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A Lower Bound of $L_p(1, \chi)$ for a Dirichlet Character χ

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Dedicated to Professor Kenkichi Iwasawa on his seventieth birthday

Let χ be a primitive Dirichlet character with conductor f, let $\tau(\chi)$ be the Gaussian sum for χ, and let $L_p(s, \chi)$ be the p-adic L-function associated with χ. Then, by the results of Brumer [3] and Leopoldt [8], the value $L_p(1, \chi)$ of this function at $s=1$ is not zero, and is given by the following formula:

$$L_p(1, \chi) = -\left(1 - \frac{\chi(p)}{p}\right)\frac{\tau(\chi)}{f}\sum_{a=1}^{f} \bar{\chi}(a) \log(1-\zeta^{-a}).$$

Since this value is related to the class numbers of cyclotomic fields, it is important to obtain a lower bound of $L_p(1, \chi)$.

Since the above formula expresses $L_p(1, \chi)$ in a linear form of p-adic logarithms of algebraic numbers, it is natural to study lower bounds of linear forms of p-adic logarithms of algebraic numbers by Baker's method. There are several results in this direction (cf. Spindzhuk [10], Kaufman [6], van der Poorten [9], *etc.*). But some results are not explicit enough for us, and some paper has (minor) mistakes so that the resulting constants must be modified (cf. Remark in 2–1). Since the values of the constants are essential for our purpose, we first study this problem. Then, calculating the relevant constants, we obtain a lower bound of $L_p(1, \chi)$.

In § 1, we improve a result of Gel'fond [4] on p-adic interpolations of p-adic normal functions by polynomials. In § 2, we calculate lower bounds of linear forms in p-adic logarithms of algebraic numbers by the method of Baker [2]. In § 3, we use the explicit formula of $L_p(1, \chi)$ and, by calculating the relevant constants, obtain a lower bound of $L(1, \chi)$.

The author first studied this problem by the method of Kaufman [6]. Then he heard the existence of van der Poorten [6] from M. Waldschmidt. So he used the method of Baker [2] and improved the lower bound. After writing this paper, the author met A. Baker and heard that Waldschmidt improved Baker's result in [11], and that a Chinese mathematician also

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studied our problem (cf. [12]).

We note that our lower bound is not best possible. It seems that we can improve the bound if we use the method of Waldschmidt [11] and calculate the relevant constants more carefully. But it seems very difficult to improve our lower bound by this method so much as examples show. Maybe, we must use the theory of $\mathbb{Z}_p$-extensions to get an essentially better lower bound.

§ 1. An improvement of a result of Gel'fond book

Let $\mathbb{Q}$ be the rational number field, let $\mathbb{R}$ be the real number field, and let $\mathbb{C}$ be the complex number field. We denote the standard valuation (the absolute value) of $\mathbb{C}$ by $|\ |_\infty$. Let p be a prime number, let $\mathbb{Z}_p$ be the ring of p-adic integers, let $\mathbb{Q}_p$ be the p-adic number field, and let $\mathbb{C}_p$ be the completion of the algebraic closure of $\mathbb{Q}_p$. We extend the standard p-adic valuation $|\ |_p$ of $\mathbb{Q}_p$ to $\mathbb{C}_p$, and denote the extended valuation by the same symbol $|\ |_p$. We fix embeddings of the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ into $\mathbb{C}$ and $\mathbb{C}_p$, and regard algebraic numbers as elements of $\mathbb{C}$ and $\mathbb{C}_p$.

We say that a formal power series $f(z)=\sum_{m=0}^{\infty} f_m z^m \in \mathbb{C}_p[[z]]$ is a *normal function* if the coefficients f_m satisfy $|f_m|_p \leq 1$ for any m and $f_m \to 0$ $(m \to \infty)$ (cf. Gel'fond [4], p. 119). Then $f(z)$ defines an analytic function on the unit disc $\{z \in \mathbb{C}_p;\ |z|_p \leq 1\}$. Further, for any point z_0 of the unit disk, $f(z-z_0)$ is also a normal function. Furthermore, if $f(z)$ is a normal function, and if $f(z)$ vanishes of order k at a point z_0 of the unit disk, then $f(z)/(z-z_0)^k$ is also a normal function (cf. ibid, for the properties of normal functions).

Let r_1 and r_2 be positive integers, let x_k $(0 \leq k \leq r_2-1)$ and $a_{s,k}$ $(0 \leq s \leq r_1-1,\ 0 \leq k \leq r_2-1)$ be elements of $\mathbb{C}_p$. Then Gel'fond has shown that the unique polynomial $P(z) \in \mathbb{C}_p[z]$ of degree $r_1 r_2 - 1$ such that

$$\left\{\left(\frac{d}{dz}\right)^s P\right\}(x_k)=a_{s,k} \qquad (0 \leq s \leq r_1-1,\ 0 \leq k \leq r_2-1)$$

is given by the following formula:

$$P(z)=\sum_{k=0}^{r_2-1}\sum_{n=0}^{r_1-1}\sum_{s=0}^{n} \frac{a_{r_1-n-1,k}}{(r_1-n-1)!\,(n-s)!} \times\left[\left(\frac{d}{dz}\right)^{n-s} \prod_{i\neq k}(z-x_k)^{-r_1}\right]_{z=x_k} (z-x_k)^{r_1-s-1} \prod_{i\neq k}(z-x_i)^{r_1}.$$

Let θ be a positive rational number. We assume

$$x_k = p^\theta k \qquad (0 \leq k \leq r_2 - 1).$$

Then we have (cf. *ibid.* p. 122, (82))

$$\frac{1}{(r_1-n-1)!\,(n-s)!}\left[\left(\frac{d}{dz}\right)^{n-s}\prod_{i\neq k}(z-x_k)^{-r_1}\right]_{z=x_k}$$
$$=\pm\frac{1}{(r_1-n-1)!}\sum_{\left\{\substack{\Sigma\nu_i=n-s\\ \nu_{k+1}=0}\right.}\frac{p^{-\theta\{(r_2-1)r_1-n+s\}}}{\{k!\,(r_2-k-1)!\}^{r_1}}\frac{(r_1+\nu_1-1)!}{(r_1-1)!\,\nu_1!}$$
$$\times\frac{(r_1+\nu_1-2)!}{(r_1-1)!\,(\nu_1-1)!}\cdots\frac{(r_1+\nu_{r_2}-1)!}{(r_1-1)!\,\nu_{r_2}!}\,\frac{1}{k^{\nu_1}(k-1)^{\nu_2}\cdots(k-r_2+1)^{\nu_{r_2}}}.$$

