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On the false discovery rates of a frequentist: Asymptotic expansions

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Abstract: Consider a testing problem for the null hypothesis $H_0 : \theta \in \Theta_0$. The standard frequentist practice is to reject the null hypothesis when the p-value is smaller than a threshold value α, usually 0.05. We ask the question how many of the null hypotheses a frequentist rejects are actually true. Precisely, we look at the Bayesian false discovery rate $\delta_n = P_g(\theta \in \Theta_0 | p - value < \alpha)$ under a proper prior density $g(\theta)$. This depends on the prior g, the sample size n, the threshold value α as well as the choice of the test statistic. We show that the Benjamini–Hochberg FDR in fact converges to δ_n almost surely under g for any fixed n. For one-sided null hypotheses, we derive a third order asymptotic expansion for δ_n in the continuous exponential family when the test statistic is the MLE and in the location family when the test statistic is the sample median. We also briefly mention the expansion in the uniform family when the test statistic is the MLE. The expansions are derived by putting together Edgeworth expansions for the CDF, Cornish–Fisher expansions for the quantile function and various Taylor expansions. Numerical results show that the expansions are very accurate even for a small value of n (e.g., $n = 10$). We make many useful conclusions from these expansions, and specifically that the frequentist is not prone to false discoveries except when the prior g is too spiky. The results are illustrated by many examples.

1. Introduction

In a strikingly interesting short note, Sorić [19] raised the question of establishing upper bounds on the proportion of fictitious statistical discoveries in a battery of independent experiments. Thus, if m null hypotheses are tested independently, of which m_0 happen to be true, but V among these m_0 are rejected at a significance level α, and another S among the false ones are also rejected, Sorić essentially suggested $E(V)/(V+S)$ as a measure of the false discovery rate in the chain of m independent experiments. Benjamini and Hochberg [3] then looked at the question in much greater detail and gave a careful discussion for what a correct formulation for the false discovery rate of a group of frequentists should be, and provided a concrete procedure that actually physically controls the groupwise false discovery rate. The problem is simultaneously theoretically attractive, socially relevant, and practically important. The practical importance comes from its obvious relation to statistical discoveries made in clinical trials, and in modern microarray experiments. The continued importance of the problem is reflected in two recent articles, Efron [7], and Storey [21], who provide serious Bayesian connections and advancements in the problem. See also Storey [20], Storey, Taylor and Siegmund [22], Storey and

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Tibshirani [23], Genovese and Wasserman [10], and Finner and Roters [9], among many others in this currently active area.

Around the same time that Sorić raised the issue of fictitious frequentist discoveries made by a mechanical adoption of the use of p-values, a different debate was brewing in the foundation literature. Berger and Sellke [2], in a thought provoking article, gave analytical foundations to the thesis in Edwards, Lindman and Savage [6] that the frequentist practice of rejecting a sharp null at a traditional 5% level amounts to a rush to judgment against the null hypothesis. By deriving lower bounds or exact values for the minimum value of the posterior probability of a sharp null hypothesis over a variety of classes of priors, Berger and Sellke [2] argued that p-values traditionally regarded as small understate the plausibility of nulls, at least in some problems. Casella and Berger [5], gave a collection of theorems that show that the discrepancy disappears under broad conditions if the null hypothesis is composite one-sided. Since the articles of Berger and Sellke [2] and Casella and Berger [5], there has been an avalanche of activity in the foundation literature on the safety of use of p-values in testing problems. See Hall and Sellinger [12], Sellke, Bayarri and Berger [18], Marden [14] and Schervish [17] for a contemporary exposition.

It is conceptually clear that the frequentist FDR literature and the foundation literature were both talking about a similar issue: is the frequentist practice of rejecting nulls at traditional p-values an invitation to rampant false discoveries? The structural difference was that the FDR literature did not introduce a formal prior on the unknown parameters, while the foundation literature did not go into multiple testing, as is the case in microarray or other emerging interesting applications. The purpose of this article is to marry the two schools together, while giving a new rigorous analysis of the interesting question: "how many of the null hypotheses a frequentist rejects are actually trues" and the flip side of that question, namely, "how many of the null hypotheses a frequentist accepts are actually falses". The calculations are completely different from what the previous researchers have done, although we then demonstrate that our formulation directly relates to *both* the traditional FDR calculations, and the foundational effort in Berger and Sellke [2], and others. We have thus a dual goal; providing a new approach, and integrating it with the two existing approaches.

In Section 2, we demonstrate the connection in very great generality, without practically any structural assumptions at all. This was comforting. As regards to concrete results, it seems appropriate to look at the one parameter exponential family, it being the first structured case one would want to investigate. In Section 3, we do so, using the MLE as the test statistic. In Section 4, we look at a general location parameter, but using the median as the test statistic. We used the median for two reasons. First, for general location parameters, the median is credible as a test statistic, while the mean obviously is not. Second, it is important to investigate the extent to which the answers depend on the choice of the test statistic; by studying the median, we get an opportunity to compare the answers for the mean and the median in the special normal case.To be specific, let us consider the one sided testing problem based on an i.i.d. sample $X_1, \ldots, X_n$ from a distribution family with parameter θ in the parameter space Ω which is an interval of R. Without loss of generality, we assume $\Omega = (\underline{\theta}, \bar{\theta})$ with $-\infty \le \underline{\theta} < \bar{\theta} \le \infty$. We consider the testing problem

$$H_0 : \theta \le \theta_0 \text{ vs } H_1 : \theta > \theta_0,$$

where $\theta_0 \in (\underline{\theta}, \bar{\theta})$. Suppose the α, $0 < \alpha < 1$, level test rejects H_0 if $T_n \in C$, where

T_n is a test statistic. We study the behavior of the quantities,

$$\delta_n = P(\theta \le \theta_0 | T_n \in C) = P(H_0 | p - value < \alpha)$$

and

$$\epsilon_n = P(\theta > \theta_0 | T_n \notin C) = P(H_1 | p - value \ge \alpha).$$

Note that δ_n and ϵ_n are inherently Bayesian quantities. By an almost egregious abuse of nomenclature, we will refer to δ_n and ϵ_n as type I and type II errors in this article. Our principal objective is to obtain third order asymptotic expansions for δ_n and ϵ_n assuming a Bayesian proper prior for θ. Suppose $g(\theta)$ is any sufficiently smooth proper prior density of θ. In the regular case, the expansion for δ_n we obtain is like

$$\delta_n = \frac{P(\theta \le \theta_0, T_n \in C)}{P(T_n \in C)} = \frac{c_1}{\sqrt{n}} + \frac{c_2}{n} + \frac{c_3}{n^{3/2}} + O(n^{-2}), \tag{1}$$

and the expansion for ϵ_n is like

$$\epsilon_n = \frac{P(\theta > \theta_0, T_n \notin C)}{P(T_n \notin C)} = \frac{d_1}{\sqrt{n}} + \frac{d_2}{n} + \frac{d_3}{n^{3/2}} + O(n^{-2}), \tag{2}$$

where the coefficients c_1, c_2, c_3, d_1, d_2, and d_3 depend on the problem, the test statistic T_n, the value of α and the prior density $g(\theta)$. In the nonregular case, the expansion differs qualitatively; for both δ_n and ϵ_n the successive terms are in powers of $1/n$ instead of the powers of $1/\sqrt{n}$. Our ability to derive a third order expansion results in a surprisingly accurate expansion, sometimes for n as small as $n = 4$. The asymptotic expansions we derive are not just of theoretical interest; the expansions let us conclude interesting things, as in Sections 3.2 and 4.5, that would be impossible to conclude from the exact expressions for δ_n and ϵ_n.

The expansions of δ_n and ϵ_n require the expansions of the numerators and the denominators of (1) and (2) respectively. In the regular case, the expansion of the numerator of (1) is like

$$A_n = P(\theta \le \theta_0, T_n \in C) = \frac{a_1}{\sqrt{n}} + \frac{a_2}{n} + \frac{a_3}{n^{3/2}} + O(n^{-2}) \tag{3}$$

and the expansion of the numerator of (2) is like

$$\tilde{A}_n = P(\theta > \theta_0, T_n \notin C) = \frac{\tilde{a}_1}{\sqrt{n}} + \frac{\tilde{a}_2}{n} + \frac{\tilde{a}_3}{n^{3/2}} + O(n^{-2}). \tag{4}$$

Then, the expansion of the denominator of (1) is

$$B_n = P(T_n \in C) = A_n + \lambda - \tilde{A}_n = \lambda - \frac{b_1}{\sqrt{n}} - \frac{b_2}{n} - \frac{b_3}{n^{3/2}} + O(n^{-2}), \tag{5}$$

where $\lambda = P(\theta > \theta_0) = \int_{\theta_0}^{\bar{\theta}} g(\theta) d\theta$ and assume $0 < \lambda < 1$, $b_1 = \tilde{a}_1 - a_1$, $b_2 = \tilde{a}_2 - a_2$ and $b_3 = \tilde{a}_3 - a_3$, and the expansion of the denominator of (2) is

$$\tilde{B}_n = P(T_n \notin C) = 1 - B_n = 1 - \lambda + \frac{b_1}{\sqrt{n}} + \frac{b_2}{n} + \frac{b_3}{n^{3/2}} + O(n^{-2}). \tag{6}$$

Then, we have

$$\begin{gathered} c_1 = \frac{a_1}{\lambda}, c_2 = \frac{a_1 b_1}{\lambda^2} + \frac{a_2}{\lambda}, c_3 = \frac{a_3}{\lambda} + \frac{a_1 b_2 + a_2 b_1}{\lambda^2} + \frac{a_1 b_1^2}{\lambda^3}, \\ d_1 = \frac{\tilde{a}_1}{1-\lambda}, d_2 = \frac{\tilde{a}_2}{1-\lambda} - \frac{\tilde{a}_1 b_1}{(1-\lambda)^2}, d_3 = \frac{\tilde{a}_3}{1-\lambda} - \frac{\tilde{a}_2 b_1 + \tilde{a}_1 b_2}{(1-\lambda)^2} + \frac{\tilde{a}_1 b_1^2}{(1-\lambda)^3}. \end{gathered} \tag{7}$$

We will frequently use the three notations in the expansions: the standard normal PDF ϕ, the standard normal CDF Φ and the standard normal upper α quantile $z_\alpha = \Phi^{-1}(1-\alpha)$.

The principal ingredients of our calculations are Edgeworth expansions, Cornish–Fisher expansions and Taylor expansions. The derivation of the expansions became very complex. But in the end, we learn a number of interesting things. We learn that typically the false discovery rate δ_n is small, and smaller than the pre-experimental claim α for quite small n. We learn that typically $\epsilon_n > \delta_n$, so that the frequentist is less vulnerable to false discovery than to false acceptance. We learn that only priors very spiky at the boundary between H_0 and H_1 can cause large false discovery rates. We also learn that these phenomena do not really change if the test statistic is changed. So while the article is technically complex and the calculations are long, the consequences are rewarding. The analogous expansions are qualitatively different in the nonregular case. We could not report them here due to shortage of space. We should also add that we leave open the question of establishing these expansions for problems with nuisance parameters, multivariate problems, and dependent data. Results similar to ours are expected in such problems.

2. Connection to Benjamini and Hochberg, Storey and Efron's work

Suppose there are m groups of iid samples $X_{i1}, \ldots, X_{in}$ for $i = 1, \ldots, m$. Assume $X_{i1}, \ldots, X_{in}$ are iid with a common density $f(x, \theta_i)$, where θ_i are assumed iid with a CDF $G(\theta)$ which does not need to have a density in this section. Then, the prior $G(\theta)$ connects our Bayesian false discovery rate δ_n to the usual frequentist false discovery rate. In the context of our hypothesis testing problem, the frequentist false discovery rate, which has been recently discussed by Benjamini and Hochberg [3], Efron [7] and Storey [21], is defined as

$$FDR = FDR(\theta_1, \ldots, \theta_m) = E_{\theta_1,\ldots,\theta_m} \left\{ \frac{\sum_{i=1}^m I_{T_{ni} \in C, \theta_i \le \theta_0}}{(\sum_{i=1}^m I_{T_{ni} \in C}) \vee 1} \right\}, \tag{8}$$

where T_{ni} is the test statistic based on the samples $X_{i1}, \ldots, X_{in}$. It will be shown below that for any fixed n as $m \to \infty$, the frequentist false discovery rate FDR goes to the Bayesian false discovery rate δ_n almost surely under the prior distribution $G(\theta)$.

