the identity (1.4.2) and its real part are written in the form

$$T_{\mathcal{L},h_{\mathbb{R}}}(\mathcal{Y}) = \int_N h^{-1}(\mathcal{Y}h)\, c_1(L,h)^n,$$
$$\mathrm{Re}(T_{\mathcal{L},h_{\mathbb{R}}}(\mathcal{Y})) = \frac{1}{2}\int_N h^{-1}(\mathcal{Y}_{\mathbb{R}}h)\, c_1(L,h)^n = (\psi_L)_*(\mathcal{Y}),$$

for all $h \in \mathcal{H}_L$ and $\mathcal{Y} \in \mathfrak{g}$, where $(\psi_L)_*\colon \mathfrak{g} \to \mathbb{R}$ is the Lie algebra homomorphism defined in [10; §1]. We now assume that $\mathcal{L}$ is *anticanonically quantized*, i.e., $\mathcal{L}$ is quantized by the anticanonical line bundle $L = K_N^{-1}$ on which G acts naturally. Throughout this note, we denote such $\mathcal{L}$ by $\mathcal{A}$ (i.e., $\mathcal{A} := (K_N^{-1})_{\mathbb{R}}$). Then for each $\omega \in \mathcal{S}_{\mathcal{A}}$ (cf. 1.2), let $F_{N,\omega}\colon \mathfrak{g} \to \mathbb{C}$ be the $\mathbb{C}$-linear map defined by

$$F_{N,\omega}(\mathcal{Y}) = \int_N (\mathcal{Y} f_\omega)\, \omega^n,$$

where $f_\omega \in C^\infty(N)_{\mathbb{R}}$ is such that

$$\sqrt{-1}\bar{\partial}\partial \log(\omega^n) - 2\pi\omega = \sqrt{-1}\partial\bar{\partial} f_\omega.$$

Note that, for ω as above, there exists an element h of $\mathcal{H}_L$ (unique up to constant multiple) such that $c_1(L, h) = \omega$. Then by the same argument as in Futaki and Morita [12; Proposition 2.3] (see also [17; Appendix A]), we obtain:

$$(1.5.1) \qquad T_{\mathcal{A},h_{\mathbb{R}}}(\mathcal{Y}) = F_{N,\omega}(\mathcal{Y}), \qquad \mathcal{Y} \in \mathfrak{g}.$$

Since $T_{\mathcal{A}} = T_{\mathcal{A},\mathbf{h}}$ does not depend on the choice of $\mathbf{h}$ in $\mathbf{H}_{\mathcal{A}}$ (cf. 1.4), the identity (1.5.1) implies that $F_{N,\omega}$ is also independent of the choice of ω in $\mathcal{S}_{\mathcal{A}}$. Hence, $F_{N,\omega}$ is often written as F_N (which is nothing but the one in the introduction). Thus, $F_N = T_{\mathcal{A}}$ if $\mathcal{S}_{\mathcal{A}} \neq \phi$ (where $T_{\mathcal{A}}$ is defined even when $\mathcal{S}_{\mathcal{A}} = \phi$, though F_N is not). Later, we shall give a nontrivial example of an Einstein non-Kähler symplectic form (cf. § 3), and show the following slight generalization of a theorem of Futaki [7]:

Theorem 1.6. *The $\mathbb{C}$-linear map $T_{\mathcal{A}} (= F_N)\colon \mathfrak{g} \to \mathbb{C}$ is a complex Lie algebra homomorphism, i.e., $T_{\mathcal{A}}$ vanishes on $[\mathfrak{g}, \mathfrak{g}]$. Moreover, if $\tilde{\mathcal{E}}$ is nonempty, then $T_{\mathcal{A}} = 0$.*

This theorem, of course, includes the important case $G = \mathrm{Aut}(N)$, and is valid even for the case where $\mathcal{S}_{\mathcal{A}}$ is empty (though in our actual proof for the former half of 1.6, we assume $\mathcal{S}_{\mathcal{A}} \neq \phi$ for simplicity).

§2. Poisson brackets for complex manifolds

Throughout this section, we fix an element ω of $\mathcal{S}_{\mathcal{L}}$ (cf. 1.4) and write it locally in the form

$$\omega = \frac{\sqrt{-1}}{2\pi} \sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}}\, dz^\alpha \wedge dz^{\bar{\beta}},$$

with a system $(z^1, z^2, \ldots, z^n)$ of holomorphic local coordinates on X. Let $C^\infty(X)_{\mathbb{C}}$ (resp. $C^\infty(X)_{\mathbb{R}}$) be the set of all complex-valued (resp. real-valued) C^∞ functions on X. Then for this ω, we can define the associated Poisson bracket $[\ ,\ ]\colon C^\infty(X)_{\mathbb{C}} \times C^\infty(X)_{\mathbb{C}} \to C^\infty(X)_{\mathbb{C}}$ by

$$[\varphi, \psi] := \sum_{\alpha,\beta=1}^{n} g^{\bar{\beta}\alpha}(\varphi_\alpha \psi_{\bar{\beta}} - \varphi_{\bar{\beta}} \psi_\alpha), \qquad \varphi, \psi \in C^\infty(X)_{\mathbb{C}},$$

where $(g^{\bar{\beta}\alpha})$ is the inverse matrix of $(g_{\alpha\bar{\beta}})$ and

$$\psi_\alpha := \frac{\partial \psi}{\partial z^\alpha}, \; \varphi_{\bar{\alpha}} := \frac{\partial \varphi}{\partial z^{\bar{\alpha}}}, \; \dots .$$

In this section, we shall give a natural realization of $\mathfrak{g}$ as a complex Lie subalgebra of $C^\infty(X)_{\mathbb{C}}$, which will later turn out to play a crucial role in our approach. Now, to each $\psi \in C^\infty(X)_{\mathbb{C}}$, associate complex C^∞ vector fields $\mathcal{V}_\psi$ and $\mathcal{W}_\psi$ on X by

$$\mathcal{W}_\psi := -\sum_{\alpha,\beta=1}^{n} g^{\bar{\beta}\alpha}\psi_{\bar{\beta}}\frac{\partial}{\partial z^\alpha},$$
$$\mathcal{V}_\psi := \mathcal{W}_\psi - \overline{\mathcal{W}_{\bar{\psi}}}.$$

We first recall the following classical fact:

Fact 2.1. *$\sqrt{-1}C^\infty(X)_{\mathbb{R}}$ is a Lie subalgebra of $C^\infty(X)_{\mathbb{C}}$. Moreover, for any $\varphi, \psi, \eta \in C^\infty(X)_{\mathbb{C}}$, we have:*

(1) *$[\mathcal{V}_\varphi, \mathcal{V}_\psi] = \mathcal{V}_{[\varphi,\psi]}$ and $[\varphi, \psi] = \mathcal{V}_\varphi \psi$.*

(2) *If X is compact, then $\int_X [\varphi,\psi]\,\eta\,\omega^n = \int_X \varphi\,[\psi,\eta]\,\omega^n$.*

