ON RANDOM WALKS AND DIFFUSIONS RELATED TO PARRONDO'S GAMES

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In a series of papers, G. Harmer and D. Abbott study the behavior of random walks associated with games introduced in 1997 by J.M.R. Parrondo. These games illustrate an apparent paradox that random and deterministic mixtures of losing games may produce winning games. In this paper, classical cyclic random walks on the additive group of integers modulo m, a given integer, are used in a straightforward way to derive the strong law limits of a general class of games that contains the Parrondo games. We then consider the question of when random mixtures of fair games related to these walks may result in winning games. Although the context for these problems is elementary, there remain open questions. An extension of the structure of these walks to a class of shift diffusions is also presented, leading to the fact that a random mixture of two fair shift diffusions may be transient to $+\infty$.

1. Introduction

The purpose of this paper is to study a family of random walks that include those arising in the games devised by J.M.R. Parrondo in 1997 to illustrate the apparent paradox that two "losing" games can result in a "winning" game when one alternates between them. We refer the reader to Harmer and Abbott (1999a,b), Harmer, Abbott and Taylor (2000) and Harmer, Abbott, Taylor and Parrondo (2000) in which Parrondo's paradox is discussed, large simulations of specific Parrondo games and mixtures thereof are presented and certain theoretical results are given. These authors also give a heuristic explanation of the paradox in terms of the Brownian ratchet, the original motivation for the suggestion of these games. Other references to the general subject are included in the above mentioned papers by Harmer and Abbott. The reader may also note the reference Durrett, Kesten and Lawler (1991) which also deals with the general question of showing that winning games can be formed by mixing fair ones.

The suggested paradox may be visualized as follows. You are about to play a two-armed slot machine. The casino that owns this two-armed bandit advertises that both arms on their two-armed machines are "fair" in the sense that any player who plays either of the arms is assured that the average cost per play approaches zero as the number of plays increase. However, the casino does not constrain you to stay with one arm; you are allowed to use

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either arm on every play. You just tell the machine before beginning how many plays you wish to make. At the end of that number of plays, the machine displays the total amount won or lost. The question of interest in this context would be whether it is possible for the casino to still make money using only "fair" games.

In this paper a random walk will refer to a Markov chain $\{S_n : n = 0, 1, 2, \dots\}$ taking values in the integers, $\mathbb{Z}$, which satisfies the discrete continuity condition

$$|S_n - S_{n-1}| = 1 \text{ a.s.} \quad \text{for each } n \geq 1.$$

Let the transition probabilities for the random walk be denoted by

$$p_j = P(S_{n+1} - S_n = 1 \mid S_n = j), \quad q_j = P(S_{n+1} - S_n = -1 \mid S_n = j)$$

and

$$r_j = 1 - p_j - q_j = P(S_{n+1} = S_n \mid S_n = j)$$

for $j \in \mathbb{Z}$. Assume that $p_j q_j \neq 0$ for all j. For fixed integer $m \geq 1$, define a *mod m random walk* to be a random walk in which the transition probabilities p_i, r_i, q_i depend only upon the congruence class mod m of the state i. Thus, these *lattice regular* or *periodic* random walks are such that for some specified integer $m > 1$, $p_j = p_{j+m}$ and $q_j = q_{j+m}$ for all $j \in \mathbb{Z}$. More generally, define a *mod m Markov chain* on the integers to be one whose parameters depend only upon the congruence classes mod m of the states, namely, $p_{ij} = p_{i+m,j}$ for all integers i, j. This paper is concerned with the case of random walks, but places where the approach applies more generally are pointed out.

A mod m random walk is determined by the $2m$ parameters p_j, q_j; $0 \leq j < m$. Write $\mathbf{p} = (p_0, p_1, \dots, p_{m-1})$ with an analogous use of $\mathbf{q}$ to specify the walk's parameters. Observe that when $m = 1$ the walk is classical simple random walk, so our main interest is in the cases of $m > 1$.

These random walks are viewed as games with the increment $X_n = S_n - S_{n-1}$ $(n \geq 1)$ denoting the gain at the n-th play. We say that the game is a winning/losing/fair game according as the almost sure limit of S_n/n is positive/negative/zero.

For given m, write $\mathbb{Z}_m := m\mathbb{Z} = \{km : k \in \mathbb{Z}\}$ for the integer lattice of span m. In the games introduced by Parrondo, it is assumed that the transition probabilities depend on the state only to the extent that the state is or is not in $\mathbb{Z}_m$. Thus, Parrondo's games are characterized by

$$P(X_{n+1} = 1 \mid S_0, S_1, \dots, S_n) = p' 1_{[S_n \in \mathbb{Z}_m]} + p 1_{[S_n \notin \mathbb{Z}_m]} \tag{1.1}$$

for some $p, p' \in [0, 1]$ and all $n \geq 0$. Write $q = 1 - p$ and $q' = 1 - p'$. We may also write $k \equiv j \mod m$ when $k \in j + \mathbb{Z}_m$.

For simplicity, we write $G(m, \mathbf{p}, \mathbf{q})$ to denote a general mod m random walk or game, but write $G(m, \mathbf{p})$ for the game when each $q_j = 1 - p_j$ (i.e. each $r_j = 0$) and write $G(m, p, p')$ for the special Parrondo random walk or game satisfying (1.1).

The required notation and preliminary structure are introduced in the following section, in which the limiting results for $G(m, p, p')$ games are given for illustration. The general case is covered in Section 3, while in Section 4 we resolve the central question about whether random mixtures of losing Parrondo's games can be winning ones. The asymptotic gain is derived in Section 5 while in Section 6 a certain expected interoccurrence time that appears in the previously obtained expression for this is also derived. The method used to solve the recursion equations in these sections makes use of an extension of results of Mihoc and Fréchet (cf. Fréchet, 1952) that are provided in the appendix to this paper. Continuous analogues to the random walks considered here are introduced in Section 7. These mod m diffusions have drift functions that are periodic step functions so that their embedded walks on the integers are $G(m, \mathbf{p}, \mathbf{q})$ walks. In Theorem 7.1 the drift rates under which the embedded walk has specified transition probabilities is determined.

2. Preliminaries and Parrondo's examples

In the games suggested by Parrondo, the transition probabilities depend on the state only to the extent that it is or is not in $\mathbb{Z}_m$; see (1.1) above. The asymptotic behavior of these games, as for any mod m random walk is determined by that of its embedded walk on the lattice $\mathbb{Z}_m$. Since this embedded walk is equivalent to simple random walk, its asymptotics are well known and dependent solely upon a single parameter, the walk's probability of "success." In this section we introduce the notation required for the general case in Section 3 below, and illustrate the approach in the special case of a Parrondo $G(m, p, p')$ walk by substituting in known results for simple random walk.

Let $T_1 < T_2 < \cdots$ be the successive transition times of the embedded walk on $\mathbb{Z}_m$. That is $T_1 = \min\{n \geq 0 : S_n \in \mathbb{Z}_m\}$ and, for $k > 1$,

$$T_{k+1} = \min\{n > T_k : S_n - S_{T_k} = \pm m\}$$

with the minimum of a null set being defined to equal $+\infty$. Set $T_0 = 0$. Write

$$J_n = m^{-1} S_{T_{n+1}}, \quad \xi_n = T_{n+1} - T_n$$

for $n \geq 0$ so that $\{(J_n, \xi_n) : n \geq 0\}$ is a (possibly delayed) Markov renewal process (MRP) in which the embedded random walk $\{J_n\}$ is simply a classical random walk with constant probability of "success,'

$$p_m^* := P(J_{n+1} - J_n = 1 \mid J_n). \tag{2.1}$$

Hence, once p_m^* is known, the winning/losing/fair nature of the walk is easily determined.

In general, for n satisfying $T_k < n \le T_{k+1}$,

$$\frac{mJ_{k+1} - 2m}{T_{k+1}} = \frac{S_{T_{k+1}} - 2m}{T_{k+1}} \le \frac{S_n}{n} \le \frac{S_{T_k} + m}{T_k} = \frac{mJ_k + m}{T_k}.$$

Thus,

$$m\left(\frac{mJ_{k+1}}{k+1} - \frac{2}{k+1}\right) \Big/ \frac{T_{k+1}}{k+1} \le \frac{S_n}{n} \le m\left(\frac{J_k}{k} + \frac{1}{k}\right) \Big/ \frac{T_k}{k}. \tag{2.2}$$

It is known for the classical random walk $\{J_n\}$ that J_k/k converges a.s. as $k \to \infty$ to $p_m^* - q_m^*$, with $q_m^* = 1 - p_m^*$. Moreover, the stopping times $\{T_k\}$ are partial sums of iid r.v.'s having finite expectations so that

$$T_k/k \xrightarrow{\text{a.s.}} E(T_2 - T_1) < \infty.$$

Upon taking limits in (2.2) one obtains that with probability one,

$$\lim_{n\to\infty} \frac{S_n}{n} = \frac{m(p_m^* - q_m^*)}{E(T_2 - T_1)}. \tag{2.3}$$

Clearly then, this limit is 0, > 0 or < 0 according as $p_m^* =, >$ or $< q_m^*$.

The quantity p_m^* is evaluated for the general $G(m, \mathbf{p}, \mathbf{q})$ walk in Lemma 3.2 below. However, for the special Parrondo $G(m, p, p')$ random walk, the evaluation is immediate once we introduce the notation and approach that is needed for the general case, and so we give it separately here as

Lemma 2.1. *The $\mathbb{Z}_m$-embedded MRP of the $G(m, p, p')$ random walk has transition probabilities determined by the "success" probability*

$$p_m^* = \frac{p'p^{m-1}}{p'p^{m-1} + q'q^{m-1}} \tag{2.4}$$

for all $p, p' \in [0, 1]$ satisfying $|p - p'| < 1$.

Proof. The first part of this proof, through (2.6) below, is general and will be needed in Section 3. The rest is substitution of known results.

Suppose $J_n = k$. That is, for the original walk suppose $S_{T_{n+1}} = km$. Since T_{n+1} is a stopping time, $P(J_{n+1} - J_n = 1 \mid J_n = k)$ is just the probability that starting at $S_0 = 0$, the random walk $\{S_n\}$ reaches m before it reaches $-m$. But S_1 equals 1 or -1 with probability p_0 or q_0, respectively. Thus if we let A denote the event that $\{S_n : n > 1\}$ reaches 0 before it reaches mS_1 then the Markov property implies that p_m^*, the success probability for the embedded walk, satisfies the following recursion relation, in which we partition the event according to whether the original walk hits zero before m or not:

$$p_m^* = P(A)p_m^* + p_0\{1 - P(A \mid S_1 = 1)\}. \tag{2.5}$$

Hence

$$p_m^* = p_0 P(A^c \mid S_1 = 1)/P(A^c). \tag{2.6}$$

Since for the special case of this lemma, the conditional probabilities given S_1 are just those that arise in the classical gambler's ruin problem, (cf. Feller, 1968, Chapter XIV) it is known that

$$P(A^c \mid S_1 = 1) = \begin{cases} (qp^{m-1} - p^m)/(q^m - p^m), & \text{if } p \neq q \\ 1/m, & \text{if } p = \frac{1}{2} \end{cases} \tag{2.7}$$

and $P(A^c \mid S_1 = -1)$ is similar but with p and q interchanged. Substitution of (2.7) into (2.6) now gives, when $p \neq q$,

$$P(A^c) = (q'q^{m-1} + p'p^{m-1})(q-p)/(q^m - p^m) \tag{2.8}$$

and, therefore, p_m^* is as required by (2.4). When $p = \frac{1}{2}$, the substitution of (2.7) yields $p_m^* = p'$ to complete the proof. □

Note that by (2.4), p_m^* is the conditional probability that S_n reaches m before $-m$ given that $S_0 = 0$ and that the first m steps of S_n are monotone. This structure is more readily seen in the general case of Lemma 3.2 below.