We will compare the numerators and the denominators of FDR in (8) and δ_n in (1) respectively. Since the comparisons are almost identical, we discuss the comparison between the numerators only. We denote $E_\theta(\cdot)$ and $V_\theta(\cdot)$ as the conditional mean and variance given the true parameter θ, and we denote $E(\cdot)$ and $V(\cdot)$ as the marginal mean and variance under the prior $G(\theta)$. Let $Y_i = I_{T_{ni} \in C, \theta_i \le \theta_0}$. Then given $\theta_1, \ldots, \theta_m$, Y_i $(i = 1, \ldots, m)$ are independent Bernoulli random variables with mean values $\mu_i = \mu_i(\theta_i) = E_{\theta_i}(Y_i)$, and marginally μ_i are iid with expected value A_n in (3). Let

$$D_m = \frac{1}{m}\sum_{i=1}^m I_{T_{ni} \in C, \theta_i \le \theta_0} - A_n = \frac{1}{m}\sum_{i=1}^m (Y_i - \mu_i) + \frac{1}{m}\sum_{i=1}^m (\mu_i - A_n).$$

Note that we assume that $\theta_1, \ldots, \theta_m$ are iid with a common CDF $G(\theta)$. The second term goes to 0 almost surely by the Strong Law of Large Numbers (SLLN) for identically distributed random variables. Note that for any given $\theta_1, \ldots, \theta_m$, $Y_1, \ldots, Y_m$ are independent but not iid, with $E_{\theta_i}(Y_i) = \mu_i$, $V_{\theta_i}(Y_i) = \mu_i(1-\mu_i)$ and

$\sum_{i=1}^{-\infty} i^{-2} V_{\theta_i}(Y_i) \leq \sum_{i=1}^{\infty} i^{-2} < \infty$. The first term also goes to 0 almost surely by a SLLN for independent but not iid random variables [15]. Therefore, D_m goes to 0 almost surely. The comparison of denominators is handled similarly. Therefore, for almost all sequences $\theta_1, \theta_2, \ldots$,

$$\frac{\sum_{i=1}^{m} I_{T_{ni} \in C, \theta_i \leq \theta_0}}{(\sum_{i=1}^{m} I_{T_{ni} \in C}) \vee 1} \to \delta_n$$

as $m \to \infty$.

Since $\sum_{i=1}^{m} I_{T_{ni} \in C, \theta_i \leq \theta_0} \leq (\sum_{i=1}^{m} I_{T_{ni} \leq C}) \vee 1$, their ratio is uniformly integrable. And so, FDR as defined in (8) also converges to δ_n as $m \to \infty$ for almost all sequences $\theta_1, \theta_2, \ldots$.

This gives a pleasant, exact connection between our approach and the established indices formulated by the previous researchers. Of course, for fixed m, the frequentist FDR does not need to be close to our δ_n.

3. Continuous one-parameter exponential family

Assume the density of the i.i.d. sample $X_1, \ldots, X_n$ is in the form of a one-parameter exponential family $f_\theta(x) = b(x) e^{\theta x - a(\theta)}$ for $x \in \mathcal{X} \subseteq R$, where the natural space Ω of θ is an interval of R and $a(\theta) = \log \int_{\mathcal{X}} b(x) e^{\theta x} dx$. Without loss of generality, we can assume Ω is open so that one can write $\Omega = (\underline{\theta}, \bar{\theta})$ for $-\infty \leq \underline{\theta} < \bar{\theta} \leq \infty$. All derivatives of $a(\theta)$ exist at every $\theta \in \Omega$ and can be derived by formally differentiating under the integral sign ([4], p. 34). This implies that $a'(\theta) = E_\theta(X_1)$, $a''(\theta) = Var_\theta(X_1)$ for every $\theta \in \Omega$. Let us denote $\mu(\theta) = a'(\theta)$, $\sigma(\theta) = \sqrt{a''(\theta)}$, $\kappa_i(\theta) = a^{(i)}(\theta)$ and $\rho_i(\theta) = \kappa_i(\theta)/\sigma^i(\theta)$ for $i \geq 3$, where $a^{(i)}(\theta)$ represents the i-th derivative of $a(\theta)$. Then, $\mu(\theta)$, $\sigma(\theta)$, $\kappa_i(\theta)$ and $\rho_i(\theta)$ all exist and are continuous at every $\theta \in \Omega$ ([4], p. 36), and $\mu(\theta)$ is non-decreasing in θ since $a''(\theta) = \sigma^2(\theta) \geq 0$ for all θ.

Let $\mu_0 = \mu(\theta_0)$, $\sigma_0 = \sigma(\theta_0)$, $\kappa_{i0} = \kappa_i(\theta_0)$ and $\rho_{i0} = \rho_i(\theta_0)$ for $i \geq 3$ and assume $\sigma_0 > 0$. The usual α $(0 < \alpha < 1)$ level UMP test ([13], p. 80) for the testing problem $H_0 : \theta \leq \theta_0$ vs $H_A : \theta \geq \theta_0$ rejects H_0 if $\bar{X} \in C$ where

$$C = \{\bar{X} : \sqrt{n} \frac{\bar{X} - \mu_0}{\sigma_0} > k_{\theta_0, n}\}, \tag{9}$$

and $k_{\theta_0,n}$ is determined from $P_{\theta_0}\{\sqrt{n}(\bar{X} - \mu_0)/\sigma_0 > k_{\theta_0,n}\} = \alpha$; $\lim_{n\to\infty} k_{\theta_0,n} = z_\alpha$. Let

$$\tilde{\beta}_n(\theta) = P_\theta \left(\sqrt{n} \frac{\bar{X} - \mu_0}{\sigma_0} > k_{\theta_0,n} \right) \tag{10}$$

Then, using the transformation $x = \sigma_0 \sqrt{n}(\theta - \theta_0) - z_\alpha$ under the integral sign below, we have

$$A_n = \int_{\underline{\theta}}^{\theta_0} \tilde{\beta}_n(\theta) g(\theta) d\theta = \frac{1}{\sigma_0 \sqrt{n}} \int_{\underline{x}}^{-z_\alpha} \tilde{\beta}_n(\theta_0 + \frac{x + z_\alpha}{\sigma_0 \sqrt{n}}) g(\theta_0 + \frac{x + z_\alpha}{\sigma_0 \sqrt{n}}) dx \tag{11}$$

and

$$\tilde{A}_n = \frac{1}{\sigma_0 \sqrt{n}} \int_{-z_\alpha}^{\bar{x}} [1 - \tilde{\beta}_n(\theta_0 + \frac{x + z_\alpha}{\sigma_0 \sqrt{n}})] g(\theta_0 + \frac{x + z_\alpha}{\sigma_0 \sqrt{n}}) dx, \tag{12}$$

where $\underline{x} = \sigma_0\sqrt{n}(\underline{\theta} - \theta_0) - z_\alpha$ and $\bar{x} = \sigma_0\sqrt{n}(\bar{\theta} - \theta_0) - z_\alpha$.

Since for an interior parameter θ all moments of the exponential family exist and are continuous in θ, we can find θ_1 and θ_2 satisfying $\bar{\theta} < \theta_1 < \theta_0$ and $\theta_0 < \theta_2 < \bar{\theta}$ such that for any $\theta \in [\theta_1, \theta_2]$, $\sigma^2(\theta)$, $\kappa_3(\theta)$, $\kappa_4(\theta)$, $\kappa_5(\theta)$, $g(\theta)$, $g'(\theta)$, $g''(\theta)$ and $g^{(3)}(\theta)$ are uniformly bounded in absolute values, and the minimum value of $\sigma^2(\theta)$ is a positive number. After we pick θ_1 and θ_2, we partition each of A_n and $\tilde{A}_n$ into two parts so that one part is negligible in the expansion. Then, the rest of the work in the expansion is to find the coefficients of the second part.

To describe these partitions, we define $\theta_{1n} = \theta_0 + (\theta_1 - \theta_0)/n^{1/3}$, $\theta_{2n} = \theta_0 + (\theta_2 - \theta_0)/n^{1/3}$, $x_{1n} = \sigma_0\sqrt{n}(\theta_{1n} - \theta_0) - z_\alpha$ and $x_{2n} = \sigma_0\sqrt{n}(\theta_{2n} - \theta_0) - z_\alpha$. Let

$$A_{n,\theta_{1n}} = \frac{1}{\sigma_0\sqrt{n}} \int_{x_{1n}}^{-z_\alpha} \tilde{\beta}_n(\theta_0 + \frac{x + z_\alpha}{\sigma_0\sqrt{n}}) g(\theta_0 + \frac{x + z_\alpha}{\sigma_0\sqrt{n}}) dx \tag{13}$$

$$\underline{R}_{n,\theta_{1n}} = \frac{1}{\sigma_0\sqrt{n}} \int_{\underline{x}}^{x_{1n}} \tilde{\beta}_n(\theta_0 + \frac{x + z_\alpha}{\sigma_0\sqrt{n}}) g(\theta_0 + \frac{x + z_\alpha}{\sigma_0\sqrt{n}}) dx, \tag{14}$$

$$\tilde{A}_{n,\theta_{2n}} = \frac{1}{\sigma_0\sqrt{n}} \int_{-z_\alpha}^{x_{2n}} [1 - \tilde{\beta}_n(\theta_0 + \frac{x + z_\alpha}{\sigma_0\sqrt{n}})] g(\theta_0 + \frac{x + z_\alpha}{\sigma_0\sqrt{n}}) dx, \tag{15}$$

and

$$\bar{R}_{n,\theta_{2n}} = \frac{1}{\sigma_0\sqrt{n}} \int_{x_{2n}}^{\bar{x}} [1 - \tilde{\beta}_n(\theta_0 + \frac{x + z_\alpha}{\sigma_0\sqrt{n}})] g(\theta_0 + \frac{x + z_\alpha}{\sigma_0\sqrt{n}}) dx. \tag{16}$$

Then, $A_n = A_{n,\theta_{1n}} + \underline{R}_{n,\theta_{1n}}$ and $\tilde{A}_n = \tilde{A}_{n,\theta_{2n}} + \bar{R}_{n,\theta_{2n}}$. In the appendix, we show that for any $\ell > 0$, $\lim_{n\to\infty} n^l \underline{R}_{n,\theta_{1n}} = \lim_{n\to\infty} n^l \bar{R}_{n,\theta_{2n}} = 0$. Therefore, it is enough to compute the coefficients of the expansions for $A_{n,\theta_{1n}}$ and $\tilde{A}_{n,\theta_{2n}}$. Among the steps for expansions, the key step is to compute the expansions of $\tilde{\beta}_n(\theta_0 + (x + z_\alpha)/(\sigma_0\sqrt{n}))$ when $x \in [x_{1n}, -z_\alpha]$ and $1 - \tilde{\beta}_n(\theta_0 + (x + z_\alpha)/(\sigma_0\sqrt{n}))$ when $x \in [-z_\alpha, x_{2n}]$ under the integral sign, since the expansion of $g(\theta_0 + (x + z_\alpha)/(\sigma_0\sqrt{n}))$ in (13) and (15) is easily obtained as

$$g(\theta_0 + \frac{x + z_\alpha}{\sigma_0\sqrt{n}}) = g(\theta_0) + g'(\theta_0)\frac{x + z_\alpha}{\sigma_0\sqrt{n}} + \frac{g''(\theta_0)}{2}\frac{(x + z_\alpha)^2}{\sigma_0^2 n} + O(n^{-2}). \tag{17}$$

After a lengthy calculation, we have

$$\begin{aligned} A_{n,\theta_{1n}} = &\frac{1}{\sigma_0\sqrt{n}} \int_{x_{1n}}^{-z_\alpha} [\Phi(x) + \frac{\phi(x)g_1(x)}{\sqrt{n}} + \frac{\phi(x)g_2(x)}{n}] \\ &\times [g(\theta_0) + g'(\theta_0)\frac{x + z_\alpha}{\sigma_0\sqrt{n}} + \frac{g''(\theta_0)}{2}\frac{(x + z_\alpha)^2}{\sigma_0^2 n}] dx + O(n^{-2}), \end{aligned} \tag{18}$$

and

$$\begin{aligned} \tilde{A}_{n,\theta_{2n}} = &\frac{1}{\sigma_0\sqrt{n}} \int_{-z_\alpha}^{x_{2n}} [1 - \Phi(x) - \frac{\phi(x)g_1(x)}{\sqrt{n}} - \frac{\phi(x)g_2(x)}{n}] \\ &\times [g(\theta_0) + g'(\theta_0)\frac{x + z_\alpha}{\sigma_0\sqrt{n}} + \frac{g''(\theta_0)}{2}\frac{(x + z_\alpha)^2}{\sigma_0^2 n}] dx + O(n^{-2}). \end{aligned} \tag{19}$$

where

$$g_1(x) = \frac{\rho_{30}}{6}x^2 + \frac{z_\alpha \rho_{30}}{2}x + \frac{z_\alpha^2 \rho_{30}}{3} \tag{20}$$

and

$$\begin{aligned} g_2(x) = {} & \frac{\rho_{30}^2}{72}x^5 - \frac{z_\alpha \rho_{30}^2}{12}x^4 + (\frac{\rho_{40}}{8} - \frac{13 z_\alpha^2 \rho_{30}^2}{72} - \frac{7\rho_{30}^2}{24})x^3 + (\frac{z_\alpha \rho_{40}}{6} - \frac{z_\alpha^3 \rho_{30}^2}{6} \\ & - \frac{z_\alpha \rho_{30}^2}{12})x^2 + [(\frac{z_\alpha^2}{4} - \frac{7}{24})\rho_{40} - \frac{z_\alpha^4 \rho_{30}^2}{18} - \frac{13 z_\alpha^2 \rho_{30}^2}{72} + \frac{4\rho_{30}^2}{9}]x \\ & + [(\frac{z_\alpha^3}{8} - \frac{z_\alpha}{24})\rho_{40} - (\frac{z_\alpha^3}{9} - \frac{z_\alpha}{36})\rho_{30}^2]. \end{aligned} \tag{21}$$

The expressions for $g_1(x)$ and $g_2(x)$ are derived in the Appendix; the derivation of these two formulae forms the dominant part of the penultimate expression and involves the use of Cornish–Fisher as well as Edgeworth expansions.

