Iteration of IID Random Maps on R^+

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Abstract

Let $\{X_n\}$ be a Markov chain on R^+ generated by the iteration scheme $X_{n+1} = C_{n+1}X_n g(X_u)$, where $\{C_n, g_n(\cdot)\}$ are i.i.d. such that $\{C_n\}$ are nonnegative r.v. with values in $[0, L]$, $L \leq \infty$, $\{g_n\}$ are continuous functions from $[0, \infty) \rightarrow [0, 1]$ with $g_n(0) = 1$. This paper presents a survey of recent results on the existence of nontrivial stationary measures, Harris irreducibility and uniqueness of stationary measures, convergence and persistence. Four well known special cases i.e. the logistic, Ricker, Hassel and Vellekoop-Högnas models are discussed.

Keywords: Markov chains, IID random maps, Stationary measures, Harris reducibility

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1 Introduction

A topic of some interest to Professor Rabi N. Bhattacharya, whom the present volume honors, and to which he has contributed substatially is the iteration of i.i.d. random quadratic maps on the unit interval $[0, 1]$. Beginning with the paper Bhattacharya and Rao [7] where they analyzed the case of i.i.d. iteration of two quadratic maps using the Dubins-Freedman [9] results on random monotone maps on an interval, Professor Bhattacharya has obtained a number of interesting results on the uniqueness and support of the stationary distribution as well as on rates of convergence. For these the reader is referred to Bhattacharya and Majumdar [6] and Bhattacharya and Waymire [8].

In the present paper we study Markov chains generated by iteration of i.i.d. random maps on R^+ that are restricted to the class of functions $f : R^+ \rightarrow R^+$ such that they possess a finite, positive derivative at 0, vanish at 0 and have a sublinear growth for large values. This class is of relevance and use in population ecology and growth models in economics. The conditions imposed on f in this class reflects two features common in ecological modelling, namely, i) for small values of the population size X_n at time n, the population size X_{n+1} at time $n + 1$ is approximately proportional to X_n with a random proportionality constant while for large values of X_n, competition sets in and the linear growth is scaled down by a factor. This class includes many of the known models in the ecology literature such as the logistic maps, Ricker maps, Hassel maps and Vellekoop-Högnas maps, as explained in the next section.

Here is an outline of the rest of the paper. In the next section we describe the basic mathematical set up and establish some results for Feller chains on R^+. In section 3 we describe a set of necessary and two sets of sufficient conditions for the existence of stationary measures with support in $(0, \infty)$. In section 4, a trichotomy into subcritical, critical and supercritical cases is introduced and convergence results for the subcritical and critical cases are provided. Section 5

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is devoted to Harris irreducibility and uniqueness of the stationary measures in the supercritical case. Some open problems are indicated at the end.

It is a great pleasure for the author to dedicate this paper to Professor Rabi N. Bhattacharya who has been a dear friend and a source of inspiration.

2   The mathematical framework

Let the collection $\mathcal{F}$ of functions $f : [0, L) \to [0, L)$, $L \leq \infty$ be such that

  i) f is continuous

 ii) $f(0) = 0$

iii) $\lim\limits_{x \downarrow 0} \frac{f(x)}{x} \equiv f'_+(0)$ exists and is positive and finite

iv) $g(x) \equiv \frac{1}{f'_+(0)} \frac{f(x)}{x}$ satisfies $0 < g(x) < 1$ for $0 < x < L$.

Let $(\Omega, \mathcal{B}, P)$ be a probability space.

Let $\{f_j(\omega, x)\}_{j \geq 1}$ be a collection of random maps from $\Omega \times [0, \infty) \to [0, \infty)$ that are jointly measurable, i.e. that are $(\mathcal{B} \times \mathcal{B}[0, \infty), \mathcal{B}[0, \infty))$ measurable and for each $j, f_j(\omega, \cdot) \in \mathcal{F}$ with probability one. Consider the random dynamical system generated by the iteration scheme:

$$\begin{aligned}
X_{n+1}(\omega, x) &\equiv f_{n+1}(\omega, X_n(\omega, x)), n \geq 0 \\
X_0(\omega, x) &\equiv x.
\end{aligned} \tag{1}$$

Since $f_j(\omega, \cdot) \in \mathcal{F}$ w.p.l. the model (1) reflects the two features common in ecological modelling i.e. for small values of X_n, X_{n+1} is proportional to X_n with proportionality constant $f'_{n+1}(0) \equiv C_{n+1}$, say, and for large values of X_t, this is reduced by the factor $g(X_n)$.

The class $\mathcal{F}$ includes the logistic, Ricker, Hassel, Vellekoop-Högnas families mentioned in the introduction, as shown below.

For the logistic family, $f_c(x) = cx(1 - x)$, $L = 1$, $f'_+(0) = c$, and $g(x) = 1 - x$ for $0 \leq x \leq 1$.

For the Ricker family [13], $L = \infty$, $f_{c,d}(x) = cxe^{-dx}$, $f'_+(0) = c$, $g(x) = e^{-dx}$, $0 \leq x < \infty$.