For the special $G(m, p, p')$ case the above result yields

Corollary 2.1 (Harmer and Abbott, 1999a). *When $|p - p'| < 1$, the game $G(m, p, p')$ is a fair, winning or losing game according as*

$$p'p^{m-1} - q'q^{m-1} = 0, \quad > 0 \quad \text{or} \quad < 0.$$

The condition in Corollary 2.1 is more clearly expressed in terms of new variables $x = p/q$ and $y = p'/q'$, namely, the game $G(m, p, p')$ is a fair, winning or losing one according as

$$y - x^{-(m-1)} = 0, \quad > 0 \quad \text{or} \quad < 0. \tag{2.9}$$

Recall that the degenerate case $q' = 0 = p$ has been excluded. Since the inverse relationships are $p = x/(1+x)$ and $p' = y/(1+y)$, it follows from (2.9) that $G(m, p, p')$ is fair if for some $x \geq 0$, p and p' are related as

$$q = 1 - p = \frac{1}{1+x} \quad \text{and} \quad p' = \frac{x^{-m+1}}{1 + x^{-m+1}} = \frac{1}{1 + x^{m-1}}.$$

Here are some examples. For $x = 1$, $G(m, \frac{1}{2}, \frac{1}{2})$ is fair for every $m \geq 1$. For $m = 4$ and $x = 2$, the game $G(4, \frac{4}{5}, 1/65)$ is seen to be fair, and for $m = 5$ and $x = 2$, $G(5, \frac{2}{3}, 1/17)$ is fair. When $m = 3$ and one chooses $x = 3$, one obtains the fair game $G(3, \frac{3}{4}, 1/10)$. The associated games $G(3, \frac{3}{4} - \varepsilon, 1/10 - \varepsilon)$ for a range of $\varepsilon > 0$ are the losing games used in the simulation study of Harmer and Abbott (1999a). The fact that these are losing games as indicated there is immediate from the following observation: If $G(m, p_0, p'_0)$ is a fair game, then $G(m, p, p')$ is a losing game whenever $0 \leq p' \leq p'_0$ and $0 \leq p \leq p_0$ with $p + p' < p_0 + p'_0$; simply observe that p'/q' and p/q are increasing functions of p' and p, respectively, so that $p'_0/q'_0 = (p_0/q_0)^{-m+1}$ implies $p'/q' \leq (p/q)^{-m+1}$ whenever $p' \leq p'_0$ and $p \leq p_0$. Since $G(3, \frac{3}{4}, 1/10)$ is a fair game the result follows.

3. General mod m random walks

Let $\{S_n : n \geq 0\}$ be a general (discretely continuous) random walk on the integers $\mathbb{Z}$ in the sense described in the Introduction above. The asymptotic behavior of $\{S_n\}$ can be described in terms of the two associated reflecting random walks on the negative and positive integers. The latter is obtained, for example, by replacing r_0 and q_0 by $\bar{r}_0 = 1 - p_0$ and $\bar{q}_0 = 0$. It is known (cf. Feller, 1968, Chapter XV.8 or Chung, 1967, Section. I.12) that the corresponding reflecting random walk on $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$ is recurrent or transient according to

$$(3.1) \qquad \sum_{i=1}^{\infty} \frac{q_1 q_2 \cdots q_i}{p_1 p_2 \cdots p_i} = \infty$$

or not. When one looks similarly at the reflecting random walk on $\mathbb{Z}^- = \{0, -1, -2, \ldots\}$, the roles of the p's and q's are interchanged so that recurrence in this case holds if and only if

$$(3.2) \qquad \sum_{i=1}^{\infty} \frac{p_{-1} p_{-2} \cdots p_{-i}}{q_{-1} q_{-2} \cdots q_{-i}} = \infty.$$

Now return to the original walk on $\mathbb{Z}$. The positive part of this walk, $\{S_n^+\}$, is a Markov renewal process in which all sojourn times are equal to one except those between successive visits to state 0. The distribution of these latter sojourn times is a possibly deficient mixture that includes with probability q_0 the distribution of the first passage time from state -1 to state 0. The latter passage time is finite with probability one only if the reflecting random walk on $\mathbb{Z}^-$ is recurrent. Hence $\{S_n\}$ is a recurrent random walk if and only if both reflecting random walks are recurrent, or equivalently, if and only if both (3.1) and (3.2) hold. Consequently, the walk is transient if and only if at least one of these series converges. Accordingly, the boundary of a transient random walk may consist of either or both of $+\infty$ and $-\infty$, depending upon which one or both of the series converge. (Cf. Karlin and McGregor (1959), Section 4 where the integral representations of the transition probabilities of the doubly infinite random walk are expressed in terms of those of the two corresponding reflecting walks.)

Consider now, for fixed integer $m \geq 1$, a *mod m random walk* as defined in Section 1. (When $m = 1$, the mod 1 random walk is just the classical random walk with constant transition probabilities.) Thus, for $i = sm + l$ for some $s \in \mathbb{Z}$ and $l = 0, 1, \dots, m-1$, we know that $(p_i, r_i, q_i) = (p_l, r_l, q_l)$. Moreover, for $s \geq 0$, the summand in (3.1) becomes

$$\frac{q_1 q_2 \cdots q_i}{p_1 p_2 \cdots p_i} = \frac{p_0}{q_0} \left\{ \frac{q_0 q_1 \cdots q_{m-1}}{p_0 p_1 \cdots p_{m-1}} \right\}^s \frac{q_0 q_1 \cdots q_l}{p_0 p_1 \cdots p_l} \tag{3.3}$$

while for $s < 0$, a similar representation holds with the p's and q's interchanged. If we define

$$\rho_m := \frac{p_0 p_1 \cdots p_{m-1}}{q_0 q_1 \cdots q_{m-1}} \tag{3.4}$$

then the divergence of (3.1) holds if and only if $\rho_m \leq 1$ while (3.2) holds if and only if $\rho_m \geq 1$. By the above discussion, the walk is then recurrent, transient to $+\infty$ or transient to $-\infty$ according as ρ_m is equal to, greater than or less than one. In analogy with Corollary 2.1, this then proves

Lemma 3.1. *For $m \geq 1$, a* mod m *random walk is recurrent, transient toward $+\infty$ or transient toward $-\infty$ according as*

$$p_0 p_1 \cdots p_{m-1} - q_0 q_1 \cdots q_{m-1} = 0, \quad > 0 \quad \text{or} \quad < 0. \tag{3.5}$$

It remains to evaluate p_m^*, the probability of "success," for the embedded walk on $\mathbb{Z}_m$. The result corresponding to Lemma 2.1 is as follows.

Lemma 3.2. *For $m \geq 1$, and a mod m random walk $G(m, \mathbf{p}, \mathbf{q})$ with parameters $\mathbf{p} = (p_0, p_1, \dots, p_{m-1})$ and $\mathbf{q} = (q_0, q_1, \dots, q_{m-1})$ satisfying $p_i q_i \neq 0$ for $i = 0, 1, \dots, m-1$, one has*

$$p_m^* = \frac{p_0 p_1 \cdots p_{m-1}}{p_0 p_1 \cdots p_{m-1} + q_0 q_1 \cdots q_{m-1}} = \frac{\rho_m}{1 + \rho_m}. \tag{3.6}$$

Proof. For this general case, set

$$v_m = P(A^c \mid S_1 = 1) \quad \text{and} \quad \bar{v}_m = P(A^c \mid S_1 = -1). \tag{3.7}$$

so that the expression for p_m^* in (2.6) becomes

$$p_m^* = p_0 v_m (p_0 v_m + q_0 \bar{v}_m)^{-1}$$

Thus (3.6) will be proved once it is established that

$$\frac{v_m}{\bar{v}_m} = \frac{p_1 p_2 \cdots p_{m-1}}{q_1 q_2 \cdots q_{m-1}}. \tag{3.8}$$

By definition, v_m ($\bar{v}_m$) is the probability (of "ruin') that starting at 1 (-1) the random walk reaches m ($-m$) before it reaches 0. Moreover, by the modulo structure of the walk, $\bar{v}_m$ is the same as the probability that starting at $m-1$, the random walk reaches 0 before m. Thus, v_m, for example is the same as ${}_0f_{1m}$ in the usual notation for these taboo probabilities; cf. Chung (1967, Section I.12) where these are derived for the random walk. Direct substitution of these exact values would then justify (3.6). Since we only require the ratio of these two taboo probabilities, the following mapping approach suffices, and may be of separate interest.

We first construct a 1–1 correspondence between the set, Γ_k, of paths that go from 1 to m without hitting 0 and the set, G_k, of paths that go from $m-1$ to 0 without hitting m. This correspondence is a simple reversal: If $\mathbf{s}_k = (s_1, s_2, \ldots, s_k, m)$ denotes a path in Γ_k so that $s_1 = 1$, $s_k = m-1$ and $1 \le s_i \le m-1$ for $1 \le i \le k$, the corresponding reversed path in G_k is

$$\mathbf{t}_k(\mathbf{s}_k) = \mathbf{t}_k = (t_1, t_2, \ldots, t_k, 0) \equiv (s_k, s_{k-1}, \ldots, s_1, 0).$$

(The reader can visualize the reversal of a path in the illustration of Figure 1. In fact, the result becomes fairly transparent once one recognizes the effect on paths of flipping the time axis.)

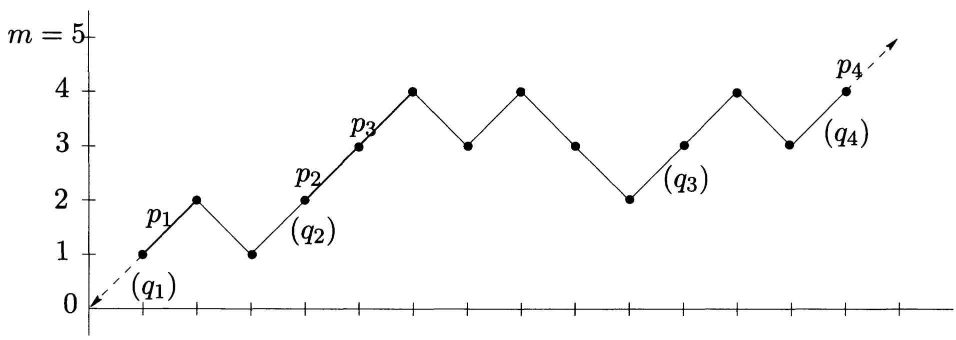

Figure 1. An illustration for $m = 5$ of the correspondence between first hitting paths from 1 to 5 and those from 4 to 0. (Probabilities in parentheses are those for the indicated segments on the reversed path.)

