The Almost Sure Number of Pairwise Sums for Certain Random Integer Subsets Considered by P. Erdös

Michael J. Klass*
Departments of Statistics and Mathematics
University of California

Abstract

Fix any $\lambda > 0$ and let $X_1, X_2, \ldots$ be independent and identically distributed 0–1 valued random variables such that

$$P(X_j = 1) = \min\left\{\sqrt{\frac{2\lambda}{\pi}\,\frac{\ln j}{j}}, 1\right\}.$$

Let $G_n = \sum_{j=1}^{\lfloor n/2 \rfloor} X_j X_{n-j}$. G_n is the number of times two numbers from the random set $S \equiv \{j : X_j = 1\}$ add to n. We evaluate the almost sure limits $\liminf_{n\to\infty} \frac{G_n}{EG_n} \equiv c_1(\lambda)$ and $c_2(\lambda) \equiv \limsup_{n\to\infty} \frac{G_n}{EG_n}$, showing that $0 \le c_1(\lambda) < 1 < c_2(\lambda) < \infty$.

Introduction

Around 1932 Sidon asked whether there exist positive integers $a_1 < a_2 < \ldots$ such that $f(n) > 0$ for all n sufficiently large and yet $\lim_{n\to\infty} \frac{f(n)}{n^{\varepsilon}} = 0$ for all $\varepsilon > 0$, where

$$f(n) = \#\{i \ge 1 : a_i + a_{j_i} = n \text{ for some } j_i \ge i\}. \tag{1}$$

Fix any $\lambda > 0$. Let $X_1, X_2, \ldots$ be independent random variables taking only values zero and one, as determined by the probabilities

$$P(X_j = 1) = \min\left\{\sqrt{\frac{2\lambda}{\pi}\,\frac{\ln j}{j}}, 1\right\} \equiv P_j. \tag{2}$$

*Supported in part by National Science Foundation Grant DMS 96–26236.

Let $G_n = \sum_{j=1}^{\lfloor n/2 \rfloor} X_j X_{n-j}$. Using the integers occuring in the random subset $S \equiv \{j : X_j = 1\}$, Paul Erdös [1956] answered Sidon's question, showing that

$$c_1 \equiv c_1(\lambda) \equiv \liminf_{n\to\infty} \frac{G_n}{EG_n} \text{ is positive almost surely iff } \lambda > 1, \tag{3}$$

$$c_2 \equiv c_2(\lambda) \equiv \limsup_{n\to\infty} \frac{G_n}{EG_n} \text{ is finite almost surely,} \tag{4}$$

and

$$EG_n \sim \lambda \ln n \text{ as } n \to \infty. \tag{5}$$

Note that G_n denotes the number of instances in which a pair of elements of S sum to n.

Erdös then wondered whether $\frac{f(n)}{\ln n}$ can ever tend to a finite, positive limit. In this paper we evaluate $c_1(\lambda)$ and $c_2(\lambda)$, showing that indeed they are distinct for almost all of the subsets S constructed here.

Results

Using exponential bounds and the convergence part of the Borel–Cantelli lemma it can be easily shown that

$$\lim_{\varepsilon \searrow 0} \limsup_{n\to\infty} \sum_{i=1}^{\lfloor n\varepsilon \rfloor} \frac{X_i X_{n-i}}{EG_n} = 0 \text{ a.s.} \tag{6}$$

and similarly that

$$\lim_{\bar{\varepsilon} \searrow 0} \limsup_{n\to\infty} \sum_{i=\lfloor \frac{n}{2} - n\bar{\varepsilon} \rfloor}^{\lfloor \frac{n}{2} \rfloor} \frac{X_i X_{n-i}}{EG_n} = 0 \text{ a.s.} \tag{7}$$

For $c > 1$ put

$$A_{n,k,\varepsilon}(c) = \left\{ \sum_{i=\lfloor \varepsilon(1+\varepsilon)^k \rfloor}^{\lfloor \frac{(1+\varepsilon)^{k-1}}{2} \rfloor} X_i X_{n-i} \geq cEG_{\lfloor (1+\varepsilon)^k \rfloor} \right\}. \tag{8}$$