On using (18), (19), (20) and (21), we have the following expansions

$$A_{n,\theta_{1n}} = \frac{a_1}{\sqrt{n}} + \frac{a_2}{n} + \frac{a_3}{n^{3/2}} + O(n^{-2}), \tag{22}$$

where

$$\begin{aligned} a_1 &= \frac{g(\theta_0)}{\sigma_0}[\phi(z_\alpha) - \alpha z_\alpha], \\ a_2 &= \frac{\rho_{30} g(\theta_0)}{6\sigma_0}[\alpha + 2\alpha z_\alpha^2 - 2 z_\alpha \phi(z_\alpha)] - \frac{g'(\theta_0)}{2\sigma_0^2}[\alpha(z_\alpha^2 + 1) - z_\alpha \phi(z_\alpha)] \\ a_3 &= \frac{g''(\theta_0)}{6\sigma_0^3}[(z_\alpha^2 + 2)\phi(z_\alpha) - \alpha(z_\alpha^3 + 3z_\alpha)] + \frac{g'(\theta_0)}{\sigma_0^2}[\frac{\alpha\rho_{30}}{3}(z_\alpha^3 + 2z_\alpha) \\ & \quad - \frac{\rho_{30}}{3}(z_\alpha^2 + 1)\phi(z_\alpha)] + \frac{g(\theta_0)}{\sigma_0}[(-\frac{z_\alpha^4 \rho_{30}^2}{36} + \frac{4 z_\alpha^2 \rho_{30}^2}{9} + \frac{\rho_{30}^2}{36} \\ & \quad - \frac{5 z_\alpha^2 \rho_{40}}{24} + \frac{\rho_{40}}{24})\phi(z_\alpha) + \alpha(-\frac{5 z_\alpha^3 \rho_{30}^2}{18} - \frac{11 z_\alpha \rho_{30}^2}{36} + \frac{z_\alpha^3 \rho_{40}}{8} + \frac{z_\alpha \rho_{40}}{8})]. \end{aligned} \tag{23}$$

Similarly,

$$\tilde{A}_{n,\theta_{2n}} = \frac{\tilde{a}_1}{\sqrt{n}} + \frac{\tilde{a}_2}{n} + \frac{\tilde{a}_3}{n^{3/2}} + O(n^{-2}), \tag{24}$$

where $\tilde{a}_1 = [g(\theta_0)/\sigma_0][\phi(z_\alpha) + (1-\alpha)z_\alpha]$, $\tilde{a}_2 = [g'(\theta_0)/(2\sigma_0^2)][(1-\alpha)(z_\alpha^2+1) + z_\alpha\phi(z_\alpha)] - [\rho_{30} g(\theta_0)/(6\sigma_0)][(1-\alpha)(1+2z_\alpha^2) + 2z_\alpha\phi(z_\alpha)]$, $\tilde{a}_3 = [g''(\theta_0)/(6\sigma_0^3)][(z_\alpha^2 + 2)\phi(z_\alpha) + (1-\alpha)(z_\alpha^3 + 3z_\alpha)] - [g'(\theta_0)\rho_{30}/(3\sigma_0^2)][(z_\alpha^2+1)\phi(z_\alpha) + (1-\alpha)(z_\alpha^3 + 2z_\alpha)] + [g(\theta_0)/\sigma_0][\phi(z_\alpha)(-z_\alpha^4\rho_{30}^2/36 + 4z_\alpha^2\rho_{30}^2/9 + \rho_{30}^2/36 - 5z_\alpha^2\rho_{40}/24 + \rho_{40}/24) - (1-\alpha)(-5z_\alpha^3\rho_{30}^2/18 - 11z_\alpha\rho_{30}^2/36 + z_\alpha^3\rho_{40}/8 + z_\alpha\rho_{40}/8)]$. The details of the expansions for $A_{n,\theta_{1n}}$ and $\tilde{A}_{n,\theta_{2n}}$ are given in the Appendix. Because the remainders $\underline{R}_{n,\theta_{1n}}$ and $\bar{R}_{n,\theta_{2n}}$ are of smaller order than n^{-2} as we commented before, the expansions in (22) and (24) are the expansions for A_n and $\tilde{A}_n$ in (3) and (4) respectively.

The expansions of δ_n and ϵ_n in (1) and (2) can now be obtained by letting $\lambda = \int_{\underline{\theta}}^{\theta_0} g(\theta)d\theta$, $b_1 = \tilde{a}_1 - a_1$, $b_2 = \tilde{a}_2 - a_2$ and $b_3 = \tilde{a}_3 - a_3$ in (7).

3.1. Examples

Example 1. Let $X_1, \ldots, X_n$ be i.i.d. $N(\theta, 1)$. Since θ is a location parameter, there is no loss of generality in letting $\theta_0 = 0$. Thus consider testing $H_0 : \theta \le 0$ vs $H_1 : \theta > 0$. Clearly, we have $\mu(\theta) = \theta$, $\sigma(\theta) = 1$ and $\rho_i(\theta) = \kappa_i(\theta) = 0$ for all $i \ge 3$.

The α $(0 < \alpha < 1)$ level UMP test rejects H_0 if $\sqrt{n}\bar{X} > z_\alpha$. For a continuously three times differentiable prior $g(\theta)$ for θ, one can simply plug the values of $\mu_0 = 0$, $\sigma_0 = 1$, $\rho_{30} = \rho_{40} = 0$ into (23) and the coefficients of the expansion in (24) to get the coefficients $a_1 = g(0)[\Phi(z_\alpha) - \alpha z_\alpha]$, $a_2 = -g'(0)[\alpha(z_\alpha^2 - 1) - z_\alpha \phi(z_\alpha)]$, $a_3 = g''(0)[(z_\alpha + 2)\phi(z_\alpha) - \alpha(z_\alpha^3 + 3z_\alpha)]/6$, $\tilde{a}_1 = g(0)[\phi(z_\alpha) + \phi(z_\alpha)]$, $\tilde{a}_2 = g'(0)[(1-\alpha)(z_\alpha^2 + 1) + z_\alpha \phi(z_\alpha)]/2$, $\tilde{a}_3 = g''(0)[(z_\alpha + 2)\phi(z_\alpha) + (1-\alpha)(z_\alpha^3 + 3z_\alpha)]/6$. Substituting a_1, a_2, a_3, $\tilde{a}_1$, $\tilde{a}_2$ and $\tilde{a}_3$ into (7), one derives the expansions of δ_n and ϵ_n as given by (1) and (2) respectively.

If the prior density function is also assumed to be symmetric, then $\lambda = 1/2$ and $g'(0) = 0$. In this case, the coefficients of the expansion of δ_n in (1) are given explicitly as follows: $c_1 = 2g(0)[\phi(z_\alpha) - \alpha z_\alpha]$, $c_2 = 4z_\alpha[g(0)]^2[\phi(z_\alpha) - \alpha z_\alpha]$, $c_3 = 2\phi(z_\alpha)\{4z_\alpha^2[g(0)]^3 + g''(0)(z_\alpha^2 + 2)/6\} - \alpha\{g''(0)(z_\alpha^3 + 3z_\alpha)/3 + 8z_\alpha^3[g(0)]^3\}$, and the coefficients of the expansions of ϵ_n in (2) are as $d_1 = 2g(0)[(1-\alpha)z_\alpha + \phi(z_\alpha)]$, $d_2 = -4z_\alpha[g(0)]^2[(1-\alpha)z_\alpha + \phi(z_\alpha)]$, $d_3 = 2\phi(z_\alpha)\{4z_\alpha^2[g(0)]^3 + g''(0)(z_\alpha^2 + 2)/6\} + (1-\alpha)\{g''(0)(z_\alpha^3 + 3z_\alpha)/3 + 8z_\alpha^3[g(0)]^3\}$.

Two specific prior distributions for θ are now considered for numerical illustration. In the first one we choose $\theta \sim N(0, \tau^2)$ and in the second example we choose $\theta/\tau \sim t_m$, where τ is a scale parameter. Clearly $g^{(3)}(\theta)$ is continuous in θ in both cases.

If $g(\theta)$ is the density of θ when $\theta \sim N(0, \tau^2)$, then $\lambda = 1/2$, $g(0) = 1/[\sqrt{2\pi}\tau]$, $g'(0) = 0$ and $g''(0) = -1/[\sqrt{2\pi}\tau^3]$.

We calculated the numerical values of c_1, c_2, c_3, d_1, d_2 and d_3 as functions of α when $\theta \sim N(0, 1)$. We note that c_1 is a monotone increasing function and d_1 is also a monotone decreasing function of α. However, c_2, d_2 and c_3, d_3 are not monotone and in fact, d_2 is decreasing when α is close to 1 (not shown), c_3 also takes negative values and d_3 takes positive values for larger values of α.

If $g(\theta)$ is the density of θ when $\theta/\tau \sim t_m$, then $\lambda = 1/2$, $g'(0) = 0$, $g(0) = \Gamma(\frac{m+1}{2})/[\tau\sqrt{m\pi}\Gamma(\frac{m}{2})]$ and $g''(0) = -\Gamma(\frac{m+3}{2})/[\tau\sqrt{m\pi}\Gamma(\frac{m+2}{2})]$. Putting those values into the corresponding expressions, we get the coefficients c_1, c_2, c_3 and d_1, d_2, d_3 of the expansions of δ_n and ϵ_n. When $m = 1$, the results are exactly the same as the Cauchy prior for θ.

Numerical results very similar to the normal prior are seen for the Cauchy case. From Figure 1, we see that for each of the normal and the Cauchy prior, *only about* 1% *of those null hypotheses a frequentist rejects with a p-value of less than* 5% *are true.* Indeed quite typically, $\delta_n < \alpha$ for even very small values of n. This is discussed in greater detail in Section 4.5. This finding seems to be quite interesting.

The true values of δ_n and ϵ_n are computed by taking an average of the lower and the upper Riemann sums in A_n, $\tilde{A}_n$, B_n and $\tilde{B}_n$ with the exact formulae for the standard normal pdf. The accuracy of the expansion for δ_n is remarkable, as can be seen in Figure 1. Even for $n = 4$, the true value of δ_n is almost identical to the expansion in (1). The accuracy of the expansion for ϵ_n is very good (even if it is not as good as that for δ_n). For $n = 20$, the true value of ϵ_n is almost identical to the expansion in (2).