For the Hassel family [11], $L = \infty$, $f_{c,d}(x) = cx(1 + x)^{-d}$, $f'_+(0) = c$ and $g(x) = (1 + x)^{-d}$.

For the Vellekoop-Högnas family [14], $L = \infty, f(x) = rx(h(x))^{-b}$, $f'_+(0) = r$, $g(x) = (h(x))^{-b}$.

From now on, suppose that $\{f_i\}_{i \geq 1}$ are *i.i.d. stochastic processes.* Then the sequence $\{X_n\}$ defined by (1) is a Markov chain with state space $S = [0, L)$ and transition function $P(\cdot, A) = P(f, (\omega, .) \in A)$ and initial value $x_0 = x$. the same is true when X_0 is chosen as a random variable (with values in S) but independently of $\{f_i\}$. Further, since f_i are continuous w.p.l., $\{X_n\}$ has the Feller property:

For each $k : S \to S$ bounded and continuous, $(Pk)(x) \equiv E(k(X_1)|x_0 = x)$ is continuous in x.

For Feller Markov Chains it is known [8] that if a probability measure Γ is the *weak* limit point of the sequence $\{\Gamma_n(\cdot)\}$ of *occupation measures*,

$$\Gamma_{n,x}(A) \equiv \frac{1}{n} \sum_0^{n-1} P(x_j \in A|x_0 = x) \tag{2}$$

then Γ is necessarily *stationary for P*, i.e.

$$\Gamma(A) = \int_S P(x, A)\Gamma(dx) \quad \forall \, A \in B(S), \tag{3}$$

the Borel σ-algebra on S. The following proposition is slightly more general.

Proposition 2.1. *Let $\{X_n\}$ be Feller with state space $S = [0, L)$. Let a sub-probability measure $\Gamma(\cdot)$ on S be a vague limit point of Γ_{n,X_0} for some initial r.v. X_0. Then Γ is stationary for P, i.e. it satisfies (3).*

For a proof see Athreya [1].
A sufficient condition for ensuring that every vague limit point Γ of $\{\Gamma_{n,x}\}$ is nontrivial on $(0, L)$, i.e. satisfies $\Gamma(0, L) > 0$ is provided by the following.

Proposition 2.2. *Suppose there exists a $V : S \equiv [0, L) \to R^+$ a set $K \subset (0, L)$ and constants $0 < \propto, M < \infty$ such that*

i) $\forall \, x \notin K, \quad E(V(x_1)|X_0 = x) \le V(x) - \propto$

ii) $\forall \, x \in S, \quad E(V(x_1)|X_0 = x) \le V(x) + M$

Then $\Gamma(K) \ge \underline{\lim}\Gamma_{n,x}(K) \ge \frac{\propto}{\propto + M} > 0$.

The proof is not difficult and may be found in Athreya [1].

3 Stationary Measures

In this section we present one set of necessary and two sets of sufficient conditions for the existence of a stationary probability measure π such that $\pi(0, L) = 1$ for the Markov Chain (1). For proofs of these see Athreya [1].

Theorem 3.1. *Let $C_j \equiv \lim\limits_{x \downarrow 0} \dfrac{f_j(x)}{x}$,*

$$g_j(x) \equiv \begin{cases} \frac{f_j(x)}{C_j x} & \text{for } x > 0 \\ 1 & \text{for } x = 0 \end{cases}$$

Let
$$E(\ln C_1)^+ < \infty. \tag{4}$$

Suppose there exists a probability measure π satisfying the stationarity condition (3) and the nontriviality condition $\pi(0, L) = 1$. Then the following hold:

i) $E(\ln C_1) < \infty,$

ii) $\int E|\ln g_1(x)|\pi(dx) < \infty,$

iii) $E \ln C_1 = -\int E(\ln g_1(x))\pi(dx) > 0.$

Corollary 3.1. *If $E \ln C_1 \le 0$ then*

   i) *The only stationary probability meausre on $[0, \infty)$ is the delta measure at 0.*

   ii) *For any $x \ge 0$, and Borel sets A such that $\overline{A} \subset (0, L)$*

$$\lim \Gamma_{n,x}(A) = 0.$$

Next we present two sets of sufficient conditions for the existence of a stationary measure π with $\pi(0, \infty) > 0$ for the Markov chain $\{X_n\}$ in (1).

Theorem 3.2. *Let $\{f_j\}, \{C_j\}, \{g_j\}$ be as in Theorem 3.1. Let $D_j(\omega) \equiv \sup_{x \ge 0} f_j(\omega, x)$.*

Assume

   i) $k(x) = -E \ln g_1(x) < \infty$ *for all* $0 < x < L.$

   ii) $\lim_{x \downarrow 0} k(x) = 0.$

   iii) $k(\cdot)$ *is nondecreasing in* (T, L) *for some* $0 \le T < L.$

   iv) $E|\ln C_1| < \infty, \ E \ln C_1 > 0.$

   v) $E(\ln D_1)^+ < \infty.$

   vi) $E|k(D_1)| < \infty.$

Then there exists a stationary distribution π for the Markov chain $\{X_n\}$ defined by (1) such that $\pi(0, L) = 1.$

Special Cases. We now apply Theorem 3.2 above to the four cases mentioned earlier.

