ON SOME CONNECTIONS BETWEEN PROBABILITY THEORY AND DIFFERENTIAL AND INTEGRAL EQUATIONS

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1. Introduction

The connections between probability theory on the one hand and differential and integral equations on the other, are so numerous and diverse that the task of presenting them in a comprehensive and connected manner appears almost impossible.

The aim of this paper is consequently much more modest. We wish mainly to illustrate, on a variety of examples, the interplay between probability problems and the analytic tools used to approach and solve them.

In the traditional approach a probability problem is often reduced to solving a differential or integral equation, and once this has been accomplished we find ourselves outside the field of probability and in a domain where methods of long standing are immediately available. This approach is far from being exhausted. We illustrate it by deriving the so called 'arc sin law' (sections 2 and 3) and certain limiting distributions arising in the study of deviations between theoretical and empirical distributions (section 6). These illustrations are taken from the domain of attraction of the normal law and we are naturally led to the diffusion theory. An attempt to extend the diffusion theory to the case when the normal law is replaced by more general limit laws is made in sections 7 and 8. We are led here to integro-differential equations which offer formidable analytic difficulties and which we were able to solve only in very few cases.

The remainder of the paper is devoted to the reversal of the traditional approach. This reversal consists in an attempt to utilize the probability theory, both rigorously and heuristically, to arrive at results about differential and integral equations. It is in this part that the interplay mentioned above is brought out most clearly.

As an example of the interplay let us mention the following results which are discussed in detail in section 10.

Let Ω be a bounded three dimensional region and let $y \in \Omega$. Let $T \equiv T_{\Omega}(y)$ be the total time that a Brownian particle starting from y spends in Ω.

It can then be shown that, as $\beta \to \infty$,

$$Pr\{T > \beta\} \sim C(y)\, e^{-\beta/\lambda_1}, \tag{1.1}$$

This paper was prepared in connection with the work on a project sponsored by the Office of Naval Research.

where λ_1 is the largest eigenvalue of the integral equation

$$\frac{1}{2\pi}\int_{\Omega}\frac{\varphi(\varrho)}{|r-\varrho|}\,d\varrho=\lambda\varphi(r) \tag{1.2}$$

and $C(y)$ is expressible in terms of the eigenfunction (or eigenfunctions) belonging to λ_1.

An examination of the method which leads to (1.1) reveals also that, for $y \in \Omega$,

$$1=\lim_{\delta\downarrow 0}\sum_{j=1}^{\infty}\frac{\lambda_j}{\lambda_j+\delta}\,\varphi_j(y)\int_{\Omega}\varphi_j(r)\,dr\,, \tag{1.3}$$

where the λ_j and φ_j are the eigenvalues and normalized eigenfunctions of (1.2). The equality (1.3) expresses the purely analytic fact that the expansion of 1 in a series of orthonormal functions $\varphi_j(y)$ is summable to 1 for every $y \in \Omega$. This fact emerges, however, as a trivial consequence of continuity of Brownian motion or, more precisely, of the fact that a Brownian particle starting at $y \in \Omega$ cannot leave Ω without spending a *positive time* inside Ω.

It is true, of course, that (1.3) is devoid of any general interest but the method by which it is derived can be applied to many other cases. In particular, the classical result of Weyl concerning the distribution of eigenvalues of the Laplace operator [section 9] and a somewhat weakened version of the so called WKB method section 5 can be thus obtained.

2. Limit theorems and functionals

Let $X_1, X_2, \ldots$ be independent random variables each having mean 0 and variance 1 and such that the central limit theorem is applicable. Let $s_k = X_1 + X_2 + \ldots + X_k$ and let $V(x)$ be a nonnegative function of the real variable x. Let furthermore $x(\tau)$, $[x(0) = 0, 0 \leqq \tau < \infty]$ be the Wiener process (Gaussian process with independent increments). Under comparatively mild assumptions on $V(x)$[1] it can be shown that

$$\lim_{n\to\infty} Pr\left\{\frac{1}{n}\sum_{k\leqq nt} V\left(\frac{s_k}{\sqrt{n}}\right)<\alpha\right\}=Pr\left\{\int_0^t V[x(\tau)]\,d\tau<\alpha\right\}. \tag{2.1}$$

Thus limiting distributions can be obtained by calculating probability distribution of Wiener functionals.

It may seem that limit theorems of type (2.1) are of a somewhat artificial nature but it can be easily seen that results of definite probabilistic interest can be obtained by specializing $V(x)$. For instance if

$$V(x)=\frac{1+\operatorname{sgn} x}{2} \tag{2.2}$$

and $t = 1$, the sum

$$N_n=\sum_{k=1}^{n} V\left(\frac{s_k}{\sqrt{n}}\right)$$

[1] The best conditions on $V(x)$ can be inferred from a general result of Dr. M. D. Donsker and will be found in a paper soon to be published in the Memoirs of the American Mathematical Society.

represents the number of positive sums among $s_1, s_2, \ldots, s_n$. The statistical behavior of N_n/n is quite curious and the result is

$$\lim_{n\to\infty} Pr\left\{\frac{N_n}{n} < \alpha\right\} = \frac{2}{\pi} \text{arc sin} \sqrt{\alpha}. \tag{2.3}$$

In particular, it follows that $\alpha = \frac{1}{2}$ is the *least likely* proportion of positive sums.

3. Functionals and differential equations

The calculation of distribution functions

$$Pr\left\{\int_0^t V[x(\tau)]\,d\tau < \alpha\right\} = \sigma(\alpha; t) \tag{3.1}$$

can be reduced to solving an appropriate differential equation. To see how this is done we restrict ourselves to the simple case,

$$0 \leqq V(x) \leqq M. \tag{3.2}$$

Define functions $Q_n(x, t)$ as follows:

$$Q_0(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}, \tag{3.3}$$

$$Q_{n+1}(x, t) = \int_0^t \int_{-\infty}^{\infty} \frac{e^{-(x-\xi)^2/2(t-\tau)}}{\sqrt{2\pi(t-\tau)}} V(\xi) Q_n(\xi, \tau)\, d\xi\, d\tau. \tag{3.4}$$