Since binomial numbers are integers, and since the inequality

$$|p^{n/(p-1)}|_p \leq |n!|_p \leq np|p^{n/(p-1)}|_p$$

holds, it is easy to study the normality of this function. Further, if z is a positive integer greater than r_2, then we have

$$\Big|\prod_{\substack{0\leq i\leq r_2-1\\ i\neq k}}(z-i)\Big|_p \leq |k!\,(r_2-k-1)!|_p \leq \left(\frac{r_2-1}{2}\right)^2 p^2 |p^{(r_2-1)/(p-1)}|_p.$$

Hence we obtain the following result (cf. *ibid.* p. 121, (80)):

Proposition 1. *Let the notation and assumption be as above. Let m be a rational number. We assume that*

$$|a_{s,k}/s!|_p \leq |p^m|_p$$

holds for any s and k. Then the interpolation polynomial $P(z)$ is normal if the inequality

$$r_1(r_2-1)\{\theta+1/(p-1)\}+(r_1-1)\log_e(r_2-1)/\log_e p \leq m$$

holds. Further, we have

$$|P(z)|_p \leq |p^{m-r_1(r_2-1)\{\theta+1/(p-1)\}-(r_1-1)\log_e(r_2-1)/\log_e p}|_p$$

for $|z|_p \leq 1$, and

$$|P(z)|_p \leq |p^{m-3r_1\log_e(r_2-1)/\log_e p - r_1\{2+1/(p-1)\}}|_p$$

for any $z \in \mathbb{Z}_p$.

Let $F(z)$ be a p-adic normal function. Let R and S be positive integers, and let m and θ be positive rational numbers. Let $Q(Z)$ be the unique polynomial of degree $RS-1$ such that

$$\left\{\left(\frac{d}{dz}\right)^s Q\right\}(p^\theta r)=\left\{\left(\frac{d}{dz}\right)^s F\right\}(p^\theta r) \qquad (0\leq r\leq R-1,\ 0\leq s\leq S-1).$$

We assume that

$$\left|\frac{1}{s!}\left\{\left(\frac{d}{dz}\right)^s F\right\}(p^\theta r)\right|_p \leq |p^m|_p$$

holds for $0\leq r\leq R-1$, $0\leq s\leq S-1$. Then it follows from Proposition 1 that $Q(z)$ is normal if the inequality

$$(R-1)S\{\theta+1/(p-1)\}+S\log_e(R-1)/\log_e p\leq m$$

holds. Further, if we write

$$F(z)=Q(z)+\prod_{0\leq r\leq R-1}(z-p^\theta r)^S G(z),$$

then the function $G(z)$ is also a normal function. Hence, if z is a p-adic integer, and if $Q(z)$ is a normal function, then we have

$$|F(z)|_p\leq \mathrm{Max}\,\{|p^{m-3S\log_e(R-1)/\log_e p-S\{2+1/(p-1)\}}|_p,\ |p^{(R-1)S\{\theta+1/(p-1)\}-S\log_e(R-1)/\log_e p-2S}|_p\}.$$

Hence we have the following:

Proposition 2. *Let $F(z)$ be a normal function, let m and θ be positive rational numbers, and let R and S be positive integers. We assume that the following two inequalities hold*:

$$\left|\frac{1}{s!}\left\{\left(\frac{d}{dz}\right)^s F\right\}(p^\theta r)\right|_p \leq |p^m|_p,$$

$$(R-1)S\{\theta+1/(p-1)\}+S\log_e(R-1)/\log_e p\leq m.$$

Then, for any $z\in\mathbb{Z}_p$, we have

$$|F(z)|_p\leq|p^{(R-1)S\{\theta+1/(p-1)\}-2S\log_e(R-1)/\log_e p-2S}|_p.$$

§ 2. Lower bounds of linear forms in p-adic logarithms of algebraic numbers

2-1. Notation and assumption. Let p be a prime number, and let $\mathbb{Q}$, $\overline{\mathbb{Q}}$, $\mathbb{Q}_p$, $\mathbb{C}$, $\mathbb{C}_p$, *etc.* be as in § 1. Let α_i $(i=1, 2, \cdots, n)$ be algebraic numbers. We assume that

$$|\alpha_i-1|_p<|2|_p \qquad (\text{if } p=2)$$
$$|\alpha_i-1|_p\leq|p|_p \qquad (\text{if } p\geq 3)$$

holds for any i. Then it follows that $|\alpha_i-1|_p<|p^{1/(p-1)}|_p$. Hence the p-adic logarithmic function $\log(z)=\sum_{n\geq 1}(-1)^{n+1}(z-1)^n/n$ converges at $z=\alpha_i$, and satisfies

$$|\log(\alpha_i)|_p=|\alpha_i-1|_p<|p^{1/(p-1)}|_p.$$

Further, for $|z|_p\leq 1$, the function $(\alpha_i)^z=\exp\{z\log(\alpha_i)\}$ is well-defined and satisfies $|(\alpha_i)^z-1|_p=|z|_p|\log(\alpha_i)|_p=|z|_p|\alpha_i-1|_p$, where $\exp(z)$ denote the p-adic exponential function $\exp(z)=\sum_{k\geqq 0}z^k/k!$.

Let β_i $(i=0, 1, \cdots, n)$ be algebraic numbers. We assume that each β_i satisfies $|\beta_i|_p\leq 1$. Let

$$\Lambda=\beta_0+\beta_1\log\alpha_1+\cdots+\beta_n\log\alpha_n.$$

If $|\beta_0|_p\geq|p^{1/(p-1)}|_p$, then $|\Lambda|_p=|\beta_0|_p$ holds. Hence *we assume* $|\beta_0|_p<|p^{1/(p-1)}|_p$.

Let $\mathbb{K}$ be the field generated over $\mathbb{Q}$ by the α_i $(i=1, \cdots, n)$ and β_i $(i=0, 1, \cdots, n)$. We *assume* that the height of each α_i is at most A_i $(A_i\geq 4)$, and the height of each β_i $(i=0, 1, \cdots, n)$ is at most B $(B\geq 4)$, and that the degree $[\mathbb{K}:\mathbb{Q}]$ is at most d. Let $A=\mathrm{Max}_{1\leq i\leq n} A_i$, and let

$$\Omega=\prod_{1\leq i\leq n}\log_e A_i \quad \text{and} \quad \Omega'=\prod_{1\leq i\leq n-1}\log_e A_i.$$

We understand that Ω' and $\log_e\Omega'$ denote 1 if $n=1$.

In this section, we obtain a lower bound of $|\Lambda|_p$ of the form $(B\Omega)^{-C\Omega\log_e\Omega'\log_e p}$.