   1. *Random Logistic Maps* [7]
      Here $f_1(x) = C_1 x(1-x), \ 0 \le x \le 1, \ 0 \le C_1 \le 4, \ g_1(x) = (1-x)$ so $k(x) = -\ln(1-x)$ and hence i), ii), and iii) of Theorem 3.2 hold. Also $D_1 = \frac{C_1}{4} \le 1$ and so v) holds. Thus i) – vi) of Theorem 3.2 reduce to

$$E|\ln C_1| < \infty, \quad E \ln C_1 > 0, \quad E\left|\ln\left(1 - \frac{C_1}{4}\right)\right| < \infty. \qquad (5)$$

      This was established by Athreya and Dai [3].

   2. *Random Ricker Maps* [13]
      Here $f_1(x) = C_1 x e^{-d_1 x}, \ 0 \le x < \infty, \ 0 \le C_1, \ d_1 < \infty.$ So $k(x) = (Ed_1)x$ and hence i), ii) and iii) of Theorem 3.2 reduce to $Ed_1 < \infty.$
      Also $D_1 = \frac{C_1}{d_1} \sup_{x>0} d_1 x e^{-d_1 x} = \frac{C_1}{d_1}.$ Thus, i) - vi) of Theorem 3.2 reduce to

$$E|\ln C_1| < \infty, \quad E \ln C_1 > 0, \quad Ed_1 < \infty, \quad E\frac{C_1}{d_1} < \infty.$$

3. *Random Hassel Maps* [11]

Here $f_1(x) = C_1 x (1+x)^{-d_1}$, $0 \leq x < \infty$, $0 \leq C_1$, $d_1 < \infty$. So $k(x) = (Ed_1) \ln(1+x)$ and hence i), ii) and iii) of Theorem 3.2 reduce to $Ed_1 < \infty$. Also,

$$D_1 = \begin{cases} C_1 \left(1 - \frac{1}{d_1}\right)^{d_1 - 1} \frac{1}{d_1} & \text{if } d_1 > 1 \\ C_1 & \text{if } d_1 = 1 \\ \infty & \text{if } d_1 < 1 \end{cases}$$

So we need $P(d_1 \geq 1) = 1$. This implies $D_1 \leq C_1$ w.p.l. and so v) is implied by $E(\ln C_1)^+ < \infty$ which in turn is implied by iv).

Finally, $|\ln(1+D_1)| \leq \ln(1+C_1)$. Thus i) - vi) of Theorem 3.2 are implied by $E|\ln C_1| < \infty$, $E(\ln C_1) > 0$, $Ed_1 < \infty$, $P(d_1 \geq 1) = 1$.

4. *Random Vellekoop-Högnas Maps* [14]

Here $f_1(x) = C_1 x (h_1(x))^{-b_1}$, $0 \leq x < \infty$ where $0 \leq C_1$, $b_1 < \infty$ and $h_1(\cdot)$ satisfies $h_1(0) = 1$, $h_1(x) \geq 1$ for $x \geq 0, h_1(\cdot)$, is continuously differentiable and $\tilde{h}_1(x) \equiv x \frac{h_1^1(x)}{h_1(x)}$ is strictly increasing.

Note that this includes all three previous cases. So $k(x) = Eb_1 \ln h_1(x)$.

Next, to find D_1 note that the function $r_1(x) = \ln(x(h_1(x))^{-b_1})$ satisfies

$$r_1^1(x) = \frac{1}{x} - b_1 \frac{h_1^1(x)}{h_1(x)} = \frac{1}{x}\left(1 - b_1 \tilde{h}_1(x)\right).$$

Since $\tilde{h}_1(x)$ is strictly increasing and is zero at $x = 0$, $r_1^1(x) > 0$ for $0 \leq x < \propto_1$, $= 0$ for $x = \propto$, and < 0 for $x > \propto$, where $\propto_1 = \inf\{x : \tilde{h}_1(x) > b_1\}$. So

$$D_1 = \begin{cases} C_1 \propto_1 (h_1(\propto_1))^{-b_1} & \text{if } \propto_1 < \infty \\ \lim_{x \uparrow \infty} C_1 x (h_1(x))^{-b_1} & \text{if } \propto_1 = \infty \end{cases}$$

Thus, i) - vi) of Theorem 3.2 are implied by

i) $Eb_1 \ln h_1(x) < \infty$ for all $0 \leq x < \infty$.

ii) $\lim_{x \downarrow v} Eb_1 \ln h_1(x) = 0$.

iii) $Eb_1 \ln h_1(x)$ is nondecreasing in (T, ∞) for some $T \geq 0$.

iv) $E|\ln C_1| < \infty$, $E \ln C_1 > 0$

v) $\exists \, 0 < \propto_1 < \infty \rightarrow \tilde{h}_1(\propto_1) = \frac{1}{b_1}$ and $E\left(\ln\left(\propto_1 (h_1(\propto))^{-b_1}\right)\right)^+ < \infty$.

vi) $E|k(D_1)| = Ek(D_1) = Eb_2 \ln h_2(D_1) < \infty$ where b_1 and $h_2(\cdot)$ are defined by $f_2(x) \equiv C_2 x (h_2(x))^{-b_2}$ with $f(\cdot)$ being i.i.d. copy of $f_1(\cdot)$.

