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Versions of de Finetti's Theorem with applications to damage models*

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Abstract: Alzaid et al. (1986) and Rao et al. (2002) have shown that several of the results on damage models have links with certain results on nonnegative matrices. Rao et al. (2002) have also shown that there is a connection between a specialized version of de Finetti's theorem for discrete exchangeable random variables and a potential theoretic result relative to nonnegative matrices. In the present article, we deal with integral equations met in damage model studies via specialized versions of de Finetti's theorem and extend further the theorems of Rao and Rubin (1964) and Shanbhag (1977) on damage models.

1. Introduction

The concept of damage models was first introduced by Rao (1963) and it has led to many interesting and illuminating characterizations of discrete distributions; among various noteworthy results in the area are those of Rao and Rubin (1964) and Shanbhag (1977). In mathematical terms, a damage model can be described by a random vector (X, Y) of non-negative integer-valued components, with the joint probability law of X and Y having the following structure:

$$P\{X = x, Y = y\} = S(y|x)g_x, \qquad y = 0, 1, 2, \ldots, x;\; x = 0, 1, 2, \ldots, \tag{1.1}$$

where $\{S(y|x) = P\{Y = y|X = x\} : y = 0, 1, 2, \ldots, x\}$ is a discrete probability law for each $x = 0, 1, 2, \ldots$ and $\{g_x = P\{X = x\} : x = 0, 1, 2, \ldots\}$ is the marginal probability law of X. In the context of damage models, the conditional probability law $\{S(y|x) : y = 0, 1, 2, \ldots, x\}$ is called the survival distribution. It is also natural to call Y the undamaged part of X and $X - Y$ the damaged part of X. Multivariate versions of the terminologies have also been dealt with in the literature. Rao and Rubin (1964) showed via Bernstein's theorem for absolutely monotonic functions that if the survival distribution is binomial with parameter vector (x, p) for almost all x (i.e. for each x with $g_x > 0$), where $p \in (0, 1)$ and fixed, and $g_0 < 1$, then the Rao-Rubin condition (RR(0))

$$P\{X = y\} = P\{Y = y|X = Y\}, \qquad y = 0, 1, 2, \ldots \tag{1.2}$$

is met if and only if X is Poisson. It was pointed out by Shanbhag (1977) that an extended version of the Rao–Rubin result can be deduced from the solution to a

*One of us has collaborated with Professor Herman Rubin on a result which is now known in statistical literature as the Rao–Rubin theorem.This theorem and another result known as Shanbhag's theorem have generated considerable research on characterization problems.Our paper on these theorems and some further results is dedicated to Professor Rubin in appreciation of his fundamental contributions to statistical inference.

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general recurrence relation of the form

$$v_n = \sum_{m=0}^{\infty} w_m v_{m+n}, \qquad n = 0, 1, 2, \ldots, \tag{1.3}$$

where $\{w_m : m \geq 0\}$ is a given sequence of nonnegative real numbers with $w_1 > 0$ and $\{v_n : n \geq 0\}$ is a sequence of nonnegative real numbers to be determined. Using essentially a renewal theoretic approach, Shanbhag obtained a complete solution to (1.3), which provided a unified approach to a variety of characterizations of discrete distributions including, in particular, those related to damage models, strong memoryless property, order statistics, record values, etc.

Shanbhag's (1977) general result on damage models states essentially (in the notation described above) that if $g_0 < 1$ and, with $\{(a_n, b_n) : n = 0, 1, \ldots\}$ as a sequence of 2-component real vectors such that $a_n > 0$ for all $n, b_0, b_1 > 0$, and $b_n \geq 0$ for all $n \geq 2$, we have, for almost all x,

$$S(y|x) \propto a_y b_{x-y}, \qquad y = 0, 1, \ldots, x,$$

then the following are equivalent:

(i) $(1 \cdot 1)$ (i.e. RR(0)) is met;

(ii) Y and $X - Y$ are independent;

(iii) $(g_x/c_x) = (g_0/c_0)\lambda^x,\ x = 0, 1, \ldots$, for some $\lambda > 0$, where $\{c_n\}$ is the convolution of $\{a_n\}$ and $\{b_n\}$.

Characterizations of many standard discrete distributions in damage model studies follow as corollaries to this latter result. In particular, taking $a_n = p^n/n!,\ n = 0, 1, \ldots$, and $b_n = (1-p)^n/n!,\ n = 0, 1, \ldots$, where $p \in (0, 1)$ and fixed, we get the Rao–Rubin (1964) theorem as a corollary to this. There are several other interesting contributions to the literature on damage models. Rao and Shanbhag (1994; Chapter 7) have reviewed and unified most of these. More recently, Rao et al. (2002) and Rao et al. (2003) have provided systematic approaches to damage models based on nonnegative matrices and Markov chains. In particular, Rao et al. (2002) have shown that several of the findings on damage models in the literature are corollaries to a potential theoretic result, appearing as Theorem 4.4.1 in Rao and Shanbhag (1994), on nonnegative matrices; these subsume some of the results in the area based on the version of de Finetti's theorem for discrete exchangeable random variables.