It is easily checked that

$$\mu_n = E\left\{\left(\int_0^t V[x(\tau)]\,d\tau\right)^n\right\} = n! \int_{-\infty}^{\infty} Q_n(x, t)\,dx \tag{3.5}$$

and

$$0 \leqq Q_n(x, t) \leqq \frac{M^n}{n!} t^n Q_0(x, t). \tag{3.6}$$

Let now

$$Q(x, t) = \sum_{n=0}^{\infty} (-1)^n u^n Q_n(x, t). \tag{3.7}$$

The series clearly converges for all x, u and $t \neq 0$ and moreover

$$|Q(x, t)| < e^{uMt} Q_0(x, t).$$

Due to (3.3) and (3.4) we have

$$Q(x, t) + \frac{u}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} \frac{e^{-(x-\xi)^2/2(t-\tau)}}{\sqrt{t-\tau}} V(\xi) Q(\xi, \tau)\, d\xi\, d\tau = Q_0(x, t). \tag{3.8}$$

If instead of expectations (3.5) we consider expectations

$$\mu_n(a, b) = E\left\{\left(\int_0^t V[x(\tau)]\,d\tau\right)^n, \quad a < x(t) < b\right\}, \tag{3.9}$$

[that is, we consider integrals not over the whole Wiener space but over that portion of it where $a < x(t) < b$], we get just as easily

$$\mu_n(a, b) = n! \int_a^b Q_n(x, t)\,dx. \tag{3.10}$$

Now, it is clear that

$$(3.11)\qquad E\left\{e^{-u\int_0^t V[x(\tau)]d\tau},\ a < x(t) < b\right\} = \int_a^b Q(x, t)\, dx,$$

and since $V(x) \geqq 0$ it follows that $Q(x, t)$ is a *decreasing* function of u. In particular,

$$Q(x, t) \leqq Q_0(x, t)$$

and thus the Laplace transform

$$(3.12)\qquad \Psi(x) \equiv \Psi(x, s) = \int_0^\infty e^{-st} Q(x, t)\, dt, \qquad s > 0,$$

exists.

If we take the Laplace transform of both sides of equation (3.8) we obtain

$$(3.13)\qquad \Psi(x) + \frac{u}{\sqrt{2s}} \int_{-\infty}^{\infty} e^{-\sqrt{2s}|x-\xi|} V(\xi) \Psi(\xi)\, d\xi = \frac{e^{-\sqrt{2s}|x|}}{\sqrt{2s}}.$$

The integral equation (3.13) is equivalent to the differential equation

$$(3.14)\qquad \tfrac{1}{2}\Psi'' - [s + uV(x)]\Psi = 0$$

and the conditions

$$(3.15)\ \text{(a)}\qquad \Psi \to 0 \quad \text{as} \quad x \to \pm\infty$$

$$\text{(b)}\qquad \Psi' \text{ continuous except at } x = 0$$

$$\text{(c)}\qquad \Psi'(-0) - \Psi'(+0) = 2.$$

This derivation may break down for unbounded $V(x)$ (the moments μ_n, for instance, may fail to exist). A derivation valid for unbounded $V(x)$ was given by the author in [1]. Recently Dr. M. Rosenblatt extended the derivation presented here by first truncating $V(x)$ and then passing to the limit.

The machinery for calculating limiting distributions of type (2.1) is thus established. We illustrate it by taking the example

$$(3.16)\qquad V(x) = \frac{1 + \operatorname{sgn} x}{2}$$

discussed in section 2.

We obtain

$$\Psi(x) = \frac{\sqrt{2}}{\sqrt{s+u} + \sqrt{s}}\, e^{-\sqrt{2(s+u)}x}, \qquad x \geqq 0$$

$$\Psi(x) = \frac{\sqrt{2}}{\sqrt{s+u} + \sqrt{s}}\, e^{\sqrt{2s}x}, \qquad x < 0.$$

Thus

$$\frac{1}{\sqrt{s(s+u)}} = \int_{-\infty}^{\infty} \Psi(x)\, dx = \int_{-\infty}^{\infty}\int_0^{\infty} e^{-st} Q(x, t)\, dt dx$$

$$= \int_0^\infty e^{-st} \int_0^\infty e^{-u\alpha} d_\alpha \sigma(\alpha; t)\, dt.$$

Inverting with respect to s and u we obtain easily

$$\sigma(\alpha; t) = \frac{2}{\pi} \arcsin \sqrt{\frac{\alpha}{t}}.$$

4. Application to the 'ruin' problem

Let $X_1 X_2, \ldots$ be independent random variables satisfying the conditions stated in section 2. It can be shown [2] that regardless of the distributions of the X's

$$(4.1)\qquad \lim_{n\to\infty} Pr\{\max_{1\leqq k\leqq nt} s_k < a\sqrt{n}\} = Pr\{\max_{0\leqq\tau\leqq t} x(\tau) < a\}.$$

The probability $Pr\{\max_{0\leqq\tau\leqq t} x(\tau) < a\}$ can be calculated by invoking the theory of section 3.

Let

$$(4.2)\qquad V_a(x) = \begin{cases} 0, & x < a \\ 1, & x > a \end{cases}$$

and note that

$$(4.3)\qquad \lim_{u\to\infty} E\left\{e^{-u\int_0^t V_a[x(\tau)]d\tau}\right\} = Pr\{\max_{0\leqq\tau\leqq t} x(\tau) < a\}.$$

By solving (3.14) for this choice of $V(x)$ we obtain

$$(4.4)\qquad \int_0^\infty e^{-st} E\left\{e^{-u\int_0^t V_a[x(\tau)]d\tau}\right\} dt = \frac{1}{s}\left\{1 - e^{-a\sqrt{2s}}\left(1 - \frac{\sqrt{s}}{\sqrt{s+u}}\right)\right\}.$$

Letting $u \to \infty$ and using (4.3) we obtain

$$\int_0^\infty e^{-st} Pr\{\max_{0\leqq\tau\leqq t} x(\tau) < a\} dt = \frac{1}{s}\{1 - e^{-a\sqrt{2s}}\}$$

and inverting with respect to s we get the classical result

$$(4.5)\qquad Pr\{\max_{0\leqq\tau\leqq t} x(\tau) < a\} = \frac{2}{\sqrt{2\pi}}\int_0^{a/\sqrt{t}} e^{-u^2/2} du.$$

If we define $V_{a,b}(x)$ by setting

$$V_{a,b}(x) = \begin{cases} 0, & -b < x < a, \\ 1, & x < -b,\ x > a \end{cases} \qquad a > 0,\ b > 0,$$

we can in a similar way calculate

$$Pr\{-b \leqq \min_{0\leqq\tau\leqq t} x(\tau) \leqq \max_{0\leqq\tau\leqq t} x(\tau) < a\}$$

which corresponds to the random walk with two absorbing barriers.