Example 2. Let $X_1, \cdots, X_n$ be iid $Exp(\theta)$, with density $f_\theta(x) = \theta e^{-\theta x}$ if $x > 0$. Clearly, $\mu(\theta) = 1/\theta$, $\sigma^2(\theta) = 1/\theta^2$, $\rho_3(\theta) = 2$ and $\rho_4(\theta) = 6$. Let $\tilde{\theta} = -\theta$. Then,

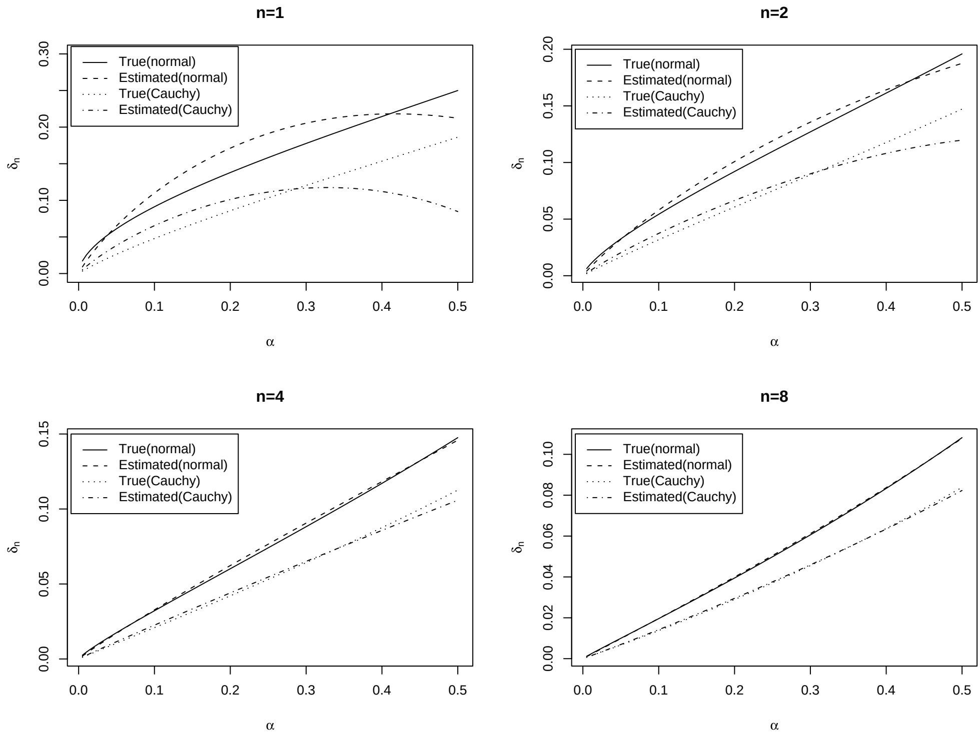

FIG 1. *True and estimated values of δ_n as functions of α for the standard normal prior and the Cauchy prior.*

one can write the density of X_1 in the standard form of the exponential family as $f_{\tilde{\theta}}(x) = e^{\tilde{\theta}x+\log|\tilde{\theta}|}$. The natural parameter space of $\tilde{\theta}$ is $\Omega = (-\infty, 0)$. If $g(\theta)$ is a prior density for θ on $(0, \infty)$, then $g(-\tilde{\theta})$ is a prior density for $\tilde{\theta}$ on $(-\infty, 0)$. Since θ is a scale parameter, it is enough to look at the case $\tilde{\theta}_0 = -1$. In terms of θ, therefore the problem considered is to test $H_0 : \theta \geq 1$ vs $H_1 : \theta < 1$. The α $(0 < \alpha < 1)$ level UMP test for this problem rejects H_0 if $\bar{X} > \Gamma_{\alpha,n,n}$, where $\Gamma_{\alpha,r,s}$ is the upper α quantile of the Gamma distribution with parameters r and s. If $g(\theta)$ is continuous and three time differentiable, then we can simply put the values $\mu_0 = 1$, $\sigma_0 = 1$, $\rho_{30} = 2$, $\rho_{40} = 6$, and $\lambda = \int_0^1 g(\theta)d\theta$ into (23) and the coefficients of the expansion in (24) to get the coefficients a_1, a_2, a_3, $\tilde{a}_1$, $\tilde{a}_2$ and $\tilde{a}_3$, and then get the expansions of δ_n and ϵ_n in (1) and (2) respectively.

Two priors are to be considered in this example. The first one is the Gamma prior with prior density $g(\theta) = s^r\theta^{r-1}e^{-s\theta}/\Gamma(r)$, where r and s are known constants. It would be natural to have the mode of $g(\theta)$ at 1, that is $s = r - 1$. In this case, $g'(1) = 0$, $g(1) = (r-1)^r e^{-(r-1)}/\Gamma(r)$ and $g''(1) = -(r-1)^{r+1}e^{-(r-1)}/\Gamma(r)$.

Next, consider the F prior with degrees of freedom $2r$ and $2s$ for θ/τ for a fixed $\tau > 0$. Then, the prior density for θ is $g(\theta) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\frac{r}{s\tau}(\frac{r\theta}{s\tau})^{r-1}(1+\frac{r\theta}{s\tau})^{-(r+s)}$. To make the mode of $g(\theta)$ equal to 1, we have to choose $\tau = r(s+1)/[s(r-1)]$. Then $g'(1) = 0$, $g(1) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}(\frac{r-1}{s+1})^r(1+\frac{r-1}{s+1})^{-(r+s)}$, and $g''(1) = -\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}(\frac{r-1}{s+1})^{r+1}(r+s)(1+\frac{r-1}{s+1})^{-(r+s+2)}$.

Exact and estimated values of δ_n are plotted in Figure 3. At $n = 20$, the expansion is clearly extremely accurate and as in example 1, we see that the false

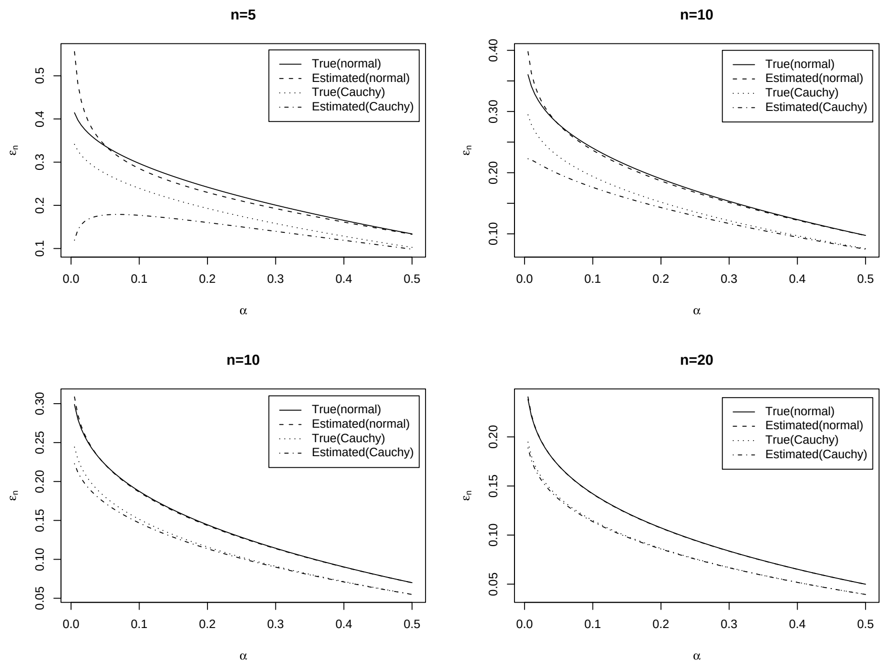

FIG 2. *True and estimated values of ϵ_n as functions of α for the standard normal prior and the Cauchy prior.*

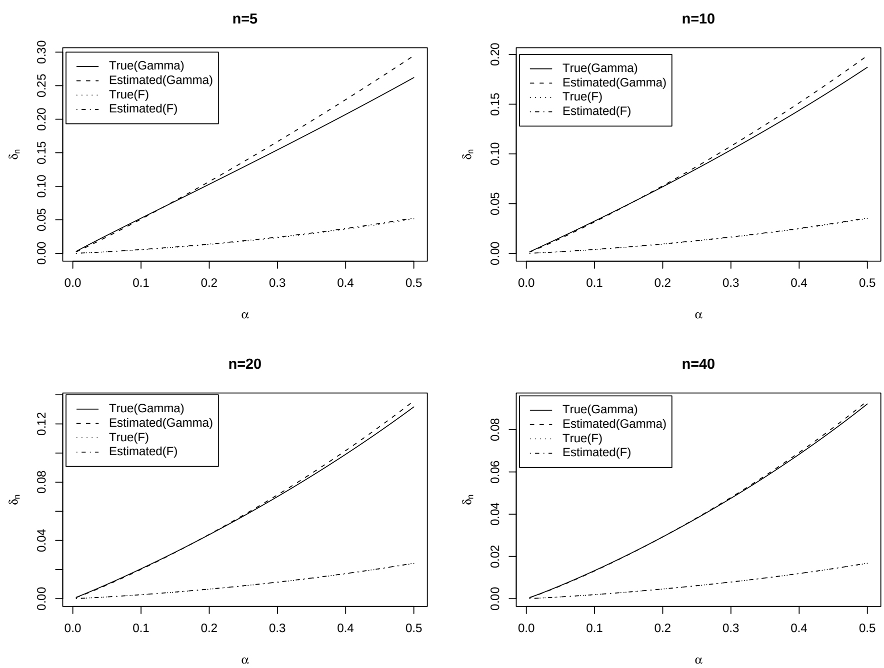

FIG 3. *True and estimated values of δ_n as functions of α under $\Gamma(2,1)$ and $F(4,4)$ priors for θ when $X \sim Exp(\theta)$.*

discovery rate δ_n is very small even for $n = 10$.

3.2. The frequentist is more prone to type II error

Consider the two Bayesian error rates

$$\delta_n = P(H_0| \text{ Frequentist rejects } H_0)$$

and

$$\epsilon_n = P(H_1| \text{ Frequentist accepts } H_0).$$

Is there an inequality between δ_n and ϵ_n? Rather interestingly, when θ is the normal mean and the testing problem is $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$, there is an approximate inequality in the sense that if we consider the respective coefficients c_1 and d_1 of the $1/\sqrt{n}$ term, then for any symmetric prior (because then $g'(0) = 0$ and $\lambda = 1 - \lambda = 1/2$), we have

$$c_1 = 2g(0)[\phi(z_\alpha) - \alpha z_\alpha] < d_1 = 2g(0)[(1-\alpha)z_\alpha + \phi(z_\alpha)]$$

for any $\alpha < 1/2$. It is interesting that this inequality holds regardless of the exact choice of $g(\cdot)$ and the value of α, as long as $\alpha < 1/2$. Thus, to the first order, the frequentist is less prone to type I error. Even the exact values of δ_n and ϵ_n satisfy this inequality, unless α is small, as can be seen, for example from a scrutiny of Figures 1 and 2. This would suggest that a frequentist needs to be more mindful of premature acceptance of H_0 rather than its premature rejection in the composite one sided problem. This is in contrast to the conclusion reached in Berger and Sellke [2] under their formulation.

4. General location parameter case

As we mentioned in Section 1, the quantities δ_n, ϵ_n depend on the choice of the test statistic. For location parameter problems, in general there is no reason to use the sample mean as the test statistic. For many non-normal location parameter densities, such as the double exponential, it is more natural to use the sample median as the test statistic.

Assume the density of the i.i.d. sample $X_1, \ldots, X_n$ is $f(x-\theta)$ where the median of $f(\cdot)$ is 0, and assume $f(0) > 0$. Then an asymptotic size α test for

$$H_0 : \theta \leq 0 \text{ vs } H_1 : \theta > 0$$

rejects H_0 if $\sqrt{n}T_n > z_\alpha/[2f(0)]$, where $T_n = X_{([\frac{n}{2}]+1)}$ is the sample median ([8], p. 89), since $\sqrt{n}(T_n - \theta) \stackrel{L}{\Rightarrow} N(0, 1/[4f^2(0)])$. We will derive the coefficients c_1, c_2, c_3 in (1) and d_1, d_2, d_3 in (2) given the prior density $g(\theta)$ for θ. We assume again that $g(\theta)$ is three times differentiable with a bounded absolute third derivative.

4.1. Expansion of type I error and type II error

To obtain the coefficients of the expansions of δ_n in (1) and ϵ_n in (2), we have to expand the A_n and $\tilde{A}_n$ given by (3) and (4). Of these,

$$(25) \quad A_n = P(\theta \leq 0, \sqrt{n}T_n > \frac{z_\alpha}{2f(0)}) = \frac{1}{\sqrt{n}} \int_{-\infty}^{0} \{1 - F_n[z_\alpha - 2xf(0)]\} g(\frac{x}{\sqrt{n}}) dx$$

where F_n is the CDF of $2f(0)\sqrt{n}(T_n-\theta)$ if the true median is θ. Reiss [16] gives the expansion of F_n as

$$F_n(t)=\Phi(t)+\frac{\phi(t)}{\sqrt{n}}R_1(t)+\frac{\phi(t)}{n}R_2(t)+r_{t,n}, \tag{26}$$

where, with $\{x\}$ denoting the fractional part of a real x, $R_1(t)=f_{11}t^2+f_{12}$, $f_{11}=f'(0)/[4f^2(0)]$, $f_{12}=-(1-2\{\frac{n}{2}\})$, and $R_2(t)=f_{21}t^5+f_{22}t^3+f_{23}t$, where $f_{21}=-[f'(0)/f^2(0)]^2/32$, $f_{22}=1/4+(1/2-\{\frac{n}{2}\})[f'(0)/(2f^2(0))]+f''(0)/[24f^3(0)]$, $f_{23}=1/4-(1-2\{\frac{n}{2}\})^2/2$. The error term $r_{1,t,n}$ can be written as $r_{t,n}=\phi(t)R_3(t)/n^{3/2}+O(n^{-2})$, where $R_3(t)$ is a polynomial.