Remark: In all the above four cases the function $g_j(x) = \frac{f(x)}{c_j x} \rightarrow 0$ as $x \rightarrow \infty$ asserting that for large x the growth is sublinear. But in some ecological context such as arising in resource management procedures it is more realistic to keep $g_j(x)$ bounded away from zero for large values of x.

Similarly, in some growth models in economics the possibility of $f_j(x) \rightarrow \infty$ as $x \rightarrow \infty$ is not unrealistic. This leads us to a second set of sufficient conditions.

Theorem 3.3. *Let $\{f_j\}$, $\{C_j\}$, $\{g\}$ be as in Theorem 3.1. Suppose*

 i) $\displaystyle\lim_{x\to 0} E \ln C_1 g_1(x) \equiv \beta_1$ *exists and is* > 0.

 ii) $\displaystyle\lim_{x\to 0} E(\ln C_1 x g_1(x))^+ = 0$.

 iii) $\displaystyle\lim_{x\to L} E \ln C_1 g_1(x) \equiv \beta_2$ *exists and is* < 0.

 iv) $\displaystyle\lim_{x\to L} E(\ln C_1 x g_1(x))^- = 0$.

 v) $\tilde{k}(x) \equiv E|\ln C_1 g_1(x)|$ *is bounded on* $[a,b]$ *for all* $0 < a < b < L$.

Then there exists a stationary measure π for the Markov chain $\{X_n\}$ defined by (1) satisfying $\pi(0, L) = 1$.

Corollary 3.2. *In the set up of Theorem 3.3, suppose:*

 i) $E|\ln C_1| < \infty$, $E \ln C_1 > 0$.

 ii) *With probability one* $\displaystyle\lim_{x\downarrow 0} g_1(x) = 1$, $\displaystyle\lim_{x\uparrow L} g_1(x) = \eta > 0$ *and there exists*
  $0 < a$ *such that* $a \le \displaystyle\inf_x g_1(x) \le \sup_x g_1(x) \le 1$.

 iii) $E \ln C_1 + E \ln \eta < 0$.

Then there exists a stationary π for $\{X_n\}$ satisfying $\pi(0, L) = 1$.

4 Convergence results

The last section dealt with the existence of stationary measures for the Markov chain $\{X_n\}$ generated by (1) or equivalently by the iteration scheme

$$X_{n+1} = C_{n+1} X_n g_{n+1}(X_n), \quad n = 0, 1, 2, \ldots \tag{6}$$

where the pair $(C_n, g_n(\cdot)_{n\ge 1})$ are i.i.d. with $0 < C_n < \infty$, $g_n(\cdot)$ being w.p.l. a continuous function as in Theorem 3.1 and independent of X_0.

 The convergence questions that we consider here are:

 i) The almost sure convergence of the sequence $\{X_n\}$ as $n \to \infty$, i.e. convergence of the trajectories,

 ii) the convergence of $\{X_n\}$ in probability and

 iii) the convergence of the distribution of $\{X_n\}$.

 Since the state space of the Markov chain $\{X_n\}$ is uncountable one has to look for results from general state space Markov chains theory. There is a body of results available for the case when the chain is Harris irreducible (see Nummelin [12]). Unfortunately, many of the iterated random maps cases turn out to be not irreducible, especially among those where the collection of functions $\mathcal{F}$ sampled from is finite or countable. In these cases if the maps are interval maps that are monotone then the Dubins-Freedman theory [9] can be appealed to. The papers by Bhattacharya and Rao [7], Bhattacharya and

Majumdar [6] and Bhattacharya and Waymire [8] have nice accounts of this in the random logistic maps case.

On the other hand, as shown in the next section, if the distribution of C_n is smooth, e.g. *absolutely continuous*, then $\{X_n\}$ turns out to be (under some more hypothesis) Harris irreducible. For the random logistics case Bhattacharya and Rao [7], Bhattacharya and Waymire [8] have some nice results under such assumptions.

Motivated by Theorem 3.1, we give the following definition.

Definition: The Markov chain $\{X_n\}$ of (1) or (6) is *subcritical, critical,* or *supercritical* according as $E \ln C_1 < 0$, $= 0$, or > 0.

In the subcritical case, $\{X_n\}$ converges to zero w.p.l. In fact, a slightly more general result holds.

For the rest of this section $\{X_n\}_{t \geq 0}$ will be as in (6).