5. An application to differential equations

If $V(x) \to \infty$, as $x \to \pm\infty$, and satisfies a few additional assumptions [3] the eigenvalue problem,

$$(5.1)\qquad \tfrac{1}{2}\Psi'' - V(x)\Psi = -\lambda\Psi,$$

where $\Psi \in L^2(-\infty, \infty)$ yields a discrete spectrum. Let

$$\lambda_1, \lambda_2, \ldots$$

be the eigenvalues and

$$\Psi_1(x), \quad \Psi_2(x), \ldots,$$

the corresponding normalized eigenfunctions. The solution of (3.14) satisfying (3.15) can be written in the form

$$\Psi(x) \sim \sum_1^\infty \frac{\Psi_j(0)\Psi_j(x)}{s+\lambda_j}. \tag{5.2}$$

(We now set $u = 1$.) Thus

$$\int_a^b \Psi(x)\,dx = \sum_1^\infty \frac{\Psi_j(0)\int_a^b \Psi_j(x)\,dx}{s+\lambda_j}. \tag{5.3}$$

Inverting with respect to s, we get

$$\int_a^b Q(x,t)\,dx = \sum_1^\infty e^{-\lambda_j t}\Psi_j(0)\int_a^b \Psi_j(x)\,dx. \tag{5.4}$$

Thus

$$E\left\{e^{-\int_0^t V[x(\tau)]d\tau},\ a < x(t) < b\right\} = \sum_1^\infty e^{-\lambda_j t}\Psi_j(0)\int_a^b \Psi_j(x)\,dx. \tag{5.5}$$

Dividing by $(b-a)$ and letting $b \to a$ we get, introducing conditional expectations,

$$\frac{e^{-a^2/2t}}{\sqrt{2\pi t}}E\left\{e^{-\int_0^t V[x(\tau)]d\tau} \mid x(t) = a\right\} = \sum_1^\infty e^{-\lambda_j t}\Psi_j(0)\Psi_j(a). \tag{5.6}$$

The appearance of $\Psi_j(0)$ on the right hand side is due to the normalization $x(0) = 0$. By a slight extension of the above argument we get

$$\frac{e^{-(a-\xi)^2/2t}}{\sqrt{2\pi t}}E\left\{e^{-\int_0^t V[x(\tau)+\xi]d\tau} \mid x(t) = a-\xi\right\} = \sum_1^\infty e^{-\lambda_j t}\Psi_j(a)\Psi_j(\xi). \tag{5.7}$$

Setting $a = \xi$ we obtain

$$\frac{1}{\sqrt{2\pi t}}E\left\{e^{-\int_0^t V[x(\tau)+\xi]d\tau} \mid x(t) = 0\right\} = \sum_1^\infty e^{-\lambda_j t}\Psi_j^2(\xi). \tag{5.8}$$

It can be shown that, as $t \to 0$,

$$E\left\{e^{-\int_0^t V[x(\tau)+\xi]d\tau} \mid x(t) = 0\right\} \to 1 \tag{5.9}$$

and consequently, as $t \to 0$,

$$\sum_1^\infty e^{-\lambda_j t}\Psi_j^2(\xi) \sim \frac{1}{\sqrt{2\pi t}}.$$

Applying the Hardy-Littlewood Tauberian theorem we obtain

$$\sum_{\lambda_j<\lambda}\Psi_j^2(\xi) \sim \frac{\sqrt{2}}{\pi}\sqrt{\lambda}, \qquad \lambda \to \infty. \tag{5.10}$$

If we integrate (5.8) with respect to ξ from $-\infty$ to ∞ we get

$$\sum_1^\infty e^{-\lambda_j t} = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^\infty E\left\{e^{-\int_0^t V[x(\tau)+\xi]d\tau} \mid x(t) = 0\right\}d\xi. \tag{5.11}$$

Arguing heuristically, we see that as $t \to 0$, $x(\tau)$ being tied down by the conditions

$$x(0) = x(t) = 0,$$

is extremely unlikely to become appreciable in the interval $(0, t)$. One is thus led to the approximation

$$\int_{-\infty}^{\infty} E\left\{e^{-\int_0^t V[x(\tau)+\xi]d\tau} \mid x(t) = 0\right\} d\xi \sim \int_{-\infty}^{\infty} e^{-tV(\xi)} d\xi$$

and hence

$$\sum_1^{\infty} e^{-\lambda_j t} \sim \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-tV(\xi)} d\xi, \qquad t \to 0. \tag{5.12}$$

If we write

$$\frac{1}{\sqrt{2\pi t}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t\eta^2/2} d\eta$$

and denote by $B(\lambda)$ the area of the region

$$\frac{\eta^2}{2} + V(\xi) \leqq \lambda$$

we can rewrite (5.12) in the form

$$\sum_1^{\infty} e^{-\lambda_j t} \sim \frac{1}{2\pi} \int_0^{\infty} e^{-\lambda t} dB(\lambda). \tag{5.13}$$

Assuming enough conditions of $V(x)$ so as to be able to apply again the Hardy-Littlewood Tauberian theorem we would be led to

$$\sum_{\lambda_j < \lambda} 1 \sim \frac{1}{2\pi} B(\lambda), \qquad \lambda \to \infty. \tag{5.14}$$

The result (5.14) is a weak version of the theorem (proved usually by the application of the so called WKB method) that for high quantum numbers the 'old' and 'new' quantum theories are asymptotically equivalent.

We considered it noteworthy however to point out that this equivalence is, at least heuristically, a consequence of the intuitively trivial fact that for small t the Wiener process $x(\tau)$ tied down by the conditions $x(0) = x(t) = 0$ is not likely to become appreciable in the interval $(0, t)$.

We shall see in the sequel that a similar heuristic argument leads to Weyl's classical theorem of the distribution of eigenvalues of the Laplace operator and that, moreover, the argument can be rendered precise.

From the formula (5.5) we also see (by putting $a = -\infty$, $b = +\infty$) that for $t \to \infty$

$$E\left\{e^{-\int_0^t V[x(\tau)]d\tau}\right\} \sim C e^{-\lambda_1 t}, \tag{5.15}$$

where

$$C = \sum_{\lambda_j = \lambda_1} \Psi_j(0) \int_{-\infty}^{\infty} \Psi_j(x)\, dx.$$

In other words

$$\lambda_1 = -\lim_{t\to\infty} \frac{\log E\left\{e^{-\int_0^t V[x(\tau)]d\tau}\right\}}{t}. \tag{5.16}$$

Thus the lowest eigenvalue can be expressed in terms of integrals over the Wiener space. It is curious that the classical variational expression

$$\lambda_1 = \min \frac{\int_{-\infty}^{\infty} \{\frac{1}{2}[\Psi'(x)]^2 + V(x)\Psi^2(x)\}dx}{\int_{-\infty}^{\infty}\Psi^2(x)\,dx} \tag{5.17}$$

does not seem to imply or be implied directly by (5.16).