Remark. In [2], Baker studied linear forms in logarithms of algebraic numbers of the form Λ. He obtained a lower bound of $|\Lambda|_\infty$ in the following form:

$$|\Lambda|_\infty>(B\Omega)^{-(16nd)^{200n}\Omega\log_e\Omega'}.$$

Further, van der Poorten has claimed in [9] that the same result holds over the field $\mathbb{C}_p$ under the conditions $|\alpha_i-1|_p<1$ $(i=1, \cdots, n)$, $|\beta_i|_p\leq 1$ $(i=0, 1, \cdots, n)$ and $|\beta_n|_p=1$. The arguments of [9] are essentially correct, but the result seems to be corrected.

In [9], the argument from p. 35, line 23 to p. 36, line 2 is not correct, because ξ is not contained in $\mathbb{K}$. Hence the resulting constant C seems to be modified so that C depends on p. Further, it seems that the normalities of the interpolation polynomials are not checked (cf. [9], p. 44, line 11–29). Since the value of the constant C is vitally important for us, we give an outline of the proof. Our constant depends on p, and it is bigger than Baker's constant.

2-2. Results under an assumption on $[\mathbb{K}(\alpha_1^{1/q}, \alpha_2^{1/q}, \cdots, \alpha_n^{1/q}): \mathbb{K}]$. We use the method of Baker [2], and use the result of § 1 to get estimates of interpolation functions.

Let n, α_i, β_i, Λ, $\mathbb{K}$, d, A_i, A, B, Ω, $\log_e \Omega'$, *etc.* be as in 2–1. We assume that $\beta_n = -1$. Let $k \in \mathbb{Z}$ be a parameter which depends on n and d. We *assume* that (1) $k \geq (30n^2d)^{6n}$ if $n \geq 2$ and $p \geq 3$, (2) $k \geq (50n^2d)^{6n}$ if $n \geq 2$ and $p=2$, (3) $k \geq (60d)^6$, $B \geq \log_e A \geq 6$, and $d \geq 6$ if $n=1$ and $p \geq 3$, and (4) $k \geq (100d)^6$, $B \geq \log_e A \geq 6$ and $d \geq 6$ if $n=1$ and $p=2$.

For any real number x, let $[x]$ denote the largest integer satisfying $[x] \leq x$. Put $\varepsilon = 1/(3n)$, $L = k\Omega \log_e \Omega'$, $h = L_{-1}+1 = [\log_e (B\Omega)]$, $L_0 = [k^{1-\varepsilon}\Omega]$, $L_i = [k^{-\varepsilon}L/\log_e A_i]$ $(1 \leq i \leq n)$.

Let q be a prime number satisfying $q \neq p$ and $7 \leq q \leq k^{1/\varepsilon}$, and let J be a non-negative integer such that $q^J \leq k\Omega' \log_e \Omega'$. This implies $q^J \leq k$ if $n=1$. Let $L_{-1}^{(J)} = L_{-1}$, $L_0^{(J)} = L_0$, and $L_j^{(J)} = [L_j/q^J]$ $(1 \leq j \leq n)$. Let $\nu(h)$ be the least common multiple of $1, 2, \cdots, h$, and, for any integers $l, m \geq 0$, let

$$\Delta(z; h, l, m) = \frac{1}{m!}\left(\frac{d}{dz}\right)^m \{(z+1)(z+2)\cdots(z+h)/h!\}^l.$$

Let $m_0, m_1, \cdots, m_{n-1}$ be non-negative integers, let $L^{(J)}$ denote the set of $n+2$-tuples $\lambda = (\lambda_{-1}, \lambda_0, \lambda_1, \cdots, \lambda_n)$ of integers satisfying $1 \leq \lambda_{-1} \leq L_{-1}^{(J)}$, $1 \leq \lambda_0 \leq L_0^{(J)}$, $1 \leq \lambda_1 \leq L_1^{(J)}$, $\cdots$, $1 \leq \lambda_n \leq L_n^{(J)}$. For any element λ of $L^{(J)}$, put $\gamma_i = \lambda_i + \lambda_n \beta_i$ $(1 \leq i \leq n-1)$,

$$\psi^{(J)}(\lambda, z) = \sum_{\mu_0=0}^{m_0} \binom{m_0}{\mu_0} \mu_0! \, \Delta\left(\frac{z}{q^J} + \lambda_{-1}; h, \lambda_0+1, \mu_0\right) (\lambda_n q^J \beta_0)^{m_0-\mu_0}.$$

Let $p(\lambda) = p^{(J)}(\lambda) \in \mathbb{Z} \cap \mathbb{C}_p$. We define two functions $f(z)$ and $g(z)$ on $\{z \in \mathbb{C}_p; |z|_p \leq 1\}$ by

$$\begin{aligned} f(z) &= f^{(J)}(z; m_0, m_1, \cdots, m_{n-1}) \\ &= \sum_{\lambda \in L^{(J)}} p^{(J)}(\lambda)\psi^{(J)}(\lambda, z) e^{\lambda_n \beta_0 z} \alpha_1^{\gamma_1 z} \alpha_2^{\gamma_2 z} \cdots \alpha_{n-1}^{\gamma_{n-1} z} \gamma_1^{m_1} \gamma_2^{m_2} \cdots \gamma_{n-1}^{m_{n-1}}, \\ g(z) &= g^{(J)}(z; m_0, m_1, \cdots, m_{n-1}) \\ &= \sum_{\lambda \in L^{(J)}} p^{(J)}(\lambda)\psi^{(J)}(\lambda, z) \alpha_1^{\lambda_1 z} \alpha_2^{\lambda_2 z} \cdots \alpha_n^{\lambda_n z} \gamma_1^{m_1} \gamma_2^{m_2} \cdots \gamma_{n-1}^{m_{n-1}}. \end{aligned}$$

Then we can prove the following proposition (cf. Baker [2], p. 16):

Proposition 3. *Let the notation and assumption be as above. We assume*

$$C \geq 0.73 \times \frac{h}{\log_e (B\Omega)} k^{3/2} \log_e p \qquad \text{if } n \geq 2,$$

and

$$C \geq 0.67 \times \frac{h}{\log_e (B\Omega)} k^{3/2} \log_e p \qquad \text{if } n = 1.$$

Then there exist integers $p(\lambda) = p^{(J)}(\lambda) \in L^{(J)}$ *such that* (i) *the* $p(\lambda)$ *are not all zero,* (ii) *the absolute value of each* $p(\lambda)$ *is at most* $\exp(10^{-9}Lh)$ *if* $n \geq 2$, *and is at most* $\exp(10^{-6}Lh)$ *if* $n = 1$, *and* (iii)

$$g^{(J)}(l; m_0, m_1, \cdots, m_{n-1}) = 0$$

holds for any integer l *with* $1 \leq l \leq hk^{\varepsilon/2}q^J$ *and for any n-tuple* $(m_0, m_1, \cdots, m_{n-1})$ *of non-negative integers with* $m_0 + m_1 + \cdots + m_{n-1} \leq Lq^{-J}$.