The purpose of the present paper is to go beyond Rao et al. (2002) and show, amongst other things, that certain specialized versions of de Finetti's theorem or the relevant moment arguments provide us with further novel approaches to arrive at the Rao–Rubin–Shanbhag theorems or their generalizations. In the process of doing this, we also establish some new results on damage models or otherwise, including, in particular, an improved version of the crucial result of Alzaid et al. (1987a).

2. Simple integral equations in damage model studies

The link between the Choquet–Deny type integral equations and exchangeability or, in particular, certain versions of de Finetti's theorem for an infinite sequence of exchangeable random variables is well-documented in Rao and Shanbhag (1994) and other places in the literature. Some specialized versions of de Finetti's theorem follow via simple arguments involving, among others, moments of probability distributions, or a potential theoretic result on nonnegative matrices; see, for example, Feller (1966, pp. 225–226) and Rao et al. (2002). A detailed account of the literature on de Finetti's theorem is provided by Aldous (1985); see, also, Chow and Teicher (1979) for an elegant proof of the theorem in the case of real-valued random variables.

Our main objective in this section though is to verify certain key results on functional equations with applications to damage models, as corollaries to specialized versions of de Finetti's theorem; the theorems and corollaries that we have dealt with in this section are obviously subumed by the relevant general results obtained via certain other techniques in Rao and Shanbhag (1994, Chapter 3) and Rao and Shanbhag (1998).

Theorem 2.1 (Shanbhag's Lemma [32]). *Let $\{(v_n, w_n) : n = 0, 1, \ldots\}$ be a sequence of 2-vectors with nonnegative real components, such that $v_n > 0$ for at least for one $n > 0$ and $w_1 > 0$. Then (1.3) is met if and only if, for some $b > 0$,*

$$v_n = v_0 b^n,\ n = 1, 2, \ldots, \qquad \text{and} \qquad \sum_{n=0}^{\infty} w_n b^n = 1. \tag{2.1}$$

Proof. The "if" part of the assertion is trivial. To prove the "only if" part of the assertion, let (1.3) be met with the stated assumptions. Since in that case we have $v_n(1 - w_0) \geq w_1 v_{n+1},\ n = 0, 1, \ldots$, it is clear that $w_0 < 1$ and $v_0 > 0$. (Note that Shanbhag (1977) observes via a slightly different argument that $v_n > 0$ for all $n \geq 0$, but, for us, it is sufficient to have that $v_0 > 0$.) Essentially from (1.3), we have then that there exists a sequence $\{X_n : n = 1, 2, \ldots\}$ of 0-1-valued exchangeable random variables satisfying

$$P\{X_1 = \cdots = X_n = 1\} = \frac{v_n}{v_0} w_1^n, \qquad n = 1, 2, \ldots. \tag{2.2}$$

(For some relevant information,see Remark 2.6.) From the corresponding specialized version of de Finetti's theorem, we have hence that $\{\frac{v_n}{v_0} w_1^n : n = 0, 1, \ldots\}$ is a moment sequence of a (bounded) nonnegative random variable, which, in turn, implies that $\{\frac{v_n}{v_0} : n = 0, 1, \ldots\}$ is a moment sequence of a (bounded) nonnegative random variable. Denoting the random variable in the latter case by Y and appealing to (1.3) in conjunction with the expression for Z, we get, in view of Fubini's theorem,or the monotone convergence theorem, that

$$E(Z) = E\big(Z^2\big) = 1, \tag{2.3}$$

where

$$Z = \sum_{n=0}^{\infty} w_n Y^n. \tag{2.4}$$

From (2.3), noting, for example, that $E\{(Z - 1)^2\} = 0$, we see that $Z = 1$ a.s.; consequently, from (2.4) and, in particular, the property that $w_0 < 1$, we get that there exists a number $b > 0$ such that $Y = b$ a.s. and $\sum_{n=0}^{\infty} w_n b^n = 1$. Since

$$\frac{v_n}{v_0} = E(Y^n),\ n = 0, 1, \ldots,$$

we then see that the "only if" part of the theorem holds. □

Theorem 2.2. *Let k be a positive integer and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $\{(v_{\underline{n}}, w_{\underline{n}}) : \underline{n} \in \mathbb{N}_0^k\}$ be a sequence of 2-vectors of nonnegative real components such that $v_{\underline{0}} > 0, w_{\underline{0}} < 1$ and $w_{\underline{n}} > 0$ whenever $\underline{n}$ is of unit length. (The notation $\underline{0}$ stands for $\underline{n}$ with all coordinates equal to zero.) Then*