For a given path $\mathbf{s}_k$, let ν_i^+ (ν_i^-) equal the number of transitions from i to $i+1$ (i to $i-1$). Then

$$(3.9) \qquad P\big((S_1, S_2, \ldots, S_{k+1}) = \mathbf{s}_k | S_1 = 1\big) = \prod_{i=1}^{m-1} p_i^{\nu_i^+} q_i^{\nu_i^-}$$
$$= \left(\prod_{i=1}^{m-1} p_i\right) \prod_{j=1}^{m-2} (p_j q_{j+1})^{\nu_j^+ - 1},$$

with the last step following since $\nu_1^- = 0$, $\nu_{m-1}^+ = 1$ and $\nu_{i+1}^- = \nu_i^+ - 1 \geq 0$ for $1 \leq i \leq m-2$. For the reversed path $\mathbf{t}_k(\mathbf{s}_k)$, where an "up" transition of s_j to s_{j+1} in $\mathbf{s}_k$ becomes a "down" transition of s_{j+1} to s_j. Write $\hat{\nu}_i^+$ and $\hat{\nu}_i^-$ for the corresponding numbers for $\mathbf{t}_k$ so that

(3.10)
$$P\big((S_1, S_2, \ldots, S_{k+1}) = \mathbf{t}_k \mid S_1 = m-1\big) = \left(\prod_{i=1}^{m-1} q_i\right) \prod_{j=2}^{m-1} (q_j p_{j-1})^{\hat{\nu}_j^- - 1}.$$

But it is clear from the correspondence that $\hat{\nu}_j^- = \nu_{j-1}^+$. Thus for every $k \geq m-1$ and every path $\mathbf{s}_k \in \Gamma_k$ the ratio of (3.9) over (3.10), namely $p_1 p_2 \cdots p_{m-1} / q_1 q_2 \cdots q_{m-1}$, is constant. It now follows immediately that (3.8) holds, thereby completing the proof. □

4. Random mixtures of Parrondo games $G(m, p, p')$

The main question of interest for these games concerns what happens to a player's fortune when two or more games are played in some alternating fashion. For example, if two different games are known to be fair, can a player create a winning game by randomly choosing between the two at each play? Observe first of all that for $\pi \in [0, 1]$, the random mixture of two games, $G(m, \mathbf{p}, \mathbf{q})$ and $G(m, \mathbf{P}, \mathbf{Q})$, in which at each play the former is chosen with probability π, is also a mod m game, namely, $G(m, \pi\mathbf{p} + (1-\pi)\mathbf{P}, \pi\mathbf{q} + (1-\pi)\mathbf{Q})$. Since Lemma 3.1 characterizes the winning or losing nature of any such game, the question of whether the random mixture of two fair games is a winning game or not has been theoretically answered. By the way, the criterion in Lemma 3.1 implies that if $\mathbf{p}$ and $\mathbf{q}$ are interchanged in a fair game $G(m, \mathbf{p}, \mathbf{q})$, it remains fair, whereas a losing game would be turned into a winning game. Moreover, the nature of the criterion is such that it should be the exception rather than the rule for a random mixture of fair games to remain fair. Thus at this stage, the existence of fair games whose mixture is winning (or losing) would appear to be less paradoxical.

A couple of general questions of interest are as follows. Suppose we say that two fair games, A and B, are *mutually supportive* if any other game consisting of a sequence of plays of game A or B is not a losing game whenever the game choices are made independently of previous outcomes. Do mutually supportive pairs of distinct games exist? Is it true that if a non-trivial random mixture (in which game A is chosen independently at each stage with constant probability) does not result in a losing game, then the two games are mutually supportive?

In this section, we give a complete answer to the structure of random mixtures in the special case of the Parrondo game, $G(m,p,p')$. Although this is done by rather elementary methods, more general questions involving mixtures appear to be quite difficult.

Consider the random mixture, $G(m, \pi p + (1-\pi)\beta, \pi p' + (1-\pi)\beta')$, of the two Parrondo games $G(m,p,p')$ and $G(m,\beta,\beta')$, in which the mixing probability is $\pi \in (0,1)$. Set

$$x = p/q, \quad y = p'/q', \quad \hat{x} = \frac{\beta}{1-\beta}, \quad \hat{y} = \frac{\beta'}{1-\beta'} \tag{4.1}$$

and

$$\bar{x} = \frac{\pi p + (1-\pi)\beta}{1 - \{\pi p + (1-\pi)\beta\}} = \frac{p + \lambda\beta}{q + \lambda(1-\beta)}, \qquad \bar{y} = \frac{p' + \lambda\beta'}{q' + \lambda(1-\beta')}, \tag{4.2}$$

where $\lambda = (1-\pi)/\pi$. Assume without loss of generality that $\beta < p$, or equivalently, $\hat{x} < x$.

The question to consider is whether the random mixture of two losing games can be a winning game. Suppose first that the two given games are fair. That is, by Corollary 2.1 in the form (2.9), our question is whether it is possible to have

$$y = x^{-m+1}, \quad \hat{y} = \hat{x}^{-m+1} \quad \text{and} \quad \bar{y} > \bar{x}^{-m+1}. \tag{4.3}$$

For simplicity, write $m - 1 = r$ so that $r = 1, 2, \ldots$. Simple algebra leads to

$$\bar{x} = \frac{x(1+\hat{x}) + \lambda\hat{x}(1+x)}{1+\lambda+\hat{x}+\lambda x}, \quad \bar{y} = \frac{y(1+\hat{y}) + \lambda\hat{y}(1+y)}{1+\lambda+\hat{y}+\lambda y}. \tag{4.4}$$

Substitution of the first two equations of (4.3) into $\bar{y}$ permits the inequality $\bar{y} > \bar{x}^{-r}$ to be written after simplification as

$$\frac{1+\lambda+\hat{x}^r+\lambda x^r}{(1+\lambda)(\hat{x}x)^r + \lambda\hat{x}^r + x^r} > \frac{(1+\lambda+\hat{x}+\lambda x)^r}{\big((1+\lambda)(\hat{x}x) + \lambda\hat{x} + x\big)^r}. \tag{4.5}$$

Clearly, this can never hold if $r = 1$, (i.e., $m = 2$). We assume, therefore, that $m > 2$ in the remainder of this section.

If one introduces functions $f(a) = a^r$ and $g(a,b) = (1+\lambda+a+\lambda b)/\big((1+\lambda)ab + \lambda a + b\big)$, then (4.5) involves a form of inverse composition, namely,

$$g\big(f(\hat{x}), f(x)\big) > f\big(g(\hat{x}, x)\big).$$

On the other hand, (4.5) may be written equivalently in terms of π as

$$\frac{1+\pi\hat{x}^r+(1-\pi)x^r}{(1+\pi\hat{x}+(1-\pi)x)^r} > \frac{1+\pi\hat{x}^{-r}+(1-\pi)x^{-r}}{1+\pi\hat{x}^{-1}+(1-\pi)x^{-1})^r}. \tag{4.6}$$

Thus, this inequality is one about norms on the simplex as may be seen as follows: If we set $\mathbf{u}=(1,\hat{x},x)$ and $\mathbf{v}=(1,1/\hat{x},1/x)$, (4.5) is equivalent to

$$\|\mathbf{u}\|_{r,\mu}/\|\mathbf{u}\|_{1,\mu} > \|\mathbf{v}\|_{r,\mu}/\|\mathbf{v}\|_{1,\mu},$$

where the norms are with respect to the measure μ that assigns masses $1,\pi,1-\pi$ to the coordinates $1,2,3$, respectively. [In this context, the special case of $\hat{x}=1$, in which the first game is the classical fair random walk, (and which is the case relevant to the examples in Harmer and Abbott (1999a)), is describable as a comparison between the r-norms of the ray projection onto the unit simplex of the vectors $(1,1,x)$ and $(1,1,x^{-1})$ (or equivalently, $(1,x,x)$). Moreover, in the case of purely random mixing ($\pi=\frac{1}{2}$), the inequality is more enticing in that it may be stated as above but for vectors $(1,1,1,x)$ and $(1,1,1,x^{-1})$ under counting measure on the coordinates.]

Fix $\hat{x}=a\geq 1$. By cross multiplying in (4.6), the inequality is equivalent to the positivity of the polynomial

$$\begin{aligned}
Q(x) &:= (1+\lambda+a^r+\lambda x^r)\big((1+\lambda)ax+\lambda a+x\big)^r \\
&\qquad - \big((1+\lambda)a^r x^r+\lambda a^r+x^r\big)(1+\lambda+a+\lambda x)^r \\
&= (1+\lambda+a^r)\big((1+\lambda)ax+x+\lambda a\big)^r - \lambda a^r(1+\lambda+a+\lambda x)^r \\
&\qquad + x^r\big\{\lambda\big((1+\lambda)ax+x+\lambda a\big)^r \\
&\qquad\qquad - \big((1+\lambda)a^r+1\big)(1+\lambda+a+\lambda x)^r\big\} \\
&= \sum_{j=0}^{r}\binom{r}{j}\{(1+\lambda+a^r)\big((1+\lambda)a+1\big)^j(\lambda a)^{r-j} \\
&\qquad\qquad - \lambda a^r(1+\lambda+a)^{r-j}\lambda^j\}x^j \\
&\qquad + \sum_{k=0}^{r}\binom{r}{k}\{\lambda\big((1+\lambda)a+1\big)^k(\lambda a)^{r-k} \\
&\qquad\qquad - \big((1+\lambda)a^r+1\big)(1+\lambda+a)^{r-k}\lambda^k\}x^{r+k}.
\end{aligned} \tag{4.7}$$

Upon writing $Q(x)=\sum_{j=0}^{2r}q_j x^j$, it follows that the coefficients are

$$q_j = \begin{cases}
\binom{r}{j}a^r\{(1+\lambda+a^r)(1+\lambda+a^{-1})^j\lambda^{r-j} \\
\qquad - (1+\lambda+a)^{r-j}\lambda^{j+1}\} & \text{for } 0\leq j<r, \\
(1+\lambda+a^r)\big((1+\lambda)a+1\big)^r \\
\qquad - \big((1+\lambda)a^r+1\big)(1+\lambda+a)^r & \text{for } j=r, \\
\binom{r}{j-r}a^r\{(1+\lambda+a^{-1})^{j-r}\lambda^{2r-j+1} \\
\qquad - (1+\lambda+a^{-r})(1+\lambda+a)^{2r-j}\lambda^{j-r}\} & \text{for } r<j\leq 2r.
\end{cases} \tag{4.8}$$

Since the expressions within the parentheses in the first and third cases are increasing in j, there can be at most one change of sign among the first r coefficients and at most one among the last r. Thus, regardless of the sign of the middle coefficient, q_r, there are at most three changes of signs in the coefficients of Q with the exact number depending upon the signs of $q_0, q_{r-1}, q_r \cdot q_{r+1}, q_{2r}$. (One may check that q_{2r} is always positive for $a > 1$, while q_0 is negative when $\lambda \leq 1$ (i.e., $\pi \geq \frac{1}{2}$) or when $a \leq 1$.) By Descartes's rule of signs, the number of positive roots of $Q(x) = 0$ does not, therefore, exceed 3.