An outline of the proof. Let k be as in the beginning of 2–2. If $n \geq 2$ and $p \geq 3$ (resp. if $n \geq 2$ and $p = 2$, resp. if $n = 1$ and $p \geq 3$, resp. if $n = 1$ and $p = 2$), then $k \geq (30n^2d)^{12} \geq (120)^{12}$, $h \geq [\log_e \{4 \times (120)^{12} \times (\log_e 4)^2 \times \log_e \log_e 4\}] = 58$, $hk^{\varepsilon/2} \geq 58 \times 120 = 6.96 \times 10^3$, $hk^{1/2} \geq 58 \times (120)^6 \geq 1.73 \times 10^{14}$ (resp. $k \geq (50n^2d)^{12} \geq (200)^{12}$, $h \geq [\log_e \{4 \times (200)^{12} \times (\log_e 4)^2 \times \log_e \log_e 4\}] = 63$, $hk^{\varepsilon/2} \geq 63 \times 200 = 1.26 \times 10^3$, $hk^{1/2} \geq 63 \times (200)^6 \geq 4.03 \times 10^{14}$, resp. $k \geq (60d)^6 \geq (360)^6$, $h \geq [\log_e \{6 \times (360)^6 \times 6\}] = 38$, $hk^{\varepsilon/2} \geq 38 \times 360 \geq 1.36 \times 10^4$, $hk^{1/2} \geq 38 \times (360)^3 \geq 1.77 \times 10^9$, resp. $k \geq (100d)^6 \geq (600)^6$, $h \geq [\log_e \{6 \times (600)^6 \times 6\}] = 41$, $hk^{\varepsilon/2} \geq 41 \times 600 = 2.46 \times 10^4$, $hk^{1/2} \geq 41 \times (600)^3 \geq 8.85 \times 10^9$). We note $1/\log_e 4 = 0.72134 \cdots \leq 0.73$ and $2/3 = 0.66666 \cdots \leq 0.67$.

We use these estimates of constants, and use the following inequality also:

$$x/\log_e x \geq y/\log_e y \qquad (x \geq y \geq e).$$

Then, following the arguments of Baker [2], Lemma 7, and using also the estimate of the constant $M/(N-M)$ in Siegel's lemma, we can prove the following lemma.

Lemma 1. *Proposition* 3 *holds for* $J = 0$.

Now, following the arguments of Baker [2], pp. 11–17, and using Proposition 2, we can prove Proposition 3 by induction on J. Namely, we assume that Proposition 3 holds for J. Then we can prove the following two lemmas:

Lemma 2. *Let the notation and assumption be as above. Then, for any integer I with $0 \leq I \leq 3n$, we have $g^{(J)}(l; m_0, m_1, \cdots, m_{n-1})=0$ for any integer l with $1 \leq l \leq hk^{(I+1)\varepsilon/2}q^J$ and for any n-tuple $(m_0, m_1, \cdots, m_{n-1})$ of non-negative integers with $m_0+m_1+\cdots+m_{n-1} \leq L(1-\varepsilon)^I q^{-J}$.*

Lemma 3. *Let the notation and assumption be as above. Then, for any integer l with $1 \leq l \leq hk^{\varepsilon/2}q^{J+1}$ and for any n-tuple $(m_0, m_1, \cdots, m_{n-1})$ of non-negative integers with $m_0+m_1+\cdots m_{n-1} \leq (1/6)Lq^{-J}$, we have $g^{(J)}(l/q; m_0, m_1, \cdots, m_{n-1})=0$.*

We assume that Proposition 3 holds for J. Then using Proposition 2 instead of using the complex contour integrals, we can prove Lemma 2. Since $(1-\varepsilon)^{3n} \geq e^{-1} \geq 1/3$, $g^{(J)}(l: m_0, m_1, \cdots, m_{n-1})=0$ holds for any integer l with $1 \leq l \leq hk^{(1+\varepsilon)/2}q^J$ and for any non-negative integers $m_0, m_1, \cdots, m_{n-1}$ with $m_0+m_1+\cdots+m_{n-1} \leq (1/6)Lq^{-J}$. Since $q \geq 7 > 6$, Proposition 3 follows easily from these two lemmas.

Note that in Proposition 2, there are two inequalities. The condition on C comes from the normality of the interpolation polynomials, and the condition on k comes from the other condition in Proposition 2.

Now we have proved Proposition 3. Hence, following the arguments of Baker [2], pp. 17–19, we can obtain a lower bound of $|\Lambda|_p$.

Let the notation and assumption be as in Proposition 3. Hence we assume (1) $k \geq (30n^2d)^{6n}$ if $n \geq 2$ and $p \geq 3$, (2) $k \geq (50n^2d)^{6n}$ if $n \geq 2$ and $p=2$, (3) $k \geq (60d)^6$, $B \geq \log_e A \geq 6$, and $d \geq 6$ if $n=1$ and $p \geq 3$, and (4) $k \geq (100d)^6$, $B \geq \log_e A \geq 6$ and $d \geq 6$ if $n=1$ and $p=2$. We also assume

$$C \geq 0.73 \times \frac{h}{\log_e(B\Omega)} \times k^{3/2} \log_e p \qquad \text{if } n \geq 2,$$

and

$$C \geq 0.67 \times \frac{h}{\log_e(B\Omega)} k^{3/2} \log_e p \qquad \text{if } n=1.$$

Then we have the following theorem:

Theorem 1. *Let $q=7$ or $q=11$ according as $p \neq 7$ or $p=7$. If $[\mathbb{K}(\alpha_1^{1/q}, \alpha_2^{1/q}, \cdots, \alpha_n^{1/q}): \mathbb{K}]=q^n$, and if $\Lambda \neq 0$, then we have*

$$|\Lambda|_p \geq (B\Omega)^{-C\Omega \log_e \Omega'}.$$

Let the notation and assumption be as in Theorem 1. Then

$$h/\log_e(B\Omega)\leq\log_e(Bk\Omega\log_e\Omega')/\log_e(B\Omega)$$
$$\leq\{\log(k)+\log_e(B\Omega\log_e\Omega)\}/\log_e(B\Omega)\leq 2+\log_e(k)/\log_e(B\Omega).$$

If $n\geq 2$ and $p\geq 3$, then $\log_e(B\Omega)\geq\log_e(4(\log_e 4)^2)\geq 2.03$ and $\log_e k\geq 12 \log_e 120\geq 57.4$. Hence it is enough to have

$$C\geq\frac{0.73}{2.03}\left(\frac{2\times 2.03}{57.4}+1\right)k^{3/2}\log_e k\log_e p.$$