To second order precision (see Lemma 1 of the Appendix)

$$P(A_{n,k,\varepsilon}(c)) \sim P(N_{g(n,k,\varepsilon)} \geq cEG_{\lfloor (1+\varepsilon)^k \rfloor}) \tag{9}$$

uniformly in $0 < \varepsilon \ll 1$ and n in $(1+\varepsilon)^k < n \leq (1+\varepsilon)^{k+1}$ as $(1+\varepsilon)^k \to \infty$, where

$$g(n,k,\varepsilon) = \sum_{i=\lfloor \varepsilon(1+\varepsilon)^k \rfloor}^{\lfloor \frac{(1+\varepsilon)^{k-1}}{2} \rfloor} EX_i EX_{n-i} \tag{10}$$

and $N_\gamma \sim \text{Poisson}(\gamma)$. Since $g(n,k,\varepsilon) \sim \lambda k(1 - O(\sqrt{\varepsilon})) \ln(1+\varepsilon)$,

$$P(A_{n,k,\varepsilon}(c)) \sim (q(c(1+O(\sqrt{\varepsilon}))))^{(1-O(\sqrt{\varepsilon}))\lambda k \ln(1+\varepsilon)} \tag{11}$$

uniformly in n and ε as $(1+\varepsilon)^k \to \infty$, where $q(c) = \frac{e^{c-1}}{c^c}$.

Notice that $q(1) = 1$ and $q(c)$ is a continuous function on $1 \leq c < \infty$ which strictly decreases to zero. By the intermediate value theorem there is a unique $c_2 = c_2(\lambda) > 1$ such that

$$(q(c_2))^\lambda = \left(\frac{e^{c_2-1}}{(c_2)^{c_2}}\right)^\lambda = e^{-1}. \tag{12}$$

Take any $\bar{c} > c_2(\lambda)$. Then there exists $\delta > 0$ such that for all sufficiently small $\varepsilon > 0$

$$(q(\bar{c}(1+O(\sqrt{\varepsilon}))))^{(1-O(\sqrt{\varepsilon}))\lambda} < e^{-1-\delta} \tag{13}$$

and so (by (11) and (13)),

$$\lim_{k_0\to\infty} \sum_{k=k_0}^{\infty} \sum_{(1+\varepsilon)^k < n \leq (1+\varepsilon)^{k+1}} P(A_{n,k,\varepsilon}(\bar{c})) \leq \lim_{k_0\to\infty} \sum_{k=k_0}^{\infty} \varepsilon(1+\varepsilon)^{k+1} e^{-k(1+\delta)\ln(1+\varepsilon)} = 0.$$

Since $\bar{c} > c_2(\lambda)$ is arbitrary,

$$\limsup_{n\to\infty} \frac{G_n}{EG_n} \leq c_2(\lambda) \text{ a.s.} \tag{14}$$

On the other hand, if $1 < \underline{c} < c_2(\lambda)$ then there exists $\delta > 0$ such that for all sufficiently small $\varepsilon > 0$

$$(q(\underline{c}(1+O(\sqrt{\varepsilon}))))^{(1-O(\sqrt{\varepsilon}))\lambda} > e^{-1+\delta}. \tag{15}$$

Let $I_{k,\varepsilon,\underline{c}}$ denote the interval of consecutive integers n such that $(1+\varepsilon)^k < n \leq n_{k,\varepsilon}(\underline{c})$, where

$$\begin{aligned} n_{k,\varepsilon}(\underline{c}) &= \text{the last } n \leq (1+\varepsilon)^{k+1} : \\ &\quad 2\pi(c_2(\lambda))^{32} EG_{\lfloor (1+\varepsilon)^k \rfloor} \sum_{j=\lceil (1+\varepsilon)^k \rceil}^{n} P(A_{j,k,\varepsilon}(\underline{c})) \leq 1. \end{aligned} \tag{16}$$

Then set

$$A^*_{k,\varepsilon}(\underline{c}) = \bigcup_{n\in I_{k,\varepsilon,\underline{c}}} A_{n,k,\varepsilon}(\underline{c}). \tag{17}$$