$$v_{\underline{n}} = \sum_{\underline{m} \in \mathbb{N}_0^k} v_{\underline{n}+\underline{m}} w_{\underline{m}}, \ \underline{n} \in \mathbb{N}_0^k \tag{2.5}$$

if and only if $\{v_{\underline{n}}/v_{\underline{0}}\}$ is the moment sequence relative to a k-component random vector $(Y_1, \ldots, Y_k)$ with Y_r's as nonnegative and bounded such that (in obvious notation)

$$\sum_{\underline{n} \in \mathbb{N}_0^k} w_{\underline{n}} \prod_{r=1}^{k} Y_r^{n_r} = 1 \quad a.s. \tag{2.6}$$

Proof. It is sufficient, as in the case of Theorem 2.1, to prove the "only if" part of the assertion. Clearly under the assumptions of the theorem taking for convenience $k \geq 2$, the validity of (2.5) implies the existence of a sequence $\{X_m : m = 1, 2, \ldots\}$ of exchangeable random variables, with values in $\{0, 1, \ldots, k\}$, satisfying (with obvious interpretation when some or all of the n_r's equal zero)

$P\{X_1, \ldots, X_{n_1+\ldots+n_k}$ are such that the first n_1 of these equal 1, the next n_2 equal 2, and so on$\}$

$$= \frac{v_{\underline{n}}}{v_{\underline{0}}} \prod_{r=1}^{k} w_{\underline{I}(r)}^{n_r}, \ \underline{n}\big(= (n_1, \ldots, n_k)\big) \in \mathbb{N}_0^k, \tag{2.7}$$

where $\underline{I}(r)$ is the rth row of the $k \times k$ identity matrix. (For some relevant information,see Remark 2.6.) Using the appropriate version of de Finetti's theorem and following a suitably modified version of the relevant part of the argument in the proof of Theorem 2.1, we see that there exists a random vector $(Y_1, \ldots, Y_k)$ as in the assertion with $\{v_{\underline{n}}/v_{\underline{0}}\}$ as the corresponding moment sequence; note especially that in this latter case (2.3) holds with Z given by the left hand side of (2.6). □

Corollary 2.1 (Hausdorff). *A sequence $\{\mu_{\underline{n}} : \underline{n} \in \mathbb{N}_0^k\}$ of real numbers represents the moment sequence of some probability distribution concentrated on $[0,1]^k$ if and only if $\mu_{\underline{0}} = 1$ and*

$$(-1)^{m_1+\ldots+m_k} \Delta_1^{m_1} \ldots \Delta_k^{m_k} \mu_{\underline{n}} \geq 0, \ (m_1, \ldots, m_k, \underline{n}) \in \mathbb{N}_0^{2k}, \tag{2.8}$$

where Δ_i is the usual difference operator acting on the ith coordinate.

Proof. Define the left hand side of the inequality under (2.8) by $v_{(m_1,\ldots,m_k,n_1,\ldots,n_k)}$. Then, we can easily verify that

$$\begin{aligned} &v_{(m_1,\ldots,m_k,n_1,\ldots,n_k)} \\ &\quad = \frac{1}{k}\big\{v_{(m_1+1,\ldots,m_k,n_1,\ldots,n_k)} + \cdots + v_{(m_1,\ldots,m_k+1,n_1,\ldots,n_k)} \\ &\qquad + v_{(m_1,\ldots,m_k,n_1+1,\ldots,n_k)} + \cdots + v_{(m_1,\ldots,m_k,n_1,\ldots,n_k+1)}\big\}, \\ &\quad (m_1, \ldots, m_k, \underline{n}) \in \mathbb{N}_0^{2k}. \end{aligned}$$

Because of (2.8),Theorem 2.2 implies then that $\{\mu_{\underline{n}} : \underline{n} \in \mathbb{N}_0^k\}$ (i.e. $\{v_{(\underline{0},\underline{n})} : \underline{n} \in \mathbb{N}_0^k\}$) is the moment sequence relative to a k-component random vector $(Y_1, \ldots, Y_k)$ with

Y_i's bounded and nonnegative. In view of (2.8), it follows further that $\{E(Y_r^{n_r}) : n_r = 0, 1, \ldots\}$ is decreasing and hence, it is obvious that the "if" part of the result holds. The "only if" part here is trivial and therefore we have the corollary. □

Remark 2.1. Although Theorem 2.1 is a corollary to Theorem 2.2, we have dealt with it separately because of its importance in characterization theory relative to univariate discrete distributions. Theorem 2.2, in turn, is a corollary to a result of Ressel (1985) and also to that of Rao and Shanbhag(1998) established via certain general versions of de Finetti's theorem, but its proof given by us here could appeal to the audience due to its simplicity. It may also be worth pointing out in this place that Chapter 3 of Rao and Shanbhag(1994) reviews and unifies,amongst other things,martingale approaches to certain generalized versions of Theorem 2.2,implied earlier;the cited chapter also shows,explicitly or otherwise,using partially a different route to ours that the following Corollaries 2.1 and 2.2 are consequences of the general results.

