Littlewood's Formulas and their Application to Representations of Classical Weyl Groups

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Introduction

The reciprocity between the representations of the general linear groups and the symmetric groups is well known. For example, in I.G. Macdonald's book [M], this reciprocity is described as a ring isomorphism between the ring \( A \) of symmetric functions in countably many variables (see [M], [K-T]) and the graded ring \( R = \bigoplus_n R(\mathfrak{S}_n) \), where \( R(\mathfrak{S}_n) \) is the free \( \mathbb{Z} \)-module generated by the irreducible characters of the symmetric group of degree \( n \) and the multiplication in \( R \) is defined for \( f \in R(\mathfrak{S}_n) \) and \( g \in R(\mathfrak{S}_m) \) by \( f \cdot g = \text{ind}_{\mathfrak{S}_m}^{\mathfrak{S}_n}(f \times g) \). In an analogous manner, we define a graded ring \( R_w = \bigoplus_n R(W(B_n)) \) using the characters of the Weyl groups \( W(B_n) \) of type \( B_n \) and a homomorphism from this ring \( R_w \) to \( A \). This homomorphism clarifies the relationship between the representations of \( GL(n) \) and the rule of decomposition (into irreducible constituents) of the representations of \( \mathfrak{S}_n \) induced by an irreducible representation of \( W(B_n) \). In this procedure, Littlewood's formulas play a crucial role. Here, Littlewood's formulas mean the expansion formulas of the following four symmetric rational functions into Schur functions:

\[
\begin{align*}
(1) & \prod_{1 \leq i < j \leq n} (1 - z_i z_j)^{-1}, \\
(2) & \prod_{1 \leq i \leq j \leq n} (1 - z_i z_j)^{-1}, \\
(3) & \prod_{1 \leq i < j \leq n} (1 - z_i z_j), \\
(4) & \prod_{1 \leq i \leq j \leq n} (1 - z_i z_j).
\end{align*}
\]

These formulas are also essential in describing the relations between the representations of \( GL(n) \) and those of \( Sp(2n) \) and \( SO(n) \) (see [K-T]).

§ 1. Littlewood's formulas

The four rational functions listed in the introduction are all \( \mathfrak{S}_n \)-invariant (where \( \mathfrak{S}_n \) acts by the permutations of variables \( \{z_i\}_{i=1}^n \), There-
Therefore if we embed the rational functions (1), (2), (3), and (4) into the formal power series ring \( C[[z_1, z_2, \ldots, z_n]] \), they can be expressed as linear combinations (finite or infinite) of Schur functions \( \chi_{GL(n)}(\lambda)(z) \)'s. Here, \( \chi_{GL(n)}(\lambda)(z) = (z_1, z_2, \ldots, z_n) \) is the irreducible character of \( GL(n, C) \) corresponding to the Young diagram (or equivalently partition) \( \lambda \), restricted to the standard maximal torus \( T = \{ \text{diag}(z_1, z_2, \ldots, z_n) \} \). We must prepare a few notations first.

For a partition \( \kappa = (k_1, k_2, \ldots, k_n) \), \( 2\kappa = (2k_1, 2k_2, \ldots, 2k_n) \). For a distinct partition \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s) \) \((\alpha_1 > \alpha_2 > \cdots > \alpha_s \geq 1)\), \( \Gamma(\alpha) \) denotes the partition \( \Gamma(\alpha) = (\alpha_1 - 1, \alpha_2 - 1, \ldots, \alpha_s - 1, \alpha_1, \alpha_2, \ldots, \alpha_s) \), using the Frobenius notation. The Frobenius notation \((\alpha_1, \alpha_2, \ldots, \alpha_r | \beta_1, \beta_2, \ldots, \beta_t)\) expresses the Young diagram \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) whose diagonal consists of \( r \) squares and the \( \alpha_i, \beta_t \) \((1 \leq i \leq r)\) and the \( \lambda_t \) \((1 \leq t \leq n)\) are combined with the relations:

\[
\alpha_i = \lambda_i - i, \quad \beta_t = \lambda'_t - i, \quad 1 \leq i \leq r,
\]

where we put \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \). Here, \( \lambda' \) denotes the transposed Young diagram of \( \lambda \). In terms of Young diagrams, \((\alpha_1, \alpha_2, \ldots, \alpha_r | \beta_1, \beta_2, \ldots, \beta_t)\) is the diagram illustrated in Figure 1a. For example, \( \Gamma(3, 2) \) is the one in Figure 1b.

The following Lemma 1.1, 1)–4) was found by D.E. Littlewood (see [L, p. 238]). Under the setting of modern terminology, I.G. Macdonald [M, p. 45] gave the detailed proof of 1) and 2). But in [M, p. 46], he gave only an outline of the proof of 3) and 4). In view of the importance of this lemma, here we give the complete proof of 3) and 4).

**Lemma 1.1** (D.E. Littlewood).

\[
(1) \quad \frac{1}{\prod_{1 \leq i < j \leq n} (1 - z_i z_j)} = \sum_{f=0}^{\infty} \sum_{\substack{|\kappa| = f \ d(\kappa) \leq n}} \chi_{GL(n)}(\Gamma(2\kappa))(z),
\]
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\begin{equation}
\prod_{1 \leq i < j \leq n} \left(1 - z_iz_j \right) = \frac{1}{\det(\theta)} \sum_{\alpha \in \Lambda^+} \chi_{GL(n)}(\alpha)(z),
\end{equation}

\begin{equation}
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\end{equation}