Let $\mathfrak{p}$ be the space of all functions $\psi \in C^\infty(X)_{\mathbb{C}}$ such that $\mathcal{W}_\psi$ is holomorphic on X. Then by comparing the holomorphic (1,0)-components of $[\mathcal{V}_\varphi, \mathcal{V}_\psi]$ and $\mathcal{V}_{[\varphi,\psi]}$, we immediately obtain:

Corollary 2.2. *$[\mathcal{W}_\varphi, \mathcal{W}_\psi] = \mathcal{W}_{[\varphi,\psi]}$ for all $\varphi, \psi \in \mathfrak{p}$, i.e., $\mathfrak{p}$ is a complex Lie subalgebra of $C^\infty(X)_{\mathbb{C}}$ and the $\mathbb{C}$-linear map: $\mathfrak{p} \ni \psi \mapsto \mathcal{W}_\psi \in H^0(X, \mathcal{O}(TX))$ is a complex Lie algebra homomorphism.*

Now, choose a norm $\mathbf{h} \in \mathbf{H}_{\mathcal{L}}$ for $\mathcal{L}$ such that $c_1(\mathcal{L}, \mathbf{h}) = \omega$ (cf. 1.2). (Note that such an $\mathbf{h}$ is unique up to constant multiple if X is compact.) Moreover, to each $\mathcal{Y} \in \mathfrak{g}$, we associate the function $\xi_{\mathbf{h}}(\mathcal{Y}) := \mathbf{h}^{-1}(\mathcal{Y}\mathbf{h}) \in C^\infty(X)_{\mathbb{C}}$. Then by setting $\tilde{\mathfrak{g}} := \text{Image}(\xi_{\mathbf{h}})$, we have:

Theorem 2.3. (1) *$\mathcal{W}_{\xi_{\mathbf{h}}(\mathcal{Y})} = \mathcal{Y}$ for all $\mathcal{Y} \in \mathfrak{g}$. In particular, $\tilde{\mathfrak{g}}$ is a subset of $\mathfrak{p}$.* (2) *The $\mathbb{C}$-linear map $\xi_{\mathbf{h}}: \mathfrak{g} \to \tilde{\mathfrak{g}}$ is a complex Lie algebra isomorphism.*

Proof. (1) Take a sufficiently fine Stein cover $X = \cup_{i \in I} U_i$ of X. Fixing an arbitrary $i \in I$, we express $\mathcal{Y}$ on U_i in the form:

$$\mathcal{Y} = \sum_{\gamma=1}^{n} a^\gamma \frac{\partial}{\partial z^\gamma} \qquad (a^\gamma \in H^0(U_i, \mathcal{O})),$$

where $(z^1, z^2, ..., z^n)$ is a system of holomorphic local coordinates on U_i. Let $\mathbf{e}_i$ be the local base for $\mathcal{L}$ over U_i as defined in 1.1, and write $\mathbf{h}$ as $\mathbf{h}_i \mathbf{e}_i^*$ on U_i for some $\mathbf{h}_i \in C^\infty(U_i)_{\mathbb{R}}$. An infinitesimal form of (2) of 1.4 yields

$$u := -(\mathcal{Y}\mathbf{e}_i)/\mathbf{e}_i \in H^0(U_i, \mathcal{O}).$$

Moreover by $c_1(\mathcal{L}, \mathbf{h}) = \omega$, we have $(\log \mathbf{h}_i)_{\gamma\bar{\beta}} = -g_{\gamma\bar{\beta}}$ for all β and γ. Since $\xi_{\mathbf{h}}(\mathcal{Y}) = \mathcal{Y}(\log \mathbf{h}_i) + u$, it now follows that:

$$\begin{aligned} \mathcal{W}_{\xi_{\mathbf{h}}(\mathcal{Y})} &= -\sum_{\alpha,\beta,\gamma} g^{\bar{\beta}\alpha}(a^\gamma(\log \mathbf{h}_i)_\gamma + u)_{\bar{\beta}} \frac{\partial}{\partial z^\alpha} \\ &= -\sum_{\alpha,\beta,\gamma} g^{\bar{\beta}\alpha} a^\gamma (\log \mathbf{h}_i)_{\gamma\bar{\beta}} \frac{\partial}{\partial z^\alpha} = \mathcal{Y}. \end{aligned}$$

(2) In view of (1) above, it suffices to show $[\xi_{\mathbf{h}}(\mathcal{Y}_1), \xi_{\mathbf{h}}(\mathcal{Y}_2)] = \xi_{\mathbf{h}}([\mathcal{Y}_1, \mathcal{Y}_2])$ for all $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathfrak{g}$. For simplicity, we put $\zeta_1 := \xi_{\mathbf{h}}(\mathcal{Y}_1)$ and $\zeta_2 := \xi_{\mathbf{h}}(\mathcal{Y}_2)$. Then by $\mathcal{Y}_1 \mathbf{h} = \zeta_1 \mathbf{h}$ and $\mathcal{Y}_2 \mathbf{h} = \zeta_2 \mathbf{h}$, we have:

$$\xi_{\mathbf{h}}([\mathcal{Y}_1, \mathcal{Y}_2])\mathbf{h} = [\mathcal{Y}_1, \mathcal{Y}_2]\mathbf{h} = \mathcal{Y}_1(\zeta_2 \mathbf{h}) - \mathcal{Y}_2(\zeta_1 \mathbf{h}) = (\mathcal{Y}_1\zeta_2 - \mathcal{Y}_2\zeta_1)\mathbf{h}.$$

This together with (1) above yields

$$\xi_{\mathbf{h}}([\mathcal{Y}_1, \mathcal{Y}_2]) = \mathcal{Y}_1\zeta_2 - \mathcal{Y}_2\zeta_1 = \mathcal{W}_{\zeta_1}(\zeta_2) - \mathcal{W}_{\zeta_2}(\zeta_1) = [\zeta_1, \zeta_2],$$

as required. Q.E.D.

Remark 2.4. Note that the kernel of the mapping

$$\mathcal{W}\colon \mathfrak{p} \to H^0(X, \mathcal{O}(TX)), \qquad \psi \mapsto \mathcal{W}_\psi$$

is exactly $H^0(X, \mathcal{O})$ (= $\mathbb{C}$ if X is compact). Suppose now that X is compact and that ω is a Kähler form. Then the Albanese map $a_X : X \to \mathrm{Alb}(X)$ of X naturally induces the Lie group homomorphism

$$\tilde{a}_X\colon \mathrm{Aut}^0(X) \to \mathrm{Aut}^0(\mathrm{Alb}(X))(\cong \mathrm{Alb}(X)),$$

where $\mathrm{Aut}^0(\cdot)$ denotes the identity component of $\mathrm{Aut}(\cdot)$. Let P_0 be the kernel of this homomorphism $\tilde{a}_X$, and $\mathfrak{p}_0$ the associated Lie subalgebra of $H^0(X, \mathcal{O}(TX))$. Then P_0 has a natural structure of a linear algebraic group (cf. Fujiki [6]), and by a theorem of Lichnerowicz [16], the image of the mapping $\mathcal{W}$ is exactly $\mathfrak{p}_0$. Hence,

$$\mathfrak{p}_0 \cong \mathfrak{p}/\,\mathrm{Ker}\ \mathcal{W} = \mathfrak{p}/\mathbb{C}.$$

Remark 2.5. In the case where $\mathcal{L}$ is anticanonically quantized with $X = N$, the Lie algebra homomorphism $\xi_{\mathbf{h}}$ was first observed by Futaki (through a definition quite different from ours) in the earlier version of [7] (see [11; 3.1]), though his original argument was later replaced by the new one in [7]. In this particular sense, our approach here is regarded as a natural generalization of forgotten Futaki's original approach to our log-harmonic bundle cases.