Remark 2.2. Corollary 2.1 can also be proved directly via de Finetti's theorem noting that there exists a sequence $\{X_n : n = 1, 2, \ldots\}$ of exchangeable random variables with values in $\{0, 1, \ldots, k\}$ and satisfying (2.7) with its right hand side replaced by $\mu_{\underline{n}} k^{-(n_1+\ldots+n_k)}$. Also, since $\{\mu_{\underline{n}}\}$ in Corollary 2.1 is the moment sequence relative to a probability distribution with compact support, it is obvious that it determines the distribution; in view of this, we can easily obtain the following result as a further corollary to Theorem 2.2

Corollary 2.2 (Bochner). *Let f be a completely monotonic function on $(0, \infty)^k$. Then f has the integral representation*

$$f(x) = \int_{[0,\infty)^k} \exp\{-\langle \underline{y}, \underline{x} \rangle\} \, d\nu(\underline{y}), \qquad \underline{x} \in (0, \infty)^k, \tag{2.9}$$

with ν as a uniquely determined measure on $[0, \infty)^k$.

Proof. Given any $\underline{x}_0 \in (0, \infty)^k$, Corollary 2.1, on taking into account the latter observation in Remark 2.2 and the continuity of f, implies after a minor manipulation that there exists a probability measure $\mu_{\underline{x}_0}$ on $[0, \infty)^k$ such that for all k-vectors $\underline{r}$ with positive rational components

$$f(\underline{x}_0 + \underline{r}) = f(\underline{x}_0) \int_{[0,\infty)^k} \exp\{-\langle \underline{y}, \underline{r} \rangle\} \, d\mu_{\underline{x}_0}(\underline{y}). \tag{2.10}$$

Since $f(\underline{x}_0 + \cdot)$ is continuous on $[0, \infty)^k$, (2.10) implies because of the dominated convergence theorem that

$$f(\underline{x}_0 + \underline{x}) = f(\underline{x}_0) \int_{[0,\infty)^k} \exp\{-\langle \underline{y}, \underline{x} \rangle\} \, d\mu_{\underline{x}_0}(\underline{y}), \qquad \underline{x} \in [0, \infty)^k.$$

In view of the arbitrary nature of $\underline{x}_0$ and the uniqueness theorem for Laplace-Stieltjes transforms, we have (2.9) to be valid with ν as unique and such that, irrespectively of what $\underline{x}_0$ is,

$$d\nu(\underline{y}) = f(\underline{x}_0) \exp\{\langle \underline{y}, \underline{x}_0 \rangle\} \, d\mu_{\underline{x}_0}(\underline{y}), \qquad \underline{y} \in [0, \infty)^k.$$

Hence, we have the Corollary. □

Remark 2.3. Bernstein's theorem for completely monotonic or absolutely monotonic functions is indeed a corollary to Corollary 2.2. Rao and Rubin (1964) have used this theorem to arrive at a characterization of Poisson distributions based on a damage model. There are also further applications of the theorem to damage models; see, for example, the next section of the present paper. Talwalker (1970) has given an extended version of the Rao–Rubin result via Corollary 2.2, while Puri and Rubin (1974) have given representations of relevance to reliability essentially via Corollaries 2.2 and 2.1, respectively; for certain observations on these latter results, see, for example, Shanbhag (1974) and Davies and Shanbhag (1987).

The following theorem of Rao and Shanbhag (1994, p.167), which is an extended version of the results of Rao and Rubin (1964) and Talwalker (1970) referred to in Remark 2.3 above as well as of the relevant result in Shanbhag (1977), is indeed a corollary to Theorem 2.2; this obviously tells us that Theorem 7.2.6 of Rao and Shanbhag (1994) is also subsumed by Theorem 2.2.