Hence it is enough to have

$$C\geq 0.386k^{3/2}\log_e k\log_e p.$$

Put $k=(30n^2d)^{6n}$. Then $\log_e k\leq 12n\log_e(6nd)\leq 72n^2d(\log_e 12)/12\leq 15.0\,(30n^2d)$. Hence it is enough to have

$$C\geq 5.80\,(30n^2d)^{9n+1}\log_e p.$$

Similarly, if $n\geq 2$ and $p=2$, then it is enough to have

$$C\geq 0.383k^{3/2}\log_e k\log_e p.$$

Put $k=(50n^2d)^{6n}$. Then it is enough to have

$$C\geq 6.40(50n^2d)^{9n+1}\log_e p.$$

If $n=1$ and $p\geq 3$, then it is enough to have

$$C\geq 0.226k^{3/2}\log_e k\log_e p.$$

Put $k=(60d)^6$. Then it is enough to have

$$C\geq 2.22\times 10^{-2}(60d)^{10}\log_e p.$$

If $n=1$ and $p=2$, then it is enough to have

$$C\geq 0.223k^{3/2}\log_e k\log_e p.$$

Put $k=(100d)^6$. Then it is enough to have

$$C\geq 1.43\times 10^{-2}(100d)^{10}\log_e p.$$

2-3. Lower bounds of linear forms. Now we obtain a lower bound of the linear form $\Lambda=\beta_0+\beta_1\log\alpha_1+\cdots+\beta_n\log\alpha_n$ without the assumption on $[\mathbb{K}(\alpha_1^{1/q}, \alpha_2^{1/q}, \cdots, \alpha_n^{1/q})\colon \mathbb{K}]$. Namely, we modify the arguments of Baker [2], pp. 19–21, and prove the following theorem:

Theorem 2. *Let the notation and assumption be as in* 2–1. *We assume that* $A_n=\mathrm{Min}_{1\le i\le n} A_i$ *and* $\Omega'=\Omega/\log_e A_n$. *Then we have*

$$|\Lambda|_p\ge(B\Omega)^{-2^{n^3/6}(300nd)^{10n+7}\Omega\log_e\Omega'\log_e p}$$

for $n\ge 2$ *and* $p\ge 3$,

$$|\Lambda|_p\ge(B\Omega)^{-2^{n^3/6}(500nd)^{10n+7}\Omega\log_e\Omega'\log_e p}$$

for $n\ge 2$ *and* $p=2$,

$$|\Lambda|_p\ge(B\Omega)^{-(60d)^{17}\Omega\log_e p}$$

for $n=1$ *and* $p\ge 3$,

$$|\Lambda|_p\ge(B\Omega)^{-(100d)^{17}\Omega\log_e p}$$

for $n=1$ *and* $p=2$. *Further, if* $n\ge 2$ *and if* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *are multiplicatively independent, then we have*

$$|\Lambda|_p\ge(B\Omega)^{-2^{n(n+1)/2}(300nd)^{10n+7}\Omega\log_e\Omega'\log_e p}$$

for $n\ge 2$ *and* $p\ge 3$, *and*

$$|\Lambda|_p\ge(B\Omega)^{-2^{n(n+1)/2}(500nd)^{10n+7}\Omega\log_e\Omega'\log_e p}$$

for $n\ge 2$ *and* $p=2$.

Remark. If $\mathbb{K}$ contains $\exp(2\pi i/q)$ ($q=7$ or 11, and $q\ne p$), then the constants 300, 500 in the above formulas can be reduced to 30, 50, respectively.

Proof. Let A_i $(i=1, \cdots, n)$, $A, B, \mathbb{K}, d, \Omega, \Omega'$, etc. be as in 2–1. We note that, by our assumption $A_n=\mathrm{Min}_{1\le i\le n} A_i$, Ω' is the largest of the $\Omega/\log_e A_i$ $(1\le i\le n)$.

If $B<\log_e A$, then put $B_1=\log_e A$. Then

$$(B_1\Omega)^{-C\Omega\log_e\Omega'}\ge(B\Omega)^{-2C\Omega\log_e\Omega'}$$

holds. Hence, replacing the constant C by $2C$, we may assume $B\ge\log_e A$. If $\log_e A<nd$, then $\log_e A_i<nd$. Put $B_2=\mathrm{Max}(B, nd)$, $\Omega_2=(nd)^n$ and $\Omega_2'=(nd)^{n-1}$. We have $(B\Omega)^d\ge 4^d(\log_e 4)^{nd}\ge nd$. Since $\Omega>1$ and $\Omega\log_e\Omega'>1/2$, we have

$$(B_2\Omega_2)^{-2C\Omega_2\log_e\Omega_2'}>B_2^{-2C(n^2-1)(nd)^{n+1}}>(B\Omega)^{-4n(nd)^{n+2}C\Omega\log_e\Omega'}.$$

Hence, replacing the constant C by $4n(nd)^{n+2}C$, we may assume $B\ge\log_e A\ge nd$.

Let $q=7$ or 11 according as $p\neq 7$ or $p=7$. Let $\mathbb{K}_1=\mathbb{K}(\exp(2\pi i/q))$. Then $\mathbb{K}_1$ contains α_i ($i=1, \cdots, n$), β_i ($i=0, 1, \cdots, n$), and $\exp(2\pi i/q)$. Further, the degree $[\mathbb{K}_1 : \mathbb{Q}]$ is at most $[\mathbb{Q}(\exp(2\pi i/q) : \mathbb{Q}] \times [\mathbb{K} : \mathbb{Q}] \leq (q-1)d \leq d_1 = 10d$. Hence, replacing d by $10d$ if necessary, we may assume that $\mathbb{K}$ contains $\exp(2\pi i/q)$. We note that this reduction is not necessary if $n=1$.

Now we rearrange the order of the indices of the A_i, and assume that $A_1 \leq A_2 \leq \cdots \leq A_n$. We assume that $[\mathbb{K}(\alpha_1^{1/q}, \cdots, \alpha_m^{1/q}) : \mathbb{K}] = q^m$ but $\alpha_{m+1}^{1/q}$ does not generate an extension of $\mathbb{K}(\alpha_1^{1/q}, \cdots, \alpha_m^{1/q})$ of degree q. Then, by the Kummer theory, there exists an element γ of $\mathbb{K}$ such that

$$\alpha_{m+1} = \alpha_1^{r_1} \alpha_2^{r_2} \cdots \alpha_m^{r_m} \gamma^q \qquad (0 \leq r_i < q).$$

By our assumption, $|\gamma^q - 1|_p \leq |p|_p$ if $p \geq 3$ and $|\gamma^q - 1|_p < |p|_p$ if $p=2$. Since $q \neq p$, the equation $X^q - 1 = 0$ is separable over a field of characteristic p. Hence the discriminant of this equation is not zero, and only one $\gamma \in \mathbb{K}$ can satisfy $X^q = \gamma^q$ and $|\gamma - 1|_p < 1$. Hence we take such an element γ. Then the condition $|\gamma - 1|_p \leq |p|_p$ ($p \geq 3$), $|\gamma - 1|_p < |p|_p$ ($p=2$) is satisfied.