§3. Factorization of the character $T_{\mathcal{L},\mathbf{h}}$

For a fixed symplectic form ω in $\mathcal{S}_{\mathcal{L}}$, we choose an element $\mathbf{h}$ of $\mathbf{H}_{\mathcal{L}}$ such that $c_1(\mathcal{L}, \mathbf{h}) = \omega$ (cf. 1.2). In this section, assuming X to be compact, we shall express $T_{\mathcal{L},\mathbf{h}}$ as a composite of two Lie algebra homomorphisms and then prove Theorem 1.6. A nontrivial example of an Einstein non-Kähler symplectic form will also be given (cf. 3.3).

3.1. We here regard $\mathbb{C}$ as an abelian one-dimensional complex Lie algebra. Let $\lambda_\omega \colon C^\infty(X)_{\mathbb{C}} \to \mathbb{C}$ be the $\mathbb{C}$-linear map defined by

$$\lambda_\omega(\varphi) := \int_X \varphi\, \omega^n, \qquad \varphi \in C^\infty(X)_{\mathbb{C}},$$

and we endow $C^\infty(X)_{\mathbb{C}}$ with the natural structure of a complex Lie algebra coming from the Poisson bracket defined in §2. Then λ_ω is a complex Lie algebra homomorphism, i.e., λ_ω vanishes on the commutator subalgebra of $C^\infty(X)_{\mathbb{C}}$, since by (2) of Fact 2.1 applied to $\eta = 1$, the identity $\int_X [\varphi, \psi]\, \omega^n = 0$ holds for all $\varphi, \psi \in C^\infty(X)_{\mathbb{C}}$. Now for $\mathcal{Y} \in \mathfrak{g}$,

$$\lambda_\omega(\xi_{\mathbf{h}}(\mathcal{Y})) = \lambda_\omega(\mathbf{h}^{-1}(\mathcal{Y}\mathbf{h})) = T_{\mathcal{L},\mathbf{h}}(\mathcal{Y}) \qquad \text{(cf. (1.4.2))},$$

and hence we have:

Theorem 3.2. *$T_{\mathcal{L},\mathbf{h}} = \lambda_\omega \circ \xi_{\mathbf{h}}$, and in particular, $T_{\mathcal{L},\mathbf{h}}$ is a complex Lie algebra homomorphism.*

Until the end of this section, we set $X = N$, and assume moreover that $\mathcal{L}$ is anticanonically quantized (cf. 1.5). Then $\mathcal{L} = \mathcal{A}$, $L = K_N^{-1}$, and we can naturally regard each $\Omega \in \tilde{\mathcal{E}}$ (cf. 1.2) as an element, denoted by h_Ω, of $\mathcal{H}_L$ via

$$h_\Omega(\ell, \ell) := \pm < \Omega,\, (\sqrt{-1})^n\, \ell \wedge \bar{\ell} >, \qquad 0 \neq \ell \in L,$$

where $<\,,\,>$ denotes the ordinary contraction of forms by vectors, and the plus or minus sign is chosen in such a way that the right-hand side is always positive. We shall now prove Theorem 1.6:

Proof of 1.6. In view of 3.2, it suffices to show the latter half of 1.6. (For this latter half, our proof down below goes through even if $\mathcal{S}_{\mathcal{A}} = \phi$.) Let $\Omega \in \tilde{\mathcal{E}}$. Then there exists an $r \in \mathbb{R}$ such that

$$r\,\Omega = (\sqrt{-1}\bar{\partial}\partial \log(\Omega))^n = (2\pi c_1(L, h_\Omega))^n.$$

Now, for any $\mathcal{Y} \in \mathfrak{g}$, we denote by $\mathcal{Y}\Omega$ the complex Lie derivative $(d \circ i_{\mathcal{Y}} + i_{\mathcal{Y}} \circ d)\Omega$ $(= d(i_{\mathcal{Y}}\Omega))$ of Ω with respect to the holomorphic vector field $\mathcal{Y}$. Then, in view of 1.5, we have:

$$\begin{aligned} T_{\mathcal{A}}(\mathcal{Y}) &= \int_N h_\Omega^{-1}(\mathcal{Y}h_\Omega)\, c_1(L, h_\Omega)^n = \int_N \{(\mathcal{Y}\Omega)/\Omega\}\, c_1(L, h_\Omega)^n \\ &= \frac{r}{(2\pi)^n}\int_N \mathcal{Y}\Omega = \frac{r}{(2\pi)^n}\int_N d(i_{\mathcal{Y}}\Omega) = 0. \end{aligned}$$

Q.E.D.

Remark 3.3. The result of Koiso and Sakane [15] on the existence of non-homogeneous Einstein-Kähler metrics is true also for the Einstein symplectic case. In fact, in the definition of "tight pair" in [17; p.731], replace the condition (2) by

"ω is an Einstein symplectic form satisfying $\sqrt{-1}\bar{\partial}\partial \log(\omega^n) = \omega$",

and moreover, in place of the assumption "$\tilde{Y}$ is a Fano manifold" in [17; Theorem 10.3], we assume the following:

(1) $(c_1(W) + t\, c_1(L_1))^e[W] \neq 0$ whenever $-n'' < t < n'$.

(2) $c_1(\tilde{Y})^{m_0}[\tilde{D}_0] \neq 0 \neq c_1(\tilde{Y})^{m_\infty}[\tilde{D}_\infty]$, where $m_0 = \dim_{\mathbb{C}} \tilde{D}_0$ and $m_\infty = \dim_{\mathbb{C}} \tilde{D}_\infty$.