\textbf{Proof of (3).} We shall use the denominator formula of H. Weyl for the Lie algebra \(\mathfrak{so}(2n, \mathbb{C})\). We define \(\mathfrak{so}(2n, \mathbb{C})\) to be \(\{X \in \mathfrak{gl}(2n, \mathbb{C}) \mid XJ_0 + J_0^t X = 0\}\), where \(J_0\) is the following matrix:

\[
J_0 = \begin{pmatrix}
0 & 1 & & \\
& 0 & 1 & \\
& & \ddots & 1 \\
& & & 0 \\
1 & & & & \ddots & 1 \\
& & & & & & \ddots & \circ & \\
& & & & & & & \ddots & \circ & \\
& & & & & & & & \ddots & \circ & \\
& & & & & & & & & \ddots & \circ & \\
\end{pmatrix}
\]

As a Cartan subalgebra we take \(\mathfrak{h} = \{H = \text{diag}(h_1, \ldots, h_n, -h_n, \ldots, -h_1)\}\). Let \(\varepsilon_i : \mathfrak{h} \to \mathbb{C}\) be defined by \(\varepsilon_i(H) = h_i\), and \(\Delta_p\) be the root system of \(\mathfrak{so}(2n, \mathbb{C})\) with respect to \(h\). Fix a set of positive roots \(\Delta_p^+ = \{\varepsilon_i \pm \varepsilon_j; i < j\}\) and let \(\rho_D\) denote the half sum of the positive roots, i.e.

\[
\rho_D = \frac{1}{2} \sum_{\alpha \in \Delta_p^+} \alpha = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + \varepsilon_{n-1}.
\]

Then the denominator formula of H. Weyl is given by

\[
\prod_{w \in \mathfrak{W}(\mathfrak{D}_n)} \det(w) e^{\rho_D} = \prod_{\alpha \in \Delta_D^+} (e^{\alpha/2} - e^{-\alpha/2}) = e^{\rho_D} \prod_{\alpha \in \Delta_D^+} (1 - e^{-\alpha}),
\]

where \(\mathfrak{W}(\mathfrak{D}_n)\) is the Weyl group of type \(\mathfrak{D}_n\) and “det” denotes the linear character of \(\mathfrak{W}(\mathfrak{D}_n)\) taking the determinant of the representation of \(\mathfrak{W}(\mathfrak{D}_n)\) on \(\mathfrak{h}_R^\mathfrak{R}\). We shall define \(\phi_i \in GL(\mathfrak{h}_R^\mathfrak{R})\), \(i = 1, 2, \ldots, n\) by \(\phi_i(\varepsilon_j) = \varepsilon_j\) if \(j \neq i\), \(\phi_i(\varepsilon_i) = -\varepsilon_i\). Then \(\mathfrak{W}(\mathfrak{D}_n) = \langle \mathfrak{S}_n, \phi_1 \phi_2 \phi_3 \cdots \phi_{i-1} \varepsilon_i \rangle\) where \(\mathfrak{S}_n\) acts on \(\mathfrak{h}_R^\mathfrak{R}\) by the permutations of the base elements \(\varepsilon_i\) of \(\mathfrak{h}_R^\mathfrak{R}\). \(\mathfrak{W}(\mathfrak{D}_n)\) has the following coset decomposition with respect to \(\mathfrak{S}_n\):

\[
\mathfrak{W}(\mathfrak{D}_n) = \sum_{1 \leq i_1 < \cdots < i_t \leq n} \mathfrak{S}_n \phi_{i_1} \phi_{i_2} \cdots \phi_{i_t},
\]
We put
\[ \rho_{D, i_1, i_2, \ldots, i_{kt}} = \phi_{i_1} \phi_{i_2} \cdots \phi_{i_{kt}} \rho_D \]
\[ = \rho_D - 2(n - i_1) \varepsilon_{i_1} - 2(n - i_2) \varepsilon_{i_2} - \cdots - 2(n - i_{kt}) \varepsilon_{i_{kt}}. \]
Since \( \phi_n(\rho_D) = \rho_D \), we have
\[
\sum_{w \in F(D_n)} \det(w) e^{w \rho_D} = \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} \sum_{w \in S_n} \det(w) e^{w \rho_D, i_1, i_2, \ldots, i_k}.
\]
We put \( e^{-t_i} = z_i \) (\( i = 1, 2, \ldots, n \)) in the denominator formula. Since
\[
e^{\rho_D} \prod_{a \in D^+} (1 - e^{-\rho}) = z_1^{(n-1)} z_2^{(n-2)} \cdots z_n^{-1} \prod_{1 \leq i < j \leq n} (1 - z_i z_j)(1 - z_i z_j^{-1})
\]
\[ = \prod_{1 \leq i < j \leq n} (1 - z_i z_j)(z_i^{-1} - z_j^{-1})
\]
\[ = \prod_{1 \leq i < j \leq n} (1 - z_i z_j) \times |z^{(n-1)}, z^{(n-2)}, \ldots, z^{-1}, 1|
\]
and
\[
\sum_{w} \det(w) e^{w \rho_{D, i_1, i_2, \ldots, i_k}}
\]
\[ = |z^{(n-1)}, z^{(n-2)}, \ldots, z^{n-t_1}, \ldots, z^{n-t_k}, \ldots, z^{-1}, 1|
\]
(the numbers \( i_t \) above the determinant signify the positions of the corresponding columns), we have
\[
\prod_{1 \leq i < j \leq n} (1 - z_i z_j)
\]
\[ = \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} \frac{|z^{(n-1)}, \ldots, z^{n-t_1}, \ldots, z^{n-t_k}, \ldots, z^{-1}, 1|}{|z^{(n-1)}, z^{(n-2)}, \ldots, z^{-1}, 1|} = **
\]
Multiplying both the denominator and the numerator on the right-hand side of the above equality by \((z_1 z_2 \cdots z_n)^{n-1}\) and permuting the columns, we have
\[
** = \prod_{1 \leq i_1 < \cdots < i_k \leq n-1} \frac{|z^{n-1}, \ldots, z^{n-s_1}, z^{n-1+s_2}, \ldots, z^{n-1+s_k}, \ldots, z, 1|}{|z^{n-1}, z^{n-2}, \ldots, z, 1|}
\]
where we have put \( s_1 = n - i_1, s_2 = n - i_2, \ldots, s_k = n - i_k \). Since \( i_k \leq n-1, 1 \leq s_k < s_{k-1} < \cdots < s_1 \leq n-1 \).