As far as possible, we construct a sequence $\gamma = \gamma_1, \gamma_2, \gamma_3, \cdots$ of elements of $\mathbb{K}$ such that $\gamma_l = \alpha_1^{r_{l1}} \alpha_2^{r_{l2}} \cdots \alpha_m^{r_{lm}} \gamma_{l+1}^q$ ($0 \leq r_{li} < q$) and express γ_l as

$$\gamma_l = \alpha_{m+1}^{1/q^l} \alpha_1^{-s_{l1}/q^l} \cdots \alpha_m^{-s_{lm}/q^l} \qquad (0 \leq s_{li} < q^l).$$

Since the height of α_i is at most A_i, the absolute value of any conjugate of α_i or α_i^{-1} is at most $A_i + 1$. Hence the absolute value of any conjugate of γ_l or γ_l^{-1} is at most $(A_1+1) \cdots (A_m+1)(A_{m+1}+1) \leq 2^{m+1} A_1 \cdots A_m A_{m+1} \leq (2A)^n$. Since the height of α_i is at most A_i, the denominator of α_i or α_i^{-1} is at most A_i. Hence the denominator of γ_l or γ_l^{-1} is at most $A_1 \cdots A_m A_{m+1}$.

Put

$$H = \{4^{n^2} (10d)^{2n} \log_e (2A)^{2n}\}^{(2n+1)^2}.$$

By our assumption $B \geq \log_e A \geq nd$,

$$H \leq \{2^{(2n^2+9n+1)} d^{2n} \log_e A\}^{(2n+1)^2} \leq B^{81n^4}.$$

If the above sequence terminates with $q^l \leq H$, then we substitute γ_l for α_{m+1}. Then Λ is expressed as

$$\begin{aligned} \Lambda = \beta_0 + (\beta_1 + s_{l1}) \log \alpha_1 + \cdots + (\beta_m + s_{lm}) \log \alpha_m \\ + q^l \log \gamma_l + \cdots + \beta_n \log \alpha_n. \end{aligned}$$

The coefficients of this linear form are in $\mathbb{K}$ with heights at most $\{2B(B+H)\}^{10d}$. We repeat this substitution at most n-times until the condition $[\mathbb{K}((\alpha_1^\#)^{1/q}, \cdots, (\alpha_n^\#)^{1/q}) : \mathbb{K}] = q^n$ is satisfied with respect to the new $\alpha_i^\#$, or the above sequence does not terminate with $q^l \leq H$. Then the coefficients $\beta_i^\#$ of the resulting linear form

$$\Lambda^\# = \beta_0^\# + \beta_1^\# \log \alpha_1^\# + \cdots + \beta_n^\# \log \alpha_n^\#$$

are in $\mathbb{K}$ with heights at most $\{2B(B+nH)\}^{10d} \leq \{2B^2(1+B^{81n^4})\}^{10d} \leq B^\# = B^{840n^4d}$. Further, the heights of the $\alpha_1^\#, \alpha_2^\#, \alpha_3^\#, \cdots, \alpha_n^\#$ are at most

$$\begin{aligned}
&\{2(A_1+1)A_1\}^{10d} \leq \{2^2 A_1^2\}^{10d} \leq A_1^{4\times 10d}, \\
&\{2(A_1+1)(A_2+1)A_1A_2\}^{10d} \leq \{2^3 A_2^4\}^{10d} \leq A_2^{8\times 10d}, \\
&\{2(A_1+1)((A_1+1)(A_2+1))(A_3+1)A_1(A_1A_2)A_3\}^{10d} \leq \{2^5 A_3^8\}^{10d} \leq A_3^{16\times 10d}, \\
&\{2(A_1+1)((A_1+1)(A_2+1))((A_1+1)(A_1+1)(A_2+1)(A_3+1))(A_4+1) \\
&\quad \times A_1(A_1A_2)(A_1A_1A_2A_3)A_4\}^{10d} \leq \{2^9 A_4^{16}\}^{10d}, \cdots, \{2^{1+2^{n-1}} A_n^{2^n}\}^{10d} \leq A_n^{2^{n+1}10d}.
\end{aligned}$$

After these substitutions, $\Omega = (\log_e A_1)(\log_e A_2) \cdots (\log_e A_n)$ is replaced by $\Omega^\# \leq 2^{n(n+1)/2} \times 20d\Omega$ and Ω' is replaced by $\Omega'^\# \leq 2^{(n+1)n/2} 20d\Omega'$.

If the condition $[\mathbb{K}((\alpha_1^\#)^{1/q}, \cdots, (\alpha_n^\#)^{1/}{}_q) : \mathbb{K}] = q^n$ is satisfied after these substitutions, then we use Theorem 1. We rearrange the order of index and assume that $|\beta_n^\#|_p \geq |\beta_i^\#|_p$ holds for any i. We consider

$$-\Lambda^\#/\beta_n^\# = (-\beta_0^\#/\beta_n^\#) + (-\beta_1^\#/\beta_n^\#) \log \alpha_1^\# + \cdots - \log \alpha_n^\#.$$

Since the denominators of the $\beta_i^\#$ and $(\beta_i^\#)^{-1}$ are at most B, the denominators of the $\beta_i^\#/\beta_n^\#$ are at most B^2. Hence the heights of the $\beta_i^\#/\beta_n^\#$ are at most $(2B^2(B^\#+1)^2)^d \leq (B^2B^\#)^{2d} \leq B^{1684n^4d^2}$. Since

$$\begin{aligned}
&\{(B^{1684n^4d^2}) 2^{n(n+1)} 20d\Omega\}^{2^{n(n+1)/2} 20d\Omega \log_e (2^{(n+1)n/2} 20d\Omega')} \\
&\leq (B^{1690n^4d^2}\Omega)^{2^{n(n+1)/2} 10d\Omega(n+1)n \log_e (40d\Omega')} \\
&\leq (B\Omega)^{2^{n(n+1)/2}(30n^2d)^4 \Omega \log_e \Omega'},
\end{aligned}$$

Theorem 2 follows from the results of 2–2 in this case. Note that, in this case, the lower bound can be taken as in the second part of the theorem.