Then [17; Theorem 10.3] is valid if we further replace (b) in that theorem by "$\tilde{Y}$ admits an Einstein symplectic form". For instance, let C_0 be an irreducible nonsingular projective algebraic curve (defined over $\mathbb{C}$) of genus $g_0 \geqq 2$ and take an ample holomorphic line bundle L_0 over C_0 satisfying $K_{C_0} = L_0^{\otimes(2g_0-2)}$, so that $c_1(L_0)$ generates $H^2(C_0, \mathbb{Z})$. We then put $W := C_0 \times C_0$, and let $p\colon L_1 \to W$ be the holomorphic line bundle $pr_1^* L_0^{\otimes k} \otimes pr_2^* L_0^{\otimes -k}$ over W $(1 \leqq k \leqq 2g_0 - 3)$, where $pr_i\colon C_0 \times C_0 \to C_0$ denotes the natural projection to the i-th factor. Now, take the Einstein-Kähler form (associated with the Poincaré metric) ω_0 on C_0 in the cohomology class $-2\pi c_1(C_0)_{\mathbb{R}}$. Then $\omega := -(pr_1^*\omega_0 + pr_2^*\omega_0)$ is an Einstein symplectic form satisfying $\sqrt{-1}\bar{\partial}\partial \log \omega = \omega$. Note that ω_0 is naturally regarded as a Hermitian (fibre) metric for the line bundle $K_{C_0}^{-1}$. Hence, we have a natural Hermitian metric h for L_0 such that

$h^{\otimes(2-2g_0)}$ coincides with ω_0. We now define $\rho : L_1 \to \mathbb{R}$ by $\rho(\ell) := \|\ell'\|_h^k \|\ell''\|_h^{-k}$ for any $\ell = \ell'^{\otimes k} \otimes \ell''^{\otimes -k}$ in the fibre $(L_1)_{(x,y)}$ of L_1 over $(x, y) \in C_0 \times C_0$ with $\ell' \in (L_0)_x$ and $\ell'' \in (L_0)_y - \{0\}$. Let Y be the projective bundle $\mathbb{P}(E^*) := (E \text{ minus zero-section})/\mathbb{C}^*$, where E is the rank 2 vector bundle $\mathcal{O}_W \oplus L_1$ over W obtained as the direct sum of the trivial line bundle $\mathcal{O}_W$ and L_1. Now for simplicity, put $\kappa := k/(2g_0 - 2)$. Since

$$\int_{-1}^{1} t\,(c_1(W) + t\,c_1(L_1))^2\,dt = c_1(W)^2[W] \int_{-1}^{1} t\,(1 - \kappa^2 t^2)\,dt = 0,$$

we have $F_Y = 0$. Hence there exists an Einstein symplectic form on Y. Actually, let $\Phi(t)$ be the polynomial

$$\Phi(t) := -\int_{-1}^{t} s\,(1 - \kappa^2 s^2)\,ds, \qquad -1 \le t \le 1,$$

and define a C^∞ function $\lambda = \lambda(\rho)$ in ρ by

$$\rho^2 = \exp\left\{ -\int_0^{\lambda} \Phi(t)^{-1}(1 - \kappa^2 t^2)\,dt \right\}.$$

Then $\eta := \sqrt{-1}\Phi(\lambda(\rho))\,\rho^{-2}(p^*\omega)^2 \wedge \partial\rho \wedge \bar{\partial}\rho$ on L_1 extends to a volume form on Y, and it is easily checked that $\sqrt{-1}\bar{\partial}\partial \log \eta$ is an Einstein non-Kähler symplectic form on Y.

Remark 3.4. Let us set $X = N$, $G = \mathrm{Aut}(N)$, and consider the case where $\mathcal{L}$ is anticanonically quantized. Assume further that $\omega \in \mathcal{EK}_{\mathcal{L}}$, i.e., ω is an Einstein-Kähler form in the class $c_1(X)_{\mathbb{R}}$. We then express ω just as in §2, using holomorphic local coordinates, and put:

$$\Box_\omega = \sum_{\alpha,\beta=1}^{n} g^{\bar{\beta}\alpha} \frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}}.$$

Let $\mathrm{Ker}_{\mathbb{C}}(\Box_\omega + 1)$ (resp. $\mathrm{Ker}_{i\mathbb{R}}(\Box_\omega + 1)$) be the space of all functions φ in $C^\infty(N)_{\mathbb{C}}$ (resp. $\sqrt{-1}C^\infty(N)_{\mathbb{R}}$) such that $(\Box_\omega + 1)\varphi = 0$. Note that in our case, we have $\mathcal{L} = (K_N^{-1})_{\mathbb{R}}$, $\mathfrak{g} = H^0(N, \mathcal{O}(TN))$, and moreover ω is naturally regarded as an element (denoted by $\mathbf{h}(\omega)$) of $\mathbf{H}_{\mathcal{L}}$. Now, the well-known Matsushima's theorem [18] asserts that:

(1) $\{\,\mathcal{W}_\varphi;\ \varphi \in \mathrm{Ker}_{\mathbb{C}}(\Box_\omega + 1)\,\} = \mathfrak{g}$ (cf. §2);
(2) $\{\,\mathcal{W}_\varphi;\ \varphi \in \mathrm{Ker}_{i\mathbb{R}}(\Box_\omega + 1)\,\}$ coincides with the $\mathbb{C}$-vector space $\mathfrak{k}$ ($\subset \mathfrak{g}$) of all Killing vector fields on the Einstein-Kähler manifold (N, ω).

By 1.6 and 3.2, this can be stated in the following slightly stronger form:

$$\xi_{\mathbf{h}(\omega)}(\mathrm{Ker}_{\mathbb{C}}(\Box_\omega + 1)) = \mathfrak{g} \quad \text{and} \quad \xi_{\mathbf{h}(\omega)}(\mathrm{Ker}_{i\mathbb{R}}(\Box_\omega + 1)) = \mathfrak{k}.$$

In view of the identity $\lambda_\omega(\mathrm{Ker}_{\mathbb{C}}(\Box_\omega + 1)) = \{0\}$, this expression has the advantage that, for $\omega \in \mathcal{EK}_{\mathcal{L}}$ as above, the vanishing of F_N is quite naturally understood (see for instance Futaki [8] for similar observations). We can now summarize Matsushima's theorem and the vanishing of F_N for $\omega \in \mathcal{EK}_{\mathcal{L}}$ just in the following one commutative diagram:

$$\begin{array}{ccc} \mathfrak{k} & \hookrightarrow & H^0(N, \mathcal{O}(TN)) \\ \cong \big\downarrow \xi_{\mathbf{h}(\omega)} & \circlearrowleft & \cong \big\downarrow \xi_{\mathbf{h}(\omega)} \\ \mathrm{Ker}_{i\mathbb{R}}(\Box_\omega + 1) & \hookrightarrow & \mathrm{Ker}_{\mathbb{C}}(\Box_\omega + 1). \end{array}$$