Theorem 2.3. *Let* $(\underline{X},\underline{Y})$ *be a random vector such that* $\underline{X}$ *and* $\underline{Y}$ *are* k*-component vectors satisfying*

$$P\{\underline{X}=\underline{n},\underline{Y}=\underline{r}\}=g_{\underline{n}}S(\underline{r}|\underline{n}),\qquad \underline{r}\in[\underline{0},\underline{n}]\cap\mathbb{N}_0^k,\ \underline{n}\in\mathbb{N}_0^k$$

with $\{g_{\underline{n}}:\underline{n}\in\mathbb{N}_0^k\}$ *as a probability distribution and, for each* $\underline{n}$ *for which* $g_{\underline{n}}>0$,

$$S(\underline{r}|\underline{n})=\frac{a_{\underline{r}}b_{\underline{n}-\underline{r}}}{c_{\underline{n}}},\qquad \underline{r}\in[\underline{0},\underline{n}]\cap\mathbb{N}_0^k,\ \underline{n}\in\mathbb{N}_0^k,$$

where $\{a_{\underline{n}}:\underline{n}\in\mathbb{N}_0^k\}$ *and* $\{b_{\underline{n}}:\underline{n}\in\mathbb{N}_0^k\}$ *are respectively positive and nonnegative real sequences with* $b_{\underline{0}}>0$ *and* $b_{\underline{n}}>0$ *if* $\underline{n}$ *is of unit length, and* $\{c_{\underline{n}}:\underline{n}\in\mathbb{N}_0^k\}$ *is the convolution of these two sequences. Then*

$$P\{\underline{Y}=\underline{r}\}=P\{\underline{Y}=\underline{r}|\underline{X}=\underline{Y}\},\ \underline{r}\in\mathbb{N}_0^k, \tag{2.11}$$

if and only if (*in obvious notation*)

$$g_{\underline{n}}/c_{\underline{n}}=\int_{[0,\infty)^k}\left(\prod_{i=1}^k\lambda_i^{n_i}\right)d\nu(\underline{\lambda}),\ \underline{n}\in\mathbb{N}_0^k, \tag{2.12}$$

with ($o^0=1$ *and*)ν *as a finite measure on* $[0,\infty)^k$ *such that it is concentrated for some* $\beta>0$ *on* $\{\underline{\lambda}:\sum_{\underline{n}\in\mathbb{N}_0^k}b_{\underline{n}}\prod_{i=1}^k\lambda_i^{n_i}=\beta\}$.

The above theorem follows on noting especially that (2.11) is equivalent to

$$g_{\underline{n}}/c_{\underline{n}}\propto\sum_{m\in\mathbb{N}_0^k}b_{\underline{m}}(g_{\underline{m}+\underline{n}}/c_{\underline{m}+\underline{n}}),\qquad \underline{n}\in\mathbb{N}_0^k.$$

To provide a further generalization of the Rao–Rubin-Shanbhag theorems, consider S to be a countable Abelian semigroup with zero element, equipped with discrete topology, and $S^*\subset S$ such that given $w:S\to[0,\infty)$ with supp $(w)(=\{x:w(x)>0\})=S^*$, any function $v:S\to[0,\infty)$ with $v(0)>0$ cannot be a solution to

$$v(x)=\sum_{y\in S}v(x+y)w(y),\qquad x\in S \tag{2.13}$$

unless it has an integral representation in terms of w-harmonic exponential functions, with respect to a probability measure. (By a w-harmonic exponential function here, we mean a function $e : S \to [0, \infty)$ such that $e(x+y) = e(x)e(y), x, y \in S$, and $\sum_{x \in S} e(x)w(x) = 1$.) Examples of such S, S^* have been dealt with by Rao and Shanbhag (1998) and studied implicitly or otherwise by Rao and Shanbhag (1994). Suppose now that $a : S \to (0, \infty)$ and $b : S \to [0, \infty)$ are such that $b(0) > 0$ and there exists $c : S \to (0, \infty)$ as the convolution of a and b, and Y and Z are random elements defined on a probability space, with values in S, such that

$$P\{Y = y, Z = z\} = g(y+z)\frac{a(y)b(z)}{c(y+z)}, \qquad y, z \in S,$$

where $\{g(x) : x \in S\}$ is a probability distribution. If $\operatorname{supp}(b) = S^*$, then it easily follows that

$$P\{Y = y\} = P\{Y = y | Z = 0\}, \qquad y \in S,$$

if and only if $g(x)/c(x), x \in S$, is of the form of a constant multiple of the solution v to (2.13) with, for some $\gamma > 0$, w replaced by γb; this latter result is clearly an extended version of Theorem 2.3.

Remark 2.4. In view of Rao et al. (2002), the link between the general result relative to a countable semigroup that we have met above and Theorem 4.4.1 of Rao and Shanbhag (1994) or its specialized version appearing in Williams (1979) is obvious. The arguments in Rao and Shanbhag (1994) for solving general integral equations on semigroups, including those involving martingales obviously simplify considerably if the semigroups are countable; we shall throw further light on these issues through a separate article.