The variational expression (5.17) yields as an important byproduct a numerical method (Raleigh-Ritz) for finding the lowest eigenvalue λ_1. Similarly the 'statistical' expression (5.16) can be made to yield a numerical method. The idea is to calculate

$$E\left\{e^{-\int_0^t V[x(\tau)]d\tau}\right\}. \tag{5.18}$$

by an appropriate sampling procedure. This can be accomplished by discretizing (5.18) and calculating by sampling

$$E\left\{e^{-1/n \sum_{k\leqq nt} V(s_k/\sqrt{n})}\right\}.$$

Initial trials [4] indicate that the method is feasible but much further study will be needed to fully test its practicality.

6. Applications to statistics

In comparing theoretical and empirical distributions one considers various measures of deviation. Let $X_1, X_2, \ldots, X_n$ be n independent random variables having the same continuous distribution $\sigma(\tau)$. The empirical distribution $\sigma_n(\tau)$ is defined by the formula

$$\sigma_n(\tau) = \frac{1}{n}\sum_{j=1}^{n} \varphi_\tau(X_j), \tag{6.1}$$

where

$$\varphi_\tau(x) = \begin{cases} 1, & x < \tau \\ 0, & x > \tau. \end{cases} \tag{6.2}$$

The measures of deviation most commonly considered are

$$D_n = \underset{-\infty<\tau<\infty}{\text{l.u.b.}} |\sigma_n(\tau) - \sigma(\tau)| \tag{6.3}$$

(Kolmogoroff [5]) and

$$\omega_n^2 = \int_{-\infty}^{\infty} [\sigma_n(\tau) - \sigma(\tau)]^2 d\sigma(\tau). \tag{6.4}$$

(Cramér [6], von Mises [7, pp. 316–336], Smirnoff [8]). Limiting distributions $\sqrt{n}D_n$ and $n\omega_n^2$ were obtained by various means.

A generalization of (6.4) is furnished by considering

$$\kappa_n = \int_{-\infty}^{\infty} V\{ \sqrt{n} [\sigma_n(\tau) - \sigma(\tau)] \} d\sigma(\tau), \tag{6.5}$$

for fairly general $V(x)$.

The distribution of D_n, ω_n^2 and κ_n are independent of $\sigma(\tau)$ [as long as $\sigma(\tau)$ is continuous] and we shall therefore consider the uniform distribution $\sigma(\tau) = \tau$, $0 \leqq \tau \leqq 1$. Set

$$y_n(\tau) = [\sigma_n(\tau) - \tau]\sqrt{n}. \tag{6.6}$$

Let $0 < \tau_1 < \tau_2 < \ldots < \tau_k \leqq 1$ and consider the joint distribution of

$$y_n(\tau_1), \ldots, y_n(\tau_k). \tag{6.7}$$

By the application of the multidimensional central limit theorem it follows that the joint distribution of the random variables (6.7) approaches, as $n \to \infty$, the joint distribution of the random variables

$$y(\tau_1), \ldots, y(\tau_k), \tag{6.8}$$

where $y(\tau)$ is a Gaussian process whose covariance function is

$$E\{y(s)\,y(t)\} = \min(s,t) - st. \tag{6.9}$$

It is worth remarking that also for each n

$$E\{y_n(s)\,y_n(t)\} = \min(s,t) - st. \tag{6.10}$$

It is intuitively appealing that

$$\lim_{n\to\infty} Pr\{\sqrt{n}\,D_n < \alpha\} = Pr\{\max_{0\leqq\tau\leqq 1} |y(\tau)| < \alpha\} \tag{6.11}$$

and that

$$\lim_{n\to\infty} Pr\{n\omega_n^2 < \alpha\} = Pr\left\{\int_0^1 y^2(\tau)\,d\tau < \alpha\right\}. \tag{6.12}$$

Doob [9] (see also [10]) who used this approach did not justify (6.11). The relatively intricate justification of this step was subsequently given by Dr. M. D. Donsker.[2]

The justification of (6.12) and the corresponding relationship for a general class of $V(x)$ is extremely simple. In fact, let

$$\begin{aligned} d_{n,k} &= \int_0^1 y_n^2(\tau)\,d\tau - \frac{1}{k}\sum_{j=1}^{k} y_n^2\left(\frac{j}{k}\right) \\ &= \sum_{j=1}^{k} \int_{j-1/k}^{j/k} \left[y_n^2(\tau) - y_n^2\left(\frac{j}{k}\right)\right] d\tau. \end{aligned} \tag{6.13}$$

By Schwarz's inequality and (6.1) we have for $\frac{j-1}{k} \leqq \tau \leqq \frac{j}{k}$,

$$E\left\{\left|y_n^2(\tau) - y_n^2\left(\frac{j}{k}\right)\right|\right\}$$

$$\leqq \sqrt{E\left\{\left[y_n(\tau) - y_n\left(\frac{j}{k}\right)\right]^2\right\}}\sqrt{E\left\{\left[y_n(\tau) + y_n\left(\frac{j}{k}\right)\right]^2\right\}} \leqq 2\sqrt{\frac{j}{k} - \tau}$$

[2] Donsker also points out that using (6.11) one can, by a process analogous to the one used in his paper cited in footnote 1, justify (6.12) and many more general cases.

and consequently

$$E\{\,|\,d_{n,k}|\,\}\leqq \frac{4}{3}\frac{1}{\sqrt{k}}. \tag{6.14}$$

It follows now almost immediately that

$$Pr\left\{\frac{1}{k}\sum_{j=1}^{k} y_n^2\left(\frac{j}{k}\right) < a-\epsilon\right\} - \frac{4}{3}\frac{1}{\epsilon\sqrt{k}} \leqq Pr\left\{\int_0^1 y_n^2(\tau)\,d\tau < a\right\}$$
$$\leqq Pr\left\{\frac{1}{k}\sum_{j=1}^{k} y_n^2\left(\frac{j}{k}\right) < a+\epsilon\right\} + \frac{4}{3}\frac{1}{\epsilon\sqrt{k}}.$$