If the above sequence does not terminate with $q^l \leq H$, let l denote the least integer such that $q^l > H$. Then, by Lemma 6 of Baker [2], there exist integers $b', b_1', \cdots, b_{m+1}'$, not all zero, with absolute value at most H such that

$$b_1' \log \alpha_1 + \cdots + b_{m+1}' \log \alpha_{m+1} + b' \log \gamma_l = 0.$$

Hence we obtain

$$b_1'' \log \alpha_1 + \cdots + b_m'' \log \alpha_m + b_{m+1}'' \log \alpha_{m+1} = 0$$
$$(b_i'' = q^l b_i' - b' s_{li},\ b_{m+1}'' = q^l b_{m+1}' + b').$$

Here the coefficients b_i'' are integers with absolute values at most $2qH^2$. Furthermore, we can write

$$b_{m+1}'' \Lambda = \beta_0' + \beta_1' \log_e \alpha_1 + \cdots + \beta_n' \log_e \alpha_n$$
$$(\beta_0 = b_{m+1}'' \beta_0,\ \beta_i' = b_{m+1}'' \beta_i - b_i'' \beta_{m+1}).$$

Here $\beta_0', \beta_1', \cdots, \beta_n'$ are elements of $\mathbb{K}$ with heights at most

$$(4(2qH^2)(B+1)B)^{10d} \leq (2^4 q B^{2\times 840 n^4 d} B^{162 n^4})^{10d} \leq B^{21(30n^2d)^2},$$

and $\beta_{m+1} = 0$.

If $b_{m+1}'' \neq 0$, then $|b_{m+1}'' \Lambda|_p \leq |\Lambda|_p$. Hence we consider this new linear form $b_{m+1}'' \Lambda$ which does not contain α_{m+1}. Then the first part of the theorem follows by induction on n, because

$$\{B^{21(30n^2d)^2} 2^{n(n+1)} 20 d \Omega^\wedge\}^{2^{m(m+1)/2} 20 d \Omega^\wedge \log_e (2^{(n-1)n/2} 20 d \Omega'^\wedge)}$$
$$\leq (B\Omega)^{2^{(n-1)n/2} 6 (30n^2d)^4 \Omega \log_e \Omega'},$$

and $\sum_{1 \leq i \leq n} (i-1)i/2 = n^3/6 - n/3$. Here $\Omega^\wedge$ and $\Omega'^\wedge$ are constructed from Ω and Ω' by deleting $\log_e A_{m+1}$, and we have used estimates $B \leq B^{840 n^4 d}$, $\Omega^\wedge \leq 2^{m(m+1)/2} 20 d \Omega$, $\Omega' \leq 2^{(n-1)n/2} 20 d \Omega'$ because we must use this induction also after the above substitutions.

If $b_{m+1}'' = 0$, then $b' = 0$ because $q^l \geq H$. Hence $b_j'' \neq 0$ for some $j \leq m$, and, eliminating $\log \alpha_j$, the first part of the theorem can be proved by induction on n. Note that, if $\alpha_1, \alpha_2, \cdots, \alpha_n$ are multiplicatively independent, then we have proved that the sequence terminates with $q^l \leq H$. Hence the second part of the theorem also holds. Therefore we have completed the proof of Theorem 2. Note also that the remark after the theorem is clear from what we have seen.

§ 3. Calculation of constants

Let p be a prime number, and let $\mathbb{Q}_p$ be the p-adic number field, and let $\mathbb{C}_p$ be the completion of the algebraic closure of $\mathbb{Q}_p$. Let χ be a non-trivial primitive Dirichlet character with conductor f, and let $f = f_0 p^e$ $(f_0, e \in \mathbb{Z}, (f_0, p) = 1)$ be the decomposition of the conductor f of χ. Since x is primitive, e is either 0 or ≥ 2 if $p = 2$. We *assume* $\chi(-1) = 1$. Let $L_p(s, \chi)$ be the p-adic L-function associated with χ.

Let $\xi = \exp(2\pi i/f)$ be the primitive f-th root of unity, let $\tau(\chi) = \sum \chi(a) \xi^a$ be the Gaussian sum associated with χ, and let log: $\{z \in \mathbb{C}_p;$

$|z-1|_p<1\} \longrightarrow \mathbb{C}_p$ be the p-adic logarithmic function $\sum_{1\leq n<\infty}(-1)^{n-1}\times (z-1)^n/n$. We extend the function log to a function on $\{z \in \mathbb{C}_p; |z|_p=1\}$ by the functional equation $\log(z^m)=m\log(z)$. Then the function $L_p(s,\chi)$ does not vanish at $s=1$, and the value $L_p(1,\chi)$ is given by the following formula (cf. Brumer [3], Leopoldt [8]):

$$L_p(1,\chi)=-\left(1-\frac{\chi(p)}{p}\right)\frac{\tau(\chi)}{f}\sum_{1\leq a\leq f}\bar{\chi}(a)\log(1-\zeta^{-a}).$$

Since χ is not trivial, $\sum_{a=1}^{f}\chi(a)=0$. Further, since $\chi(-1)=1$, $\sum_{1\leq a\leq f/2}\chi(a)=0$. Hence

$$L_p(1,\chi)=-\left(1-\frac{\chi(p)}{p}\right)\frac{\tau(\chi)}{f}\sum_{1\leq a\leq f/2}\bar{\chi}(a)\log\left(\frac{1-\zeta^{-a}}{1-\zeta^{-1}}\,\frac{1-\zeta^{+a}}{1-\zeta^{+1}}\right).$$

Let $E(a)=(1-\zeta^{-a})/(1-\zeta^{-1})$ for any integer a. Since ζ is a root of unity, $\log\zeta=0$. Since $\chi(a)=0$ for $(a,f)\neq 1$, it is enough to consider only $E(a)$ for $1<a\leq f/2$, $(a,f)=1$. Then it is well-known that the $E(a)$ are units of the field $\mathbb{Q}(\zeta)$, and that they are multiplicatively independent. Since $\overline{E(a)}=\zeta^{a-1}E(a)$, the $E(a)\overline{E(a)}$ for $1<a\leq f/2$, $(a,f)=1$ are also multiplicatively independent.

Let

$$L(\chi)=\sum\bar{\chi}(a)\log(E(a)\overline{E(a)}),$$

where a runs over all integers satisfying $1<a\leq f/2$, $(a,f)=1$. Then we have

$$L_p(1,\chi)=-\left(1-\frac{\chi(p)}{p}\right)\frac{\tau(\chi)}{f}L(\chi).$$

If f is prime to p, then $|1-(\chi(p)/p)|_p=|p^{-1}|_p=p$. Otherwise, $|1-(\chi(p)/p)|_p=1$. Since $\tau(\chi)\overline{\tau(\chi)}=f$, $|\tau(\chi)/f|_p=|\overline{(\tau(\chi))}^{-1}|_p\geq 1$. Hence $|L_p(1,\chi)|_p\geq|L(\chi)|_p$. Hence, to obtain a lower bound of $L_p(1,\chi)$, it is enough to obtain a lower bound of $L(\chi)$.