By restricting $A^*_{k,\varepsilon}(\underline{c})$ to a union over only some of the integers $(1+\varepsilon)^k < n \leq (1+\varepsilon)^{k+1}$, we will be able to compute the order of magnitude of $P(A^*_{k,\varepsilon}(\underline{c}))$. Applying Lemma 4 of the Appendix to the probability of pairwise intersections of events whose union comprises $A^*_{k,\varepsilon}(\underline{c})$ demonstrates by means of the Bonferroni inequality

$$\sum_{n\in I_{k,\varepsilon,\underline{c}}} P(A_{n,k,\varepsilon}(\underline{c})) - \frac{1}{2} \sum_{\{n\neq n':n,n'\in I_{k,\varepsilon,\underline{c}}\}} P(A_{n,k,\varepsilon}(\underline{c}) \cap A_{n',k,\varepsilon}(\underline{c})) \leq P(A^*_{k,\varepsilon}(\underline{c})) \tag{18}$$

that the correct order of magnitude of $P(A^*_{k,\varepsilon}(\underline{c}))$ is given by Boole's inequality:

$$P(A^*_{k,\varepsilon}(\underline{c})) \leq \sum_{n\in I_{k,\varepsilon,\underline{c}}} P(A_{n,k,\varepsilon}(\underline{c})). \tag{19}$$

Actually, for all $\varepsilon > 0$ sufficiently small and $\lfloor (1+\varepsilon)^k \rfloor$ sufficiently large

$$P(A^*_{k,\varepsilon}(\underline{c})) \geq \frac{(5\pi(c_2(\lambda))^{32})^{-1}}{EG_{\lfloor (1+\varepsilon)^k \rfloor}}. \tag{20}$$

For $k' \geq k + \varepsilon^{-2}$, $A_{k,\varepsilon}(\underline{c})$ and $A_{k',\varepsilon}(\underline{c})$ are independent. Moreover, by (5) and (20), $\sum_{k=1}^{\infty} P(A^*_{\lfloor k\varepsilon^{-2}\rfloor,\varepsilon}(\underline{c}))$ diverges. Hence $\limsup_{n\to\infty} \frac{G_n}{EG_n} \geq \underline{c}$ and so

$$\limsup_{n\to\infty} \frac{G_n}{EG_n} = c_2(\lambda) \text{ a.s.} \tag{21}$$

As for the almost sure lower bound, Erdös showed in 1956 that $c_1 \equiv c_1(\lambda) = 0$ if $\lambda \leq 1$. In fact, Erdös showed that $G_n = 0$ infinitely often if $\lambda < 1$. Suppose, therefore, that $\lambda > 1$. By a zero-one law followed by application of Fatou's lemma,

$$\begin{aligned} L \equiv L(\lambda) &\equiv \liminf_{n\to\infty} \frac{G_n}{EG_n} = E \liminf_{n\to\infty} \frac{G_n}{EG_n} \\ &\leq \liminf_{n\to\infty} E\left(\frac{G_n}{EG_n}\right) = 1. \end{aligned}$$

Hence the $\liminf_{n\to\infty}$ and $\limsup_{n\to\infty}$ of $\frac{G_n}{EG_n}$ are indeed distinct. In an effort to identify $L(\lambda)$, let $c_1 \equiv c_1(\lambda)$ denote the smallest positive root of the equation

$$\left(\frac{e^{c_1-1}}{(c_1)^{c_1}}\right)^{\lambda} = e^{-1}. \tag{22}$$

Since $q(c)$ is continuous on $[0,1]$, strictly increasing from e^{-1} to 1, it is clear that $0 < c_1(\lambda) < 1$ for $\lambda > 1$. Set

$$B_{n,k,\varepsilon}(c) = \left\{ \sum_{i=\lfloor \varepsilon(1+\varepsilon)^k \rfloor}^{\lfloor \frac{(1+\varepsilon)^{k-1}}{2} \rfloor} X_i X_{n-i} \le cEG_{\lfloor (1+\varepsilon)^k \rfloor} \right\}. \tag{23}$$

Reasoning much as before, if $0 < \underline{c} < c_1(\lambda)$ and $\lambda > 1$, then

$$\lim_{k_0 \to \infty} \sum_{k=k_0}^{\infty} \varepsilon(1+\varepsilon)^k P(B_{\lfloor (1+\varepsilon)^{k+1} \rfloor, k, \varepsilon}(\underline{c})) = 0, \tag{24}$$

which implies $P(B_{n,k,\varepsilon}(\underline{c})$ i.o. $(n)) = 0$. Since $\varepsilon > 0$ and $0 < \underline{c} < c_1(\lambda)$ are arbitrary,

$$\liminf_{n \to \infty} \frac{G_n}{EG_n} \ge c_1(\lambda) \text{ a.s.} \tag{25}$$

As for the reverse inequality, it is proved by applying an analogue of Lemma 4 of the Appendix to the analogous Bonferroni inequality for all fixed $\bar{c} > c_1(\lambda)$ and then using the divergence part of the Borel–Cantelli lemma as before. Consequently, $\liminf_{n \to \infty} \frac{G_n}{EG_n} \le c_1(\lambda)$ a.s. and therefore $\liminf_{n \to \infty} \frac{G_n}{EG_n} = c_1(\lambda)$ a.s.