Claim.
\[
|z^{n-1}, \ldots, z^{n-s_1}, z^{n-1+s_2}, \ldots, z^{n-1+s_k}, \ldots, z, 1| \]
\[ \frac{|z^{n-1}, z^{n-2}, \ldots, z, 1|}{|z^{n-1}, z^{n-2}, \ldots, z, 1|} = (-1)^{i_t} \chi_{GL(n)}(\Gamma(s)),
\]
where \( s = (s_1, s_2, \ldots, s_k) \) and \( |s| = s_1 + s_2 + \cdots + s_k \).

**Proof of (3).** We use induction on \( k \).

If \( k = 1 \), the numerator on the left-hand side of the claim equals

\[
|z^{n-1}, \ldots, z^{n-1+s_1}, \ldots, z, 1|^{(s_1)}.
\]

If we exchange the columns, we have

\[
|z^{n-1}, \ldots, z^{n-1+s_1}, \ldots, z, 1|
= (-1)^{s_1} |z^{n-1+s_1}, z^{n-1}, \ldots, z^{n-s_1-2}, \ldots, 1|.
\]

Owing to H. Weyl's character formula (see [W, p. 201, Theorem 7.5B]) it follows that

\[
\frac{|z^{n-1}, \ldots, z^{n-1+s_1}, \ldots, z, 1|}{|z^{n-1}, \ldots, z, 1|} = (-1)^{s_1} \chi_{GL(n)}(\Gamma(s_1)).
\]

Assume that the claim holds for \( k - 1 \). If we put \( s' = (s_1, s_2, \ldots, s_{k-1}) \) and exchange the columns, we have

\[
\text{(the numerator of the claim)}
= (-1)^{s'} |z^{n-1+s_1}, \ldots, z^{n-1+s_{k-1}}, z^{n-1}, \ldots, z^{n-1+s_k}, \ldots, 1|^{(k+s_k)}.
\]

Moreover if we move the \((k+s_k)\)-th column to just behind the column \( z^{n-1+s_{k-1}} \), we have

\[
\text{(the numerator of the claim)}
= (-1)^{|s'|} |z^{n-1+s_1}, \ldots, z^{n-1+s_{k-1}}, z^{n-1+s_k}, z^{n-1}, \ldots, 1|.
\]

In the above determinant, we denote the set of exponents of \( z \) by

\[
I_s = (n-1+s_1, \ldots, n-1+s_k, n-1, \ldots, \overbrace{n-1-s_k, \ldots, n-1-s_1, 1, 0}^{n-1-s_k, \ldots, n-1-s_1, 1, 0}),
\]

and also the exponents of \( z \) in the denominator of the claim by \( \partial = (n-1, n-2, \ldots, 1, 0) \). Then if we put \( \lambda = I_s - \partial \), according to the character formula, the left-hand side of the claim exactly expresses \((-1)^{|s|} \chi_{GL(n)}(\lambda)\). On the other hand, if we use the induction hypothesis for \( s' = (s_1, s_2, \ldots, s_{k-1}) \) we have

\[
I_{s'} = (n-1+s_1, \ldots, n-1+s_{k-1}, n-1, \ldots, \overbrace{n-1-s_{k-1}, \ldots, n-1-s_1, 1, 0}^{n-1-s_{k-1}, \ldots, n-1-s_1, 1, 0}).
\]
and $I_s - \delta = \Gamma(s')$. Comparing $I_s$ with $I_{s'}$, the variation of exponents is exactly caused by exchanging the $(k+s_k)$-th exponent of $I_{s'}$ for $n-1+s_k$ and moving the $(k+s_k)$-th column $z^{n-1+s_k}$ to right behind the column $z^{n-1+s_k-1}$. But if we refer to the case $k=1$, this variation corresponds to adding the hook of Fig. 2 diagonally to the Young diagram

$$\Gamma(s') = (s_1-1, s_2-1, \cdots, s_{k-1}-1 | s_1, s_2, \cdots, s_{k-1}).$$

Hence the claim is proved.

![Fig. 2](image)

(3) follows immediately from the above claim.

Proof of (4). We use Weyl’s denominator formula for $\mathfrak{sp}(2n) = \{X \in \mathfrak{sl}(2n) | XJ_{sp} + J_{sp}X = 0\}$, where $J_{sp}$ is the following matrix:

$$J_{sp} = \begin{pmatrix}
0 & \cdots & 1 \\
& \ddots & 1 \\
& & -1 \\
& & & \ddots & 0 \\
& & & & -1
\end{pmatrix}$$

$h$ and the $\varepsilon_i$ are defined in the same manner as in the proof of (3). Let $A_0 = \{\varepsilon_i \pm \varepsilon_j \ (i > j), 2\varepsilon_i\}$ be a set of positive roots of $\mathfrak{sp}(2n)$ and let $\rho_0 = 1/2 \sum_{\alpha \in A_0^+} \alpha$ be the half sum of the positive roots, then $\rho_0$ is given by $\rho_0 = n\varepsilon_1 + (n-1)\varepsilon_2 + \cdots + \varepsilon_n$. Let us recall Weyl’s denominator formula:

$$\sum_{w \in W(\mathfrak{c}_n)} \det (w)e^{w\rho_0} = e^{\rho_0} \prod_{\alpha \in A_0^+} (1 - e^{-\alpha}),$$
where \( W(C_n) = \langle \mathfrak{S}_n, \phi_i \ (1 \leq i \leq n) \rangle \). As before \( W(C_n) \) has the coset decomposition with respect to \( \mathfrak{S}_n \) as follows:

\[
W(C_n) = \bigcup_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \mathfrak{S}_n \phi_{i_1} \phi_{i_2} \cdots \phi_{i_k}.
\]