Theorem 4.1. *Suppose*

$$\overline{\lim} \frac{1}{n} \sum_1^n \ln C_j(\omega) \equiv d(\omega) < 0 \quad w.p.l. \tag{7}$$

Then

$$X_n(\omega) = O(\rho^n) \quad w.p.l. \tag{8}$$

for any $\rho > e^{d(\omega)}$ and hence $X_n(\omega) \to 0$ w.p.l.

Proof. Since $f_j \in F$

$$
\begin{aligned}
X_{n+1} &= C_{n+1} X_n g_{n+1}(X_n) \leq C_{n+1} X_n \\
&\leq C_{n+1} C_n \ldots C_1 X_0
\end{aligned}
$$

Thus

$$\frac{1}{n} \ln X_n \leq \frac{1}{n} \ln X_0 + \frac{1}{n} \sum_1^n \ln C_j.$$

Now $(7) \Rightarrow (8)$.

$\square$

Corollary 4.1. *If $E \ln C_1 < 0$ then (7) and hence (8) holds, provided $\{C_1\}_{n \geq 1}$ are i.i.d.*

Remark: In this theorem the hypothesis $\{C_n\}_{n \geq 1}$ are independent is not needed. The geometric decay of $\{X_n\}$ can be exploited to establish the log normality of X_n, a common hypothesis proposed in the ecology literature.

Theorem 4.2. *Assume*

i) $g_j(\cdot)$ is nonincreasing in $[0, \delta]$ w.p.l. for some $\delta > 0$.

ii) $E \ln C_1 < 0$, $E(\ln C_1)^2 < \infty$.

iii) $0 \leq k(x) = -E \ln g_1(x) < \infty$ for all x and nondecreasing.

iv) $\sum_1^\infty k(\propto \lambda^j) < \infty$ for some $0 < \propto < \infty$ and $e^{E \ln C_1} < \lambda < 1$.

Then

$$\frac{\ln X_n - nE\ln C_1}{\sigma\sqrt{n}} \tag{9}$$

where $\sigma^2 = V(\ln C_1)$.

Proof. From (6)

$$\ln X_n - \ln X_0 = \sum_1^n \ln C_j + \sum_1^n \ln g_1(X_{j-1}) \tag{10}$$

Since g_1 is nonincreasing in $[0, \delta]$ w.p.l. and (8) holds, $1 \geq g_j(X_{j-1}) \geq g_j(\alpha\lambda^j)$ for j large, some constant α and $0 < \lambda < 1$.

But $-E\ln g_j(\alpha\lambda^j) \leq k(\alpha\lambda^j)$ and so

$$E(-\sum_{j=1}^\infty \ln g_j(\alpha\lambda^j)) \leq \sum_1^\infty k(\alpha\lambda^j)$$

which is finite by (iii). Thus,

$$-\sum_{j=1}^\infty \ln g_j(\alpha\lambda^j) < \infty \text{ w.p.l. } \Rightarrow \frac{1}{\sqrt{n}}\sum_{j=1}^n \ln g_j(X_{j-1}) \to 0 \text{ w.p.l.} \tag{11}$$

By the central limit theorem

$$\frac{\sum_1^n \ln C_j - nE\ln C_1}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1).$$

Now (10) and (11) yield (9).

$\square$

Next we turn to the critical case.

In the critical case the occupation measures $\mu_{n,x}(\cdot)$ defined by (2) all converge in distribution to δ_0. This implies that for every $\in > 0$,

$$\frac{1}{n}\sum_0^{n-1} P_x(X_j \geq \in) \to 0,$$

i.e. $a_n \equiv P_x(X_n \geq \in) \to 0$ in the Cesaro sense. A natural question is whether it can be improved to full convergence or equivalently does $X_n \to 0$ in probability for all $0 < x < \infty$? For the logistic case, i.e. when f_1 is a logistic map w.p.l. Athreya and Dai [3] have shown this by comparison argument. This is extended below to the present context assuming that w.p.l., f_1 is unimodal with a common nonrandom mode α such that f_1 is nondecreasing in $[0, \alpha]$ and nonincreasing in $[\alpha, \infty)$.

Theorem 4.3. *Let $E(\ln C_1)^+ < \infty$ and $E\ln C_1 = 0$. Assume further that there exists a nonrandom α in $(0, \infty)$ such that w.p.l. f_1 is nondecreasing in $[0, \alpha]$ and nonincreasing in $[\alpha, \infty)$.*
   Then

$$X_n \xrightarrow{p} 0 \text{ for any initial value } X_0 = x. \tag{12}$$

The proof makes use of the following.