Letting $n \longrightarrow \infty$, while keeping k and ϵ fixed, we get

$$Pr\left\{\frac{1}{k}\sum_{1}^{k} y^2\left(\frac{j}{k}\right) < a-\epsilon\right\} - \frac{4}{3}\frac{1}{\epsilon\sqrt{k}} \leqq \liminf_{n\to\infty} Pr\left\{\int_0^1 y_n^2(\tau)\,d\tau < a\right\}$$
$$\leqq \limsup_{n\to\infty} Pr\left\{\int_0^1 y_n^2(\tau)\,d\tau < a\right\} \leqq Pr\left\{\frac{1}{k}\sum_{1}^{k} y_n^2\left(\frac{j}{k}\right) < a+\epsilon\right\} + \frac{4}{3}\frac{1}{\epsilon\sqrt{k}}.$$

Letting $k \longrightarrow \infty$ and observing that $\epsilon > 0$ is arbitrary we get

$$\lim_{n\to\infty} Pr\left\{\int_0^1 y_n^2(\tau)\,d\tau < a\right\} = Pr\left\{\int_0^1 y^2(\tau)\,d\tau < a\right\},$$

for every a which is a continuity point of the distribution function of $\int_0^1 y^2(\tau)\,d\tau$. The fact that

$$\lim_{k\to\infty} Pr\left\{\frac{1}{k}\sum_{1}^{k} y^2\left(\frac{j}{k}\right) < \beta\right\} = Pr\left\{\int_0^1 y^2(\tau)\,d\tau < \beta\right\}$$

follows immediately by observing that the sample functions of the $y(\tau)$ process are continuous with probability 1. The easiest way to see this is to note, following Doob [9], that $y(\tau)$ is related to the Wiener process by the simple relation

$$y(\tau) = (\tau - 1)\,x\left(\frac{\tau}{1-\tau}\right), \quad 0 \leqq \tau < 1,\ y(1) = 0. \tag{6.15}$$

The calculation of the distribution function of $\int_0^1 V[y(\tau)]\,d\tau$ can be again reduced to the differential equation (3.14). Restricting oneself to the case of bounded $V(x)$ it is easy to check that [10]

$$E\left\{\left(\int_0^1 V[y(\tau)]\,d\tau\right)^n\right\} = n!\,\sqrt{2\pi}\,Q_n(0,1), \tag{6.16}$$

and consequently by inverting $\sqrt{2\pi}\Psi(0)$ with respect to s and setting $t = 1$ we obtain

$$E\left\{e^{-u\int_0^1 V[y(\tau)]d\tau}\right\}.$$

Inverting with respect to u we obtain the distribution function of $\int_0^1 V[y(\tau)]\,d\tau$. The result is still valid for a wide class of unbounded $V(x)$ although the proof is somewhat more involved. Let us illustrate this theory by considering $V(x) = x^2$ [see (6.12)]. Equation (3.14) assumes now the form

$$\tfrac{1}{2}\Psi'' - (s + ux^2)\Psi = 0 \tag{6.17}$$

and the Green's function satisfying (3.5) can be shown to be

$$\Psi(x) = \frac{1}{2a\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{D_n\left(\frac{x}{a}\right) D_n(0)}{n!\,[s + (n+\frac{1}{2})\sqrt{2u}]}, \tag{6.18}$$

where

$$a^2 = \frac{1}{2\sqrt{2u}}$$

and the $D_n(x)$ are the Hermite functions. We have

$$\Psi(0) = \frac{1}{2a\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{D^2(0)}{n!\,[s + (n+\frac{1}{2})\sqrt{2u}]}.$$

Inverting with respect to s and setting $t = 1$ we get

$$E\left\{e^{-\int_0^1 y^2(\tau)d\tau}\right\} = \frac{1}{2a} \sum_{n=0}^{\infty} \frac{D_n^2(0)}{n!} e^{-(n+1/2)\sqrt{2u}} \tag{6.19}$$

$$= \left(\frac{\sin h\sqrt{2u}}{\sqrt{2u}}\right)^{-1/2}.$$

Inversion with respect to u is somewhat tedious but can be accomplished by elementary means.

The special case $V(x) = x^2$ discussed above can be approached more elegantly by the use of integral equation. Consider the kernel

$$K(s, t) = \min(s, t) - st, \qquad 0 \leqq s, t \leqq 1, \tag{6.20}$$

and the eigenvalue problem

$$\int_0^1 K(s, t) f(t)\, dt = \lambda f(s). \tag{6.21}$$

The kernel is positive definite (being a covariance function) and hence all its eigenvalues are positive. Denote then by

$$\lambda_1, \lambda_2, \ldots$$

and the corresponding normalized eigenfunctions by

$$f_1(t), f_2(t), \ldots.$$

Let $G_1, G_2, \ldots$ be independent, normally distributed random variables each having mean 0 and variance 1. Consider the process

$$\sum_1^{\infty} \sqrt{\lambda_j} G_j f_j(t), \qquad 0 \leqq t \leqq 1.$$

It is clearly Gaussian and its covariance function is by Mercer's theorem, $K(s, t) = \min(s, t) - st$. Thus we can put

$$y(t) = \sum_{1}^{\infty} \sqrt{\lambda_j} G_j f_j(t) \tag{6.22}$$

and by Parseval's relation

$$\int_0^1 y^2(t)\, dt = \sum_{1}^{\infty} \lambda_j G_j^2$$

and

$$E\left\{ e^{-u\int_0^1 y^2(\tau) d\tau} \right\} = \prod_j \frac{1}{\sqrt{1+2u\lambda_j}}. \tag{6.23}$$

The integral equation (6.21) is equivalent to the eigenvalue problem

$$f'' + \frac{1}{\lambda} f = 0,\ f(0) = f(1) = 0, \tag{6.24}$$

so that

$$\lambda_j = \frac{1}{\pi^2 j^2}$$

and by (6.23)

$$E\left\{ e^{-u\int_0^1 y^2(\tau) d\tau} \right\} = \left(\frac{\sin h\sqrt{2u}}{\sqrt{2u}} \right)^{-1/2}$$

in agreement with (6.19).

The method of integral equations can be extended to the calculation of the distribution of

$$\int_0^1 p(\tau)\, y^2(\tau)\, d\tau, \qquad p(\tau) > 0.$$

We consider the kernel,

$$K(s, t) = \frac{\min(s, t) - st}{\sqrt{p(s)}\sqrt{p(t)}}$$

and the eigenvalue problem

$$\int_0^1 K(s, t)\, f(t)\, dt = \lambda f(s). \tag{6.25}$$

Instead of the representation (6.22) we have now

$$y(t) = \frac{1}{\sqrt{p(t)}} \sum_{1}^{\infty} \sqrt{\lambda_j} G_j f_j(t)$$

and

$$\int_0^1 p(\tau)\, y^2(\tau)\, d\tau = \sum_{1}^{\infty} \lambda_j G_j^2.$$

The integral equation (6.25) is now equivalent to the eigenvalue problem

$$\frac{d^2}{dt^2}\left(\frac{f(t)}{\sqrt{p(t)}} \right) + \frac{1}{\lambda} p(t) \left(\frac{f(t)}{\sqrt{p(t)}} \right) = 0, \qquad f(0) = f(1) = 0.$$

This approach is applicable only to quadratic functionals but the processes can be quite general. It was first introduced by A. J. F. Siegert and the present writer in connection with the theory of random noise [11], [12].