If we put \( e^{-z_i} = z_i \ (1 \leq i \leq n) \) and take the sum for every coset, we have

\[
\prod_{1 \leq i_j \leq n} (1 - z_i z_j) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{[z^{-n}, \ldots, z^{n+1-i_1}, \ldots, z^{n+1-i_k}, \ldots, z^{-1}]}{[z^{n-1}, z^{n-2}, \ldots, z, 1]}
\]

Multiplying both the denominator and the numerator by \((z_1 z_2 \cdots z_n)^n\) and permuting the columns, we have

\[
\prod_{1 \leq i_j \leq n} (1 - z_i z_j) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{[z^{-1}, \ldots, z^{n+i_k}, \ldots, z^{n+s_1}, \ldots, z, 1]}{[z^{n-1}, z^{n-2}, \ldots, z, 1]},
\]

where \( s_1 = n+1-i_1, s_2 = n+1-i_2, \ldots, s_k = n+1-i_k \) and \( 1 \leq s_k < s_{k-1} < \cdots < s_1 \leq n \).

Therefore we have only to prove the next claim.

**Claim.**

\[
\frac{[z^{-1}, \ldots, z^{n+i_k}, \ldots, z^{n+s_1}, \ldots, z, 1]}{[z^{n-1}, z^{n-2}, \ldots, z, 1]} = (-1)^{1+n} \chi_{GL(n)}(\Gamma(s))(z).
\]

But the proof is similar to that of (3), so we omit it.

§ 2. Relations between the classical Weyl groups and the Universal Character Ring

In this section we deal with the relations between the Weyl group \( W(B_n) = W(C_n) \), referred to as \( W_n \) hereafter, and the Universal Character Ring \( A \). First, let us recall the definition of the ring \( A \) (cf. [M]).

Let \( A_n = Z[t_1, t_2, \ldots, t_n]^{\mathfrak{S}_n} = R_+ (GL(n)) \) be the graded algebra consisting of the symmetric polynomials in \( n \) variables and let \( \tilde{\rho}_{m,n} : Z[t_1, \ldots, t_m] \rightarrow Z[t_1, \ldots, t_n] \) be the homomorphism of graded algebras defined by \( \tilde{\rho}_{m,n}(t_i) = t_i \) if \( 1 \leq i \leq n \) and \( \tilde{\rho}_{m,n}(t_i) = 0 \) if \( n < i \). \( \tilde{\rho}_{m,n} \) induces a homomorphism \( \rho_{m,n} : A_m \rightarrow A_n \). Then \( (A_n, \rho_{m,n}) \) becomes a projective system and the projective limit of this system in the category of graded algebras is denoted by \( A \), i.e. \( A = \lim A_n \). We call \( A \) the Universal Character Ring.

By definition \( A \) is also a graded algebra: \( A = \sum_{k \geq 0} A^k \), where \( A^k = \lim A^k_n \). \((A^k_n)\) is the homogeneous part of degree \( k \) of \( A_n \). Note that \( A \) can be considered as the ring consisting of symmetric functions in countably
many variables $t_1, t_2, \ldots, t_n, \ldots$. Let $\pi_n : A \to A_n$ be the natural projection.

As is well known, $\{\chi_{GL(n)}(\lambda)_{\lambda: \text{partition}, d(\lambda) \leq n} \ (d(\lambda) \text{ denotes the depth of the Young diagram } \lambda)\}$ is a $\mathbb{Z}$-base of $A_n = R_+^{GL(n)}$. (Here we are using $t_1, t_2, \ldots, t_n$ as variables of $\chi_{GL(n)}(\lambda)$, instead of $z_1, z_2, \ldots, z_n$.) It is known that for $m \geq n \geq d(\lambda)$ we have $\rho_{m,n}(\chi_{GL(m)}(\lambda)) = \chi_{GL(n)}(\lambda)$ and for $d(\lambda) > k$ we have $\rho_{n,k}(\chi_{GL(n)}(\lambda)) = 0$. Hence the $\chi_{GL(n)}(\lambda)$'s form a projective system and we may define $\chi_{GL(n)}(\lambda) = \frac{\chi_{GL(n)}(\lambda)}{\chi_{GL(n)}(\lambda)}$ if $n \geq d(\lambda)$ and $\pi_n(\chi_{GL(n)}(\lambda)) = 0$ if $n < d(\lambda)$. \{\chi_{GL(n)}(\lambda)\}_{\lambda: \text{partition}} becomes a $\mathbb{Z}$-linear base of $A$. If we take $\lambda = (f)$, we also denote $\chi_{GL}(\lambda) = \chi_{GL}(f))$ by $p_f$. $\pi_n(p_f)$ is the sum of all monomials with coefficient 1 in $t_1, \ldots, t_n$ of degree $f$. If we take $\lambda = (1^f) = (1, 1, \ldots, 1)$ ($f$ times), then we also denote $\chi_{GL}(\lambda) = \chi_{GL((1^f))}$ by $e_f$. If $n \geq f$, $\pi_n(e_f)$ is the $f$-th elementary symmetric polynomial in $t_1, \ldots, t_n$.