It follows directly from the definition of Q that $Q(a) = 0$. However, one may check that $x = a$ is in fact a double root for all positive a. To see this, compute directly that

$$\begin{aligned} (4.9) \quad Q'(x) = {} & rx^{r-1}\lambda\big((1+\lambda)ax + x + \lambda a\big)^r \\ & + r\big((1+\lambda)a+1\big)(1+\lambda+a^r+\lambda x^r)\big((1+\lambda)ax+x+\lambda a\big)^{r-1} \\ & - \big((1+\lambda)a^r+1\big)rx^{r-1}(1+\lambda+a+\lambda x)^r \\ & - r\lambda\big((1+\lambda)a^r x^r + x^r + \lambda a^r\big)(1+\lambda+a+\lambda x)^{r-1} \end{aligned}$$

so that after simplification

$$\begin{aligned} Q'(a) = r(1+\lambda)^r a^{r-1}(a+1)^{r-1}\{ & \lambda a^r(a+1) + \big((1+\lambda)a+1\big)(1+a^r) \\ & - \big((1+\lambda)a^r+1\big)(1+a) - \lambda a(a^r+1)\} = 0 \end{aligned}$$

for any a. Since this implies that $x-a$ is a double root of Q, it follows from Descartes's rule of signs that Q has either two or three positive roots. In either case, we need to know that the root at $x = a$ is the largest positive root. To show this, differentiate (4.9) to obtain

$$\begin{aligned} r^{-1}Q''(x) = {} & \lambda(r-1)x^{r-2}\big((1+\lambda)ax+\lambda a+x\big)^r \\ & + 2\lambda r x^{r-1}\big((1+\lambda)a+1\big)\big((1+\lambda)ax+\lambda a+x\big)^{r-1} \\ & + (r-1)\big((1+\lambda)a+1\big)^2(1+\lambda+a^r+\lambda x^r)\big((1+\lambda)ax+\lambda a+x\big)^{r-2} \\ & - \big((1+\lambda)a^r+1\big)\{(r-1)x^{r-2}(1+\lambda+a+\lambda x)^r \\ & \qquad + 2\lambda r x^{r-1}(1+\lambda+a+\lambda x)^{r-1}\} \\ & - \lambda^2(r-1)\big((1+\lambda)a^r x^r + x^r + \lambda a^r\big)(1+\lambda+a+x)^{r-2} \end{aligned}$$

from which

$$Q''(a) = r(1+\lambda)^{r-1}a^{r-2}(a+1)^{r-2} \times \{\lambda(1+\lambda)(r-1)a^r(a+1)^2 + 2\lambda r a^r\big((1+\lambda)a+1\big)(a+1) + (r-1)\big((1+\lambda)a+1\big)^2(1+a^r) - (1+\lambda)(r-1)(a+1)^2\big((1+\lambda)a^r+1\big) - 2\lambda r a(a+1)\big((1+\lambda)a^r+1\big) - \lambda^2(r-1)a^2(a^r+1)\}.$$

By grouping the terms within the parentheses here according to powers of a, this becomes

$$Q''(a) = (1+\lambda)^{r-1}ra^{r-2}(a+1)^{r-2} \times \{\lambda(r-1)(a^{r+2}-1) + 2\lambda r(a^{r+1}-a) + \lambda(r+1)(a^r-a^2)\}.$$

Thus, for $r \geq 2$ $(m \geq 3)$, $Q''(a)$ is positive, negative or zero according as $a > 1$, $a < 1$ or $a = 1$. This implies in particular that when $a = 1$, $x = 1$ is a triple root, and hence the only root by Descartes's rule of signs. Thus, when $a = 1$, $x = 1$ is the only positive root, insuring that $Q(x) > 0$ for all $x > 1$. For $a > 1$, the fact that $Q''(a) > 0$ shows that this double root at $x = a$ is a local minimum. Since by (4.8) the leading coefficient, q_{2r}, is positive for all λ and all $a > 1$, this insures again that $x = a$ is the largest real root of $Q(x) = 0$, thereby establishing that $Q(x) > 0$ for all $x > a$ whenever $a \geq 1$. This completes the proof of

Theorem 4.1. *The random mixture, $G(m, \pi p + (1-\pi)\beta, \pi p' + (1-\pi)\beta')$, of two fair games, $G(m, \beta, \beta')$ and $G(m, p, p')$, is a winning game whenever $m \geq 3$ and $\frac{1}{2} \leq \beta < p \leq 1$.*

Corollary 4.1. *There exist losing games, the random mixture of which is a winning game.*

Proof. By Corollary 2.1, the expression whose sign determines whether a game is winning, losing or fair, is a continuous function of its variables. It is therefore clear that for the games appearing in the statement of Theorem 4.1, one may make a sufficiently small change in the parameters (β, β') and (p, p') to make the associated fair games become losing ones, while preserving the inequality that ensures that the random mixture of the two remains a winning game. □

The example presented in Harmer and Abbott (1999a) may now be described as follows. Take $m = 3$, $\beta = \frac{1}{2} = \beta'$, $p = \frac{3}{4}$ and $p' = 1/10$. The games $G(3, \frac{1}{2}, \frac{1}{2})$ and $G(3, \frac{3}{4}, 1/10)$ are fair by Corollary 2.1, so that by Theorem 4.1, the mixture $G(3, \frac{5}{8}, 3/10)$ is a winning game. Consider now the games used by these authors, $G(3, \frac{3}{4}-\varepsilon, 1/10-\varepsilon)$ and $G(3, \frac{1}{2}-\varepsilon, \frac{1}{2}-\varepsilon)$, and their random mixture $G(3, \frac{5}{8}-\varepsilon, 3/10-\varepsilon)$. It is clear that the first two are losing games for each positive $\varepsilon < 1/10$ and that there would be some positive value $\varepsilon_0 \leq 1/10$ for which the mixture remains a winning game whenever $0 < \varepsilon < \varepsilon_0$, as postulated in Harmer and Abbott (1999a).

In this section we have considered the random mixing of two $G(m, \mathbf{p})$ walks. One is also interested in deterministic mixtures. Simulations in Harmer and Abbott (1999a) indicate that deterministic mixtures of the two games proposed by Parrondo turn their separate losing nature into a winning combination. It is difficult in general to analyze such deterministic mixtures since it requires computing the stationary probabilities of the product of the associated stochastic matrices. To expand upon this, suppose one has two distinct $G(m, \mathbf{p})$ games called A and B with parameters $a_j, 0, 1-a_j$ and $b_j, 0, 1-b_j$, respectively. By Lemma 3.2, the probabilities $p_m^*(A)$ and $p_m^*(B)$ for the two games would equal $\frac{1}{2}$ (i.e., the games would be fair) if and only if

$$\prod_{j=0}^{m-1} \frac{a_j}{1-a_j} = 1 = \prod_{j=0}^{m-1} \frac{b_j}{1-b_j}. \tag{4.10}$$

Consider now the random walk formed by alternating the transition probabilities of these two. Then the two-step process is also a random walk, though one with jumps of two units and with non-zero probabilities r_i of zero jumps. That is, the alternation of two $G(m, \mathbf{p})$ games is equivalent to a $G(m, \mathbf{p}, \mathbf{q})$ game. This 2-step process is then reducible with two classes, the odd and the even integers. If the walk starts in state "0," for example, the corresponding quotient of relevant parameters is

$$\frac{(a_0 b_1)(a_2 b_3)\cdots(a_{m-2} b_{m-1})}{(1-a_0)(1-b_1)\cdots(1-a_{m-2})(1-b_{m-1})}. \tag{4.11}$$

Since only half of the parameters enter here, it is clear that this ratio may be greater or less than or equal to 1 even when the separate games are fair. This implies that when m is even, the alternation of two fair games may be either fair, winning or losing. Notice that even if one imposes the natural restriction that a fair game must be fair for all starting states one gains nothing more since, for example, the condition for fairness starting in state "1," namely,

$$\frac{(a_1 b_2)(a_3 b_4)\cdots(a_{m-1} b_0)}{(1-a_1)(1-b_2)\cdots(1-a_{m-1})(1-b_0)} = 1, \tag{4.12}$$

is equivalent under (4.10) to the expression in (4.11) being set equal to 1.

When m is odd, the alternation of fair games is fair as can be seen by considering the two-step game as a mod $2m$ game for which fairness requires by Lemma 3.1 that the product of (4.11) and the quotient on the right-hand side of (4.12) must be equal to 1, which follows from (4.10). Thus the alternation of these fair games cannot result in winning ones when m is odd.

The story is different, however, for $[AABB]$, the mixture in which two plays of game A are alternated with two plays of B. In view of the previous paragraphs, this game is equivalent when m is odd to an alternating $[AB]$ game *but* one in which both A and B are $G(m,\mathbf{p},\mathbf{q})$ games. For $m=3$ this is reasonably tractable. In particular, if one of the games is the classical simple random walk one can show that the mixture is indeed a winning game under a natural restriction on the second game. For the special case of $AABB$ in which A and B are the fair games $G(3,\frac{1}{2},\frac{1}{2})$ and $G(3,\frac{3}{4},1/10)$ corresponding to Parrondo's example, one can show that the asymptotic average gain is $0.0218363 > 0$.

5. Direct calculation of the asymptotic expected average gain for a $G(m,\mathbf{p})$ game

By (2.3), since p_m^* has been evaluated, the asymptotic average gain (or loss) would be known once $E(T_2-T_1)$ is computed. A closed form for this expected inter-occurrence time is discussed below since it is of interest in its own right for these processes. However, the asymptotic average gain, $\lim_{n\to\infty} S_n/n$, being a limit of bounded r.v.'s, may also be derived directly by obtaining the limit of the corresponding expectations. We do this as follows.

Consider the game $G(m,\mathbf{p})$. Define

$$\begin{aligned}\mu_k^{(j)} &:= E(S_{n+k}-S_n \mid S_n \equiv j \bmod m)\\ &= E\Bigl(\sum_{i=1}^{k} E(X_i \mid S_0 \equiv j \bmod m)\Bigr),\end{aligned} \tag{5.1}$$

emphasizing by the notation the fact that the expectation depends only upon the congruence class of S_n modulo m and not upon the actual value of S_n nor of n. In fact, the random walk S_n is equivalent to the random walk on the circular group of integers mod m where a positive move is taken to be in the clockwise direction. Clearly,

$$\begin{aligned}\mu_{k+1}^{(0)} &= p_0\bigl(1+\mu_k^{(1)}\bigr) + q_0\bigl(-1+\mu_k^{(m-1)}\bigr)\\ &= p_0 - q_0 + p_0\mu_k^{(1)} + q_0\mu_k^{(m-1)}.\end{aligned}$$

Similarly, for $j = 1,2,\dots,m-1$,

$$\mu_{k+1}^{(j)} = p_j - q_j + p_j\mu_k^{(j+1)} + q_j\mu_k^{(j-1)} \tag{5.2}$$

where we equate $\mu_k^{(m)} = \mu_k^{(0)}$ and $\mu_k^{(-1)} = \mu_k^{(m-1)}$. To express this conveniently in matrix form, write $\boldsymbol{\mu}_k = (\mu_k^{(0)},\dots,\mu_k^{r})'$ and $\mathbf{b} = (p_0-q_0, p_1-$

$q_1, \ldots, p_r - q_r)'$ as $m \times 1$ column vectors and set

$$\mathbb{C} = \begin{bmatrix} 0 & p_0 & 0 & \cdots & 0 & q_0 \\ q_1 & 0 & p_1 & \cdots & 0 & 0 \\ 0 & q_2 & 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{r-1} & 0 & p_{r-1} \\ p_r & 0 & 0 & \cdots & q_r & 0 \end{bmatrix} \tag{5.3}$$

where again $r = m - 1$. Since $\boldsymbol{\mu}_1 = \mathbf{b}$, it is clear from (5.2) that

$$\boldsymbol{\mu}_k = \mathbf{b} + \mathbb{C}\mathbf{b} + \mathbb{C}^2\mathbf{b} + \cdots + \mathbb{C}^{k-1}\mathbf{b}.$$

which implies that

$$\lim_{n\to\infty} \boldsymbol{\mu}_n / n = \left(\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{C}^i \right) \mathbf{b}. \tag{5.4}$$

The reader should note that if $\{S_n : n \geq 0\}$ were a more general mod m Markov chain, the vector $\mathbf{b}$ would be given by

$$b_i \equiv E(X_{n+1} \mid S_n = i \bmod m) = \sum_j (j - i) p_{ij}$$

and $\mathbb{C}$ would be determined by

$$C_{ij} = P(S_{n+1} = j \bmod m \mid S_n = i \bmod m) = \sum_{k \in \mathbb{Z}} p_{i,j-i+km}.$$

That is, the transition matrix $\mathbb{C}$ for the Markov chain of congruence classes of $\{S_n\}$ is formed from the original chain's transition matrix $\mathbf{P}$ by summing over all states in the appropriate congruence class. With these definitions, the limit of (5.4) applies to a general mod m Markov chain. We shall continue, however, with the $G(m, \mathbf{p})$ case in order to obtain explicit values.