By letting $y=2xf(0)-z_\alpha$ in (25), we have

$$\begin{aligned} A_n=&\frac{1}{2f(0)\sqrt{n}}\int_{-\infty}^{-z_\alpha}\{\Phi(y)-\frac{\phi(y)}{\sqrt{n}}R_1(-y)-\frac{\phi(y)}{n}R_2(-y)-r_{-y,n}\}\\ &\times[g(0)+g'(0)\frac{y+z_\alpha}{2f(0)\sqrt{n}}+\frac{g''(0)}{2}\frac{(y+z_\alpha)^2}{4f^2(0)n}+\frac{(y+z_\alpha)^3}{48f^3(0)n^{3/2}}g^{(3)}(y^*)]dy,\end{aligned} \tag{27}$$

where y^* is between 0 and $(y+z_\alpha)/[2f(0)\sqrt{n}]$.

Hence, assuming $\sup_\theta|g^{(3)}(\theta)|<\infty$, on exact integration of each product of functions in (27) and on collapsing the terms, we get

$$A_n=\frac{a_1}{\sqrt{n}}+\frac{a_2}{n}+\frac{a_3}{n^{3/2}}+O(n^{-2}), \tag{28}$$

where

$$a_1=\frac{g(0)}{2f(0)}[\phi(z_\alpha)-\alpha z_\alpha], \tag{29}$$

$$a_2=\frac{g'(0)}{8f^2(0)}[z_\alpha\phi(z_\alpha)-\alpha(z_\alpha^2+1)]-\frac{g(0)}{2f(0)}\{f_{11}[z_\alpha\phi(z_\alpha)+\alpha]+f_{12}\alpha\} \tag{30}$$

and

$$\begin{aligned} a_3=&\frac{g''(0)}{48f^3(0)}[(z_\alpha^2+2)\phi(z_\alpha)-\alpha(z_\alpha^3+3z_\alpha)]\\ &-\frac{g'(0)}{4f^2(0)}\{f_{11}[\alpha z_\alpha-2\phi(z_\alpha)]+f_{12}[\alpha z_\alpha-\phi(z_\alpha)]\}\\ &-\frac{g(0)}{2f(0)}\{f_{21}[(z_\alpha^4+4z_\alpha^2+8)\phi(z_\alpha)]+f_{22}[(z_\alpha^2+2)\phi(z_\alpha)]+f_{23}\phi(z_\alpha)\}.\end{aligned} \tag{31}$$

We claim the error term in (28) is $O(n^{-2})$. To prove this, we need to look at its exact form, namely,

$$-\frac{O(n^{-2})}{2f(0)}\int_{-\infty}^{-z_\alpha}\phi(y)R_3(-y)g(\theta_0+\frac{y+z_\alpha}{2f(0)\sqrt{n}})dy+O(n^{-2})\int_{-\infty}^{0}g(\theta_0+y)dy.$$

Since $g(\theta)$ is absolutely uniformly bounded, the first term above is bounded by $O(n^{-2})$. The second term is $O(n^{-2})$ obviously. This shows that the error term in (28) is $O(n^{-2})$.

As regards $\tilde{A}_n$ given by (4), one can similarly obtain

$$\begin{aligned} \tilde{A}_n &= P(\theta > 0, \sqrt{T}_n \le \frac{z_\alpha}{2f(0)}) = \frac{1}{\sqrt{n}} \int_0^\infty F_n[z_\alpha - 2f(0)x] g(\frac{x}{\sqrt{n}}) dx \\ &= \frac{\tilde{a}_1}{\sqrt{n}} + \frac{\tilde{a}_2}{n} + \frac{\tilde{a}_3}{n^{\frac{3}{2}}} + O(n^{-2}), \end{aligned} \tag{32}$$

where y^* is between 0 and $(z_\alpha - y)/[2f(0)\sqrt{n}]$, $\tilde{a}_1 = [g(0)/(2f(0))][(1-\alpha)z_\alpha + \phi(z_\alpha)]$, $\tilde{a}_2 = [g'(0)/(8f^2(0))][(1-\alpha)(z_\alpha^2+1) + z_\alpha\phi(z_\alpha)] + [g(0)/(2f(0))]\{f_{11}[(1-\alpha) - z_\alpha\phi(z_\alpha)] + f_{12}(1-\alpha)\}$, $\tilde{a}_3 = [g''(0)/(48f^3(0))][(z_\alpha^2+2)\phi(z_\alpha) + (1-\alpha)(z_\alpha^3 + 3z_\alpha)] + [g'(0)/(4f^2(0))]\{f_{11}[(1-\alpha)z_\alpha + 2\phi(z_\alpha)] + f_{12}[(1-\alpha)z_\alpha + \phi(z_\alpha)]\} - [g(0)/(2f(0))] \times \{f_{21}[(z_\alpha^4 + 4z_\alpha^2 + 8)\phi(z_\alpha)] + f_{22}[(z_\alpha^2+2)\phi(z_\alpha)] + f_{23}\phi(z_\alpha)\}$. The error term in (32) is still $O(n^{-2})$ and this proof is omitted.

Therefore, we have the the expansions of B_n given by (5) $B_n = \lambda - b_1/\sqrt{n} - b_2/n - b_3/n^{3/2} + O(n^{-2})$ where $\lambda = \int_0^\infty g(\theta)d\theta$ as before, $b_1 = \tilde{a}_1 - a_1 = z_\alpha g(0)/[2f(0)]$, $b_2 = \tilde{a}_2 - a_2 = g'(0)(z_\alpha^2+1)/[8f^2(0)] + g(0)(f_{11}+f_{12})/[2f(0)]$, $b_3 = \tilde{a}_3 - a_3 = g''(0)(z_\alpha^3 + 3z_\alpha)/[48f^3(0)] + z_\alpha g'(0)(f_{11}+f_{12})/[4f^2(0)]$. Substituting a_1, a_2, a_3, $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{a}_3$, b_1, b_2 and b_3 into (7), we get the expansions of δ_n and ϵ_n for the general location parameter case given by (1) and (2).

4.2. Testing with mean vs. testing with median

Suppose $X_1, \ldots, X_n$ are i.i.d. observations from a $N(\theta, 1)$ density and the statistician tests $H_0 : \theta \le 0$ *vs.* $H_1 : \theta > 0$ by using either the sample mean $\bar{X}$ or the median T_n. It is natural to ask the choice of which statistic makes him more vulnerable to false discoveries. We can look at both false discovery rates δ_n and ϵ_n to make this comparison, but we will do so only for the type I error rate δ_n here.

We assume for algebraic simplicity that g is symmetric, and so $g'(0) = 0$ and $\lambda = 1/2$. Also, to keep track of the two statistics, we will denote the coefficients c_1, c_2 by $c_{1,\bar{X}}$, c_{1,T_n}, $c_{2,\bar{X}}$ and c_{2,T_n} respectively. Then from our expansions in section 3.1 and section 4.1, it follows that

$$c_{1,T_n} - c_{1,\bar{X}} = g(0)(\phi(z_\alpha) - \alpha z_\alpha)(\sqrt{2\pi} - 2) = a(\text{say}),$$

and

$$\begin{aligned} c_{2,T_n} - c_{2,\bar{X}} &= g^2(0) z_\alpha(\phi(z_\alpha) - \alpha z_\alpha)(2\pi - 4) - g(0)\sqrt{2\pi} f_{12}\alpha \\ &\ge g^2(0) z_\alpha(\phi(z_\alpha) - \alpha z_\alpha)(2\pi - 4) = b(\text{say}) \text{ as } f_{12} \le 0. \end{aligned}$$

Hence, there exist positive constants a, b such that $\liminf_{n\to\infty} \sqrt{n}(\sqrt{n}(\delta_{n,T_n} - \delta_{n,\bar{X}}) - a) \ge b$, i.e., the statistician is *more* vulnerable to a type I false discovery by using the sample median as his test statistic. Now, of course, as a point estimator, T_n is less efficient than the mean $\bar{X}$ in the normal case. Thus, the statistician is more vulnerable to a false discovery if he uses the less efficient point estimator as his test statistic. We find this neat connection between efficiency in estimation and false discovery rates in testing to be interesting. Of course, similar connections are well known in the literature on Pitman efficiencies of tests; see, e.g., van der Vaart ([24], p. 201).

4.3. Examples

In this subsection, we are going to study the exact values and the expansions for δ_n and ϵ_n in two examples. One example is $f(x) = \phi(x)$ and $g(\theta) = \phi(\theta)$; for the other example, f and g are both densities of the standard Cauchy. We will refer to them as normal-normal and Cauchy-Cauchy for convenience of reference. The purpose of the first example is comparison with the normal-normal case when the test statistic was the sample mean (Example 2 in Section 3); the second example is an independent natural example.

For exact numerical evaluation of δ_n and ϵ_n, the following formulae are necessary. The pdf of the standardized median $2f(0)\sqrt{n}(T_n - \theta)$ is

$$(33) \quad f_n(t) = \frac{\sqrt{n}\binom{n-1}{[\frac{n}{2}]-1}}{2f(0)} f\Big(\frac{t}{2f(0)\sqrt{n}}\Big) F^{[\frac{n}{2}]-1}\Big(\frac{t}{2f(0)\sqrt{n}}\Big)\Big(1 - F\Big(\frac{t}{2f(0)\sqrt{n}}\Big)\Big)^{n-[\frac{n}{2}]}.$$

We are now ready to present our examples.

Example 3. Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. $N(\theta, 1)$ and $g(\theta) = \phi(\theta)$. Then, $g(0) = f(0) = 1/\sqrt{2\pi}$, $g'(0) = f'(0) = 0$ and $g''(0) = f''(0) = -1/\sqrt{2\pi}$. Then, we have $f_{11} = 0$, $f_{12} = -(1 - 2\{\frac{n}{2}\})$, $f_{21} = 0$, $f_{22} = 1/4 - \pi/12$ and $f_{23} = 1/4 - (1/2 - \{\frac{n}{2}\})^2$. Plugging these values for $f_{11}, f_{12}, f_{21}, f_{22}, f_{23}$ into (29), (30), (31) and (7), we obtain the expansions for δ_n, and similarly for ϵ_n in the normal-normal case.

Next we consider the Cauchy-Cauchy case, i.e., $X_1, \ldots, X_n$ are i.i.d. with density function $f(x) = 1/\{\pi[1 + (x-\theta)^2]\}$ and $g(\theta) = 1/[\pi(1+\theta^2)]$. Then, $f(0) = 1/\pi$, $f'(0) = 0$ and $f''(0) = -2/\pi$. Therefore, $f_{11} = 0$, $f_{12} = -(1 - 2\{\frac{n}{2}\})$, $f_{21} = 0$, $f_{22} = 1/4 - \pi^2/12$, and $f_{23} = 1/4 - (1/2 - \{\frac{n}{2}\})^2$. Plugging these values for $f_{11}, f_{12}, f_{21}, f_{22}, f_{23}$ in (29), (30), (31), we obtain the expansions for δ_n, and similarly for ϵ_n in the Cauchy-Cauchy case.

The true and estimated values of δ_n for selected n are given in Figure 4 and Figure 5. As before, the true values of δ_n and ϵ are computed by taking an average of the lower and the upper Riemann sums in A_n, $\tilde{A}_n$, B_n and $\tilde{B}_n$ with the exact formulae for f_n as in (33). It can be seen that the two values are almost identical when $n = 30$. By comparison with Figure 1, we see that the expansion for the median is not as precise as the expansion for the sample mean.

The most important thing we learn is how small δ_n is for very moderate values of n. For example, in Figure 4, δ_n is only about 0.01 if $\alpha = 0.05$, when $n = 20$. Again we see that even though we have changed the test statistic to the median, the frequentist's false discovery rate is very small and, in particular, smaller than α. More about this is said in Sections 4.4 and 4.5.

4.4. Spiky priors and false discovery rates

We commented in Section 4.1 that if the prior density $g(\theta_0)$ is large, it increases the leading term in the expansion for δ_n (and also ϵ_n) and so it can be expected that spiky priors cause higher false discovery rates. In this section, we address the effect of spiky and flat priors a little more formally.

Consider the general testing problem $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$, where the natural parameter space $\Omega = (\underline{\theta}, \bar{\theta})$.