Our arguments here are based on the following theorem due to H. Weyl. Let $V = C^m$ be the natural $GL(m)$-space. The symmetric group $\mathfrak{S}_k$ naturally acts on $\otimes^k V$, that is, $\sigma \in \mathfrak{S}_k$ acts on $x_1 \otimes x_2 \otimes \cdots \otimes x_k \in \otimes^k V$ by

$$\sigma(x_1 \otimes x_2 \otimes \cdots \otimes x_k) = x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes \cdots \otimes x_{\sigma^{-1}(k)}.$$  

On the other hand, $A \in GL(m)$ acts on $\otimes^k V$ by

$$A \cdot (x_1 \otimes x_2 \otimes \cdots \otimes x_k) = Ax_1 \otimes Ax_2 \otimes \cdots \otimes Ax_k,$$

and this action commutes with that of $\mathfrak{S}_k$ defined above.

**Theorem 2.1** (H. Weyl’s reciprocity). If we regard $\otimes^k V$ as a $GL(m) \times \mathfrak{S}_k$-module, it decomposes as

$$\otimes^k V = \sum_{\lambda: \text{partition} \atop d(\lambda) \leq m \atop |\lambda| = k} V_{\lambda}^{GL(m)} \otimes V_{\lambda}^{\mathfrak{S}_k} \ (\text{direct sum}).$$

Here $V_{\lambda}^{GL(m)}$ is the irreducible $GL(m)$-module corresponding to the character $\chi_{GL(m)}(\lambda)$, and $V_{\lambda}^{\mathfrak{S}_k}$ is the irreducible $\mathfrak{S}_k$-module corresponding to the Young diagram $\lambda$. (For the parametrization of the irreducible representations of $\mathfrak{S}_k$, see [J-K, Chap. 2]).

Since the equivalence classes of irreducible representations of $\mathfrak{S}_k$ are parametrized by the partitions of size $k$, we denote by $\chi_{\mathfrak{S}_k}(\lambda)$ the irreducible representation of $\mathfrak{S}_k$ or its character corresponding to a partition $\lambda$ with $|\lambda| = k$.

Let $R^k$ denote the character ring of $\mathfrak{S}_k$ (over $\mathbb{Z}$). Their module
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Let $R = \bigoplus_{k \geq 0} R^k$ (where $R^0 = \mathbb{Z}$) can be made into a graded algebra over $\mathbb{Z}$ with the multiplication $\cdot$ defined by

$$f \cdot g = \text{Ind}_{S_n \times S_n}^{S_{m+n}}(f \times g) \in R^{m+n} \quad \text{for } f \in R^m, \ g \in R^n.$$  

The $\mathbb{Z}$-linear map defined by

$$\text{ch} : R \longrightarrow A$$

$$\omega \longrightarrow \chi_{\omega}(\lambda)$$

gives an isomorphism of graded algebras, in virtue of the above theorem (H. Weyl's reciprocity). (See [M, p. 61, (7.3)])

$W_n$ is embedded into $\mathfrak{S}_{2n}$ as the centralizer:

$$W_n = C_{\mathfrak{S}_{2n}}((1, 2)(3, 4) \cdots (2n-1, 2n)).$$

$C_{\mathfrak{S}_{2n}}(1, 2)(3, 4) \cdots (2n-1, 2n)$. More precisely, if we define an injective homomorphism $\Delta : \mathfrak{S}_n \rightarrow \mathfrak{S}_{2n}$ by

$$\Delta(\tau) : \begin{cases} 2i-1 & \longrightarrow 2\tau(i)-1 \\ 2i & \longrightarrow 2\tau(i) \end{cases} \quad (i = 1, 2, \ldots, n)$$

for $\tau \in \mathfrak{S}_n$ and put $\sigma_i = (2i-1, 2i)$, $i = 1, 2, \ldots, n$, then we have

$$W_n = \langle \Delta(\mathfrak{S}_n), \sigma_1, \ldots, \sigma_n \rangle = \Delta(\mathfrak{S}_n) \times H,$$

where $H = \langle \sigma_1, \sigma_2, \ldots, \sigma_n \rangle \simeq \mathbb{Z}_{2n}^n$.

For each $i = 0, 1, 2, \ldots, n$, define a representation $\rho_i$ of $H$ by

$$\rho_i(\sigma_j) = \begin{cases} 1 & \text{if } 1 \leq j \leq i, \\ -1 & \text{if } i+1 \leq j \leq n. \end{cases}$$

Noting that $W_n/H \simeq \mathfrak{S}_n$, we denote by $\chi_{\mathfrak{S}_n}(\lambda, \phi)$ the pull-back of the character $\chi_{\mathfrak{S}_n}(\lambda)$ to $W_n$ ($\phi$ denotes the empty diagram). On the other hand, the representation $\rho_0$ of $H$ can be extended to that of $W_n$ by letting $\Delta(\mathfrak{S}_n)$ act trivially, since $\rho_0$ is $\Delta(\mathfrak{S}_n)$-invariant. Denote this character by $\chi_{\mathfrak{S}_n}(\phi, (n))$ and put $\chi_{W_n}(\phi, \lambda) = \chi_{\mathfrak{S}_n}(\lambda, \phi) \otimes \chi_{W_n}(\phi, (n))$. Corresponding to each representation $\rho_i$ of $H$, a subgroup $W_i \times W_{n-i}$ is defined by

$$W_i = \langle \Delta(\mathfrak{S}_i), \sigma_1, \sigma_2, \ldots, \sigma_i \rangle$$

and $W_{n-i} = \langle \Delta(\mathfrak{S}_{n-i}), \sigma_{i+1}, \sigma_{i+2}, \ldots, \sigma_n \rangle$, where

$$\mathfrak{S}_i = \langle (1, 2), (2, 3), \ldots, (i-1, i) \rangle$$

and

$$\mathfrak{S}_{n-i} = \langle (i+1, i+2), (i+2, i+3), \ldots, (n-1, n) \rangle$$

are subgroups of $\mathfrak{S}_{2n}$. Then, according to so-called "Mackey-Wigner's
little group method” (See [S, p. 62, Proposition 25]), we have an irreducible representation $\chi_{W_n}(\mu, \nu)$ by putting

$$\chi_{W_n}(\mu, \nu) = \text{Ind}_{W_{n-1} \times W_{n-1}}^{W_n}(\chi_{\mu}(\phi) \times \chi_{\nu}(\phi)).$$