§4. The moment map

Let $\omega \in \mathcal{S}_{\mathcal{L}}$, and choose an element $\mathbf{h}$ of $\mathbf{H}_{\mathcal{L}}$ such that $c_1(\mathcal{L}, \mathbf{h}) = \omega$. We then define the *moment map* $\mu^{\mathcal{L}}_{\mathfrak{g},\mathbf{h}}\colon X \to \mathfrak{g}^*$ associated with the quasi-holomorphic G-action (cf. 1.4) on $\mathcal{L}$ by

$$(\mu^{\mathcal{L}}_{\mathfrak{g},\mathbf{h}}(x))(\mathcal{Y}) = (\xi_{\mathbf{h}}(\mathcal{Y}))(x), \qquad x \in X,$$

where $\mathfrak{g}^*$ denotes the space of $\mathbb{C}$-linear functionals on $\mathfrak{g}$. Note that, in our definition of $\mu^{\mathcal{L}}_{\mathfrak{g},\mathbf{h}}$, there is no ambiguity of translations even if G is not semisimple. Let $\mathfrak{g}'$ be a (possibly real) Lie subalgebra of $\mathfrak{g}$ and G' be the corresponding connected Lie subgroup of G. If $\mathfrak{g}'$ is a complex Lie subalgebra of $\mathfrak{g}$, then we again have the moment map $\mu^{\mathcal{L}}_{\mathfrak{g}',\mathbf{h}}\colon X \to \mathfrak{g}'^*$ associated with the natural quasi-holomorphic G'-action on $\mathcal{L}$. For the remaining case where $\mathfrak{g}'$ is not a complex Lie subalgebra, we can still define $\mu^{\mathcal{L}}_{\mathfrak{g}',\mathbf{h}}$ as follows. In this case, let $\mathfrak{g}'^*$ be the space of all $\mathbb{C}$-linear functionals on $\{\mathfrak{g}'\}_{\mathbb{C}}$, where $\{\mathfrak{g}'\}_{\mathbb{C}}$ denotes the complex Lie subalgebra of $\mathfrak{g}$ spanned by $\mathfrak{g}'$. We then put $\mu^{\mathcal{L}}_{\mathfrak{g}',\mathbf{h}} := p' \circ \mu^{\mathcal{L}}_{\mathfrak{g},\mathbf{h}}$ $(= \mu^{\mathcal{L}}_{\{\mathfrak{g}'\}_{\mathbb{C}},\mathbf{h}})$, where $p'\colon \mathfrak{g}^* \to \mathfrak{g}'^*$ is the natural projection induced by $\mathfrak{g}' \hookrightarrow \mathfrak{g}$. Then, in any case, it is easy to check

$$(1) \qquad (\mu^{\mathcal{L}}_{\mathrm{Ad}(g)\mathfrak{g}',g\mathbf{h}}(gx))(\mathrm{Ad}(g)\mathcal{Y}) = (\mu^{\mathcal{L}}_{\mathfrak{g}',\mathbf{h}}(x))(\mathcal{Y})$$

for all $g \in G$, $x \in X$, $\mathcal{Y} \in \mathfrak{g}'$.

Remark 4.1. Suppose G' is such that $g' \cdot \mathbf{h} = \mathbf{h}$ for all $g' \in G'$ (and this condition is satisfied if both X and G' are compact and G' preserves the symplectic form ω). Then, in view of (1.4.1), we have the inclusion $\xi_{\mathbf{h}}(\mathfrak{g}') \subset \sqrt{-1}C^{\infty}(X)_{\mathbb{R}}$. Now by (1) above, $\mu^{\mathcal{L}}_{\mathfrak{g}',\mathbf{h}}\colon X \to \sqrt{-1}\mathfrak{g}'^{*}_{\mathbb{R}}$ is G'-equivariant, where $\mathfrak{g}'^{*}_{\mathbb{R}}$ denotes the space of $\mathbb{R}$-linear functionals on $\mathfrak{g}'$ endowed with the natural coadjoint G'-action. Hence, in this case, our $\mu^{\mathcal{L}}_{\mathfrak{g}',\mathbf{h}}$ is nothing but the ordinary moment map (cf. Guillemin and Sternberg [14]).

Definition 4.2. Recall that the (n,n)-form ω^n is naturally regarded as a signed measure on X, where either ω^n or $-\omega^n$ is a positive measure. If $\mu^{\mathcal{L}}_{\mathfrak{g},\mathbf{h}}$ is proper, then the push-forward $(\mu^{\mathcal{L}}_{\mathfrak{g},\mathbf{h}})_*(\omega^n)$ of the measure ω^n by $\mu^{\mathcal{L}}_{\mathfrak{g},\mathbf{h}}$ is a well-defined signed measure on $\mathfrak{g}^*$, and is called the *Duistermaat-Heckman's measure* associated with the moment map $\mu^{\mathcal{L}}_{\mathfrak{g},\mathbf{h}}$. Note that $(\mu^{\mathcal{L}}_{\mathfrak{g},\mathbf{h}})_*(\omega^n)$ is zero outside the closure of the image of $\mu^{\mathcal{L}}_{\mathfrak{g},\mathbf{h}}$. If X is compact, then we denote by $\theta_{\mathcal{L},\mathfrak{g},\mathbf{h}} \in \mathfrak{g}^*$ the barycenter

$$\frac{\int_{\chi\in\mathfrak{g}^*} \chi \cdot \{(\mu^{\mathcal{L}}_{\mathfrak{g},\mathbf{h}})_*(\omega^n)\}(\chi)}{\int_{\mathfrak{g}^*} (\mu^{\mathcal{L}}_{\mathfrak{g},\mathbf{h}})_*(\omega^n)} \quad \left(= \frac{\int_{\chi\in\mathfrak{g}^*} \chi \cdot \{(\mu^{\mathcal{L}}_{\mathfrak{g},\mathbf{h}})_*(\omega^n)\}(\chi)}{(c_1(\mathcal{L})^n[X])} \right)$$

of the Duistermaat-Heckman's measure $(\mu^{\mathcal{L}}_{\mathfrak{g},\mathbf{h}})_*(\omega^n)$. Now, we shall show the following (cf. [17]; see also Futaki [9]):

Theorem 4.3. *Suppose X is compact. Then we have the identity $\theta_{\mathcal{L},\mathfrak{g},\mathbf{h}} = (c_1(\mathcal{L})^n[X])^{-1}T_{\mathcal{L},\mathbf{h}}$, and in particular for any (possibly real) Lie subalgebras $\mathfrak{g}_1$, $\mathfrak{g}_2$ of $\mathfrak{g}$ with $\mathfrak{g}_1 \subset \mathfrak{g}_2$,*

$$\theta_{\mathcal{L},\mathfrak{g}_1,\mathbf{h}}(\mathcal{Y}) \; (= (c_1(\mathcal{L})^n[X])^{-1}T_{\mathcal{L},\mathbf{h}}(\mathcal{Y})) = \theta_{\mathcal{L},\mathfrak{g}_2,\mathbf{h}}(\mathcal{Y}) \qquad \textit{for all } \mathcal{Y} \in \mathfrak{g},$$

i.e., $p_{12}(\theta_{\mathcal{L},\mathfrak{g}_2,\mathbf{h}}) = \theta_{\mathcal{L},\mathfrak{g}_1,\mathbf{h}}$, where $p_{12}\colon \mathfrak{g}_2^* \to \mathfrak{g}_1^*$ denotes the natural projection induced by $\mathfrak{g}_1 \subset \mathfrak{g}_2$.