The value of this limit depends upon the periodicity of $\mathbb{C}$. Suppose first that m is *odd*. In this case, $\mathbb{C}$ is an irreducible aperiodic stochastic matrix provided only that $p_j q_j \neq 0$ for each j. Thus the limit exists and is a stochastic matrix, each of whose rows is the row vector of stationary probabilities associated with $\mathbb{C}$, $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_{m-1})$, say. It is a known result of G. Mihoc (cf. Fréchet, 1952, pp. 114–116) that the entries in $\boldsymbol{\pi}$ are proportional to the diagonal cofactors of $\mathbb{I} - \mathbb{C}$. (See Appendix A below for this and other results to be used below.)

Let γ_{im} denote the (i,i)th cofactor of $\mathbb{I} - \mathbb{C}$. These are tractable for reasonable values of m. Due to the cyclic structure underlying the matrix

$\mathbb{C}$ it is necessary only to obtain the first cofactor for each m. The first few values are:

$$(5.5)\qquad \gamma_{13} = 1 - p_1q_2, \qquad \gamma_{14} = 1 - p_1q_2 - p_2q_3,$$
$$\gamma_{15} = 1 - p_1q_2 - p_2q_3 - p_3q_4 + p_1q_2p_3q_4$$

and

$$\begin{aligned}\gamma_{16} &= (1 - p_1q_2 - p_2q_3)(1 - p_3q_4 - p_4q_5) - p_2p_3q_3q_4 \\ &= 1 - p_1q_2 - p_2q_3 - p_3q_4 - p_4q_5 + p_1q_2p_3q_4 + p_1q_2p_4q_5 + p_2q_3p_4q_5\end{aligned}$$

The remaining diagonal cofactors are then obtained for each m by successively applying the cyclic permutation of $(p_0, p_1, \ldots, p_{m-1})$ into $(p_1, p_2, \ldots, p_{m-1}, p_0)$. For the case of a Parrondo $G(m, p, p')$ game with $m = 3$, the situation studied in Harmer and Abbott (1999a), (5.5) implies that

$$\gamma_{13} = 1 - pq, \quad \gamma_{23} = 1 - pq', \quad \gamma_{33} = 1 - p'q.$$

A general formula, presumably known, is possible for these cofactors, namely,

$$(5.6)\quad \gamma_{1m} = 1 - \sum_{i=1}^{m-2} p_iq_{i+1} + \sum_{1 \le i < j-1 \le m-3} p_iq_{i+1}p_jq_{j+1} - \sum_{1 \le i < j-1 < k-2 \le m-4} p_iq_{i+1}p_jq_{j+1}p_kq_{k+1} + \cdots$$

with the series continuing as long as the largest subscript does not exceed $m - 1$. Thus for $l = [(m-1)/2]$, the last term has sign $(-1)^l$ and involves l subscripts $i_1, \ldots, i_l$ satisfying

$$1 \le i_1 < i_2 - 1 < i_3 - 2 < \cdots i_l - l + 1 \le m - l.$$

As indicated by its appearance, (5.6) follows from an inclusion-exclusion argument based on the number of pairs of adjacent diagonal 1's used in the evaluation of the cofactor's determinant. (All diagonal cofactors are of course equal for each value of $m \ge 3$ whenever the parameters p_j and q_j do not depend on j.)

As mentioned earlier, the stationary probabilities associated with $\mathbb{C}$ are proportional to these diagonal cofactors so that in our previous notation $\pi_i = \gamma_{i+1,m}/\gamma_{\cdot m}$ where $\gamma_{\cdot m} = \gamma_{1m} + \cdots + \gamma_{mm}$.

An early reference for the study of the general cyclical random walk on the integers modulo m, the one whose transition matrix is $\mathbb{C}$, is Fréchet (1952, pp. 122–125). This is in effect a 1938 reference for this random walk, called by Fréchet, "mouvement circulaire", since the material is present in the 1938 first edition of his book. He works out as an example the stationary probabilities for the case of $m = 4$. He obtains γ_{14} as $p_2p_3 + q_1q_2$ which is easily seen to agree with the expression given above in (5.5).

The asymptotic average gain given by (5.4) now follows directly from the above for the case when m is odd. It is of the form $\lambda_m(1, 1, \ldots, 1)'$ with

$$\lambda_m = \boldsymbol{\pi}_m \mathbf{b} \equiv \frac{1}{\gamma_{\cdot m}} \sum_{i=1}^{m} \gamma_{im}(p_{i-1} - q_{i-1}). \tag{5.7}$$

Consider now the case of m *even*, say $m = 2k$ for $k \geq 2$. Then $\mathbb{C}$ is the stochastic matrix of a periodic Markov chain of period 2. By clustering the even and odd rows and columns, it may be written in the form

$$\mathbb{C} = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \tag{5.8}$$

in which A and B are $k \times k$ stochastic matrices. Consequently,

$$\mathbb{C}^2 = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix}, \quad \mathbb{C}^{2s} = \begin{pmatrix} (AB)^s & 0 \\ 0 & (BA)^s \end{pmatrix},$$
$$\mathbb{C}^{2s+1} = \begin{pmatrix} 0 & A(BA)^s \\ B(AB)^s & 0 \end{pmatrix}$$

in which both AB and BA are irreducible aperiodic recurrent stochastic matrices. If $\boldsymbol{\delta}$, $\boldsymbol{\rho}$ represent the vectors of limiting stationary probabilities for AB and BA, respectively, and if D and R are the matrices all of whose rows are $\boldsymbol{\delta}$ and $(\boldsymbol{\rho})$, respectively, then

$$\lim_{s\to\infty} \mathbb{C}^{2s} = \begin{pmatrix} D & 0 \\ 0 & R \end{pmatrix}, \quad \lim_{s\to\infty} \mathbb{C}^{2s+1} = \begin{pmatrix} 0 & R \\ D & 0 \end{pmatrix}$$

and so (5.4) becomes in the case of m even,

$$\lim_{n\to\infty} \boldsymbol{\mu}_n/n = \frac{1}{2} \begin{pmatrix} 0 & R \\ D & 0 \end{pmatrix} \mathbf{b}. \tag{5.9}$$

By the result of Mihoc, the elements of the common rows $\boldsymbol{\delta}$ and $\boldsymbol{\rho}$ of D and R are proportional to the diagonal cofactors of AB and BA, respectively. However, as shown in the appendix below, the diagonal cofactors of $\mathbb{I} - \mathbb{C}$ are made up of those of $\mathbb{I}_{m/2} - AB$ and $\mathbb{I}_{m/2} - BA$ and that the column sums of the latter are equal and equal to $\frac{1}{2}$ of the sum of the diagonal cofactors of $\mathbb{I} - \mathbb{C}$; cf. (A.6) below. In view of (5.9) it follows that (5.7) holds true as well when m is even. We summarize this as

Theorem 5.1. *For the general $G(m, \mathbf{p})$ game, with probability one,*

$$\lim_{n\to\infty} \frac{S_n}{n} \equiv \lambda_m = \boldsymbol{\pi}_m \mathbf{b} \equiv \frac{1}{\gamma_{\cdot m}} \sum_{i=1}^{m} \gamma_{im}(p_{i-1} - q_{i-1}). \tag{5.10}$$

in which the γ_i are the diagonal cofactors of $\mathbb{I} - \mathbb{C}$ and $\gamma_{\cdot m}$ is their sum.

For the special case of a $G(m, p, p')$ walk, the limit of interest in (5.10) becomes

$$\begin{aligned} \lambda_m &= \{(p' - q')\gamma_{1m} + (p - q)(\gamma_{\cdot m} - \gamma_{1m})\}/\gamma_{\cdot m} \\ &= 2p - 1 + 2(p' - p)\gamma_{1m}/\gamma_{\cdot m}. \end{aligned} \tag{5.11}$$

From (5.5), the first few values of $\gamma_{\cdot m}$ for a $G(m, p, p')$ walk are

$$\begin{aligned} \gamma_{\cdot 3} &= 3 - pq - pq' - p'q = 2 + p'p^2 + q'q^2 \\ \gamma_{\cdot 4} &= 4 - 4pq - 2pq' - 2p'q = 2(1 - pq) + 2(p'p^2 + q'q^2) \\ \gamma_{\cdot 5} &= 5 - 9pq - 3pq' - 3p'q + pq(pq + 2pq' + 2p'q). \end{aligned}$$

For Game B of Harmer and Abbott (1999a), in which $m = 3$, $p = 3/4 - \varepsilon$ and $p' = 1/10 - \varepsilon$, one obtains

$$\gamma_{13} = 13/16 - \varepsilon/2 + \varepsilon^2, \quad \gamma_{\cdot 3} = \frac{169}{80} - \frac{\varepsilon}{5} + 3\varepsilon^2,$$

from which the limit in (5.11) becomes

$$\lambda_3 = -2\varepsilon \frac{147 - 24\varepsilon + 240\varepsilon^2}{169 - 16\varepsilon + 240\varepsilon^2} \cong -1.74\varepsilon - .16\varepsilon^2 + O(\varepsilon^2) \tag{5.12}$$

This value appears to differ from the one implied by the simulated curve for Game B shown in Fig. 3 of Harmer and Abbott (1999a). The value for the curve given there for $n = 100$ is approximately -1.35, whereas for $\varepsilon = .005$ and $n = 100$, the value from (5.12) is approximately $n\lambda_3 \cong -1.74/2 = -.87$. The difference is that the slope of the simulated curve is affected by the early transient behavior; in a private communication, Harmer and Abbott confirm the agreement with this theoretical limit of their simulated slope when the first 100 plays are excluded. The analogous value for their Game A (where $p = p' = \frac{1}{2} - \varepsilon$) is $n\lambda_3 = (-2\varepsilon)n = -1$ which agrees with the curve for Game A given in their Fig. 3.

For the randomized game that chooses between Games A and B with probability $\frac{1}{2}$, one obtains $p = \frac{5}{8} - \varepsilon$ and $p' = 3/10 - \varepsilon$ for which

$$\gamma_{13} = \frac{49}{64} - \frac{\varepsilon}{4} + \varepsilon^2, \quad \gamma_{\cdot 3} = \frac{709}{320} - \frac{\varepsilon}{10} + 3\varepsilon^2.$$

Thus in this randomized case the asymptotic slope of S_n/n is by (5.11)

$$\begin{aligned} (5.13) \qquad \lambda_3 &= \frac{1}{4} - \frac{13 \times 49}{4 \times 709} - \varepsilon\left\{2 - \frac{52 \times 611}{(709)^2}\right\} + O(\varepsilon^2) \\ &\cong .0254 - 1.9368\varepsilon + O(\varepsilon^2); \end{aligned}$$

the expansion used in the first step requires only that $\varepsilon < .876$. For the parameters $n = 100$ and $\varepsilon = .005$ used in Fig. 3 of Harmer and Abbott (1999a) the asymptotic approximation becomes $n\lambda_3 \cong 2.54 - .97 = 1.57$. This differs from their simulated value of about 1.26, again due to early outcome effects. The reader might note that the graphs in the insert of Fig. 3 seem to be closer to those of (5.12) and (5.13).