The calculation of

$$Pr\{\max_{0\leq\tau\leq 1}|y(\tau)| < a\}$$

can be performed in a manner analogous to that used in section 4. In fact, we note that

$$Pr\{\max_{0\leq\tau\leq 1}|y(\tau)| < a\} = \lim_{u\to\infty} E\left\{e^{-u\int_0^1 V[y(\tau)]d\tau}\right\},$$

where

$$V(x) = \begin{cases} 1, & |x| > a, \\ 0, & |x| < a. \end{cases}$$

This leads to Kolmogoroff's limiting distribution of $\sqrt{n}D_n$ but it should be borne in mind that the justification of (6.11) is of crucial importance. In ending this section we should like to call attention to the interesting problem of extending the theory to the bivariate (or multivariate) case. One can still reduce the theory to the study of a certain process $y(u, v)$ but because 'time' is now two dimensional no analogue of the diffusion theory seems to exist. The analogue of $n\omega_n^2$ can be treated by the integral equation method.

7. Extension to some non-Gaussian processes with independent increments

Let $x(\tau)$ $[x(0) = 0]$ be now a process with independent increments obeying a symmetric stable law with exponent $\alpha(0 < \alpha < 2)$. We thus have

$$E\{e^{i\xi x(\tau)}\} = e^{-\tau|\xi|^\alpha}. \tag{7.1}$$

The most important distinction between these processes and the Wiener process, is that the sample functions of these processes are discontinuous with probability 1. Let

$$\rho(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\eta x} e^{-t|\eta|^\alpha} d\eta \tag{7.2}$$

and consider again the problem of calculating the distribution function of the functional $\int_0^t V[x(\tau)]d\tau$.

The considerations of section 3 can be imitated step by step if we put

$$Q_0(x, t) = \rho(x, t) \tag{7.3}$$

and

$$Q_{n+1}(x, t) = \int_0^t \int_{-\infty}^{\infty} \rho(x-\xi, t-\tau) V(\xi) Q_n(\xi, \tau)\, d\xi\, d\tau. \tag{7.4}$$

The analogue of (3.13) is now

$$\Psi(x) + u\int_{-\infty}^{\infty} R(x-\xi) V(\xi) \Psi(\xi)\, d\xi = R(x), \tag{7.5}$$

where

$$R(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\eta x}}{s+|\eta|^\alpha} d\eta. \tag{7.6}$$

It will be more convenient to multiply both sides of (7.5) by $e^{ix\zeta}$ and integrate with respect to x from $-\infty$ to ∞. This yields after a few simple transformations

$$(7.7)\qquad (s+|\zeta|^{\alpha})\int_{-\infty}^{\infty}\Psi(x)\,e^{ix\zeta}dx+u\int_{-\infty}^{\infty}\Psi(x)\,V(x)\,e^{ix\zeta}dx=1\,.$$

The only case in which we can get an explicit result is when

$$V(x)=\begin{cases}0, & x<0\,,\\ 1, & x>0\,.\end{cases}$$

Although the result in this case can be inferred from a general theorem of E. S. Andersen [13] we present here an analytic argument because of its simplicity and elegance. The argument, except for a minor simplification, is due to H. Pollard.

We rewrite (7.7) in the form

$$(7.8)\quad (s+|\zeta|^{\alpha})\int_{-\infty}^{0}\Psi(x)\,e^{i\zeta x}dx+(s+u+|\zeta|^{\alpha})\int_{0}^{\infty}\Psi(x)\,e^{i\zeta x}dx=1$$

and set, for complex z,

$$F_1(z)=\int_0^{\infty}e^{izx}\Psi(x)\,dx\,,$$

$$F_2(z)=\int_{-\infty}^{0}e^{izx}\Psi(x)\,dx\,.$$

$F_1(z)$ is analytic in the upper half plane and $F_2(z)$ in the lower. Define now the function $\Phi(z)$ as follows:

$$\Phi(z)=\begin{cases}[1-uF_1(z)]\,[1+uF_2(-z)]\,, & z \text{ in the upper half plane}\\ [1-uF_1(-z)]\,[1+uF_2(z)]\,, & z \text{ in the lower half plane}\,.\end{cases}$$

We see that $\Phi(z)$ is analytic in both the upper and lower half planes and because of (7.8) it is easily checked that it is continuous across the real axis. Thus $\Phi(z)$ is analytic in the whole plane. It is also seen to be bounded and hence it must be a constant. Since for real z both $F_1(z)$ and $F_2(z)$ approach 0 as $z\to\infty$ we infer that

$$\Phi(z)\equiv 1$$

and in particular,

$$[1-uF_1(0)]\,[1+uF_2(0)]=1\,.$$

From (7.8) it follows (setting $\zeta=0$)

$$sF_2(0)+(s+u)\,F_1(0)=1$$

and finally

$$F_1(0)=\frac{-s+\sqrt{s(s+u)}}{us}\,,\qquad F_2(0)=\frac{u+s-\sqrt{s(s+u)}}{us}\,.$$

Thus

$$\int_{-\infty}^{\infty}\Psi(x)\,dx=F_1(0)+F_2(0)=\frac{1}{\sqrt{s(s+u)}}\,.$$

Inverting with respect to s and u we obtain that the distribution function of $\int_0^t V[x(\tau)]\,d\tau$ is

$$\frac{2}{\pi}\arc\sin\sqrt{\frac{\alpha}{t}}\,,$$

just as in the Gaussian case.

Unfortunately this method fails already in the slightly more complicated case,

$$V(x) = \begin{cases} 0, & x < a, \\ 1, & x > a, \end{cases}$$

$a \neq 0$, and we were, so far, unable to find any other way to calculate the distribution function of $\int_0^t V[x(\tau)]\,d\tau$ in this case.