Proof. It suffices to show $\int_{\chi\in\mathfrak{g}^*} \chi \cdot \{(\mu^{\mathcal{L}}_{\mathfrak{g},\mathbf{h}})_*(\omega^n)\}(\chi) = T_{\mathcal{L},\mathbf{h}}$. Let $\mathcal{Y} \in \mathfrak{g}$. Then this required identity follows immediately from

$$\begin{aligned}\int_{\chi\in\mathfrak{g}^*} \chi(\mathcal{Y}) \cdot \{(\mu^{\mathcal{L}}_{\mathfrak{g},\mathbf{h}})_*(\omega^n)\}(\chi) &= \int_X \mu^{\mathcal{L}}_{\mathfrak{g},\mathbf{h}}(\mathcal{Y})\,\omega^n \\ &= \int_X \xi_{\mathbf{h}}(\mathcal{Y})\,\omega^n = T_{\mathcal{L},\mathbf{h}}(\mathcal{Y}).\end{aligned}$$

Q.E.D.

§5. $(\mathbb{C}^*)^r$-actions and the theorem of stationary phase

In this section, we consider the case where X is compact with $G = (\mathbb{C}^*)^r$ for some $0 < r \in \mathbb{Z}$. Let $K\,(\cong (S^1)^r)$ be the maximal compact subgroup of G, and $\mathfrak{k}$ the corresponding Lie subalgebra of $\mathfrak{g}$. Moreover, by $\mathcal{S}_{\mathcal{L}}^K$, $\mathbf{H}_{\mathcal{L}}^K$, we denote the set of all K-invariant elements in $\mathcal{S}_{\mathcal{L}}$, $\mathbf{H}_{\mathcal{L}}$, respectively. Then for any $\omega \in \mathcal{S}_{\mathcal{L}}^K$, there exists an $\mathbf{h} \in \mathbf{H}_{\mathcal{L}}^K$, unique up to constant multiple, such that $c_1(\mathcal{L}, \mathbf{h}) = \omega$. The purpose of this section is to obtain, as a corollary of the theorem of stationary phase, the independence of $T_{\mathcal{L},\mathbf{h}}$ on the choice of $\mathbf{h}$ in $\mathbf{H}_{\mathcal{L}}^K$ (cf. Remark 5.3).

Let X^G be the fixed point set of the G-action on X, and write X^G as a union $\cup_{i=1}^p X_i$ of the connected components. Recall the classical fact (due to Atiyah, Guillemin and Sternberg) that the image of the moment map $\mu_{\mathfrak{k},\mathbf{h}}^{\mathcal{L}}\colon X \to \sqrt{-1}\,\mathfrak{k}_{\mathbb{R}}^*$ is the convex hull of the finite set $\mu_{\mathfrak{k},\mathbf{h}}^{\mathcal{L}}(X^G)$ (see for instance [13]). Now, the isotropy representation of G along each X_i induces a natural infinitesimal action of $\mathfrak{g}$ on the normal bundle (denoted by E_i) of X_i in X. In particular, E_i splits into a direct sum of holomorphic vector subbundles (possibly with $n_i = 1$):

$$E_i = \oplus_j E_{ij}, \qquad j = 1, 2, \dots, n_i,$$

to which we can associate nontrivial characters $\chi_{ij} \in \mathfrak{g}^*$ such that every $\mathcal{Y} \in \mathfrak{g}$ acts on E_{ij} as scalar multiplication by $\sqrt{-1}\chi_{ij}(\mathcal{Y})$. Choose a K-invariant Hermitian connection for each E_{ij} and let Ω_{ij} be the corresponding curvature form. Then the theorem of stationary phase asserts that (cf. Duistermaat and Heckman [4], [5], Atiyah and Bott [1]):

Fact 5.1. *Let $\mathcal{Y} \in \mathfrak{k}$ be such that $\chi_{ij}(\mathcal{Y}) \neq 0$ for any i and j. Moreover, put $\phi_i := \prod_{j=1}^{n_i} \det\left\{(2\pi\sqrt{-1})^{-1}(\Omega_{ij} + \chi_{ij}(\mathcal{Y})\,\mathrm{id}_{E_{ij}})\right\}$. Then*

$$\int_X \exp\left(\frac{<\mu_{\mathfrak{k},\mathbf{h}}^{\mathcal{L}}, \mathcal{Y}>}{2\pi}\right)\frac{\omega^n}{n!} = \sum_{i=1}^{p}\int_{X_i} \exp\left(\frac{<\mu_{\mathfrak{k},\mathbf{h}}^{\mathcal{L}}, \mathcal{Y}>}{2\pi}\right)\phi_i^{-1}\exp(\omega).$$

We now observe that $H^0(X_i, |\mathcal{O}^*|^2) \cong \mathbb{C}^*$. Moreover, the restriction $\mathcal{L}_i$ of $\mathcal{L}$ to X_i admits a natural bundle action of G induced from $\mathcal{L}$. We then have real Lie group homomorphisms

$$\kappa_i\colon G \to \mathbb{R}_+, \qquad i = 1, 2, \dots, p,$$

such that every $g \in G$ acts on $\mathcal{L}_i$ as scalar multiplication by $\kappa_i(g)$. Let $(\kappa_i)_*\colon \mathfrak{g}_{real}\,(\cong \mathfrak{g}) \to \mathbb{R}$ be the corresponding Lie algebra homomorphism.

Then for any $x \in X_i$, $\mathcal{Y} \in \mathfrak{k}$ and $\mathbf{h} \in \mathbf{H}_{\mathcal{L}}^K$,

$$\{\mathbf{h}^{-1}(\mathcal{Y}\mathbf{h})\}(x) = \frac{1}{2}\{\mathbf{h}^{-1}(\mathcal{Y}_{\mathbb{R}} - \sqrt{-1}J \cdot \mathcal{Y}_{\mathbb{R}})\mathbf{h}\}(x) = \tfrac{1}{2}\sqrt{-1}(\kappa_i)_*(J \cdot \mathcal{Y}_{\mathbb{R}}),$$

where the left-hand side is nothing but $(\mu^{\mathcal{L}}_{\mathfrak{k},\mathbf{h}}(x))(\mathcal{Y})$. Hence, for each i, the image $\mu^{\mathcal{L}}_{\mathfrak{k},\mathbf{h}}(X_i)$ is a single point independent of $\mathbf{h}$. We now choose a general $\mathbb{R}$-basis $\{\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_r\}$ for $\mathfrak{k}$ such that $\chi_{ij}(\Sigma_{k=1}^r a_k \mathcal{Y}_k) \neq 0$ for any i, j when $0 \neq (a_1, a_2, \dots, a_r) \in \mathbb{Z}^r$. Further, define a system $(y_1, y_2, \dots, y_r)$ of real linear coordinates on $\sqrt{-1}\mathfrak{k}^*_{\mathbb{R}}$ by

$$y_k(\eta) = < (2\pi\sqrt{-1})^{-1}\eta, \mathcal{Y}_k >, \qquad \eta \in \sqrt{-1}\mathfrak{k}^*_{\mathbb{R}}.$$

We then have the following consequence of Fact 5.1:

Corollary 5.2. *The Duistermaat-Heckman's measure* $(\mu^{\mathcal{L}}_{\mathfrak{k},\mathbf{h}})_*(\omega^n)$ *on* $\sqrt{-1}\mathfrak{k}^*_{\mathbb{R}}$ *is independent of the choice of* $\omega \in \mathcal{S}_{\mathcal{L}}$ *and* $\mathbf{h} \in \mathbf{H}_{\mathcal{L}}$.