As an illustration for even m, consider $m = 4$ for which the matrices become

$$A = \begin{pmatrix} p' & q' \\ q & p \end{pmatrix}, \quad B = \begin{pmatrix} q & p \\ p & q \end{pmatrix},$$

and

$$AB = \begin{pmatrix} p'p + q'q & -p'p - q'q \\ 2pq - 1 & 1 - 2pq \end{pmatrix}.$$

Hence $\delta_{14} = (1 - 2pq)(1 - 2pq + p'p + q'q)^{-1}$ and thus

$$(5.14) \qquad \lambda_4 = \frac{(p' - p)(1 - 2pq)}{p'p^2 + q'q^2 + 1 - pq} + p - q = \frac{2(p'p^3 - q'q^3)}{p'p^2 + q'q^2 + 1 - pq};$$

see also (6.5) below.

In this section, we restricted consideration to $G(m, \mathbf{p})$ games. The approach applies as well to $G(m, \mathbf{p}, \mathbf{q})$ games but with the simplifying zero diagonal of $\mathbb{C}$ being replaced with the r_j's.

6. Expected interoccurrence times of visits to $\mathbb{Z}_m$

Set $\tau_j = E(T_1 \mid S_0 = j)$ for $j = 0, \pm 1, \ldots, \pm(m-1)$ to denote the expected time of the first visit to $\mathbb{Z}_m$ of a $G(m, \mathbf{p})$ walk $\{S_n\}$ starting at j. In the expression (2.3) for the asymptotic average gain, the denominator $E(T_2 - T_1)$ is equal to τ_0. Hence, an alternate derivation of the asymptotic average gain would be, in view of Lemma 3.2, to derive τ_0. This may be done by solving the recursion relations satisfied by the τ_j's, namely,

$$(6.1) \qquad \tau_j = p_j \tau_{j+1} + q_j \tau_{j-1} + 1, \quad \text{for } j = 0, \pm 1, \ldots, \pm(m-1),$$

with boundary conditions $\tau_{-m} = \tau_m = 0$, where for negative j we have $p_j = p_{j+m}$ and $q_j = q_{j+m}$ for a mod m walk. The solution of (6.1) is given for example in Chung (1967, I.12.(8)) in which the reader should note that the ρ_i's in this reference are related to the *reciprocals* of those used here.

The expression that one obtains in this way is quite complicated even in the case of $m = 3$ and difficult to simplify into the more tractable expressions that can be obtained by direct solution of (6.1) by matrix inversion. For if $\boldsymbol{\tau} := (\tau_{m-1}, \dots, \tau_1, \tau_0, \tau_{-1}, \dots, \tau_{-m+1})'$ is the $(2m-1)$-dimensional column vector of expected occurrence times, $\mathbf{1}$ is the $(2m-1)$-dimensional column vector of ones and $\mathbb{G}$ denotes the $(2m-1) \times (2m-1)$ matrix of coefficients in (6.1) then the system (6.1) may be expressed as $\boldsymbol{\tau} = \mathbb{G}\boldsymbol{\tau} + \mathbf{1}$ whose solution, with $\mathbb{H} \equiv \mathbb{I} - \mathbb{G}$ is expressible by

$$\boldsymbol{\tau} = (\mathbb{I} - \mathbb{G})^{-1}\mathbf{1} = \mathbb{H}^{-1}\mathbf{1}. \tag{6.2}$$

Thus the expected interoccurrence times of $\mathbb{Z}_m$ are given as the row sums of the matrix $\mathbb{H}^{-1}$. The matrix $\mathbb{H}$ whose inverse is needed is a Jacobi matrix with $-p_i$'s below a diagonal of 1's and $-q_i$'s above it, namely,

$$\mathbb{H} = \begin{bmatrix} 1 & -q_{m-1} & 0 & & & & 0 & 0 & 0 \\ -p_{m-2} & 1 & -q_{m-2} & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ 0 & & & 1 & -q_1 & 0 & & & 0 \\ 0 & & & -p_0 & 1 & -q_0 & & & 0 \\ 0 & & & 0 & -p_{m-1} & 1 & & & 0 \\ & & & & & \ddots & \ddots & \ddots & \\ 0 & & & & & & -p_2 & 1 & -q_2 \\ 0 & & & & & & 0 & -p_1 & 1 \end{bmatrix}.$$

In particular, by (2.3) the required quantity, $E(T_2 - T_1) = \tau_0$, in the computation of p_m^*, is the sum of the middle row of $\mathbb{H}^{-1}$. Thus, if $H_{i,j}$ denotes the $\{i,j\}$-cofactor of $\mathbb{H} = \mathbb{I} - \mathbb{G}$ and $|\mathbb{H}|$ denotes the determinant of $\mathbb{H}$, then $\tau_0 = H_{\cdot m}/|\mathbb{H}|$ where $H_{\cdot m} = H_{1m} + \cdots + H_{2m-1,m}$.

When $m = 3$, $\mathbb{H}$ is a 5×5 matrix whose middle cofactors are straightforwardly shown to be

$$H_{13} = p_1 p_0 (1 - p_1 q_2), \quad H_{23} = p_0(1 - p_1 q_2), \quad H_{33} = (1 - p_1 q_2)^2,$$
$$H_{43} = q_0(1 - p_1 q_2) \quad \text{and} \quad H_{53} = q_0 q_2 (1 - p_1 q_2).$$

Hence

$$H_{\cdot 3} = (1 - p_1 q_2)(3 - p_1 q_2 - p_2 q_0 - p_0 q_1)$$

and

$$|\mathbb{H}| = (1 - p_1 q_2)(1 - p_1 q_2 - p_2 q_0 - p_0 q_1) = (1 - p_1 q_2)(p_0 p_1 p_2 + q_0 q_1 q_2).$$

Therefore, for $m = 3$,

$$\tau_0 = E(T_2 - T_1) = 1 + 2/(p_0 p_1 p_2 + q_0 q_1 q_2).$$

By (2.3) and (2.4), this implies that with probability one,

$$\lambda_3 = \lim_{n\to\infty} \frac{S_n}{n} = \frac{3(p_0p_1p_2 - q_0q_1q_2)}{2 + p_0p_1p_2 + q_0q_1q_2}. \tag{6.3}$$

The reader may check that this agrees with the expression given for λ_3 in (5.10).

For $m = 4$, $\mathbb{G}$ is a 7×7 matrix, and the middle column's cofactors for the corresponding $\mathbb{H} \equiv (\mathbb{I} - \mathbb{G})^{-1}$ are easily computed to be

$$H_{14} = p_0p_1p_2|\mathbb{K}|, \quad H_{24} = p_0p_1|\mathbb{K}|, \quad H_{34} = p_0(1 - p_2q_3)|\mathbb{K}|, \quad H_{44} = |\mathbb{K}|^2,$$
$$H_{54} = q_0(1 - p_1q_2)|\mathbb{K}|, \quad H_{64} = q_0q_3|\mathbb{K}|, \quad H_{74} = q_0q_3q_2|\mathbb{K}|.$$

where $\mathbb{K}$ is the upper left (*and* lower right) $(m-1) \times (m-1)$ corner matrix of $\mathbb{H}$. This gives

$$H_{\cdot 4} = |\mathbb{K}|[3 - p_0q_1 - p_1q_2 - p_2q_3 - p_3q_0 + (p_1 - q_3)(p_2 - q_0)]$$

and, by expansion along the middle column, the determinant of $\mathbb{H}$ is

$$|\mathbb{H}| = |\mathbb{K}|[p_0p_1p_2p_3 + q_0q_1q_2q_3].$$

Therefore, after simplification,

$$\tau_0 = \frac{\mathbb{H}_{\cdot 4}}{|\mathbb{H}|} = \frac{2(p_0p_1 + p_2p_3 + q_0q_3 + q_2q_1)}{p_0p_1p_2p_3 + q_0q_1q_2q_3}. \tag{6.4}$$

So that by (2.3) and (2.4) the asymptotic slope of the random walk for $m = 4$ is

$$\lambda_4 = \lim_{n\to\infty} \frac{S_n}{n} = \frac{2(p_0p_1p_2p_3 - q_0q_1q_2q_3)}{p_0p_1 + p_2p_3 + q_0q_3 + q_2q_1} \tag{6.5}$$

with probability one. This is consistent with the result obtained by the methods of Section 4; see (4.11).

The above discussion focuses on $G(m, \mathbf{p})$ games rather than the more general $G(m, \mathbf{p}, \mathbf{q})$ games. Only minor modifications for the latter are needed. The term $r_j\tau_j$ is added to the right-hand side of the equations (6.1). This results in a substitution of $p_j/(p_j + q_j)$ and $q_j/(p_j + q_j)$ for the parameters of the walk, and, more significantly, a replacement of the vector $\mathbb{I}$ in the solution (6.2) by the vector of the reciprocals, $p_j + q_j$. A benefit of working out the more general case would be that whenever m is even, one could reduce the problem to one of order $m/2$ by observing that the embedded walk on $\mathbb{Z}_m$ is equivalent in its asymptotic behavior to that of the 2-step random walk in which the parameters would become the products, p_0p_1, q_0q_{m-1}, etc. One can see this already in the example of $m = 4$ above, which the reader may compare to the case of $m = 2$ for the associated 2-step case.

7. A diffusion analogue of a general random walk

Partition the real line into intervals $J_j = (j, j+1] = j + (0,1]$, for $j = 0, \pm 1, \pm 2, \ldots$. Let $\boldsymbol{\mu} = \{\mu_j : j = 0, \pm 1, \ldots\}$ be given constants. For real x set

$$\mu(x) = \sum_j \mu_j 1_{J_j}(x). \tag{7.1}$$

Now define a diffusion $\{W_t : t \geq 0\}$ in terms of a standard Brownian motion $\{B_t : t \geq 0\}$ by

$$dW_t = dB_t + \mu(W_t)dt, \tag{7.2}$$

for $t > 0$. For this process, introduce the probabilities of transition between consecutive integers, namely,

$$p_j = p_j(\mu_j, \mu_{j-1}) = P[W_\cdot \text{ hits } j+1 \text{ before hitting } j-1 \mid W_0 = j] \tag{7.3}$$

and let $q_j = 1 - p_j$. Observe that $q_j(\mu_j, \mu_{j-1}) = p_j(-\mu_{j-1}, -\mu_j)$ by reflection.

To obtain expressions for the p_j in terms of the pertinent drift rates, μ_j and μ_{j-1}, we will use the scale function of the diffusion. For this, fix constants $a < b$ and define for $x \in [a, b]$ the first passage probabilities

$$u(x) = P[W_\cdot \text{ hits } b \text{ before } a \mid W_0 = x]. \tag{7.4}$$

The backward equations for the Markov process $W_\cdot$ imply that u satisfies the second order differential equation $u'' + 2\mu u' = 0$, the solution of which is of the form

$$u(x) = c \int_a^x \exp\left\{-2 \int_a^y \mu(z)\, dz\right\} dy + b.$$

The boundary conditions, $u(a) = 0$, $u(b) = 1$ then give

$$u(x) = \frac{\int_a^x \exp\left\{-2 \int_a^y \mu(z)\, dz\right\} dy}{\int_a^b \exp\left\{-2 \int_a^y \mu(z)\, dz\right\} dy}. \tag{7.5}$$

Note that in the case of $\mu_j = \mu$ for every j, this becomes the formula of Anderson (1960, Theorem 4.1); for $a < 0 < b$

$$P[B_t + \mu t \text{ hits } b \text{ before } a \mid B_0 = 0] = \frac{1 - e^{2a\mu}}{1 - e^{-2(b-a)\mu}} \tag{7.6}$$

when $\mu \neq 0$, and equals $\frac{1}{2}$ when $\mu = 0$.