Theorem 4.4. COMPARISON LEMMA *Let $\{f_i\}_{i \geq 1}$ be i.i.d. and unimodal as in the above theorem. Let X_0 be independent of $\{f_i\}_{i \geq 1}$.*

Let $\{X_n\}$, $\{Y_n\}$, $\{\tilde{Y}_n\}$ and $\{Z_n\}$, $n \geq 0$, be defined by

$$
\begin{aligned}
X_{n+1} &= f_{n+1}(X_n) \\
Y_{n+1} &= \min\{f_{n+1}(Y_n), \alpha\}, \; Y_0 = \min\{X_0, \alpha\} \\
\tilde{Y}_{n+1} &= \min\{f_{n+1}(\tilde{Y}_n), \alpha\}, \; \tilde{Y}_0 = \alpha \\
Z_n &= \min\{X_n, \alpha\}
\end{aligned}
$$

Then for all $n \geq 0$, $\tilde{Y}_n \geq Y_n \geq Z_n$ w.p.l.

Proof. Since $Y_0 \leq \tilde{Y}_0 = \alpha$, and f_1 is nondecreasing in $[0, \alpha]$, $f_1(Y_0) \leq f_1(\tilde{Y}_0)$ implying $Y_1 \equiv \min(f_1(Y_0), \alpha) \leq \min(f_1(Y_0), \alpha) \equiv \tilde{Y}_1$. Now induction yields $\tilde{Y}_n \geq Y_n$ for all n.

If $X_0 \leq \alpha$, then $Y_0 = X_0$ and so

$$f_1(Y_0) = f_1(X_0) = X_1$$

implying $Y_1 = \min\{f_1(Y_0), \alpha\} = \min\{X_1, \alpha\} = Z_1$.

If $X_0 > \alpha$, then $Y_0 = \alpha$ so

$$f_1(Y_0) = f_1(\alpha) \geq f_1(x_0) = X_1$$

implying $Y_1 = \min\{f_1(Y_0), \alpha\} \geq \min\{X_1, \alpha\} = Z_1$. Thus $Y_1 \geq Z_1$. Induction yields $Y_n \geq Z_n$ for all n.

$\square$

Remark: This comparison lemma does not require any conditions as $E \ln C_1$.

Corollary 4.2. *For any $0 < \epsilon < \alpha$, and $n \geq 1$*

i)

$$
\begin{aligned}
P_x(X_n \geq \epsilon) &\leq P_x(X_n \geq \epsilon) \\
&\leq P_x(Y_n \geq \epsilon) \leq P(\tilde{Y}_n \geq \epsilon)
\end{aligned}
$$

ii)

$$P(\tilde{Y}_{n+1} > \epsilon) \leq P(\tilde{Y}_n \geq \epsilon)$$

Proof. Clearly i) follows from the comparison lemma. Next, by the Markov property of $\{\tilde{Y}_n\}$

$$P(\tilde{Y}_{n+1} \geq \epsilon) = E \, P(\tilde{Y}_n \geq \epsilon \mid \tilde{Y}_1 \geq \epsilon) \leq P(\tilde{Y}_n \leq \epsilon).$$

Proof of Theorem 4.3 By Corollary 4.2 i) it suffices to show that $P(\tilde{Y}_n \geq \epsilon) \to 0$. But since this is nondecreasing in n this is equivalent to showing

$$\frac{1}{n} \sum_{0}^{n-1} P(\tilde{Y}_j \geq \epsilon) \to 0 \tag{13}$$

But the occupation measure sequence $\mu_n^{\tilde{Y}}(\cdot)$ defined by

$$\mu_n^Y(\cdot) \equiv \frac{1}{n} \sum_{0}^{n-1} P(\tilde{Y}_j \geq \epsilon)$$

can be shown to have a nontrivial limit point only if $E \ln C_1 > 0$ (as in the proof of Theorem 3.1). Thus $\mu_n^Y([\epsilon, \infty)) \to 0$ implying (13).

A natural question prompted by Theorem 4.3 is whether in the critical case the convergence of X_n to zero in probability could be strengthened to convergence w.p.l. Athreya and Schuh [5] showed that in the logistic case this is not possible.

Theorem 4.5. *Let $E \ln C_1 = 0$, $P(C_1 = 1) < 1$ and $\gamma \equiv \sup\{x : P(C_1 < x) < 1\}$. Then:*

i) *There exists a level β, $0 < \beta < 1$ and an atmost countable set Δ such that for any $x \in (0,1) - \Delta$, P_x ($X_n \geq \beta$ for infinitely many $n \geq 1) = 1$ where P_x stands for the intial condition $X_0 = x$. Further, Δ is empty if $P(C_1 = 4) = 0$.*

ii) *If $1 < \gamma \leq 2$, i.e. $P(C_1 \leq 2) = 1$ then for all $x \in (0,1) - \Delta$*

$$P_x \left(\overline{\lim} X_n = 1 - \frac{1}{\gamma} \right) = 1.$$

iii) *If $\gamma > 2$, i.e. $P(C_1 > 2) > 0$ then*

$$P_x \left(\overline{\lim} X_n \geq 1 - \frac{1}{\gamma} \right) = 1.$$

iv) *For any initial value of X_0, the empirical distribution*

$$L_n(A) \equiv \frac{1}{n} \sum_0^{n-1} I_A(X_j), \ A \in B[0,1]$$

converges weakly to δ_0 w.p.l.