Appendix

Lemma 1. *Let* $(1+\varepsilon)^k < n \le (1+\varepsilon)^{k+1}$ *and define* $A_{n,k,\varepsilon}(c)$ *as in* (8). *Then* (A.9) *holds for fixed* $c > 1$ *and* $\lambda > 0$.

Proof. Let $Y_{i,n} = X_i X_{n-i}$. For each fixed n in the indicated interval and all $\lfloor \varepsilon(1+\varepsilon)^k \rfloor \le i \le \frac{(1+\varepsilon)^{k-1}}{2}$, the random variables $Y_{i,n}$ are independent Bernoulli's. Letting

$$e^{-\lambda_{i,n}} = 1 - P_i P_{n-i} \tag{A.1}$$

and introducing independent random variables

$$W_{i,n} = \mathrm{Pois}(\lambda_{i,n}), \tag{A.2}$$

it is obvious that

$$\mathcal{L}(Y_{i,n}) = \mathcal{L}(\min\{W_{i,n}, 1\}).$$

Hence we may assume

$$Y_{i,n} = \min\{W_{i,n}, 1\}. \tag{A.3}$$

Let

$$\lambda_n = \sum_{i=\lfloor \varepsilon(1+\varepsilon)^k \rfloor}^{\lfloor \frac{(1+\varepsilon)^{k-1}}{2} \rfloor} \lambda_{i,n}, \tag{A.4}$$

$$W_n = \sum_{i=\lfloor \varepsilon(1+\varepsilon)^k \rfloor}^{\lfloor \frac{(1+\varepsilon)^{k-1}}{2} \rfloor} W_{i,n}, \tag{A.5}$$

and

$$Y_n = \sum_{i=\lfloor \varepsilon(1+\varepsilon)^k \rfloor}^{\lfloor \frac{(1+\varepsilon)^{k-1}}{2} \rfloor} Y_{i,n}. \tag{A.6}$$

Then

$$W_n \sim \mathrm{Pois}(\lambda_n)$$

and

$$P(Y_n \ne W_n) \le \sum_{i=\lfloor \varepsilon(1+\varepsilon)^k \rfloor}^{\lfloor \frac{(1+\varepsilon)^{k-1}}{2} \rfloor} \frac{(\lambda_{i,n})^2}{2}.$$

Since

$$\begin{aligned}\lambda_{i,n} &= -\ln(1 - P_i P_{n-i}) \\ &= P_i P_{n-i} + \theta_{i,n}(P_i P_{n-i})^2,\end{aligned}$$

where $\frac{1}{2} \le \theta_{i,n} \le 1$ for all i sufficiently large

$$(A.7) \qquad \lambda_n = EY_n + \theta_{n,k,\varepsilon} \frac{4\lambda^2 k^2 \varepsilon^2 \ln \frac{1}{\varepsilon}}{\pi^2 n}$$

where $|\theta_{n,k,\varepsilon}| \le \frac{1}{2} + O(\varepsilon)$ for all $(1+\varepsilon)^k$ sufficiently large and

$$(A.8) \qquad P(Y_n \ne W_n) \le \frac{4\lambda^2 k^2 \varepsilon^2 \ln \frac{1}{\varepsilon}}{\pi^2 n}$$

for all $(1+\varepsilon)^k$ sufficiently large and $0 < \varepsilon \ll \frac{1}{4}$. Note that $g(n,k,\varepsilon) = EY_n$.