8. The "ruin" problem

We shall now discuss an approach to the calculation of the probability

$$(8.1)\qquad Pr\{-b < \underset{0\leq\tau\leq t}{\text{g.l.b.}}\, x(\tau) \leq \underset{0\leq\tau\leq t}{\text{l.u.b.}}\, x(\tau) \leq a\}.$$

Although this approach can be carried out explicitly only for $\alpha = 1$ (Cauchy process) it throws considerable light on the whole problem. For the sake of simplicity we shall assume that $b = a$ and thus consider

$$(8.2)\qquad Pr\{\underset{0\leq\tau\leq t}{\text{l.u.b.}}\,|x(\tau)| < a\}.$$

We follow the idea of section 4 and write

$$(8.3)\qquad Pr\{\underset{0\leq\tau\leq t}{\text{l.u.b.}}\,|x(\tau)| < a\} = \lim_{u\to\infty} E\left\{e^{-u\int_0^t V[x(\tau)]d\tau}\right\},$$

where

$$(8.4)\qquad V(x) = \begin{cases} 1, & |x| > a, \\ 0, & |x| < a. \end{cases}$$

We must now make a distinction between the cases $\alpha > 1$, $\alpha = 1$ and $\alpha < 1$. Let us consider first the case $\alpha > 1$.

Let $g(x)$ be a function such that

$$(8.5)\qquad \begin{aligned} &g(x) = 0 \qquad\qquad |x| > a, \\ &g(a) = g(-a) = g'(a) = g'(-a) = 0, \\ &g''(x) \in L^2, \end{aligned}$$

and let

$$h(\zeta) = \frac{1}{2\pi}\int_{-a}^{a} g(x)\, e^{-ix\zeta} dx.$$

We multiply both sides of the equation (7.7) by $R(\zeta)$ and integrate with respect to ζ from $-\infty$ to $+\infty$. We obtain

$$(8.6)\qquad \int_{-\infty}^{\infty} (s + |\zeta|^{\alpha})\, h(\zeta) \int_{-\infty}^{\infty} \Psi(x)\, e^{ix\zeta} dx\, d\zeta = g(0).$$

It is quite easy to verify that (8.6) yields after a few transformations

$$(8.7)\qquad \int_{-\infty}^{\infty} \Psi(x) \left[s\, g(x) - D(\alpha) \int_{-a}^{a} \frac{g''(\xi)}{|x - \xi|^{\alpha-1}}\, d\xi \right] dx = g(0),$$

where

$$D(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\cos\zeta}{|\zeta|^{2-\alpha}}\, d\zeta.$$

As $u \to \infty$ (increasingly) $\Psi(x)$ form a nonincreasing sequence of functions whose limit we denote by $\Psi_0(x)$.

Moreover,

$$0 \leqq \Psi(x) \leqq \int_0^\infty e^{-st} p(x, t)\, dt$$

and it can be easily shown that

$$\Psi_0(x) = 0, \quad |x| > a.$$

Thus we infer from (8.7) that

$$\int_{-a}^{a} \Psi_0(x) \left[s g(x) - D(\alpha) \int_{-a}^{a} \frac{g''(\xi)}{|x-\xi|^{\alpha-1}} d\xi \right] dx = g(0) \tag{8.8}$$

for *every* $g(x)$ satisfying conditions (8.5). From (8.3) we have

$$\int_0^\infty e^{-st} Pr\{\underset{0 \leqq \tau \leqq t}{\text{l.u.b.}} |x(\tau)| < a\} dt = \int_{-a}^{a} \Psi_0(x)\, dx \tag{8.9}$$

and the question is, does (8.8) determine $\Psi_0(x)$ uniquely. The answer is undoubtedly 'no' but we have no proof. The basis for this belief is as follows: If we were to carry out the above calculations for the Wiener process we would be led to

$$\int_{-a}^{a} \Psi_0(x) [s g(x) - \tfrac{1}{2} g''(x)]\, dx = g(0) \tag{8.10}$$

for *every* g satisfying (8.5). This does not determine $\Psi_0(x)$ uniquely. One can, however, show by a separate argument that in addition to (8.10) $\Psi_0(x)$ satisfies

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{-a}^{-a+\epsilon} \Psi_0(x)\, dx = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{a-\epsilon}^{a} \Psi_0(x)\, dx = 0. \tag{8.11}$$

Now (8.10) and (8.11) determine $\Psi_0(x)$ uniquely and it turns out that

$$\Psi_0(x) = \int_0^\infty e^{-st} P(x, t)\, dt,$$

where $P(x, t)$ is the Green's function of

$$\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial^2 P}{\partial x^2}$$

with the singularity at $x = 0$ as $t \to 0$ and subject to the boundary conditions

$$P(-a, t) = P(a, t) = 0.$$

This, of course, is in complete accord with the well known facts from diffusion theory. Now, as long as $\alpha > 1$, we can still prove (8.11) and hence it is natural to conjecture that (8.11) together with (8.8) determine $\Psi_0(x)$ uniquely.

The situation changes radically when $\alpha = 1$ (Cauchy process). The analogue of (8.8) is now

$$\int_{-a}^{a} \Psi_0(x) \left[s g(x) - \frac{1}{\pi} \text{P.V.} \int_{-a}^{a} \frac{g'(\xi)}{\xi - x} d\xi \right] dx = g(0) \tag{8.12}$$

for every $g(x)$ satisfying the conditions

$$g(a) = g(-a) = 0, \qquad g'(x) \in L^2(-a, a). \tag{8.13}$$

It has been shown by H. Pollard and the present writer [14] that (8.12) yields the *unique* solution

$$\Psi_0(x) \sim \frac{1}{a}\sum_{1}^{\infty} \frac{g_j(0)\, g_j\left(\frac{x}{a}\right)}{s + \frac{\pi}{2a\lambda_j}}, \tag{8.14}$$

where the λ's are the eigenvalue and the $g_j(x)$ the normalized eigenfunctions of the integral equation

$$\int_{-1}^{1} K(x, y)\, g(y)\, dy = \lambda g(x) \tag{8.15}$$

with

$$K(x, y) = \tfrac{1}{4} \log \frac{1 - xy + \sqrt{(1 - x^2)(1 - y^2)}}{1 - xy - \sqrt{(1 - x^2)(1 - y^2)}}. \tag{8.16}$$

In particular, we obtain in this case

$$Pr\{\operatorname*{l.u.b.}_{0 \leqq \tau \leqq t} |x(\tau)| < a\} = \frac{1}{a}\sum_{j=1}^{\infty} e^{-\pi t/2a\lambda_j} g_j(0) \int_{-a}^{a} g_j\left(\frac{x}{a}\right) dx. \tag{8.17}$$

For stable processes with $\alpha < 1$ the analogue of (8.8) and (8.12) is

$$\int_{-a}^{a} \Psi_0(x) \left[s g(x) + \mathcal{E}(\alpha) \int_{-a}^{a} \frac{g'(\xi)\, \operatorname{sgn}(x - \xi)}{|x - \xi|^{\alpha}}\, d\xi \right] dx = g(0) \tag{8.18}$$

for every $g(x)$ satisfying (8.13), where

$$\mathcal{E}(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\zeta|^{\alpha}}{\zeta} \sin \zeta\, d\zeta.$$

In this case we conjecture that like in the case of the Cauchy process, (8.18) alone should yield a unique $\Psi_0(x)$.