Remark: The above result has an interesting interpretation. In the critical case even though for large n the population size X_n is small with a high probability the population does not die out. Indeed w.p.l. the trajectory of X_n rises to heights β and beyond again and again. This may be referred to as the *persistence of the critical logistic process.*

5   Harris irreducibility

A Markov chain $\{X_n\}$ with a measurable state space $(S, \mathcal{S})$ and transition function $P(\cdot, \cdot)$ is *Harris irreducible* with reference measure ϕ if for every $x \in \mathcal{S}$, $\phi(A) > 0 \Rightarrow P(X_n \in A$ for some $n \geq 1 | X_0 = x) > 0$. Here ϕ is assumed to be a σ-finite nonzero measure.

In this section we find sufficient conditions for Markov chains on $S \subset R^+$ generated by the iteration of maps of the form $f(x) = \theta h(x)$ where $h(\cdot)$ is a continuous function. All the results of this section are from Athreya [2] where the reader will find full details.

Let $S = [0, L]$, $L \leq \infty$, $\theta = [0, k]$, $k \leq \infty$ and $h : S \to [0, \infty)$ be continuous and strictly positive on $(0, L)$. Let $\{\theta_i\}_{i \geq 1}$ be i.i.d. r.v. with values in $[0, k]$. Let $\{X_n\}_{n \geq 0}$ be the Markov chain defined by

$$X_{n+1} = \theta_{n+1} h(X_n) \tag{14}$$

where X_0 is independent of $\{\theta_i\}$. It is assumed here that for all θ in $[0, k]$, $\theta h(x) \in S = [0, L]$.

The following provides a sufficient condition for Harris irreducibility of $\{X_n\}$.

Theorem 5.1. *Suppose:*

i) $\exists\ 0 < \propto < k,\ \delta > 0$ *and a strictly positive Borel function* Ψ *in* $J \equiv (\propto -\delta,\ \propto +\delta) \subset (0, k)$ *such that for all Borel sets*

$$B \subset J, \quad Q(B) \equiv P(\theta, \in B) \geq \int_B \Psi(\theta)\, d\theta.$$

ii) $\exists\ 0 < p < L$ *and* $m \geq 1$ *such that* $f^{(m)}(p, \propto) = p$ *where* $f^{(m)}(\cdot, \propto)$ *is the* m^{th} *iterate of* $f(\cdot, \propto) = \propto h(\cdot)$. *Then, (a):* $\exists\ \eta > 0$ *such that* $\forall\ x \in I \equiv (p - \eta,\ p + \eta)$, *and Borel set* $A \subset I$ *with* $m(A) > 0$

$$P(X_m \in A | X_0 = x) > 0.$$

If, in addition to i) and ii), suppose the following holds:

iii) $\forall\ 0 < x < L,\ \exists$ *a finite set* $\{\propto_1, \propto_2, \ldots, \propto_n\}$ *contained in support of* $Q(\cdot) = P(\theta_1 \in \cdot)$ *such that* $Y_n \in I$ *where*

$$Y_0 = x, \quad Y_{j+1} = f(Y_j, \propto_{j+1}), \quad i = 0, 1, 2, \ldots, n - 1.$$

Then $(b) : \{X_n\}$ *is Harris irreducible on* $(0, L)$ *with reference measure* $\phi(\cdot) \equiv m(\cdot \cap I)$.

Remark: Condition i) is a smoothness hypothesis on the distribution of θ_1. Without this, one could provide examples where the chain is not Harris irreducible. For example, if θ_1 has a finite support and $\{X_n\}$ admits a stationary distribution π that is nonatomic then it cannot be Harris irreducible since for any initial value x, the distribution of X_n is discrete and hence cannot converge in the Cesaro sense and in variation norm to π. But Harris irreducibility and the existence of a stationary distribution π would imply such a convergence.

Condition ii) is the existence of a periodic point.

The first conclusion (a) is a *local irreducibility result* while (b) is a *global irreducibility result*. The next result exploits the fact that a sufficient condition for iii) of Theorem 5.1 to hold in the case when $h(\cdot)$ is S-unimodal on $[0, 1]$ (see definition below) is for the pair $(p, \propto)$ to be such that p is a *stable periodic point* for the map $f(\cdot, \propto) = \propto h(\cdot)$.

Definition: A map $f : [0, 1] \to [0, 1]$ is called S-unimodal if

i) f is three times continuously differentiable,

ii) f is unimodal with a mode at c in $(0, 1)$ such that $f''(c) < 0$ and f is strictly increasing in $(0, c)$ and strictly decreasing in $(c, 1)$,

iii) $f(0) = f(1) = 0$ and

iv) the Schwartzian derivative of f

$$(Sf)(x) \equiv \begin{cases} \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 & \text{if } f'(x) \neq 0 \\ -\infty & \text{if } f'(x) = 0. \end{cases}$$

is < 0 for all $0 < x < 1$.