Remark 2.5. Modifying the proof of Theorem 2.1 slightly, involving in particular a further moment argument, a proof based on the version of de Finetti's theorem relative to 0-1-valued exchangeable random variables can be produced for Corollary 2.2.3 appearing on page 31 in Rao and Shanbhag (1994). (Note that the version of (1.3) in this case implies that there exists a nonnegative bounded random variable Y such that $E(Y^{mn}) = \frac{v_{mn}}{v_0}$, $n = 0, 1, \ldots$, for each m with $w_m > 0$.) This latter result is indeed a corollary to the Lau–Rao theorem ([13], [20]), and, in turn, is essentially a generalization of Shanbhag's lemma. As pointed out by Rao and Shanbhag (2004), in view of Alzaid et al. (1987b), there exists a proof for the Lau–Rao theorem based, among other things, on the version of de Finetti's theorem just referred to; there also exist possibilities of solving integral equations via this or other versions of de Finetti's theorem, elsewhere.

Remark 2.6. Suppose S is a countable Abelian semigroup with zero element, equipped with discrete topology, and v and w are nonnegative real-valued functions on S such that $v(0) > 0$, $w(0) < 1$, and (2.13) is met. Then there exists an infinite sequence $\{X'_n : n = 1, 2, \ldots\}$ of exchangeable random elements with values in S for which for each positive integer n and $x'_1, \ldots, x'_n \in S$,

$$P\big\{X'_1 = x'_1, X'_2 = x'_2, \ldots, X'_n = x'_n\big\} = \big(v\big(x'_1 + \cdots + x'_n\big)/v(0)\big) \prod_{i=1}^{n} w\big(x'_i\big). \quad (2.14)$$

If s_i, $i = 1, \ldots, k$ (with $k \geq 1$), are distinct nonzero members of S such that $w(s_i) > 0$, $i = 1, \ldots, k$, taking for example, X_n, $n = 1, 2, \ldots$, such that

$$X_n = \begin{cases} i & \text{if } X'_n = s_i,\ i = 1, \ldots, k, \\ 0 & \text{if } X'_n \notin \{s_1, \ldots, s_k\}, \end{cases}$$

we can now see that there exists a sequence $\{X_n : n = 1, 2, ...\}$ of exchangeable random variables with values in $\{0, 1, \dots, k\}$ for which (2.7) (when its left hand side is read as that of (2.2) with n_1 in place of n if $k = 1$) is valid, provided its right hand side is now replaced by $\frac{v(n_1 s_1 + \cdots + n_k s_k)}{v(0)} \prod_{i=1}^{k} (w(s_i))^{n_i}$. Consequently, in view of the relevant version of de Finetti's theorem, it follows that even when s_i, $i = 1, \dots, k$, are not taken to be distinct or nonzero, provided $w(s_i) > 0$, $i = 1, \dots, k$, we have $\{\frac{v(n_1 s_1 + \cdots + n_k s_k)}{v(0)} : n_1, n_2, \dots, n_k = 0, 1, ...\}$ to be the moment sequence of a probability distribution on $\mathbb{R}^k$, with support as a compact subset of $[0, \infty)^k$.

3. Spitzer's integral representation theorem and relevant observations

This section is devoted mainly to illustrate as to how Bernstein's theorem on absolutely monotonic functions,referred to in Remark 2.3, in conjunction with Yaglom's theorem mentioned on page 18 in Athreya and Ney (1972), leads us to an improved version of the key result of Alzaid et al. (1987a) and certain of its corollaries.

Suppose $\{Z_n : n = 0, 1, \dots\}$ is a homogeneous Markov chain with state space $\{0, 1, \dots\}$, such that the corresponding one-step transition probabilities are given by

$$\begin{aligned} p_{ij} &= P\{Z_{n+1} = j | Z_n = i\} \\ &= \begin{cases} c p_j^{(i)}, & \text{i=0,1,\dots; j=1,2,\dots,} \\ 1 - c + c p_0^{(i)}, & \text{i=0,1,\dots; j=0,} \end{cases} \end{aligned}$$

where $c \in (0, 1]$ and $\{p_j^{(i)} : j = 0, 1, \dots\}$ is the i-fold convolution of some probability distribution $\{p_j\}$ for which $p_0 \in (0, 1)$, for $i = 1, 2, \dots,$ and the degenerate distribution at zero if $i = 0$. Clearly, this is an extended version of a Bienaymé-Galton-Watson branching process; indeed, we can view the latter as a special case of the former with $c = 1$.

Under the condition that $m = \sum_{j=1}^{\infty} j p_j < 1$ with $m^* = \sum_{j=1}^{\infty} (j \log j) p_j < \infty$, Alzaid et al. (1987a) have given an integral representation for stationary measures of the general process referred to above. A specialized version of this representation in the case of $c = 1$ was essentially established earlier by Spitzer (1967); this latter result appears also as Theorem 3 in Section 2 of Chapter II of Athreya and Ney (1972). The general representation theorem as well as its specialized version follow via Martin boundary related approaches or their alternatives involving specific tools such as Bernstein's theorem on absolutely monotonic functions, see, for example, Alzaid et al. (1987a) and Rao et al. (2002) for some relevant arguments or observations in this connection.