Suppose the α $(0 < \alpha < 1)$ level test rejects H_0 if $T_n \in C$, where T_n is the test statistic. Let $P_n(\theta) = P_\theta(T_n \in C)$. Let $g(\theta)$ be any fixed density function for θ and

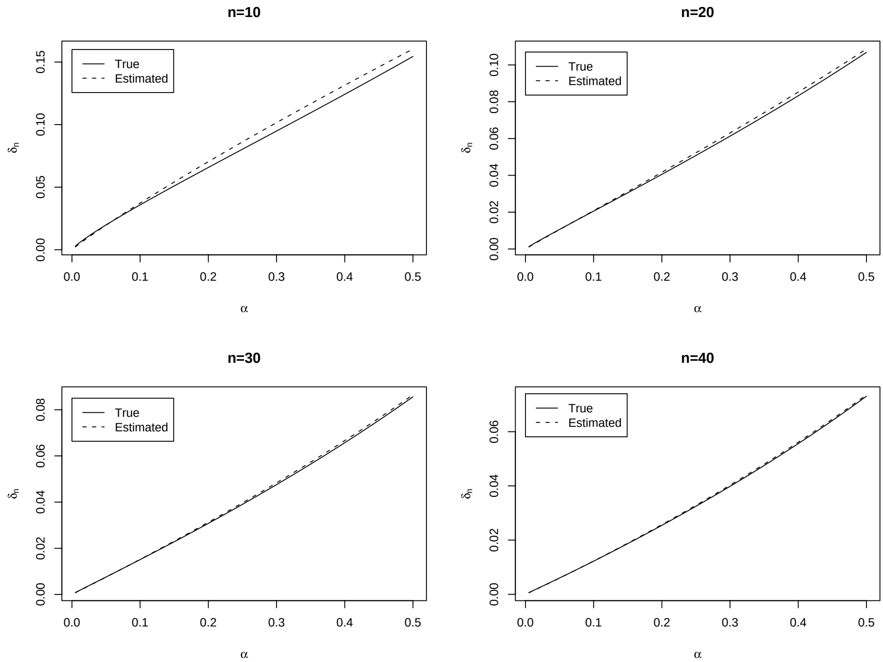

FIG 4. *True and estimated values of* δ_n *when the test statistic is the median for the normal-normal case.*

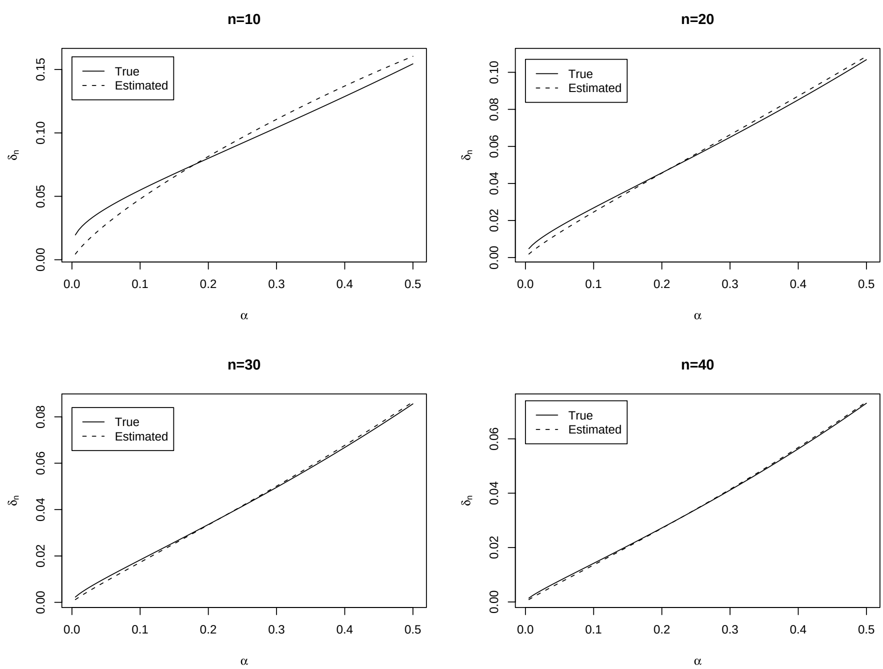

FIG 5. *True and estimated values of* δ_n *when the test statistic is the median for the Cauchy-Cauchy case.*

let $g_\tau(\theta) = g(\theta/\tau)/\tau$, $\tau > 0$. Then $g_\tau(\theta)$ is spiky at 0 for small τ and $g_\tau(\theta)$ is flat for large τ. When $\theta_0 = 0$, under the prior $g_\tau(\theta)$,

$$\delta_n(\tau) = P(\theta \le 0 | T_n \in C) = \frac{\int_{\underline{\theta}/\tau}^{0} P_n(\tau y) g(y) dy}{\int_{\underline{\theta}/\tau}^{\bar{\theta}/\tau} P_n(\tau y) g(y) dy}, \tag{34}$$

and

$$\epsilon_n(\tau) = P(\theta > 0 | T_n \notin C) = \frac{\int_{\underline{\theta}/\tau}^{0} [1 - P_n(\tau y)] g(y) dy}{\int_{\underline{\theta}/\tau}^{\bar{\theta}/\tau} [1 - P_n(\tau y)] g(y) dy}. \tag{35}$$

Let as before $\lambda = \int_{-\infty}^{0} g(\theta) d\theta$, the numerator and denominator of (34) be denoted by $A_n(\tau)$ and $B_n(\tau)$ and the numerator and denominator of (35) be denoted by $\tilde{A}_n(\tau)$ and $\tilde{B}_n(\tau)$. Then, we have the following results.

Proposition 1. *If $P_n^-(\theta_0) = \lim_{\theta \to \theta_0 -} P_n(\theta)$ and $P_n^+(\theta_0) = \lim_{\theta \to \theta_0 +} P_n(\theta)$ both exist and are positive, then*

$$\lim_{\tau \to 0} \delta_n(\tau) = \frac{\lambda P_n^-(0)}{\lambda P_n^-(0) + (1-\lambda) P_n^+(0)} \tag{36}$$

and

$$\lim_{\tau \to 0} \epsilon_n(\tau) = \frac{(1-\lambda)[1 - P_n^+(0)]}{\lambda[1 - P_n^-(0)] + (1-\lambda)[1 - P_n^+(0)]}. \tag{37}$$

Proof. Because $0 \le P_n(\tau y) \le 1$ for all y, by simply applying the Lebesgue Dominated Convergence Theorem, $\lim_{\tau \to 0} A_n(\tau) = \lambda P_n^-(0)$, $\lim_{\tau \to 0} B_n(\tau) = \lambda P_n^-(0) + (1-\lambda) P_n^+(0)$, $\lim_{\tau \to 0} \tilde{A}_n(\tau) = (1-\lambda)[1 - P_n^+(0)]$ and $\lim_{\tau \to 0} \tilde{B}_n(\tau) = \lambda[1 - P_n^-(0)] + (1-\lambda)[1 - P_n^+(0)]$. Substituting in (34) and (35), we get (36) and (37). □

Corollary 1. *If $0 < \lambda < 1$, $\lim_{\tau \to \infty} P_n(\tau y) = 0$ for all $y < 0$, $\lim_{\tau \to \infty} P_n(\tau y) = 1$ for all $y > 0$, then $\lim_{\tau \to \infty} \delta_n(\tau) = \lim_{\tau \to \infty} \epsilon_n(\tau) = 0$.*

Proof. Immediate from (36) and (37). □

It can be seen that $P_n^-(0) = P_n^+(0)$ in most testing problems when the test statistic T_n has a continuous power function. It is true for all the problems we discussed in Sections 3 and 4. If moreover $g(\theta) > 0$ for all θ, then $0 < \lambda < 1$. As a consequence, $\lim_{\tau \to 0} \delta_n(\tau) = \lambda$, $\lim_{\tau \to 0} \epsilon_n(\tau) = 1 - \lambda$, and $\lim_{\tau \to \infty} \delta_n(\tau) = \lim_{\tau \to \infty} \epsilon_n(\tau) = 0$. If θ is a location parameter, $\theta_0 = 0$ and $g(\theta)$ is symmetric about 0, then $\lim_{\tau \to 0} \delta_n(\tau) = \lim_{\tau \to 0} \epsilon_n(\tau) = 1/2$.

In other words, the false discovery rates are very small for any n for flat priors and roughly 50% for any n for very spiky symmetric priors. This is a qualitatively informative observation.

4.5. Pre-experimental promise and post-experimental honesty

We noticed in our example in Section 4.4 that for quite small values of n, the post-experimental error rate δ_n was smaller than the pre-experimental assurance, namely α. For any given prior g, this is true for all large n; but clearly we cannot achieve this uniformly over all g, or even large classes of g. In order to remain honest, it seems reasonable to demand of a frequentist that δ_n be smaller than

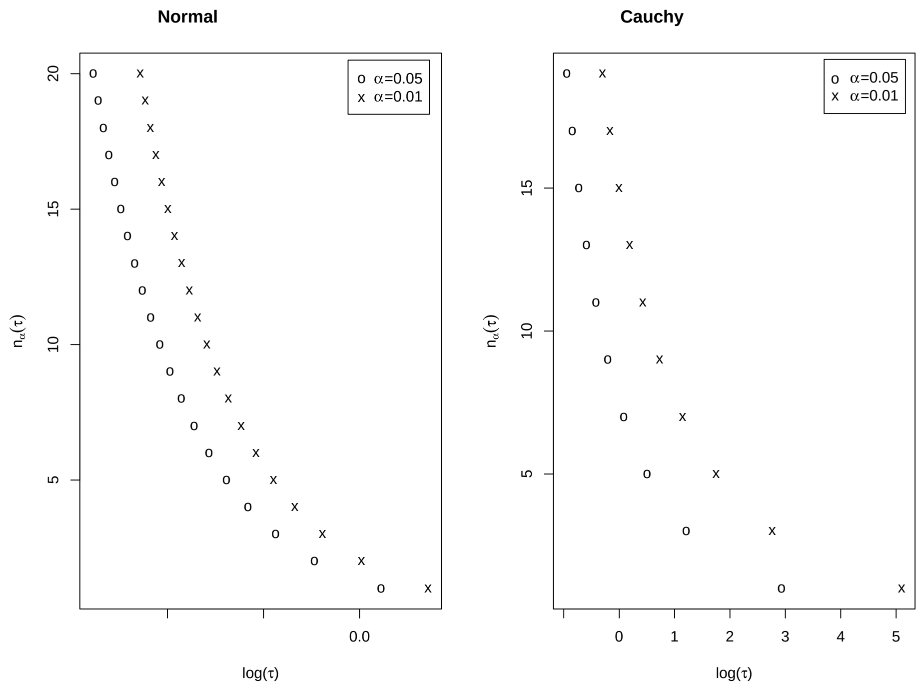

FIG 6. *Plots of $n_\alpha(\tau)$ as functions of τ for normal-normal test by mean and Cauchy-Cauchy test by median for selected α.*

α. The question is, typically for what sample sizes can the frequentist assert his honesty.

Let us then consider the prior $g_\tau(\theta) = g(\theta/\tau)/\tau$ with fixed g, and consider the minimum value of the sample size n, denoted by $n_\alpha(\tau)$, such that $\delta_n \le \alpha$. It can be seen from (36) that $n_\alpha(\tau)$ goes to ∞ as τ goes to 0. This of course was anticipated. What happens when τ varies from small to large values?

Plots of $n_\alpha(\tau)$ as functions of τ when the population CDF is $F_\theta(x) = \Phi(x-\theta)$, $g(\theta) = \phi(\theta)$ and the test statistic is $\bar{X}$ are given in the left window of Figure 6. It is seen in the plot that $n_\alpha(\tau)$ is non-increasing in τ for the selected α-values 0.05 and 0.01. Plots of $n_\alpha(\tau)$ when $F_\theta(x) = C(x-\theta)$ and $g(\theta) = c(\theta)$, where $C(\cdot)$ and $c(\cdot)$ are standard Cauchy CDF and PDF respectively, are given in the right window of Figure 6.

In both examples, a modest sample size of $n = 15$ suffices for ensuring $\delta_n \le \alpha$ if $\tau \ge 1$. For somewhat more spiky priors with $\tau \approx 0.5$, in the Cauchy-Cauchy case, a sample of size just below 30 will be required. In the normal-normal case, even $n = 8$ still suffices.

The general conclusion is that unless the prior is very spiky, a sample of size about 30 ought to ensure that $\delta_n \le \alpha$ for traditional values of α.

Appendix: Detailed expansions for the exponential family

We now provide the details for the expansions of $A_{n,\theta_{1n}}$ in (13) and $\tilde{A}_{n,\theta_{2n}}$ in (15) and we also prove that $\underline{R}_{n,\theta_{1n}}$ in (14) and $\bar{R}_{n,\theta_{2n}}$ in (16) are smaller order terms.