By virtue of $(A.7)$ and $(A.8)$, for all $\varepsilon > 0$ sufficiently small and $(1+\varepsilon)^k$ sufficiently large,

$$(A.9) \qquad |P(A_{n,k,\varepsilon}(c)) - P(N_{g(n,k,\varepsilon)} \ge cEG_{\lfloor (1+\varepsilon)^k \rfloor})| \le \frac{8\lambda^2 k^2 \varepsilon^2 \ln \frac{1}{\varepsilon}}{\pi^2 (1+\varepsilon)^k}$$

for all $n \in \left(\lfloor (1+\varepsilon)^k \rfloor, \lfloor (1+\varepsilon)^{k+1} \rfloor\right]$.

Lemma 2. *Let $N_\gamma \sim$ Pois γ. Take any $1 < \underline{c} < \bar{c} < \infty$. For $\underline{c} \le c \le \bar{c}$*

$$(A.10) \qquad P(N_\gamma \ge c\gamma) \sim \sqrt{\frac{c}{2\pi\gamma(c-1)}} \left(\frac{e^{c-1}}{c^c}\right)^\gamma .$$

uniformly in c as $\gamma \to \infty$. For purposes of comparison, the best possible exponential upper bound of this probability is

$$(A.11) \qquad \inf_{t>0} Ee^{t(N_\gamma - c\gamma)} = \left(\frac{e^{c-1}}{c^c}\right)^\gamma ,$$

using $t = t_c = \ln c$. Hence if $\underline{c} \le c \le \bar{c}$

$$(A.12) \qquad \inf_{t>0} Ee^{t(N_\gamma - c\gamma)} \le \sqrt{2\pi\gamma} P(N_\gamma \ge c\gamma)$$

for all γ sufficiently large.

Secondly, take any $0 < c_- < c_-^* < 1$. For $c_- \le c \le c_-^*$,

$$(A.13) \qquad P(N_\gamma \le c\gamma) \sim \frac{1}{(1-c)\sqrt{2\pi c\gamma}} \left(\frac{e^{c-1}}{c^c}\right)^\gamma .$$

The best possible exponential upper bound of this probability is

$$(A.14) \qquad \inf_{t>0} Ee^{t(c\gamma - N_\gamma)} = \left(\frac{e^{c-1}}{c^c}\right)^\gamma ,$$

using $t = t_c = -ln\ c$.

Hence for $c_- \le c \le c_-^*$

$$\inf_{t>0} Ee^{t(c\gamma - N_\gamma)\gamma} \le \sqrt{\gamma} P(N_\gamma \le c\gamma) \tag{A.15}$$

for all sufficiently large γ *(since* $(1-c)^2 2\pi c \le 1$*).*

Lemma 3. *Let* $(1+\varepsilon)^k < n < n' \le (1+\varepsilon)^{k+1}$ *and* $J_k = \{l : \lfloor \varepsilon(1+\varepsilon)^k \rfloor \le l \le \lfloor \frac{(1+\varepsilon)^{k-1}}{2} \rfloor\}$*. Then*

$$P\left(\sum_{l\in J_k} X_l X_{n-l} X_{n'-l} \ge 30\right) \le \frac{1}{(1+\varepsilon)^{3k}}$$

for all $(1+\varepsilon)^k$ *sufficiently large (uniformly in* n *and* n'*).*

Proof. The set J_k can be partitioned into three disjoint subsets (and sometimes two) $J_{k,1}$, $J_{k,2}$ and $J_{k,3}$ such that the variates $\{X_l X_{n-l} X_{n'-l} : l \in J_{k,i}\}$ are independent for each $1 \le i \le 3$.

Letting l_k denote the smallest integer in J_k, the set $J_{k,i}$ can be constructed as follows. Let $\tilde{J}_{k,i} = \{l \in J_k$ of the form $l_k + (i-1)(\lfloor \frac{n'-n}{2} \rfloor + 1) + i' + j'(n' - n + \lfloor \frac{n'-n}{2} \rfloor + 1)$ such that $0 \le i' \le \lfloor \frac{n'-n}{2} \rfloor$ and $j' \ge 0\}$. Then let $J_{k,1} = \tilde{J}_{k,1}$, $J_{k,2} = \tilde{J}_{k,2}$, and $J_{k,3} = \tilde{J}_{k,3} \backslash J_{k,1}$

$$P\left(\sum_{l\in J_k} X_l X_{n-l} X_{n'-l} \ge 30\right) \le \sum_{i=1}^{3} P\left(\sum_{l\in J_{k,i}} X_l X_{n-l} X_{n'-l} \ge 10\right).$$