From a minute scrutiny of the proof provided by Alzaid et al. (1987a) for the general representation theorem, i.e. Theorem 2 in the cited reference, it has now emerged that the theorem referred to holds even when the constraint that $m^* < \infty$ is dropped. Indeed, Yaglom's theorem mentioned on page 18 in Athreya and Ney (1972) implies (in obvious notation) that if $m < 1$, then, irrespective of whether or not $m^* < \infty, \{\mathcal{B}_n\}$ converges pointwise to $\mathcal{B}$; essentially, the argument on page 1212 in Alzaid et al. (1987a) to show that a certain function, U^*, is the generating function of a nonnegative sequence then remains valid and gives us specifically the sequence to be that corresponding to a stationary measure of the process with $p_0 = 1 - m$ and $p_1 = m$, without requiring that $m^* < \infty$. (One can also, obviously, give the argument implied here in terms of f_n, the nth iterates of f, directly without involving Q_n; note that we use, as usual, the notation f for the generating function of $\{p_j\}$.)

The original form of Spitzer's theorem, involving, amongst other things, the parameter $Q(0)$, requires the assumption of $m^* < \infty$. [Note that $f_n(s) = \mathcal{B}^{-1}(1 - m^n + m^n\mathcal{B}(s))$ and hence $Q_n(0) = \dfrac{(f_n(0) - 1)}{m^n} = (\mathcal{B}^{-1}(1 - m^n) - 1)/m^n$ has a nonzero limit $Q(0)$ as $n \to \infty$ only of $\mathcal{B}'(1-) < \infty$ and hence only if $m^* < \infty$; see the proof of the theorem on page 70, in conjunction with the remark on page 18, in Athreya and Ney (1972).] However, from what we have observed above, it is clear that this latter theorem holds even when the assumption mentioned is deleted, provided "-1" is taken in place of "$Q(0)$" in the statement of the theorem.

As a by-product of the revelation that we have made above, it follows that if $m < 1, U(\cdot)$ is the generating function of a stationary measure of the process if and only if it is of the form $U^*(\mathcal{B}(\cdot))$ with U^* as the generating function of a stationary measure in the special case where $p_0 = 1 - m$, $p_1 = m$. This is obviously a consequence of Yaglom's theorem, in light of the extended continuity theorem of Feller (1966, page 433). The example given by Harris, appearing on page 72 of Athreya and Ney (1972), to prove the existence of stationary measures does not require $m^* < \infty$ and is of the form that we have met here; clearly it is not covered by Spitzer's original representation theorem. As implied in Alzaid et al. (1987a), a representation for U^* itself in our general case follows essentially as a consequence of Bernstein's theorem on absolutely monotonic functions or the Poisson-Martin integral representation theorem for a stationary measure; see, also, Rao et al. (2002) for some relevant observations.

Taking into account our observations, it is hence seen that the following modified version of the main result of Alzaid et al. (1987a) holds.

Theorem 3.1. *If $m < 1$, then every sequence $\{\eta_j : j = 1, 2, \ldots\}$ is a stationary measure if and only if, for some non-null finite measure ν on $[0, 1)$,*

$$\eta_j = \sum_{n=-\infty}^{\infty} c^n \int_{[0,1)} \exp\{-m^{n-t}\} \left(\sum_{k=1}^{j} \frac{m^{(n-t)k}}{k!} b_j^{(k)} \right) d\nu(t), \qquad j = 1, 2, \ldots, \tag{3.1}$$

where, for each k, $\{b_j^{(k)} : j = 1, 2, \ldots\}$ (with $b_0^{(k)} = 0$) denotes the distribution relative to the probability generating function $(\mathcal{B}(\cdot))^k$ with $\mathcal{B}(\cdot)$ as implied earlier (to be a unique probability generating function satisfying $\mathcal{B}(0) = 0$ and $\mathcal{B}(f(s)) = 1 - m + m\mathcal{B}(s)$, $s \in [-1, 1]$.) Moreover, if (3.1) is met with $m < 1$, then $\{\eta_j\}$ is a stationary measure satisfying $\sum_{j=1}^{\infty} \eta_j p_0^j = 1$. i.e. with generating function U such that $U(p_0) = 1$, if and only if, for some probability measure μ on $[0, 1)$,

$$d\nu(t) = K\ d\mu(t), \qquad t \in [0, 1), \tag{3.2}$$

with K such that

$$K^{-1} = \begin{cases} 1 & \text{if } c = 1 \\ \left(\dfrac{1-c}{c} \right) \displaystyle\sum_{n=-\infty}^{\infty} c^n \int_{[0,1)} \exp\{-m^{n-t}\}\, d\mu(t) & \text{if } c \in (0, 1). \end{cases}$$

The following theorem is of relevance to the topic of damage models especially in view of the results on damage models appearing in Talwalker (1980), Rao et al. (1980) and Alzaid et al. (1987a); this theorem is indeed a variation of Theorem 1 of Alzaid et al. (1987a).