Suppose $g(\theta)$ is a three times differentiable proper prior for θ. The expansions are considered for those θ_0 so that the exponential family density has a positive variance at θ_0. Then, we can find two values θ_1 and θ_2 such that $\underline{\theta} < \theta_1 < \theta_0 < \theta_2 < \bar{\theta}$ and the minimum value of $\sigma^2(\theta)$ is positive when $\theta_1 \le \theta \le \theta_2$. That is if we let $m_0 = \min_{\theta_1<\theta<\theta_2} \sigma^2(\theta)$, then $m_0 > 0$. Since $\sigma^2(\theta)$, $k_i(\theta)$, $\rho_i(\theta)$ and $g^{(3)}(\theta)$ are all continuous in θ, each of them is uniformly bounded in absolute value for $\theta \in [\theta_1, \theta_2]$. We denote M_0 as the common upper bound of the absolute values of $\sigma^2(\theta)$, $\kappa_i(\theta)$ $(i = 3, 4, 5)$, $\rho_i(\theta)$ $(i = 3, 4, 5)$, $g(\theta)$, $g'(\theta)$, $g''(\theta)$ and $g^{(3)}(\theta)$.

In the rest of this section, we denote $\theta_{1n} = \theta_0 + (\theta_1 - \theta_0)/n^{1/3}$, $\theta_{2n} = \theta_0 + (\theta_2 - \theta_0)/n^{1/3}$, $x_1 = \sigma_0\sqrt{n}(\theta_1 - \theta_0) - z_\alpha$, $x_2 = \sigma_0\sqrt{n}(\theta_2 - \theta_0) - z_\alpha$, $x_{1n} = \sigma_0\sqrt{n}(\theta_{1n} - \theta_0) - z_\alpha$ and $x_{2n} = \sigma_0\sqrt{n}(\theta_{1n} - \theta_0) - z_\alpha$. As in (13), (14), (15) and (16), we define $A_{n,\theta_{1n}} = P(\theta_{1n} \le \theta \le \theta_0, \bar{X} \in C)$, $\underline{R}_{n,\theta_{1n}} = A_n - A_{n,\theta_{1n}}$, $\tilde{A}_{n,\theta_{2n}} = P(\theta_0 < \theta \le \theta_{2n}, \bar{X} \notin C)$ and $\bar{R}_{n,\theta_{2n}} = \tilde{A}_n - \tilde{A}_{n,\theta_{2n}}$, where A_n and $\tilde{A}_n$ are given by (3) and (4) respectively. We write $B_{n,\theta_1} = P(\theta \ge \theta_{1n}, \bar{X} \in C)$ and $\tilde{B}_{n,\theta_2} = P(\theta \le \theta_{2n}, \bar{X} \notin C)$. Then, one can also see that $\underline{R}_{n,\theta_{1n}} = B_n - B_{n,\theta_1}$ and $\bar{R}_{n,\theta_{2n}} = \tilde{B}_n - \tilde{B}_{n,\theta_2}$ from definition, where B_n and $\tilde{B}_n$ are given by (5) and (6) respectively.

The following Proposition and Corollary claim that $\underline{R}_{n,\theta_{1n}}$ and $\bar{R}_{n,\theta_{2n}}$ are the smaller order terms. Therefore, the coefficients of the expansions of A_n and $\tilde{A}_n$ are exactly the same as those of $A_{n,\theta_{1n}}$ and $\tilde{A}_{n,\theta_{2n}}$.

Proposition 2. *Let $\theta_{1,\tau,n} = \theta_0 + (\theta_1 - \theta_0)/n^\tau$ and $\theta_{2,\tau,n} = \theta_0 + (\theta_2 - \theta_0)/n^\tau$. If $0 \le \tau < 1/2$, then for any $\ell < \infty$, $\lim_{n\to\infty} n^l \tilde{\beta}_n(\theta_{1,\tau,n}) = \lim_{n\to\infty} n^l[1 - \tilde{\beta}_n(\theta_{2,\tau,n})] = 0$.*

Proof. A proof of this can be obtained by simply using Markov's inequality. We omit it. □

Corollary 2. *For any $l > 0$, $\lim_{n\to\infty} n^l \underline{R}_{n,\theta_{1n}} = \lim_{n\to\infty} n^l \bar{R}_{n,\theta_{2n}} = 0$.*

Proof. Since $\tilde{\beta}_n(\theta)$ is nondecreasing in θ, we have

$$n^l \underline{R}_{n,\theta_{1n}} = n^l \int_{\underline{\theta}}^{\theta_{1n}} \tilde{\beta}_n(\theta) g(\theta) d\theta \le n^l \tilde{\beta}_n(\theta_{1,1/3,n}) \int_{\underline{\theta}}^{\theta_{1n}} g(\theta) d\theta \le n^l \tilde{\beta}_n(\theta_{1,1/3,n})$$

and similarly $n^l \bar{R}_{n,\theta_{2n}} \le n^l[1 - \tilde{\beta}_n(\theta_{2,1/3,n})]$. The conclusion is drawn by taking $\tau = 1/3$ in Proposition 2. □

In the rest of this section, we will only derive the expansion of $A_{n,\theta_{1n}}$ in detail since the expansion of $\tilde{A}_{n,\theta_{2n}}$ is obtained exactly similarly.

Using the transformation $x = \sigma_0\sqrt{n}(\theta - \theta_0) - z_\alpha$ in the following integral, we have

$$A_{n,\theta_{1n}} = \frac{1}{\sigma_0\sqrt{n}} \int_{x_{1n}}^{-z_\alpha} \tilde{\beta}_n(\theta_0 + \frac{x + z_\alpha}{\sigma_0\sqrt{n}}) g(\theta_0 + \frac{x + z_\alpha}{\sigma_0\sqrt{n}}) dx. \tag{38}$$

Note that

$$\tilde{\beta}_n(\theta_0 + \frac{x + z_\alpha}{\sigma_0\sqrt{n}}) = P_{\theta_0 + \frac{x+z_\alpha}{\sigma_0\sqrt{n}}} \left(\sqrt{n} \frac{\bar{X} - \mu(\theta_0 + \frac{x+z_\alpha}{\sigma_0\sqrt{n}})}{\sigma(\theta_0 + \frac{x+z_\alpha}{\sigma_0\sqrt{n}})} \ge \tilde{k}_{\theta_0,x,n} \right), \tag{39}$$

where

$$\tilde{k}_{\theta_0,x,n} = \left[\sqrt{n} \frac{\mu_0 - \mu(\theta_0 + \frac{x+z_\alpha}{\sigma_0\sqrt{n}})}{\sigma_0} + k_{\theta_0,n} \right] \frac{\sigma_0}{\sigma(\theta_0 + \frac{x+z_\alpha}{\sigma_0\sqrt{n}})}. \tag{40}$$

We obtain the coefficients of the expansions of $A_{n,\theta_{1n}}$ in the following steps:

1. The expansion of $g(\theta_0 + \frac{x+z_\alpha}{\sigma_0\sqrt{n}})$ for any fixed $x \in [x_{1n}, -z_\alpha]$ is obtained by using Taylor expansions.
2. The expansion of $\tilde{k}_{\theta_0,x,n}$ for any fixed $x \in [x_{1n}, -z_\alpha]$ is obtained by jointly considering the Cornish-Fisher expansion of $k_{\theta_0,n}$, the Taylor expansion of $\sqrt{n}[\mu_0 - \mu(\theta_0 + \frac{x+z_\alpha}{\sigma_0\sqrt{n}})]/\sigma_0$ and the Taylor expansion of $\sigma_0/\sigma(\theta_0 + \frac{x+z_\alpha}{\sigma_0\sqrt{n}})$.
3. Write the CDF of $\bar{X}$ in the form of $P_\theta[\sqrt{n}\frac{\bar{X}-\mu(\theta)}{\sigma(\theta)} \le u]$. Formally substitute $\theta = \theta_0 + \frac{x+z_\alpha}{\sigma_0\sqrt{n}}$ and $u = \tilde{k}_{\theta_0,x,n}$ in the Edgeworth expansion of the CDF of $\bar{X}$. An expansion of $\tilde{\beta}_n(\theta_0 + \frac{x+z_\alpha}{\sigma_0\sqrt{n}})$ is obtained by combining it with Taylor expansions for a number of relevant functions (see (47)).
4. The expansion of $A_{n,\theta_{1n}}$ is obtained by considering the product of the expansions of $g(\theta_0 + \frac{x+z_\alpha}{\sigma_0\sqrt{n}})$ and $\tilde{\beta}_n(\theta_0 + \frac{x+z_\alpha}{\sigma_0\sqrt{n}})$ under the integral sign.
5. Finally prove that all the error terms in Steps 1, 2, 3 and 4 are smaller order terms.

We give the expansions in steps 1, 2, 3 and 4 in detail. For the error term study in step 5, we omit the details due to the considerably tedious algebra.

Step 1: The expansion of $g(\theta_0 + \frac{x+z_\alpha}{\sqrt{n}})$ is easily obtained by using a Taylor expansion:

$$g(\theta_0 + \frac{x+z_\alpha}{\sigma_0\sqrt{n}}) = g(\theta_0) + g'(\theta_0)\frac{x+z_\alpha}{\sigma_0\sqrt{n}} + \frac{g''(\theta_0)}{2}\frac{(x+z_\alpha)^2}{\sigma_0^2 n} + r_{g,x,n}. \tag{41}$$

where $r_{g,x,n}$ is the error term.

Step 2: The Cornish–Fisher expansion of $k_{\theta_0,n}$ ([1], p. 117) is given by

$$k_{\theta_0,n} = z_\alpha + \frac{(z_\alpha^2 - 1)\rho_{30}}{6\sqrt{n}} + \frac{1}{n}\left[\frac{(z_\alpha^3 - 3z_\alpha)\rho_{40}}{24} - \frac{(2z_\alpha^3 - 5z_\alpha)\rho_{30}^2}{36}\right] + r_{1,n}, \tag{42}$$

where $r_{1,n}$ is the error term.

The Taylor expansion of the first term inside the bracket of (40) is

$$-(x+z_\alpha) - \frac{\rho_{30}(x+z_\alpha)^2}{2\sqrt{n}} - \frac{\rho_{40}(x+z_\alpha)^3}{6n} + r_{2,x,n} \tag{43}$$

and the Taylor expansion of the term outside of the bracket of (40) is

$$1 - \frac{\rho_{30}(x+z_\alpha)}{2\sqrt{n}} + \frac{1}{n}\left(\frac{3\rho_{30}^2}{8} - \frac{\rho_{40}}{4}\right)(x+z_\alpha)^2 + r_{3,x,n}, \tag{44}$$

where $r_{2,x,n}$ and $r_{3,x,n}$ are error terms.

Plugging (42), (43) and (44) into (40), we get the expansion of $\tilde{k}_{\theta_0,x,n}$ below:

$$\tilde{k}_{\theta_0,x,n} = -x + \frac{1}{\sqrt{n}}f_1(x) + \frac{1}{n}f_2(x) + r_{4,x,n}, \tag{45}$$

where $r_{4,x,n}$ is the error term, $f_1(x) = f_{11}x + f_{10}$ and $f_2(x) = f_{23}x^3 + f_{22}x^2 + f_{21}x + f_{20}$, and the coefficients for $f_1(x)$ and $f_2(x)$ are $f_{10} = -(2z_\alpha^2 + 1)\rho_{30}/6$, $f_{11} = -z_\alpha\rho_{30}/2$, $f_{20} = (z_\alpha^3 + 2z_\alpha)\rho_{30}^2/9 - (z_\alpha^3 + z_\alpha)\rho_{40}/8$, $f_{21} = (7z_\alpha^2/24 + 1/12)\rho_{30}^2 - z_\alpha^2\rho_{40}/4$, $f_{22} = 0$, $f_{23} = \rho_{40}/12 - \rho_{30}^2/8$.