Using an exponential upper bound,

$$\begin{aligned} P\left(\sum_{l\in J_{k,i}} X_l X_{n-l} X_{n'-l} \ge 10\right) &\le E\exp\left(-10t + \sum_{l\in J_{k,i}} tX_l X_{n-l} X_{n'-l}\right) \\ &= e^{-10t} \prod_{l\in J_{k,i}} Ee^{tX_l X_{n-l} X_{n'-l}} \\ &= e^{-10t} \prod_{l\in J_{k,i}} (1 + P_l P_{n-l} P_{n'-l}(e^t - 1)) \\ &\le e^{-10t} \exp \sum_{l\in J_{k,i}} P_l P_{n-l} P_{n'-l}(e^t - 1). \end{aligned}$$

Set $t = \ln 2(1+\varepsilon)^{k/3}$. Then

$$\sum_{l\in J_{k,i}} P_l P_{n-l} P_{n'-l}(e^t - 1) \to 0$$

as $k \to \infty$ and the result holds. □

Lemma 4. *Fix any* $1 < \underline{c} < c_2(\lambda)$. *Put* $g_k \equiv g_{k,\varepsilon} = EG\lfloor(1+\varepsilon)^k\rfloor$. *Then take* $\varepsilon > 0$ *sufficiently small. Using the same notations and assumptions as given elsewhere in the paper,*

$$P(A_{n,k,\varepsilon}(\underline{c}) \cap A_{n',k,\varepsilon}(\underline{c})) \leq (1+\varepsilon)^{-3k} + (c_2(\lambda))^{31} 2\pi EG_{\lfloor(1+\varepsilon)^k\rfloor} P(A_{n,k,\varepsilon}(\underline{c}))P(A_{n',k,\varepsilon}(\underline{c})).$$

Proof.

$$\begin{aligned}
P(A_{n,k,\varepsilon}(\underline{c}) \cap A_{n',k,\varepsilon}(\underline{c})) &\leq P\left(\sum_{l\in J_k} X_l X_{n-l} X_{n'-l} \geq 30\right) \\
&\quad + P\left(\sum_{l\in J_k} X_l X_{n-l} X_{n'-l} \leq 30, e^{t_1(\sum_{l\in J_k} X_l X_{n-l} - \underline{c} g_k)}\right. \\
&\quad \left. \times e^{t_2(\sum_{l\in J_k} X_l X_{n'-l} - \underline{c} g_k)} \geq 1\right) \\
&\leq \frac{1}{(1+\varepsilon)^{3k}} + E e^{t_1(\sum_{l\in J_k} X_l X_{n-l} - \underline{c} g_k)} \\
&\quad \times e^{t_2(\sum_{l\in J_k} X_l X_{n'-l} - \underline{c} g_k)} e^{t_2(30 - \sum_{l\in J_k} X_l X_{n-l} X_{n'-l})} \\
&\leq (1+\varepsilon)^{3k} + T_2.
\end{aligned}$$

Let $l_k = \min\{l \in J_k\}$ and $l_k^* = \max\{l \in J_k\}$. Taking conditional expectations given $\{X_{n-l}X_{n'-l} : l \in J_k\}$, rewriting the resultant expression and then upper bounding that,