A scale function for the diffusion, a function, S say, which satisfies $u(x) = \{S(x) - S(a)\}/\{S(b) - S(a)\}$, may be deduced from (7.5) to be

$$S(x) = 2\int_0^x \exp\{-2\int_0^y \mu(z)\}\,dy, \tag{7.7}$$

the scalar 2 being inserted for later simplicity.

For the step function μ considered here, the above may be integrated out for all x. However, our interests here require S only for integer values of $x = n$, and in this case, $S(0) = 0$ and

$$u(x) = \begin{cases} \sum_{k=0}^{n-1} r(\mu_k)\exp\{-2\sum_{j=0}^{k}\mu_j\} & \text{if } n > 0, \\ -\sum_{k=n}^{-1} r(\mu_k)\exp\{2\sum_{j=k+1}^{-1}\mu_j\} & \text{if } n < 0. \end{cases} \tag{7.8}$$

The desired transition probabilities p_j follow directly now from (7.8). It suffices to consider $j = 0$. Since $p_0 = u(0)$ when $b = 1 = -a$, (7.8) implies that

$$p_0 \equiv p(\mu_0, \mu_{-1}) \equiv \frac{S(0) - S(-1)}{S(1) - S(-1)} = \frac{r(\mu_{-1})}{r(\mu_0)e^{-2\mu_0} + r(\mu_{-1})} \tag{7.9}$$

where $r(u) = (e^{2u} - 1)/u$ for $u \neq 0$ and $r(0) = 2$. Note that $p(0,0) = \frac{1}{2}$ as required for standard Brownian motion. Using the fact that $r(u)\exp(-2u) = r(-u)$, we summarize this as follows:

Lemma 7.1. *For the diffusion defined by* (7.2), *the transition probabilities of the embedded random walk on the integers that are defined by* (7.3) *are given by*

$$p_j = \frac{\mu_j(e^{2\mu_{j-1}} - 1)}{\mu_j(e^{2\mu_{j-1}} - 1) + \mu_{j-1}(1 - e^{-2\mu_j})} = \frac{r(\mu_{j-1})}{r(\mu_{j-1}) + r(-\mu_j)} \tag{7.10}$$

for $j = 0, \pm1, \pm2, \ldots$.

It is clear that the recurrence or transience of this diffusion agrees with that of the embedded random walk. By Section 6, this in turn depends upon the quotients, $p_1p_2\cdots p_k/q_1q_2\cdots q_k$. From (7.10),

$$\frac{p_j}{q_j} = \frac{r(\mu_{j-1})}{r(\mu_j)}e^{2\mu_j}. \tag{7.11}$$

Hence for any $k \geq 1$,

$$\prod_{j=1}^{k}\frac{p_j}{q_j} = \frac{r(\mu_0)}{r(\mu_k)}\exp\left\{2\sum_{j=1}^{k}\mu_j\right\} \tag{7.12}$$

with a similar expression for negative indices. Substitution of these into (3.1) and (3.2) would then determine recurrence or not.

It is of interest to point out that the p_j's may be evaluated directly from (7.7) without finding the scale function. To see this, set $b = 1 = -a$ and let $x \in [-1, 1]$. By partitioning the event [W. hits 1 before -1] according to hitting 0 or not before 1 and -1, the Markov property and Anderson's result (7.6) yield

$$u(x) = \begin{cases} (1 - e^{-2x\mu_1})/(1 - e^{-2\mu_1}) \\ \qquad + (e^{-2x\mu_1} - e^{-2\mu_1})/\big(1 - e^{-2\mu_1}u(0)\big), & \text{if } x > 0, \\ (1 - e^{-2(1-x)\mu_2})/\big(1 - e^{-2\mu_2}u(0)\big), & \text{if } x < 0. \end{cases} \tag{7.13}$$

It therefore remains to derive $u(0)$.

For $\alpha \in (0, 1]$ let $v(\alpha)$ denote the value of $u(0)$ when the barriers at ± 1 are replaced by $\pm\alpha$. That is, $v(\alpha)$ is the probability of hitting α before $-\alpha$ given the process starts at zero. By partitioning the event of hitting 1 before -1 according to which of α or $-\alpha$ is hit first, one obtains

$$u(0) \equiv v(1) = v(\alpha)u(\alpha) + [1 - v(\alpha)]u(-\alpha). \tag{7.14}$$

Upon substitution of (7.13) and then solving for $v(1)$ one obtains

$$v(1) = \frac{f(\alpha)}{1/v(\alpha) + f(\alpha) - 1} \tag{7.15}$$

with

$$f(\alpha) = \frac{(1 - e^{-2\alpha\mu_0})(1 - e^{-2\mu_{-1}})}{(e^{-2(1-\alpha)\mu_{-1}} - e^{-2\mu_{-1}})(1 - e^{-2\mu_0})}.$$

Observe that the limit of $f(\alpha)$ as $\alpha \searrow 0$ is

$$f(0+) = \left(\frac{\mu_0}{\mu_{-1}}\right)\frac{e^{2\mu_{-1}} - 1}{1 - e^{-2\mu_0}} = \frac{r(\mu_{-1})}{r(-\mu_0)}.$$

By scaling, $p(\alpha)$ is the same as $v(1)$, *but* with μ_1, μ_2 replaced by $\alpha\mu_1, \alpha\mu_2$. Thus one concludes that $p(0+) = \frac{1}{2}$. Substitution of these limits into the right-hand side of (7.13) leads to

$$v(1) \equiv p(\mu_0, \mu_{-1}) = p_0 = \frac{r(\mu_{-1})}{r(\mu_2) + r(-\mu_0)} \tag{7.16}$$

as desired.

For our interests here, consider the mod m *shift diffusions* in which $\mu_j = \mu_l$ whenever $j \equiv l \bmod m$. In this case, Lemma 7.1 implies that

$$\rho_m = \prod_{j=0}^{m-1} \frac{p_j}{q_j} = \exp\Big\{2\sum_{j=0}^{m-1} \mu_j\Big\} \tag{7.17}$$

so that by Lemma 3.1, the embedded mod m random walk, and hence the mod m shift diffusion, is recurrent, transient toward $+\infty$ or transient toward $-\infty$ according as

$$\sum_{j=0}^{m-1} \mu_j = 0, \qquad > 0 \quad \text{or} \quad < 0. \tag{7.18}$$

Then, by Lemma 3.2, the constant probability of "success" on $\mathbb{Z}_m$ is

$$p_m^* = 1 \Big/ \left(1 + \exp\left\{-2\sum_{j=0}^{m-1} \mu_j\right\}\right). \tag{7.19}$$

Observe that the walk is fair ($p_m^* = \frac{1}{2}$) if and only if $\mu_0 + \mu_1 + \cdots + \mu_{m-1} = 0$.

If one is given the p_j's, one may solve the system of equations given by (7.10) for $j = 0, 1, \ldots, m-1$ to find the shift rates $\mu_0, \ldots, \mu_{m-1}$ for the associated mod m shift diffusion. For example, for the random walk related to Game B of Harmer and Abbott (1999a) in which $m = 3$, $p_0 = 1/10$ and $p_1 = p_2 = \frac{3}{4}$, the drift rates are

$$\mu_0 = -.687032, \quad \mu_1 = 2.748128, \quad \mu_2 = -2.06109.$$

Note that these are proportional to $(-1, 4, -3)$. In fact, for any fair game $G(3, p, p')$, the associated drift rates (μ_0, μ_1, μ_2) are equal to $\mu_1(-q, 1, -p)$ as given by

Theorem 7.1. *If the transition probabilities, p_0, p_1, p_2, of a recurrent* mod *(m) shift diffusion are known, the associated drift rates may be determined uniquely as follows:*

(i) *If each p_i equals $\frac{1}{2}$, then each $\mu = 0$;*

(ii) *If exactly one of the p_i's, say p_2, is equal to $\frac{1}{2}$ then $(\mu_0, \mu_1, \mu_2) = (0, x, -x)$ with x being the solution of $p_0/q_0 = (1 - e^{-2x})/2x$;*

(iii) *If none of the p_i's are equal to $\frac{1}{2}$, then*

$$(\mu_0, \mu_1, \mu_2) = \left(\tfrac{1}{2}\ln w\right)\left(-(1-\theta), 1, -\theta\right)$$

in which $\theta = (1 - q_1/p_1)/(1 - p_0/q_0)$ and $w \equiv e^{2\mu_1}$ is the positive solution other than 1 of the equation

$$\alpha w - w^\theta + (1 - \alpha) = 0 \tag{7.20}$$

where $\alpha = (q_2/p_2)\theta$.

Proof. We prove case (iii) first. Write $x = \mu_1$ and $y = \mu_2$. Set $a = p_1p_2/q_1q_2$ and $b = p_2/q_2$, neither of which equals 1. By (7.11) the equations to be solved are

$$(7.21) \qquad a = r(x+y)/r(y) \quad \text{and} \quad b = r(x)e^{2y}/r(y) = r(x)/r(-y).$$

Observe that

$$\frac{x+y}{y}a = \frac{e^{2(x+y)}-1}{e^{2y}-1} = 1 + \frac{e^{2x}-1}{e^{2y}-1}e^{2y} = 1 + \frac{xb}{y},$$

noting that the arguments of r are not zero in this case. Hence, if we set $u = x+y$, we must have $x = cu$ and $y = (1-c)u$ with $c = (a-1)/(b-1)$. Substitution into the second equation of (7.21) gives

$$b = (\frac{1-c}{c})\frac{e^{2cu}-1}{1-e^{2(c-1)u}}.$$

By setting $w = e^{2x} = e^{2cu}$ this equation becomes

$$\frac{\theta}{b}w - w^\theta + \left(1 - \frac{\theta}{b}\right) = 0$$

which completes the proof of (iii).