Step 3: The Edgeworth expansion of the CDF of $\bar{X}$ is (Barndorff-Nielsen and Cox ([1], p. 91) and Hall ([11], p. 45)) given below:

$$
\begin{aligned}
&P_{\theta_0+\frac{x+z_\alpha}{\sigma_0\sqrt{n}}}\left(\sqrt{n}\frac{\bar{X}-\mu(\theta_0+\frac{x+z_\alpha}{\sigma_0\sqrt{n}})}{\sqrt{\sigma(\theta_0+\frac{x+z_\alpha}{\sigma_0\sqrt{n}})}} \le u\right)\\
&\quad =\Phi(u)-\frac{\phi(u)}{\sqrt{n}}\frac{(u^2-1)}{6}\rho_3(\theta_0+\frac{x+z_\alpha}{\sigma_0\sqrt{n}})-\frac{\phi(u)}{n}[\frac{(u^3-3u)}{24}\\
&\quad\quad \times\rho_4(\theta_0+\frac{x+z_\alpha}{\sigma_0\sqrt{n}})+\frac{(u^5-10u^3+15u)}{72}\rho_3^2(\theta_0+\frac{x+z_\alpha}{\sigma_0\sqrt{n}})]+r_{5,n},
\end{aligned} \tag{46}
$$

where $r_{5,n}$ is an error term. If we take $\mu=\tilde{k}_{\theta_0,x,n}$ in (46), then the left side is $1-\tilde{\beta}_n(\theta_0+\frac{x+z_\alpha}{\sqrt{n}})$ and so

$$
\begin{aligned}
\tilde{\beta}(\theta_0+\frac{x+z_\alpha}{\sigma_0\sqrt{n}}) &= \Phi(-\tilde{k}_{\theta_0,x,n})+\frac{\phi(\tilde{k}_{\theta_0,x,n})}{\sqrt{n}}\frac{(\tilde{k}^2_{\theta_0,x,n}-1)}{6}\rho_3(\theta_0+\frac{x+z_\alpha}{\sigma_0\sqrt{n}})\\
&\quad+\frac{\phi(\tilde{k}_{\theta_0,x,n})}{n}[\frac{(\tilde{k}^3_{\theta_0,x,n}-3\tilde{k}_{\theta_0,x,n})}{24}\rho_4(\theta_0+\frac{x+z_\alpha}{\sigma_0\sqrt{n}})\\
&\quad+\frac{(\tilde{k}^5_{\theta_0,x,n}-10\tilde{k}^3_{\theta_0,x,n}+15\tilde{k}_{\theta_0,x,n})}{72}\rho_3^2(\theta_0+\frac{x+z_\alpha}{\sigma_0\sqrt{n}})]-r_{5,n}.
\end{aligned} \tag{47}
$$

Plug the Taylor expansion of $\rho_3(\theta_0+\frac{x+z_\alpha}{\sigma_0\sqrt{n}})$

$$
\rho_3(\theta_0+\frac{x+z_\alpha}{\sigma_0\sqrt{n}})=\rho_{30}+\frac{(x+z_\alpha)}{\sqrt{n}}\left(\rho_{40}-\frac{3}{2}\rho_{30}^2\right)+r_{6,x,n} \tag{48}
$$

in (47), where $r_{6,x,n}$ is an error term, and then consider the Taylor expansions of the three terms related to $\tilde{k}_{\theta_0,x,n}$ in (47) and also use the expansion (45). On quite a bit of calculations, we obtain the following expansion:

$$
\begin{aligned}
\tilde{\beta}_n(\theta_0+\frac{x+z_\alpha}{\sigma_0\sqrt{n}}) &= \Phi(x)-\phi(x)\left[\frac{f_1(x)}{\sqrt{n}}+\frac{f_2(x)}{n}\right]-x\phi(x)\left[\frac{f_1(x)}{\sqrt{n}}\right]^2\\
&\quad+\frac{\phi(x)(x^2-1)}{6\sqrt{n}}[\rho_{30}+\frac{(x+z_\alpha)}{\sqrt{n}}(\rho_{40}-\frac{3}{2}\rho_{30}^2)]+\frac{\rho_{30}}{6n}\phi(x)(x^3-3x)f_1(x)\\
&\quad+\frac{\phi(x)}{n}[\frac{(x^3-3x)}{24}\rho_{40}+\frac{(x^5-10x^3+15x)}{72}\rho_{30}^2]+r_{7,x,n}\\
&=\Phi(x)+\frac{\phi(x)}{\sqrt{n}}g_1(x)+\frac{\phi(x)}{n}g_2(x)+r_{7,x,n},
\end{aligned} \tag{49}
$$

where $r_{7,x,n}$ is an error term, $g_1(x)=g_{12}x^2+g_{11}x+g_{10}$, $g_2(x)=g_{20}+g_{21}x+g_{22}x^2+g_{23}x^3+g_{24}x^4+g_{25}x^5$, and the coefficients of $g_1(x)$ and $g_2(x)$ are $g_{12}=\rho_{30}/6$, $g_{11}=z_\alpha\rho_{30}/2$, $g_{10}=z_\alpha^2\rho_{30}/3$, $g_{25}=\rho_{30}^2/72$, $g_{24}=-z_\alpha\rho_{30}^2/12$, $g_{23}=\rho_{40}/8-13z_\alpha^2\rho_{30}^2/72-7\rho_{30}^2/24$, $g_{22}=z_\alpha\rho_{40}/6-z_\alpha^3\rho_{30}^2/6-z_\alpha\rho_{30}^2/12$, $g_{21}=(z_\alpha^2/4-7/24)\rho_{40}-z_\alpha^4\rho_{30}^2/18-13z_\alpha^2\rho_{30}^2/72+4\rho_{30}^2/9$, $g_{20}=(z_\alpha^3/8-z_\alpha/24)\rho_{40}-(z_\alpha^3/9-z_\alpha/36)\rho_{30}^2$.

Step 4: The expansion of $A_{n,\theta_{1n}}$ is obtained by plugging the expansions of $\tilde{\beta}(\theta_0+\frac{x+z_\alpha}{\sigma_0\sqrt{n}})$ and $g(\theta_0+\frac{x+z_\alpha}{\sigma_0\sqrt{n}})$. On careful calculations,

$$
A_{n,\theta_{1n}}=\frac{a_1}{\sqrt{n}}+\frac{a_2}{n}+\frac{a_3}{n^{3/2}}+r_{8,n}, \tag{50}
$$

where $r_{8,n}$ is an error term, $a_1 = (g(\theta_0)/\sigma_0)[\phi(z_\alpha) - \alpha z_\alpha]$, $a_2 = \rho_{30} g(\theta_0)[\alpha + 2\alpha z_\alpha^2 - 2z_\alpha \phi(z_\alpha)]/(6\sigma_0) - g'(\theta_0)[\alpha(z_\alpha^2+1) - z_\alpha\phi(z_\alpha)]/(2\sigma_0^2)$, and $a_3 = [h_{11}\phi(z_\alpha) + \alpha h_{12}][(g''(\theta_0)/(6\sigma_0^3)] + [h_{21}\phi(z_\alpha) + \alpha h_{22}][g'(\theta_0)/\sigma_0^2] + [h_{31}\phi(z_\alpha) + \alpha h_{32}][g(\theta_0)/\sigma_0]$, where $h_{11} = z_\alpha^2 + 2$, $h_{12} = -(z_\alpha^3 + 3z_\alpha)$, $h_{21} = -(\rho_{30}/3)(z_\alpha^2+1)$, $h_{22} = (\rho_{30}/3)(z_\alpha^3 + 2z_\alpha)$, $h_{31} = -z_\alpha^4\rho_{30}^2/36 + 4z_\alpha^2\rho_{30}^2/9 + \rho_{30}^2/36 - 5z_\alpha^2\rho_{40}/24 + \rho_{40}/24$, $h_{32} = -5z_\alpha^3\rho_{30}^2/18 - 11z_\alpha\rho_{30}^2/36 + z_\alpha^3\rho_{40}/8 + z_\alpha\rho_{40}/8$. These a_1, a_2 and a_3 are the coefficients in the expansion of (23).

The computation of the coefficients of the expansions of $A_{n,\theta_{1n}}$ is now complete. The rest of the work is to prove that all the error terms are smaller order terms. But first we give the results for the expansion of $\tilde{A}_{n,\theta_{2n}}$. The details for the expansions of $\tilde{A}_{n,\theta_{2n}}$ are omitted.

Expansion of $\tilde{A}_{n,\theta_{2n}}$: The expansion of $\tilde{A}_{n,\theta_{2n}}$ can be obtained similarly by simply repeating all the steps for $A_{n,\theta_{1n}}$. The results are given below:

$$\tilde{A}_{n,\theta_{2n}} = \frac{\tilde{a}_1}{\sqrt{n}} + \frac{\tilde{a}_2}{n} + \frac{\tilde{a}_3}{n^{3/2}} + r_{9,n}, \tag{51}$$

where $r_{9,n}$ is an error term, $\tilde{a}_1 = g(\theta_0)[\phi(z_\alpha) + (1-\alpha)z_\alpha]/\sigma_0$, $\tilde{a}_2 = g'(\theta_0)[(1-\alpha)(z_\alpha^2+1) + z_\alpha\phi(z_\alpha)]/(2\sigma_0^2) - \rho_{30}g(\theta_0)[(1-\alpha)/6 + (1-\alpha)z_\alpha^2/3 + z_\alpha\phi(z_\alpha)/3]/\sigma_0$, and $\tilde{a}_3 = (g''(\theta_0)/6\sigma_0^3)[h_{11}\phi(z_\alpha) - (1-\alpha)h_{12}] - (g'(\theta_0)/\sigma_0^2)[-h_{21}\phi(z_\alpha) + (1-\alpha)h_{22}] - (g(\theta_0)/\sigma_0)[-h_{31}\phi(z_\alpha) + (1-\alpha)h_{32}]$, where h_{11}, h_{12}, h_{21}, h_{22}, h_{31} and h_{32} are the same as defined in Step 3. These $\tilde{a}_1$, $\tilde{a}_2$ and $\tilde{a}_3$ are the coefficients in the expansion of (24).

Remark. The coefficients of expansions of δ_n and ϵ_n are obtained by simply using formula (7) with a_1, a_2 and a_3 in (23) and also the coefficient $\tilde{a}_1$, $\tilde{a}_2$ and $\tilde{a}_3$ in (24) respectively.

Step 5: (Error term study in the expansions of $A_{n,\theta_{1n}}$). We only give the main steps because the details are too long. Recall from equation (38) that the range of integration corresponding to $A_{n,\theta_{1n}}$ is $x_{1n} \le x \le -z_\alpha$. In this case, we have $\lim_{n\to\infty} x_{1n} = \infty$ and $\lim_{n\to\infty} x_{1n}/\sqrt{n} = -z_\alpha$. This fact is used when we prove the error term is still a smaller order term when we move it out of the integral sign.

(I) In (41), since $g^{(3)}(\theta)$ is uniformly bounded in absolute values, $r_{g,x,n}$ is absolutely bounded by a constant times $n^{-3/2}(x+z_\alpha)^2$

(II) From Barndorff-Nielsen and Cox [4.5, pp 117], the error term $r_{1,n}$ in (42) is absolutely uniformly bounded by a constant times $n^{-3/2}$.

(III) In (43) and (44), since $\rho_i(\theta)$ and $\kappa_i(\theta)$ $(i = 3,4,5)$ are uniformly bounded in absolute values, the error term $r_{2,x,n}$ is absolutely bounded by a constant times $n^{-3/2}(x+z_\alpha)^4$ and the error term $r_{3,x,n}$ is absolutely bounded by a constant times $n^{-3/2}(x+z_\alpha)^3$.

(IV) The exact form of the error term $r_{4,x,n}$ in (45) can be derived by considering the higher order terms and their products in (42), (43) and (44) for the derivation of expression (45). The computation is complicated but straightforward. However, still, since $\rho_i(\theta)$ and $\kappa_i(\theta)$ $(i = 3,4,5)$ are uniformly bounded in absolute values, $r_{4,x,n}$ is absolutely bounded by $n^{-3/2}P_1(|x|)$, where $P_1(|x|)$ is a seventh degree polynomial and its coefficients do not depend on n.

(V) Again, from Barndorff-Nielsen and Cox ([1], p. 91), the error term $r_{5,n}$ in (46) is absolutely bounded by a constant times $n^{-3/2}$.

(VI) The error term $r_{6,x,n}$ in (48) is absolutely bounded by a constant times $n^{-1}(x+z_\alpha)^2$ since $\rho_i(\theta)$ and $\kappa_i(\theta)$ $(i = 3,4,5)$ are uniformly bounded in absolute values.

(VII) This is the critical step for the error term study since we need to prove that the error term is still a smaller order term when it is moved out of the integral in (50). We need to study the behaviors of $\Phi(-\tilde{k}_{\theta_0,x,n})$ and $\phi(\tilde{k}_{\theta_0,x,n})$ as $n \to \infty$ for all $x \in [x_{1n}, -z_\alpha]$ uniformly (see (49) in detail). This also explains why we choose $\theta_{1n} = \theta_0 + (\theta_1 - \theta_0)/n^{1/3}$ and $x_{1n} = \sigma_0\sqrt{n}(\theta_{1n} - \theta_0) - z_\alpha$ at the beginning of this section, since in this case $|\tilde{k}_{\theta_0,x,n} + x|$ is uniformly bounded by $|x|/2 + 1$ for a sufficiently large n. Then for sufficiently large n, the error term $|r_{7,x,n}|$ in (49) is uniformly bounded by $|r_{7,x,n}| \le \phi(x/2+1)P_2(|x|)$ where $P_2(|x|)$ is a twelveth degree polynomial of $|x|$ and its coefficients do not depend on n.

(VIII) Finally, we can show that the error term $r_{8,n}$ in (50) in $O(n^{-2})$. This is tedious but straightforward. It is proven by considering each of the ten terms in $r_{8,n}$ separately.

Remark. We can similarly prove that the error term $r_{9,n}$ in (51) corresponding to $\tilde{A}_{n,\theta_{2n}}$ is $O(n^{-2})$. Since the steps are very similar, we do not mention them.

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