Proof. Let $(\omega', \mathbf{h}') \in \mathcal{S}_{\mathcal{L}} \times \mathbf{H}_{\mathcal{L}}$ be another pair such that $c_1(\mathcal{L}, \mathbf{h}') = \omega'$. Replacing $(\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_r)$ by its constant multiple, if necessary, we may assume that both $\mu^{\mathcal{L}}_{\mathfrak{k},\mathbf{h}}(X)$ and $\mu^{\mathcal{L}}_{\mathfrak{k},\mathbf{h}'}(X)$ are contained in $V := \{\eta \in \sqrt{-1}\mathfrak{k}^*_{\mathbb{R}};\ |y_k(\eta)| < 1 \text{ for all k}\}$. Since $\mu^{\mathcal{L}}_{\mathfrak{k},\mathbf{h}}(X_i) = \mu^{\mathcal{L}}_{\mathfrak{k},\mathbf{h}'}(X_i)$, and since $\int_{X_i} \phi_i^{-1} \exp(\omega) = \int_{X_i} \phi_i^{-1} \exp(\omega')$, the identity in Fact 5.1 implies

$$\begin{aligned}&\int_{X_i} \exp(\frac{1}{2\pi} < \mu^{\mathcal{L}}_{\mathfrak{k},\mathbf{h}'}, \Sigma_{k=1}^r m_k \mathcal{Y}_k) >) (\omega')^n \\ &= \int_{X_i} \exp(\frac{1}{2\pi} < \mu^{\mathcal{L}}_{\mathfrak{k},\mathbf{h}}, \Sigma_{k=1}^r m_k \mathcal{Y}_k >) \,\omega^n\end{aligned}$$

for all i and all $\mathbf{m} = (m_1, m_2, \dots, m_r) \in \mathbb{Z}^r - \{0\}$. Hence by setting $d\nu := (\mu^{\mathcal{L}}_{\mathfrak{k},\mathbf{h}})_*(\omega^n)$, $d\nu' := (\mu^{\mathcal{L}}_{\mathfrak{k},\mathbf{h}'})_*((\omega')^n)$, $\varphi_{\mathbf{m}} := \exp(\sqrt{-1}\,\Sigma_{k=1}^r m_k y_k)$, and $T := \sqrt{-1}\mathfrak{k}^*_{\mathbb{R}}$, we have:

$$\int_T \varphi_{\mathbf{m}}\, d\nu = \int_T \varphi_{\mathbf{m}}\, d\nu'$$

for all $\mathbf{m} \in \mathbb{Z}^r$. Since every continuous function on V of compact support is uniformly approximated by finite linear combinations of the $\varphi_{\mathbf{m}}$'s, we have $d\nu = d\nu'$, as required. Q.E.D.

Remark 5.3. By Theorem 4.3, $(c_1(\mathcal{L})^n[X])^{-1}T_{\mathcal{L},\mathbf{h}}$ is the barycenter $\theta_{\mathcal{L},\mathfrak{g},\mathbf{h}}$ of the Duistermaat-Heckman's measure $(\mu^{\mathcal{L}}_{\mathfrak{k},\mathbf{h}})_*(\omega^n)$. This together with Corollary 5.2 shows that $T_{\mathcal{L},\mathbf{h}}$ is independent of the choice of

$\mathbf{h}$ in $\mathbf{H}_{\mathcal{L}}^{K}$. Actually, 5.1 and 5.2 assert the stronger fact that $(\mu^{\mathcal{L}}_{\mathfrak{t},\mathbf{h}})_*(\omega^n)$ is completely determined by the data on the fixed point locus X^G and its normal bundle via the Fourier transform (see Guillemin and Sternberg [14; §34] for another characterization of such a measure).

Let G' be a connected linear algebraic group, defined over $\mathbb{C}$, and H' ($\cong (\mathbb{C}^*)^r$ for some r) its maximal torus. Then the Lie algebra $\mathfrak{g}'$ of G' is written as a direct sum of vector spaces

$$\mathfrak{g}' = \mathfrak{h}' + \sum_{\chi} \mathbb{C}\mathcal{Y}_{\chi}, \tag{5.4}$$

where $\mathfrak{h}'$ is the Cartan subalgebra corresponding to H', and we have a finite subset Δ of $\mathfrak{h}'^*$ such that each $\mathcal{Y}_\chi \in \mathfrak{g}'$ is related to $\chi \in \Delta$ by

$$\mathrm{Ad}(g)\mathcal{Y}_{\chi} = \chi(g)\mathcal{Y}_{\chi}, \qquad g \in H'.$$

Note that $\mathbb{C}\mathcal{Y}_\chi$'s are Lie algebras associated with 1-dimensional unipotent subgroups of G'. In §5, we obtained a fairly good description of the Lie algebra character $T_{\mathcal{L}}$ on Cartan subalgebras of $\mathfrak{g}$. Now, in view of the decomposition (5.4), it remains to study the behaviour of $T_{\mathcal{L}}$ on Lie algebras associated with unipotent subgroups of G, which we shall discuss in detail in the next section.

§6. $\mathbb{G}_a$-actions and the character $T_{\mathcal{L}}$

In this section, we assume that $X = N$ with $\mathcal{L}$ quantized by L (cf. 1.5), and let $c_1(L)_{\mathbb{R}} > 0$, so that N is projective algebraic. We moreover assume that G is a linear algebraic group, defined over $\mathbb{C}$, which acts biregularly on N. Let U be an arbitrary 1-dimensional unipotent subgroup of G (assuming such a subgroup exists), and by $\mathfrak{u} = \mathbb{C}\mathcal{Y}$, we denote the corresponding Lie subalgebra of $\mathfrak{g}$, where $\mathcal{Y}$ is a $\mathbb{C}$-base for $\mathfrak{u}$. We choose $0 \ll q \in \mathbb{Z}$ such that $L^{\otimes q}$ is generated by global sections. Let $\{\sigma_0, \sigma_1, \dots, \sigma_m\}$ be a $\mathbb{C}$-basis for $S := H^0(N, \mathcal{O}(L^{\otimes q}))$. Note that, via the U-action on L, the unipotent group U acts naturally on S, which induces an infinitesimal action of $\mathfrak{u}$ on S. Since U is unipotent, Jordan's normal form of $\mathcal{Y}$ allows us to assume without loss of generality that