Case (i) is clear. For (ii), the constant b above is equal to 1. Since r is an increasing function, this means $x = -y$. The first equation then becomes $a = r(0)/r(y) = 2/r(y)$ which is equivalent to the equation given in the statement of case (ii). □

When p is rational, the equation (7.20) of Theorem 7.1 becomes a polynomial. Here are two other examples: For the fair game $G(3, \frac{2}{3}, \frac{1}{5})$, $\theta = \frac{2}{3}$ and $\alpha = \theta/2$. The equation that determines $\mu_1 = \frac{1}{2}\log w$ is by (7.20), $w - 3w^{2/3} + 2 = 0$. Upon setting $z = w^{1/3}$, the equation becomes $z^3 - 3z^2 + 2 = 0$, or, equivalently, after factoring out $z = 1$, $z^2 - 2z - 2 = 0$. Its desired positive solution is $z = 1 + \sqrt{3}$ so that $\mu_1 = \frac{3}{2}\log(1+\sqrt{3})$. This implies by Theorem 7.1 that the drift rates are

$$\mu_0 = -.502526, \quad \mu_1 = 1.507579, \quad \mu_2 = -1.005053.$$

For the fair Parrondo game $G(3, \frac{4}{5}, 1/17)$, $\theta = \frac{4}{5}$ and $\alpha = \frac{1}{5}$ so that (7.20) becomes $w - 5w^{4/5} + 4 = 0$. With $z = w^{1/5}$, this becomes, after factoring out $z = 1$, $z^4 - 4z^3 - 4z^2 - 4z - 4 = 0$. The unique positive root (by Mathematica) is $z = 4.99357$ so that $\mu_1 = \frac{5}{2}\log z = 4.020378$ so that by Theorem 7.1 the drift rates are

$$\mu_0 = -.804076, \quad \mu_1 = 4.020378, \quad \mu_2 = -3.216302.$$

APPENDIX

A. Results about stationary probabilities of Markov chains

We begin with a 1934 result of G. Mihoc that expresses stationary probabilities of a finite state Markov chain in terms of cofactors: See Fréchet (1952, pp. 114–116). (Mihoc's original paper was in Romanian, and Fréchet elaborated upon it in his 1938 first edition of the cited reference.) Let $\mathbb{P}$ be any $k \times k$ stochastic matrix. For any $j \in \{1, 2, \ldots, k\}$, the following two determinants are equal, since the matrix in the second is obtained from that in the first by replacing the jth column with the sum of all columns: For $0 \leq s < 1$,

$$\Delta(s) := |s\mathbb{I} - \mathbb{P}| = \begin{vmatrix} s - p_{11} & -p_{12} & -p_{1j-1} & s-1 & p_{1j+1}, & \cdots, & -p_{1k} \\ -p_{21} & s - p_{22} & & s-1 & & & \\ \vdots & & & \vdots & & & \\ -p_{k1} & & & s-1 & & & s - p_{kk} \end{vmatrix}$$

and so

$$\lim_{s \nearrow 1} \frac{\Delta(s)}{s-1} = \begin{vmatrix} 1 - p_{11} & -p_{12} & 1 & -p_{1k} \\ & & 1 & \\ & & \vdots & \\ -p_{k1} & & 1 & 1 - p_{kk} \end{vmatrix}. \tag{A.1}$$

Observe that the left-hand side does not depend upon j. Hence the right-hand side evaluated by expanding along the jth column does not depend upon j. That is, if Δ_{ij} denotes the (i, j)th cofactor of $\Delta(1)$,

$$\Delta_{\cdot j} := \Delta_{1j} + \cdots + \Delta_{kj} = \Delta_{\cdot 1}, \quad j = 1, 2, \ldots, k. \tag{A.2}$$

On the other hand, direct evaluation of $\Delta(1) = |\mathbb{I} - \mathbb{P}|$ by expansion along the j-th column gives

$$\Delta(1) = \sum_{i=1}^{k} \Delta_{ij}(\delta_{ij} - p_{ij}).$$

Since $\Delta(1) = 0$ this shows that

$$\Delta_{jj} = \sum_{i=1}^{k} \Delta_{ij} p_{ij}. \tag{A.3}$$

If 1 is a *simple* root of $\Delta(s) = 0$, (when the corresponding Markov chain has a single recurrent class) the derivative in (A.1) is non-zero so that the common sums $\Delta_{\cdot j}$ are non-zero. In this case, (A.3) implies that for each j, $(\Delta_{1j}, \Delta_{2j}, \ldots, \Delta_{kj})/\Delta_{\cdot 1}$ is a solution in $\mathbf{x}$ of

$$\mathbf{x} = \mathbf{x}\mathbb{P}, \quad \sum_{i=1}^{k} x_i = 1. \tag{A.4}$$

Thus if $\mathbb{P}$ is also such that (A.4) has a unique solution, which is the case of $\mathbb{P}^n$ converging as $n \to \infty$, these solutions must all agree (with the common row elements of that limit) so that the numbers $\Delta_{ij}/\Delta_{\cdot j} \equiv \Pi_j$ say, do not depend upon j. Equivalently, the cofactors of $\mathbb{I} - \mathbb{P}$ form a matrix all of which columns are equal whenever (A.4) has a unique solution. (The reader will note the relationship to Cramer's rule for solving simultaneous linear equtions.)

Even when $\mathbb{P}^n$ does not converge the columns of cofactors are still all the same as long as the corresponding Markov chain has only one recurrent class. Here is the case of a periodic chain of period 2 which is needed for this paper.

Suppose the stochastic matrix $\mathbb{P}$ in the above discussion is a periodic matrix $\mathbb{C}$ of period 2 of the form given in (5.8), namely,

$$\mathbb{C} = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix},$$

in which A is $r \times t$ and B is $t \times r$ with $k = r + t$. (The non-square nature of A and B makes this slightly different from the $\mathbb{C}$ of (5.7).) Then,

$$\Delta(s) = |s\mathbb{I}_k - \mathbb{C}| = \begin{vmatrix} s\mathbb{I}_r & -A \\ -B & s\mathbb{I}_t \end{vmatrix}. \tag{A.5}$$

It is known (e.g., Rao, 1973, p. 32) that determinants of this form can be evaluated in two ways giving

$$\Delta(s) = s|s\mathbb{I}_t - s^{-1}BA| = s|s\mathbb{I}_r - s^{-1}AB|.$$

Therefore, for $u = s^2$,

$$\Delta(\sqrt{u}) = |u\mathbb{I}_t - BA| = |u\mathbb{I}_r - AB|$$

and so

$$\lim_{u \nearrow 1} \frac{|u\mathbb{I}_t - BA|}{u-1} = \lim_{u \nearrow 1} \frac{|u\mathbb{I}_r - AB|}{u-1}.$$

By (A.1) and (A.2) above, this means that the common column sum of cofactors of $\mathbb{I}_t - BA$ is equal to that of the column sums of cofactors of $\mathbb{I}_r - AB$. Moreover, since

$$\lim_{s \nearrow 1} \frac{\Delta(s)}{s-1} = 2 \lim_{u \nearrow 1} \frac{\Delta(\sqrt{u})}{u-1},$$

each of these column sums is exactly half of the equal column sums of cofactors of $\mathbb{I}_k - \mathbb{C}$.

It is possible also to show that the set of diagonal cofactors of $\mathbb{I} - \mathbb{C}$ is made up of the diagonal cofactors of $\mathbb{I}_t - BA$ and $\mathbb{I}_r - AB$. Write $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ for the first row and column of A and B, respectively, so that

$$A = \begin{pmatrix} \boldsymbol{\alpha} \\ A^* \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} \boldsymbol{\beta} & B^* \end{pmatrix}.$$

Then the cofactor Δ_{11} of $\mathbb{I}_k - \mathbb{C}$ is

$$\Delta_{11} = \begin{vmatrix} \mathbb{I}_{r-1} & A^* \\ B^* & \mathbb{I}_t \end{vmatrix} = |\mathbb{I}_{r-1} - A^* B^*|.$$

But since

$$\mathbb{I}_r - AB = \begin{pmatrix} 1 - \boldsymbol{\alpha\beta} & -\alpha B^* \\ -A^*\beta & \mathbb{I}_{r-1} - A^* B^* \end{pmatrix}$$

it is clear that its first diagonal cofactor is also $|\mathbb{I}_{r-1} - A^* B^*|$. On the other hand if we use instead the partitioning

$$A = \begin{pmatrix} \boldsymbol{\alpha}^* & A^{**} \end{pmatrix}, \quad B = \begin{pmatrix} \boldsymbol{\beta}^* \\ B^{**} \end{pmatrix}$$

in which $\boldsymbol{\alpha}^*$ and $\boldsymbol{\beta}^*$ are the first column and first row of A and B, respectively, then

$$\mathbb{I}_k - \mathbb{C} = \begin{pmatrix} \mathbb{I}_r & \begin{pmatrix} \boldsymbol{\alpha}^* & A^{**} \end{pmatrix} \\ \begin{pmatrix} \boldsymbol{\beta}^* \\ B^{**} \end{pmatrix} & \mathbb{I}_t \end{pmatrix}.$$

Therefore, the $(r+1)$th diagonal cofactor of $\mathbb{I} - \mathbb{C}$ is

$$\begin{vmatrix} \mathbb{I}_r & A^{**} \\ B^{**} & \mathbb{I}_{t-1} \end{vmatrix} = |\mathbb{I}_{t-1} - B^{**} A^{**}|.$$

But since

$$\mathbb{I}_t - BA = \begin{pmatrix} 1 - \boldsymbol{\beta}^* \boldsymbol{\alpha}^* & -\boldsymbol{\beta}^* A^{**} \\ -B^{**} \boldsymbol{\alpha}^* & \mathbb{I}_{t-1} - B^* A^* \end{pmatrix}$$

its first diagonal cofactor is $|\mathbb{I}_{t-1} - B^* A^*|$ as well.

By cyclically permuting the first j columns and rows when $1 \le j \le r$, or the $(r+1)$th through j-th columns and rows when $r < j \le k$, the above arguments prove that the first r diagonal cofactors of $\mathbb{I}_k - \mathbb{C}$ are those of $\mathbb{I}_r - AB$ and the last t of them are the diagonal cofactors of $\mathbb{I}_t - BA$.

In view of the above results, the stationary probability vectors $\boldsymbol{\delta}$ and $\boldsymbol{\rho}$ for AB and BA, respectively, that were introduced for (5.8) may be expressed in the notation of Section 4 as

$$\text{(A.6)} \qquad \boldsymbol{\delta} = \frac{2}{\gamma_{\cdot m}} (\gamma_{1m}, \dots, \gamma_{km}) \quad \text{and} \quad \boldsymbol{\rho} = \frac{2}{\gamma_{\cdot m}} (\gamma_{k+1,m}, \dots, \gamma_{mm}).$$

In particular, this verifies the equivalence of (5.7) and (5.9), showing that (5.9) applies for all m, whether even or odd.

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REFERENCES

Anderson, T.W. (1960). A modification of the sequential probability ratio test to reduce the sample size. *Ann. Math. Statist.* 31, 165–197.

Chung, K.L. (1967). *Markov Chains.* Springer-Verlag, New York.

Durrett, R., Kesten, H. and Lawler, G. (1991). Making money from fair games. In *Random Walks, Brownian Motion, and Interacting Particle Systems*, pp. 255–267. Progr. Probab., Vol. 28. Birkhäuser, Boston.

Feller, W. (1968). *An Introduction to Probability Theory and Its Applications.* Third Edition. J. Wiley and Sons, New York.

Fréchet, M. (1952). *Recherches théoriques modernes sur le calcul des probabilités. Volume 2. Méthode des fonctions arbitraires. Théorie des événements en chaîne dans le cas d'un nombre fini d'états possibles.* Second Edition, Gauthier-Villars, Paris.

Harmer, G.P. and Abbott, D. (1999a). Parrondo's paradox. *Statist. Sci.* 14, 206–213.

Harmer, G.P. and Abbott, D. (1999b). Parrondo's paradox: losing strategies cooperate to win. *Nature* 402, 864.

Harmer, G.P., Abbott, D. and Taylor, P. (2000). The paradox of Parrondo's games. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* 456, 247–259.

Harmer, G.P., Abbott, D., Taylor, P.G. and Parrondo, J.M.R. (2000). Parrondo's paradoxical games and the discrete Brownian ratchet. In *Unsolved Problems of Noise and Fluctuations (UPoN'99)* (D. Abbott and L.B. Kiss, eds.), pp. 189-200. American Inst. Physics Press, Melville.

Karlin, S. and McGregor, J. (1959). Random walks. *Illinois J. Math.* 3 66–81.

Rao, C.R. (1973). *Linear Statistical Inference and its Applications.* Second Edition. J. Wiley and Sons, New York.

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