Then the $\chi_{W_n}(\xi, \psi)$ with $|\xi| + |\psi| = n$ constitute a complete set of representatives of the equivalence classes of irreducible representations of $W_n$. Just as we did for $S_n$, we shall write $V_{\chi_{W_n}(\xi, \psi)}$ for the irreducible $W_n$-module with character $\chi_{W_n}(\xi, \psi)$.

If we denote the character ring of $W_n$ (over $\mathbb{Z}$) by $R_{W}$, then their module direct sum $R_{W} = \bigoplus_{n \geq 0} R_{W}^{n}$, with $R_{W}^{0} = \mathbb{Z}$, can be made into a graded algebra over $\mathbb{Z}$ with the multiplication defined by

$$f \cdot g = \text{Ind}_{W_{n} \times W_{m}}^{W_{n+m}}(f \times g) \in R_{W}^{n+m} \quad \text{for} \quad f \in R_{W}^{n}, \ g \in R_{W}^{m}.$$

Now we define another (noncommutative) multiplication in $A$. Recall that an element of $A$ can be regarded as a certain infinite $\mathbb{Z}$-linear combination of monomials in countably many variables $t_1, t_2, \ldots$. Let $f, g \in A$ and put $g = \sum u_\lambda t^\lambda$, where $u_\lambda \in \mathbb{Z}$ and $\lambda$ runs through multi-indices, namely the infinite sequences of nonnegative integers with finitely many nonzero terms. Suppose all $u_\lambda \geq 0$. If we bring $u_\lambda$ copies of each $t^\lambda$, then altogether we have a collection of countably many monomials. Arrange them in one sequence and label them $s_1, s_2, \ldots$ (the order is arbitrary). If we substitute $s_\lambda$ for each variable $t_\lambda$ in $f$, we get a new symmetric function in $A$, which is denoted by $f \circ g$. This multiplication $\circ$ is extended for all $g \in A$ by $\mathbb{Z}$-linearity. This notion has been introduced and called plethysm by D.E. Littlewood (see [L]).

$\pi_n(\chi_{GL}(\lambda) \circ \chi_{GL}(\mu))$ is the character of the representation of $GL(n)$ obtained as the composite of the following two homomorphisms:

$$GL(n) \xrightarrow{\rho_p, GL(n)} GL(V_\rho) \xrightarrow{\rho_\lambda, GL(V_\rho)} GL(V_\lambda^{GL(V_\rho)}).$$

We define a $\mathbb{Z}$-linear map $c_{W} : R_{W} \rightarrow A$ by

$$c_{W}(\chi_{W_n}(\lambda, \mu)) = (\chi_{GL}(\lambda) \circ p_\lambda)(\chi_{GL}(\mu) \circ e_\lambda).$$

It is easy to see that $c_{W}$ is an algebra homomorphism.

**Caution.** In general, $c_{W}$ is not injective. However, the significance of $c_{W}$ lies in the following fact.

**Proposition 2.2.** The decomposition coefficients of

$$(\chi_{GL}(\lambda) \circ p_\lambda)(\chi_{GL}(\mu) \circ e_\lambda)$$
into Schur functions coincide with those of \( \chi_{W_n}(\lambda, \mu) \uparrow^\mathbb{S}_n \) into irreducible constituents. That is, if we put

\[
(\chi_{GL}(\lambda) \circ p_\lambda)(\chi_{GL}(\mu) \circ e_\lambda) = \sum d^\nu_{\lambda, \mu} \chi_{GL}(\nu)
\]

and

\[
\chi_{W_n}(\lambda, \mu) \uparrow^\mathbb{S}_n \mid_{W_n} = \sum d^\nu_{\lambda, \mu} \chi_{\mathbb{S}_n}(\nu),
\]

then we have \( d^\nu_{\lambda, \mu} = d^\nu_{\lambda, \mu} \) for all \( \lambda, \mu, \) and \( \nu. \)

**Proof.** Fix an \( m \geq 2n, \) and put \( |\lambda| = i. \) We shall show that \( d^\nu_{\lambda, \mu} \) and \( d^\nu_{\lambda, \mu} \) are both equal to the multiplicity of the irreducible \( GL(m) \times W_i \times W_{n-i} \)-module

\[
(*) V^{GL(m)} \otimes V^{W_i}_{(\lambda, \mu)} \otimes V^{W_{n-i}}_{(\phi, \mu)}
\]

in \( \otimes^{2n} C^m. \) Here \( \otimes^{2n} C^m \) is regarded as a \( GL(m, C) \times \Sigma_{2n} \)-modules as in H. Weyl's reciprocity, and \( W_i \times W_{n-i} \subseteq W_n \subseteq \Sigma_{2n}. \)

By H. Weyl's theorem, \( \otimes^{2n} C^m \) decomposes as

\[
\otimes^{2n} C^m = \sum_{|\nu| = 2n} V^{GL(m)}_{\nu} \otimes V^{\mathbb{S}_n}_{\nu}. 
\]