Theorem 3.2. *Let $c \in (0,1)$ and $\{(v_n, h_n) : n = 0, 1, \ldots\}$ be a sequence of 2-vectors with nonnegative real components such that at least one v_n is nonzero and h_0 is nonzero and $h_1 < 1$. Then*

$$c \sum_{k=0}^{\infty} v_k h_j^{(k)} = v_j, \qquad j = 0, 1, \ldots, \tag{3.3}$$

where, for each $k > 0, \{h_j^{(k)}\}$ is the k-fold convolution of $\{h_j\}$, and $\{h_j^{(0)}\}$ is the probability distribution that is degenerate at zero, if and only if, for some $s_0 > 0$,

$$p_j = h_j s_0^{j-1}, \qquad j = 0, 1, \ldots, \tag{3.4}$$

is a nondegenerate probability distribution, $\{v_j s_0^j : j = 1, 2, \ldots\}$ is a stationary measure (not necessarily normalized as in Alzaid et al. (1987a)) relative to the general branching process with $\{p_j\}$ as in (3.4), and $v_0 = c(1-c)^{-1} \sum_{k=1}^{\infty} v_k h_0^k$.

Theorem 3.2 is easy to establish.

Remark 3.1. If $\{h_n\}$ of Theorem 3.2 satisfies a further condition that $h_n = 0$ for $n \geq 2$, then the assertion of the theorem holds with $s_0 = \frac{h_0}{(1-h_1)}$ and the stationary measure in it satisfying (3.1) with $b_1 = 1$ and $m = h_1$. Additionally, if we are given a priori that $\{v_j\}$ is of the form

$$v_j = g_j \alpha^j, \qquad j = 0, 1, \ldots$$

with $\{g_j\}$ as a probability distribution and $\alpha > 0$, then it is clear that (3.3) holds if and only if

$$g_j \propto \sum_{n=-\infty}^{\infty} c^n \int_{[0,1)} \exp\{-h_1^{n-t}\} \frac{h_1^{(n-t)j}}{j!} \left(\frac{1-h_1}{h_0 \alpha}\right)^j d\mu(t), \qquad j = 0, 1, \ldots$$

with μ as a probability measure on $[0,1)$. As an immediate consequence of the latter result, Theorem 3 of Alzaid et al. (1987a) now follows.

Remark 3.2. One can extend the main result of Alzaid et al. (1986) based on the Perron–Frobenius theorem in an obvious way involving (in usual notation)

$$\begin{aligned} P\{Y = r\} &= P\{Y' = r | X' - Y' = k_0\} \\ &= P\{Y'' = r | X'' - Y'' = k_0 + k_1\}, \qquad r = 0, 1, \ldots \end{aligned}$$

with $k_0 \geq 0$ and $k_1 > 0$, such that the survival distributions corresponding to $(X, Y), (X', Y')$ and (X'', Y'') are not necessarily the same but $X \overset{d}{=} X' \overset{d}{=} X''$. This provides us with further insight into Theorem 3 of Alzaid et al. (1987a). (For an account of the Perron–Frobenius theorem with applications to Markov chains, see Seneta (1981).)

Remark 3.3. Most of the results dealt with in this article also follow via alternative arguments based on Choquet's theorem; for the details of this theorem, see Phelps (1966).

Remark 3.4. If we agree to rewrite the notation U^* as $U^*_{(c)}$, to take into account the value of the parameter c of the process, it easily follows (in obvious notation) that,given $c < 1$ and $U^*_{(c)}$, there exists an $U^*_{(1)}$ such that

$$\frac{d}{ds} U^*_{(1)}(s) \propto \left(\frac{d}{ds} U^*_{(c)}(s)\right) / (1-s)^{(\ln c)/(\ln m)}, \qquad s \in (-1, 1). \tag{3.5}$$

However, it is worth noting here that there exist cases of $U^*_{(1)}$ (such as those with $U^*_{(1)}(s) = (\ln(1-s))/(\ln(m))$, $s \in (-1,1)$) for which (3.5) with $c \in (0,1)$ is not met.

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This list includes, amongst others, some items that are not cited explicitly, although implied, in the text; these deal with aspects of functional equations of relevance to the present study.