Let $\varphi\colon \mathbb{N}\longrightarrow\mathbb{N}$ be the Euler function, and let a be an integer satisfying $1\leq a\leq f/2$, $(a,f)=1$. Since any conjugate of $1-\zeta^{-a}$ has the form $1-\zeta^{-b}$ with a positive integer b, the absolute value of any conjugate of $1-\zeta^{-a}$ is at most two. Since any conjugate of ζ^{-1} is also a primitive f-th root of unity, the absolute value of any conjugate of $1-\zeta^{-1}$ is at least $2\sin(\pi/f)\geq 4/f$. Hence the height of $E(a)E(a)$ is at most

$$\{2(2/2\sin(\pi/f))^2\}^{\varphi(f)/2}\leq(f/2^{1/2})^f.$$

Let K be the totally real subfield $\mathbb{Q}(\sin(2\pi/f))$ of $\mathbb{Q}(\zeta)$. Then $[K: \mathbb{Q}]=\varphi(f)/2\leq f/2$. Let $\mathfrak{p}$ be a prime ideal of K which divides p, and let μ and ν be the residue degree of p and the ramification index of p, respectively. Then $\mu, \nu\leq\varphi(f)/2\leq f/2$, and the norm $N(\mathfrak{p})$ of $\mathfrak{p}$ is given by p^{μ}. Further, since K is the totally real subfield of the cyclotomic field $\mathbb{Q}(\exp(2\pi/f_0p^e))$, the ramification index ν is equal to p^{e-2} if $p=2$ and $f_0=1$, and ν is equal to $p^{e-1}(p-1)$ otherwise.

Let ε be a unit of K. Then, by Fermat's theorem, $\eta=\varepsilon^{p^{\mu}-1}$ is congruent to 1 modulo $\mathfrak{p}$. Hence $|\eta-1|_p\leq|p^{1/\nu}|_p\leq|p^{1/p^{e-1}(p-1)}|_p$. Hence $|\eta^{p^{e-1}}-1|_p\leq|p^{1/(p-1)}|_p$. Hence $|\eta^{p^e}-1|_p<|p^{1/(p-1)}|_p$, and $<|p|_p$.

Let $\alpha_i=(E(i)\overline{E(i)})^{(p^{\mu}-1)p^e}$ $(1<i\leq f/2,\ (i,f)=1)$. Then the α_i are units of the fields $\mathbb{Q}(\sin(2\pi/f))$. Further, the height of each α_i is at most

$$(2^{-1/2}\sin(\pi/f))^{-\varphi(f)(p^{\mu}-1)p^e}\leq(f/2^{1/2})^{f^2pf/2}.$$

Let $\beta_i=\bar{\chi}(i)$ for any integer i with $(i,f)=1$. Then the β_i are $\varphi(f)/2$-th root of unity. Hence the height of each β_i is at most $2^{\varphi(\varphi(f)/2)}\leq 2^{f/2}$.

Let $q=7$ if $p\neq 7$, and let $q=11$ if $p=7$. Let $\mathbb{K}=\mathbb{Q}(\alpha_i, \beta_i;\ 1<i\leq f/2, (i,f)=1)$. Then $[\mathbb{K}:\mathbb{Q}]\leq(\varphi(f)/2)\times\varphi(\varphi(f)/2)\leq f^2/4$.

Let $d=f^2/4$, let $n=(\varphi(f)-2)/2\leq f/2$, let $B=2^{f/2}$, and let $A=A_i=(f/2^{1/2})^{f^2pf/2}$. Then we have $\Omega\leq(f^2p^{f/2}\log_e(f/2^{1/2}))^{f/2}$. If $f\neq 3, 4, 5, 8, 12$, then $n\geq 2$ and $f\geq 7$. Hence $2\Omega\leq(f^3p^{f/2})^{f/2}\leq p^{\{(3\log_e 7)/(7\log_e 2)+1/2\}ff/2}\leq p^{f^2}$. Hence

$$\begin{aligned}
&\{2^{f/2}(f^2p^{f/2}\log_e(f/2^{1/2}))^{f/2}\}^{-2(f/2)((f+2)/2)/2\{500(f/2)(f^2/4)\}^{10(f/2)+7}\times(f^2p^{f/2}\log_e(f/2^{1/2}))^{f/2}(f/2)\log_e(f^2p^{f/2}\log_e(f/2^{1/2}))\log_e p}\\
&\geq p^{-2((1/8)f^2+(121/4)f+42)f(15f+25)pf^2(\log_e p)^2}\\
&\geq p^{-p^{((9/8)f^2+(121/4)f+44)f(15f+25)}}\\
&\geq p^{-p^{7.2f^2+40.3f+44}}.
\end{aligned}$$

Therefore, by the second assertion of Theorem 2, we obtain the following theorem:

Theorem 3. *Let the notation and assumption be as above. We assume further that* $f\neq 3, 4, 5, 8, 12$. *Then we have*

$$\begin{aligned}
|L_p(1, \chi)|_p&\geq p^{-2((1/8)f^2+(121/4)f+42)f(15f+25)pf^2(\log_e p)^2}\\
&\geq p^{-p^{7.2f^2+40.3f+44}}.
\end{aligned}$$

References

[1] A. Baker, Transcendental number theory, Cambridge Univ. Press, Cambridge, 1975.
[2] ——, The theory of linear forms in logarithms, In: Transcendence theory, Advances and Applications: 1–27, Academic Press, London–New York–San Francisco, 1977.
[3] A. Brumer, On the units of algebraic number fields, Mathematika, **14** (1967), 121–124.
[4] A. O. Gel'fond, Transcendental & algebraic numbers, Dover Publ. Inc. New York, 1960 (English translation).
[5] K. Iwasawa, Lectures on p-adic L-functions, Ann. Math. Studies, **74** (1972).
[6] R. M. Kaufman, An estimate of linear forms in the logarithms of algebraic numbers in the p-adic metric, Vestnik Moskov. Matem., **26** (1971), 57–63 (English translation).
[7] T. Kubota and H. W. Leopoldt, Eine p-adische Theorie der Zetawerte, I, J. Reine Angew. Math., **214/215** (1964), 328–339.
[8] H. W. Leopoldt, Eine p-adische Theorie der Zetawerte, II, J. Reine Angew. Math., **274/275** (1975), 224–239.
[9] A. J. van der Poorten, Linear forms in logarithms in the p-adic case. In: Transcendence Theory, Advances and Applications: 29–57, Academic Press, London–New York–San Francisco, 1977.
[10] V. G. Spindzhuk, Estimates of linear forms with p-adic logarithms of algebraic numbers, Izv. Akad. Nauk BSSR. ser. Fiz.-Matem., **4** (1968), 5–14.
[11] M. Waldschmidt, A lower bound for linear forms in logarithms, Acta Arith., **37** (1980), 257–283.
[12] Kunrui Yu, Linear forms in the p-adic logarithms, to appear.

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