On the other hand, the definition of \( d^\nu_{\lambda, \mu} \) can be read as follows. Since \( \chi_{W_n}(\lambda, \mu) = \text{Ind}_{W_i \times W_{n-i}}^{W_n}(\chi_{W_i}(\lambda, \phi) \times \chi_{W_{n-i}}(\phi, \mu)), \) we have

\[
\chi_{W_n}(\lambda, \mu) \uparrow^\mathbb{S}_n \mid_{W_n} = \chi_{W_i}(\lambda, \phi) \times \chi_{W_{n-i}}(\phi, \mu) \uparrow^\mathbb{S}_n \mid_{W_i \times W_{n-i}}.
\]

Frobenius' reciprocity shows that \( V^{\mathbb{S}_n}_{\nu}, \) regarded as a \( W_i \times W_{n-i} \)-module, contains \( d^\nu_{\lambda, \mu} \) times of \( V^{W_i}_{(\lambda, \phi)} \otimes V^{W_{n-i}}_{(\phi, \mu)}. \) So the multiplicity of \( (*) \) in \( \otimes^{2n} C^m \) is also \( d^\nu_{\lambda, \mu}. \)

Next we consider \( d^\nu_{\lambda, \mu}. \) Any submodule of \( \otimes^{2n} C^m \) isomorphic to \( (*) \) is contained in the \( \rho_i \)-isotypical component of \( \otimes^{2n} C^m \) with respect to \( H \subseteq W_n, \) because \( \chi_{W_i}(\lambda, \phi) \times \chi_{W_{n-i}}(\phi, \mu) \downarrow_H \) is a multiple of \( \rho_i. \) The \( \rho_i \)-isotypical component is:

\[
(\#) (\otimes^i S_2(C^m)) \otimes (\otimes^{n-i} A^i(C^m)),
\]

where \( S_2 [\text{resp. } A^i] \) denotes the symmetric [resp. alternating] product of rank 2. \( W_i \times W_{n-i} \) stabilizes this space, since it stabilizes \( \rho_i. \) We are going to take out its \( \chi_{W_i}(\lambda, \phi) \times \chi_{W_{n-i}}(\phi, \mu) \)-isotypical component.

\( \Delta(\mathbb{S}_i) \) acts on \( (\#) \) as the permutations of the \( i \) tensor factors of \( \otimes^i S_2(C^m). \) So if we regard it as a \( GL(S_i(C^m)) \times \Delta(\mathbb{S}_i) \)-module, then we have
\( \otimes^i S_\phi(C^m) = \sum_{|\lambda| = i} V^G(\mathfrak{sl}_2(C^m)) \otimes V^J(\mathfrak{gl}_n). \)

Taking the action of \( \langle \sigma_1, \sigma_2, \ldots, \sigma_i \rangle \) into account, we see that \( V^J(\mathfrak{gl}_n) \) becomes a \( W_i \)-module \( V_{\mu_i}^{W_i}. \)

On the other hand, \( \Delta(\mathfrak{g}_{n-i}) \) acts (\#) as the permutations of the \( n-i \) tensor factors of \( \otimes^{n-i} A^n(C^m). \) So if we regard it as a \( GL(A^n(C^m)) \times \Delta(\mathfrak{g}_{n-i}) \)-module, we have

\( \otimes^{n-i} A^n(C^m) = \sum_{|\mu'| = n-i} V^G(\mathfrak{gl}_n) \otimes V_{\mu'}^{W_{n-i}}. \)

Taking the action of \( \langle \sigma_{i+1}, \sigma_{i+2}, \ldots, \sigma_n \rangle \) into account, we see that \( V_{\mu'}^{W_{n-i}} \) becomes a \( W_{n-i} \)-module \( V^G(\mathfrak{gl}_n) \).

Hence the \( \chi_{\mathfrak{g}_n}(\lambda, \phi) \times \chi_{\mathfrak{g}_{n-i}}(\phi, \mu) \)-isotypical component of \( \otimes^{2n} C^m \) is

\( V^G(\mathfrak{sl}_2(C^m)) \otimes V^G(\mathfrak{gl}_n) \otimes V_{(\lambda, \phi) \times (\phi, \mu)}^{W_{n-i}}. \)

As a \( GL(m, C) \)-module, \( V^G(\mathfrak{sl}_2(C^m)) \otimes V^G(\mathfrak{gl}_n) \) affords \( (\chi_{\mathfrak{gl}}(\lambda) \circ p_2)(\chi_{\mathfrak{gl}}(\mu) \circ e_2). \) By the definition of \( d_{\mu \nu}^\lambda \), it contains \( d_{\mu \nu}^\lambda \) times of \( V^G(\mathfrak{gl}(m)). \) So the multiplicity of \((*)\) in \( \otimes^{2n} C^m \) is also \( d_{\mu \nu}^\lambda. \)

By the definition of plethysm, the following four equalities hold:

i) \( \prod_{1 \leq i < j \leq n} \frac{1}{1 - z_i z_j} = \sum_{f=0}^{\infty} \pi_n(p_f \circ e_2), \)

ii) \( \prod_{1 \leq i < j \leq n} \frac{1}{1 - z_i z_j} = \sum_{f=0}^{\infty} \pi_n(p_f \circ p_2), \)

iii) \( \prod_{1 \leq i < j \leq n} (1 - z_i z_j) = \sum_{f=0}^{\infty} \pi_n(e_f \circ e_2), \)

iv) \( \prod_{1 \leq i < j \leq n} (1 - z_i z_j) = \sum_{f=0}^{\infty} \pi_n(e_f \circ p_2). \)

Comparing these with D.E. Littlewood’s Lemma 1.1, we have:

**Proposition 2.3.**

i) \( p_f \circ e_2 = \sum_{\text{partition} \ |x| = f} \chi_{\mathfrak{gl}}(2x), \)

ii) \( p_f \circ p_2 = \sum_{\text{partition} \ |x| = f} \chi_{\mathfrak{gl}}(2x), \)

iii) \( e_f \circ e_2 = \sum_{\sigma=(a_1, a_2, \ldots, a_2) \ a_1 > a_2 > \ldots > a_2 > 0} \chi_{\mathfrak{gl}}(\Gamma(\alpha)), \)

iv) \( e_f \circ p_2 = \sum_{\sigma=(a_1, a_2, \ldots, a_2) \ a_1 > a_2 > \ldots > a_2 > 0} \chi_{\mathfrak{gl}}(\Gamma(\alpha)). \)
Applying Proposition 2.2 to \( \chi_{\mathfrak{S}_n}(\phi, (n)) \), \( \chi_{\mathfrak{S}_n}(n, \phi) \), \( \chi_{\mathfrak{S}_n}(\phi, (1^n)) \), and \( \chi_{\mathfrak{S}_n}((1^n), \phi) \), we have:

**Proposition 2.3'.**

i) \( \chi_{\mathfrak{S}_n}(\phi, (n)) \),

\[
\sum_{|\alpha| = n} \chi_{\mathfrak{S}_n}(\iota(2\alpha)),
\]

ii) \( \chi_{\mathfrak{S}_n}(n, \phi) \),

\[
\sum_{|\alpha| = n} \chi_{\mathfrak{S}_n}(2\alpha),
\]

iii) \( \chi_{\mathfrak{S}_n}(\phi, (1^n)) \),

\[
\sum_{|\alpha| = n} \chi_{\mathfrak{S}_n}(\iota(\Gamma(\alpha))),
\]

iv) \( \chi_{\mathfrak{S}_n}((1^n), \phi) \),

\[
\sum_{|\alpha| = n} \chi_{\mathfrak{S}_n}(\iota'(\Gamma(\alpha))).
\]

The formula ii) has been applied to the projective geometry over finite fields by J.G. Thompson [T].

Moreover, it should be noted that we can derive an algorithm to decompose the representation of \( \mathfrak{S}_n \) induced by an arbitrary irreducible representation of \( W_n \). For a partition \( \lambda=(\lambda_1, \lambda_2, \ldots, \lambda_k) \) of depth \( k \), we have

\[
\chi_{\mathfrak{S}_n}(\lambda) = \det \begin{vmatrix}
\sum_{|\alpha| = \lambda_1} \chi_{\mathfrak{S}_n}(2\alpha) & \cdots & \sum_{|\alpha| = \lambda_k + (k-1)} \chi_{\mathfrak{S}_n}(2\alpha) \\
\sum_{|\alpha| = \lambda_2} \chi_{\mathfrak{S}_n}(2\alpha) & \cdots & \sum_{|\alpha| = \lambda_k + (k-2)} \chi_{\mathfrak{S}_n}(2\alpha) \\
\vdots & \ddots & \vdots \\
\sum_{|\alpha| = \lambda_k} \chi_{\mathfrak{S}_n}(2\alpha) & \cdots & \sum_{|\alpha| = \lambda_k} \chi_{\mathfrak{S}_n}(2\alpha)
\end{vmatrix}.
\]

Hence, by ii) of Proposition 2.3, \( \chi_{\mathfrak{G}_L}(\lambda) \circ p_2 \) is given by

\[
\chi_{\mathfrak{G}_L}(\lambda) \circ p_2 = \det \begin{vmatrix}
\sum_{|\alpha| = \lambda_1} \chi_{\mathfrak{G}_L}(2\alpha) & \cdots & \sum_{|\alpha| = \lambda_k + (k-1)} \chi_{\mathfrak{G}_L}(2\alpha) \\
\sum_{|\alpha| = \lambda_2} \chi_{\mathfrak{G}_L}(2\alpha) & \cdots & \sum_{|\alpha| = \lambda_k + (k-2)} \chi_{\mathfrak{G}_L}(2\alpha) \\
\vdots & \ddots & \vdots \\
\sum_{|\alpha| = \lambda_k} \chi_{\mathfrak{G}_L}(2\alpha) & \cdots & \sum_{|\alpha| = \lambda_k} \chi_{\mathfrak{G}_L}(2\alpha)
\end{vmatrix}.
\]

Similarly,

\[
\chi_{\mathfrak{G}_L}(\lambda) \circ e_2 = \det \begin{vmatrix}
\sum_{|\alpha| = \lambda_1} \chi_{\mathfrak{G}_L}(\iota(2\alpha)) & \cdots & \sum_{|\alpha| = \lambda_k + (k-1)} \chi_{\mathfrak{G}_L}(\iota(2\alpha)) \\
\sum_{|\alpha| = \lambda_2} \chi_{\mathfrak{G}_L}(\iota(2\alpha)) & \cdots & \sum_{|\alpha| = \lambda_k + (k-2)} \chi_{\mathfrak{G}_L}(\iota(2\alpha)) \\
\vdots & \ddots & \vdots \\
\sum_{|\alpha| = \lambda_k} \chi_{\mathfrak{G}_L}(\iota(2\alpha)) & \cdots & \sum_{|\alpha| = \lambda_k} \chi_{\mathfrak{G}_L}(\iota(2\alpha))
\end{vmatrix}.
\]
For partitions $\mu$ and $\nu$, $\chi_{AL}(\mu)\chi_{AL}(\nu)$ can be computed using Littlewood-Richardson's rule. Combining these facts with Proposition 2.2, we obtained an algorithm to write down the irreducible constituents of the representation of $S_{2n}$ induced by any irreducible representation of $W_n$.

References


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