Examples of S unimodal maps are

$$f(x) = x(1-x), \quad f(x) = x^2 \sin \pi x.$$

A result of Guckenheimer [10] is that if $f(\cdot)$ is a S-unimodal with a stable periodic point p, i.e. for some $m \geq 1$, $f^{(m)}(p) = p$ and $\left| f^{(m)'}(p) \right| < 1$, then for almost all x in $(0,1)$ (with respect to Lebesgue measure) the limit point set $\omega(x)$ of the orbit $0_x \equiv \{ f^{(n)}(x), n \geq 0 \}$ of x under f coincides with the orbit $\gamma(p)$ of p under f, i.e. the set $\{ p, f(p), \dots, f^{(m-1)}(p) \}$.

Theorem 5.2. *Let* $S = [0,1]$, $\theta = [0,L]$, $f(x,\theta) = \theta h(x)$ *with* $f : S \to S$ *for each* $\theta \in \theta$. *Suppose:*

  i) $h(\cdot)$ *is S-unimodal*

  ii) $\exists \; (p, \propto) \in Sx\Theta$, $p \neq 0$ *and for some* $m \geq 1$, $f^{(m)}(p, \propto) = p$ *and* $\left| f^{(m)'}(p, \propto) \right| < 1$ *(ù p is a stable periodic point of $f(\cdot, \propto)$).*

  iii) $\exists \; \delta > 0$ *and a stricty positive function* Ψ *on* $J \equiv (\propto -\delta, \; \propto +\delta)$ *a subset of* Θ *such that for all* $B \subset J$,

$$Q(B) \equiv P(\theta_i \in B) \geq \int_B \Psi(\theta) m(d\theta)$$

  where $\{\theta_i\}_{i \geq 1}$ *are i.i.d. r.v. with values in* Θ *and* $m(\cdot)$ *is Lebesgue measure.*

  iv) $X_{n+1} = \theta_{n+1} h(X_n)$, $n \geq 0$, *where* X_0 *is independent of* $\{\theta_i\}_{i \geq 1}$ *with values in* $(0,1)$.

Then $\{X_n\}$ *is Harris irreducible.*

A special case of the above is the case of i.i.d. random logistic maps.

Theorem 5.3. *Let* $S = [0,1]$, $\theta = [0,4]$, $X_{n+1} = \theta_{n+1} X_n (1 - X_n)$ *with* $\{\theta_n\}_{n \geq 1}$ *i.i.d. r.v. with values in* $[0,4]$ *and* X_0 *an independent r.v. with values in* $[0,1]$. *Suppose* $\exists$ *an open interval* $J \subset (0,4)$ *and a strictly positive function* Ψ *on* J *such that for all* $B \subset J$

$$Q(B) \equiv P(\theta_i \in B) \geq \int_B \Psi(\theta) M(d\theta)$$

where $m(\cdot)$ *is Lebesgue measure. If* $J \cap (1,4) = Q$, *then assume in addition that there exists a* $\beta > 1$ *in the support of* $Q(\cdot)$ *such that the map* $f(x, \beta) \equiv \beta x (1-x)$ *admits a stable periodic point* p *in* $(0,1)$.
  Then $\{X_n\}$ *is Harris irreducible.*
  Suppose further that $E \ln C_1 > 0$ *and* $E \left| \ln \left(1 - \frac{C_1}{4} \right) \right| < \infty$. *Then there exists a unique ergodic absolutely continuous stationary measure* π *such that the occupation measure*

$$\mu_{n,x}(\cdot) = \frac{1}{n} \sum_0^{n-1} P_x(x_j \in \cdot)$$

converges to π *in total variation norm.*

Corollary 5.1. *In the set up of Theorem 5.3 suppose* θ_1 *has the uniform* $[0,4]$ *distribution. Then* $\exists$ *a unique absolutely stationary probability* π *such that* $\pi(0,1) = 1$ *and for any* $0 < x < 1$, $\| P_x(X_n \in \cdot) - \pi(\cdot) \| \to 0$ *where* $\| \cdot \|$ *is total variation.*

6   Some open questions

1) *Persistence in the critical case.* Extend the Athreya-Schuh [5] results to the present more general setting.

2) *Nonuniqueness.* Extend the nonuniqueness result of Athreya and Dai [4] for the logistic case to the present setting.

3) *The condition* $E \left| \ln \left(1 - \frac{C_1}{4} \right) \right| < \infty$. For the random logistic case in the supercritical case this is a sufficient condition for the existence of a nontrivial stationary measure. However, it is known that if $P(C_1 = 4) = 1$ then von Neumann and Ulam [15] showed that the arcsine law is the unique ergodic has absolutely continuous stationary distribution. It is worth investigating whether this condition could be dropped.

4) *The lognormal limit law in the critical case.* It has been shown here that in the subcritical case the distribution of $\ln X_n$ is approximately normal. Extend this to the critical case.

5) *Statistical inference.* Suppose the sequence $\{X_j\}$ is observed for $0 \le j \le n$. Can one estimate the distribution of C_1 and $g_1(\cdot)$?

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