(1) $\mathcal{Y}\sigma_0 = 0$;
(2) $\mathcal{Y}\sigma_i = e_i\sigma_{i-1}, \qquad 1 \le i \le m$,

where $e_i \in \mathbb{Z}$ is 0 or 1. For $0 < \varepsilon \in \mathbb{R}$, we define a Hermitian metric $h_\varepsilon \in \mathcal{H}_L$ for L by

$$h_\varepsilon := \left(\Sigma_{i=0}^m \, \varepsilon^{2i}\, \sigma_i\bar{\sigma}_i\right)^{-1} = \left\{\Sigma_{i=0}^m \,(\varepsilon^i\sigma_i)(\varepsilon^i\,\bar{\sigma}_i)\right\}^{-1}.$$

Now, the infinitesimal action of $\mathcal{Y}$ on h_ε (cf. (1.4)) is written as

$$\mathcal{Y}h_\varepsilon = -h_\varepsilon^2 \left\{\Sigma_{i=0}^m \varepsilon^{2i} (\mathcal{Y}\sigma_i)\, \bar{\sigma}_i\right\} = -\varepsilon\, h_\varepsilon^2 \left\{\Sigma_{i=1}^m (e_i\varepsilon^{i-1}\, \sigma_{i-1})(\varepsilon^i\bar{\sigma}_i)\right\}.$$

Put $v_i := e_i\varepsilon^{i-1}\sigma_{i-1}$ and $w_i := \varepsilon^i\sigma_i$. Then by the Cauchy-Schwarz inequality, the absolute value $|h_\varepsilon^{-1}\mathcal{Y}h_\varepsilon|$ of $h_\varepsilon^{-1}\mathcal{Y}h_\varepsilon$ is estimated as follows:

$$|h_\varepsilon^{-1}\mathcal{Y}h_\varepsilon|^2 = \varepsilon^2 \frac{|\Sigma_{i=1}^m v_i\bar{w}_i|^2}{(\Sigma_{i=0}^m w_i\bar{w}_i)^2} \le \varepsilon^2 \frac{|\Sigma_{i=1}^m v_i\bar{w}_i|^2}{(\Sigma_{i=1}^m v_i\bar{v}_i)(\Sigma_{i=1}^m w_i\bar{w}_i)} \le \varepsilon^2.$$

Note that $c_1(L, h_\varepsilon)$ is positive semi-definite as a pull-back of the Fubini-Study form on $\mathbb{P}^m(\mathbb{C})$. Therefore, for every $0 < \varepsilon \in \mathbb{R}$,

$$\begin{aligned} |T_{\mathcal{L}}(\mathcal{Y})| = |T_{\mathcal{L},(h_\varepsilon)_{\mathbb{R}}}(\mathcal{Y})| &\le \int_N |h_\varepsilon^{-1}\mathcal{Y}h_\varepsilon|\, c_1(L, h_\varepsilon)^n \\ &\le \int_N \varepsilon\, c_1(L, h_\varepsilon)^n = \varepsilon\, c_1(L)^n[N]. \end{aligned}$$

Let ε tend to 0. It then follows that $T_{\mathcal{L}}(\mathcal{Y}) = 0$, i.e., $T_{\mathcal{L}}$ vanishes on $\mathfrak{u}$. Thus we obtain:

Lemma 6.1. *For any unipotent subgroup U of G (of arbitrary dimension), the complex Lie algebra homomorphism $T_{\mathcal{L}}\colon \mathfrak{g} \to \mathbb{C}$ vanishes on the corresponding Lie subalgebra $\mathfrak{u}$ of $\mathfrak{g}$. In particular, if $T_{\mathcal{L}} \neq 0$, then G contains an algebraic subgroup isomorphic to $\mathbb{G}_m$ $(= \mathbb{C}^*)$.*

Let $N (= X)$, G, L be as above. We moreover use the notation $\beta \in \mathbb{Q}$, $\psi_L\colon G \to \mathbb{R}_+$ and $\rho_M\colon G \to \mathbb{G}_m$ in [10]. Then, under the same assumption as in [10; (5.1)], the following equality holds for all $g \in G$:

Formula 6.2. $\psi_L(g) = |\det(\rho_M(g))|^\beta \quad (g \in G).$

Proof. By the Chevalley decomposition, we can express the identity component G^0 of G as a semidirect product $R_0 \ltimes U_0$ of a reductive algebraic subgroup R_0 of G^0 and the unipotent radical U_0 of G_0. Let G_1, S_1 be the same as in [10; §5]. Since G is linear algebraic, G^0 coincides with the identity component of G_1, and hence R_0 is regarded as the identity component of S_1. In particular, by [10; (5.1)], the formula 6.2 is true for $g \in R_0$. Recall that the Lie algebra homomorphism $(\psi_L)_*\colon \mathfrak{g} \to \mathbb{R}$ associated with $\psi_L\colon G \to \mathbb{R}_+$ is nothing but $\mathrm{Re}(T_{\mathcal{L}})$ (cf. 1.5). Hence, by Lemma 6.1, ψ_L is trivial on U_0. Moreover, the algebraic group homomorphism $\rho_M\colon G \to \mathbb{G}_m$ is trivial on U_0. Therefore, the formula 6.2 is true for $g \in U_0$, and consequently, also for $g \in G^0$. Since for any $g \in G$

there exists $0 < \nu \in \mathbb{Z}$ such that $g^\nu \in G^0$, it now follows that

$$\psi_L(g) = (\psi_L(g^\nu))^{\frac{1}{\nu}} = |\det(\rho_M(g^\nu))|^{\frac{\beta}{\nu}} = |\det(\rho_M(g))|^{\beta},$$

as required. Q.E.D.

Note that, by taking the infinitesimal form of 6.2, we obtain the identity $T_{\mathcal{L}} = \beta\,(\det \circ \rho_M)_*$ on $\mathfrak{g}$. Now, consider the case where $\mathcal{L}$ is anticanonically quantized with $c_1(N)_{\mathbb{R}} > 0$. Then Lemma 6.1 and Formula 6.2 above immediately prove Theorem 0.1 and (0.2) in the introduction respectively. Finally, recall the following conjecture of Futaki:

Conjecture 6.3. *Let N be a compact complex connected manifold with $c_1(N)_{\mathbb{R}} > 0$. If moreover $F_N = 0$, then N admits an Einstein-Kähler metric.*

If 6.3 is affirmative, then Matsushima's theorem and 6.1 above show that any compact complex connected manifold N with $c_1(N)_{\mathbb{R}} > 0$ admits a nontrivial biregular $\mathbb{G}_m$-action unless $\mathrm{Aut}(N)$ is finite. At present, however, we can find neither strong reasons for 6.3, nor counterexamples to it.

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