$$\begin{aligned}
T_2 &= e^{30t_2-(t_1+t_2)\underline{c}g_k} E\prod_{l\in J_k}(1+P_l(e^{t_1X_{n-l}+t_2X_{n'-l}-t_2X_{n-l}X_{n'-l}}-1)) \\
&= e^{30t_2-(t_1+t_2)\underline{c}g_k} E\prod_{l\in J_k}(1+P_l(e^{t_1}-1))^{X_{n-l}}(1+P_l(e^{t_2}-1))^{X_{n'-l}(1-X_{n-l})} \\
&\le e^{30t_2-(t_1+t_2)\underline{c}g_k} E\left(\left(\prod_{l\in n-J_k}(1+P_{n-l}(e^{t_1}-1))^{X_l}\right)\left(\prod_{l\in n'-J_k}(1+P_{n'-l}(e^{t_2}-1))^{X_l}\right)\right) \\
&= e^{30t_2-(t_1+t_2)\underline{c}g_k} E\left(\left(\prod_{l=n-l_k^*}^{n'-l_k^*-1}(1+P_{n-l}(e^{t_1}-1))^{X_l}\right)\right. \\
&\qquad \prod_{l=n'-l_k^*}^{n-l_k}\{(1+P_{n-l}(e^{t_1}-1))(1+P_{n'-l}(e^{t_2}-1))\}^{X_l} \\
&\qquad \left.\times\prod_{l=n-l_k+1}^{n'-l_k}(1+P_{n'-l}(e^{t_2}-1))^{X_l}\right) \\
&= e^{30t_2-(t_1+t_2)\underline{c}g_k}\prod_{l=n-l_k^*}^{n'-l_k^*-1}(1+P_lP_{n-l}(e^{t_1}-1)) \\
&\qquad \times\prod_{l=n'-l_k^*}^{n-l_k}(1+P_lP_{n-l}(e^{t_1}-1)+P_lP_{n'-l}(e^{t_2}-1)+P_lP_{n-l}P_{n'-l}(e^{t_1}-1)(e^{t_2}-1)) \\
&\qquad \times\prod_{l=n-l_k+1}^{n'-l_k}(1+P_lP_{n-l}(e^{t_2}-1)) \\
&\le \exp\left\{30t_2-(t_1+t_2)\underline{c}g_k+(e^{t_1}-1)\sum_{j\in J_k}P_jP_{n-j}\right. \\
&\qquad \left.+(e^{t_2}-1)\sum_{j\in J_k}P_jP_{n'-j}+(e^{t_1}-1)(e^{t_2}-1)\sum_{j\in J_k}P_jP_{n-j}P_{n'-j}\right\}.
\end{aligned}$$

$\sum_{j\in J_k}P_jP_{n-j}=g(n,k,\varepsilon)$ and $\sum_{j\in J_k}P_jP_{n'-j}=g(n',k,\varepsilon)$, each of which is asymptotic to $EG_{\lfloor(1+\varepsilon)^k\rfloor}$ uniformly in n,n' as $(1+\varepsilon)^k\to\infty$. Letting $e^{t_1}=\frac{\underline{c}EG_{\lfloor(1+\varepsilon)^k\rfloor}}{g(n,k,\varepsilon)}$ and $e^{t_2}=\frac{\underline{c}EG_{\lfloor(1+\varepsilon)^k\rfloor}}{g(n',k,\varepsilon)}$, $(A.12)$ of Lemma 2 gives

$$e^{-t_1\underline{c}g_k+(e^{t_1}-1)g(n,k,\varepsilon)}\le\sqrt{2\pi g(n,k,\varepsilon)}P(N_{g(n,k,\varepsilon)}\ge\underline{c}g_k)$$

and

$$e^{-t_2 \underline{c} g_k + (e^{t_2}-1) g(n',k,\varepsilon)} \le \sqrt{2\pi g(n',k,\varepsilon)}\, P(N_{g(n',k,\varepsilon)} \ge \underline{c} g_k).$$

Note that

$$(e^{t_1}-1)(e^{t_2}-1) \sum_{j \in J_k} P_j P_{n-j} P_{n'-j} \to 0$$

as $(1+\varepsilon)^k \to \infty$. Incorporating Lemma 1 as well as the formula for e^{t_2}, etc.,

$$T_2 \le (\underline{c})^{31} 2\pi E G_{\lfloor (1+\varepsilon)^k \rfloor} P(A_{n,k,\varepsilon}(\underline{c})) P(A_{n',k,\varepsilon}(\underline{c}))$$

for all $(1+\varepsilon)^k$ sufficiently large. □

Acknowledgment. I would like to express my gratitude to Professor Anant P. Godbole who kindly suggested this problem to me and thereby also acquainted me with some of Professor Paul Erdös' classic problems, solved and unsolved.

Dedication. It is a pleasure and an honor for me to be able to seize this opportunity to dedicate a paper to the life of Professor Thomas S. Ferguson. By introducing me to the "S_n/n" Problem, Tom attracted me to problems involving almost sure convergence and paved the way for whatever I have been able to do in probability theory. Thanks for a priceless gift, Tom.

References

[1] Erdös, P. (1955), *Problems and results in additive number theory*, editor George Thone, Liege, Colloque sur la Theorie des Nombres, Bruxelles, pp. 127–137. Published by Masson & Cie, Paris.

[2] Halberstam, H. and Roth, K.F. (1983), "Sequences", 2nd edition, vol. I, Springer-Verlag, xviii+292 pages.