It might be mentioned that the method[3] by which one obtains (8.11) for $\alpha > 1$, fails for $\alpha \leqq 1$. This is perhaps another indication that (8.18) like (8.12) may be sufficient.

Equations (8.8) and (8.18) were first derived (by a different method) in collaboration with H. Pollard.

9. The multidimensional absorption problem and the theorems of Weyl and Carleman

Let Ω be a two dimensional region bounded by the curve Γ and consider the eigenvalue problem

$$\tfrac{1}{2}\Delta u + \lambda u = 0, \qquad u = 0 \text{ on } \Gamma. \tag{9.1}$$

If $\lambda_1, \lambda_2, \ldots$ are the eigenvalues and $u_1(x, y), u_2(x, y), \ldots$ the corresponding normalized eigenfunctions then according to a classical result of H. Weyl

$$\sum_{\lambda_j < \lambda} 1 \sim \frac{|\Omega|}{2\pi} \lambda, \qquad \lambda \to \infty, \tag{9.2}$$

[3] Not reproduced here because it is somewhat lengthy.

and according to a later result of Carleman

$$\sum_{\lambda_j<\lambda} u_j^2(x, y) \sim \frac{\lambda}{2\pi}, \qquad \lambda \to \infty, \quad (x, y) \in \Omega. \tag{9.3}$$

Here $|\Omega|$ denotes the area of the region Ω.

These results can be derived heuristically by considerations very similar to those used in section 5. Let $P(x_0, y_0|x, y; t)dxdy$ denote the probability that a Brownian particle starting from (x_0, y_0) will be at (x, y) (within $dxdy$) at time t without having crossed Γ in the meantime. From the early work on Brownian motion by Einstein and Smoluchowski it was known that $P(x_0, y_0|x, y; t)$ is the fundamental solution of the differential

$$\frac{\partial P}{\partial t} = \frac{1}{2} \Delta P \tag{9.4}$$

becoming singular at (x_0, y_0) as $t \to 0$ and subject to the boundary condition

$$P = 0 \quad \text{on } \Gamma. \tag{9.5}$$

Under sufficient restrictions on Γ it is known that

$$P(x_0, y_0|x, y; t) = \sum_{1}^{\infty} e^{-\lambda_j t} u_j(x_0, y_0)\, u_j(x, y). \tag{9.6}$$

As $t \to 0$ the Brownian particle has had no time to 'feel' the boundary and consequently one might suspect that $P(x_0, y_0|x, y; t)$ is well approximated (as $t \to 0$) by the unrestricted fundamental solution. Thus

$$P(x_0, y_0|x, y; t) \sim \frac{1}{2\pi t} e^{-[(x-x_0)^2+(y-y_0)^2]/2t} \tag{9.7}$$

or upon setting $x = x_0$, $y = y_0$

$$P(x_0, y_0|x_0, y_0; t) \sim \frac{1}{2\pi t}, \qquad t \to 0. \tag{9.8}$$

Using (9.6) we obtain

$$\sum_{1}^{\infty} e^{-\lambda_j t} u_j^2(x_0, y_0) \sim \frac{1}{2\pi t}, \qquad t \to 0, \tag{9.9}$$

or by applying the Hardy-Littlewood Tauberian theorem

$$\sum_{\lambda_j<\lambda} u_j^2(x_0, y_0) \sim \frac{\lambda}{2\pi}, \qquad \lambda \to \infty. \tag{9.10}$$

This is Carleman's result (9.3).

To obtain Weyl's theorem we integrate (9.9) over Ω obtaining

$$\sum_{1}^{\infty} e^{-\lambda_j t} \sim \frac{|\Omega|}{2\pi t}, \qquad t \to 0, \tag{9.11}$$

and applying again the Hardy-Littlewood Tauberian theorem we obtain (9.2). This reasoning which is crude and heuristic can be made precise along the following lines. Accepting for the moment the probabilistic interpretation of $P(x_0, y_0|x, y; t)$ we have the immediate inequality

$$P(x_0, y_0|x, y; t) \leq \frac{1}{2\pi t} e^{-[(x-x_0)^2+(y-y_0)^2]/2t}$$

or, in particular

$$P(x_0, y_0 \mid x_0, y_0; t) \leqq \frac{1}{2\pi t}. \tag{9.12}$$

Surround now (x_0, y_0) by a square $S_\epsilon \subset \Omega$ of side ϵ with sides parallel to the coordinate axes, and denote by Γ_ϵ its boundary. Consider now the probability $P_\epsilon(x_0, y_0 \mid x, y; t)dxdy$, $(x, y) \epsilon S_\epsilon$, that a Brownian particle starting from (x_0, y_0) will be at (x, y) (within dx, dy) without having crossed Γ_ϵ in the meantime. On probabilistic grounds it is again obvious that

$$P_\epsilon(x_0, y_0 \mid x, y; t) \leqq P(x_0, y_0 \mid x, y; t)$$

or, in particular

$$P_\epsilon(x_0, y_0 \mid x_0 y_0; t) \leqq P(x_0, y_0 \mid x_0, y_0; t). \tag{9.13}$$

Now, $P_\epsilon(x_0, y_0 \mid x_0, y_0; t)$ can be calculated explicitly and one obtains

$$P_\epsilon(x_0, y_0 \mid x_0, y_0; t) = \frac{4}{\epsilon^2} \sum_{m,n=1}^{\infty} e^{-[(m^2+n^2)\pi^2/2\epsilon^2]t} \sin^2 \frac{m\pi(x_0 - \xi)}{\epsilon} \sin^2 \frac{n\pi(y_0 - \eta)}{\epsilon}, \tag{9.14}$$

where (ξ, η) is the lower left corner of S_ϵ.

By an elementary computation one shows that as $t \to 0$

$$P_\epsilon(x_0, y_0 \mid x_0, y_0; t) \sim \frac{1}{2\pi t} \tag{9.15}$$

for *every* $\epsilon > 0$.

From (9.15) and the inequalities (9.12) and (9.13) we have

$$P(x_0, y_0 \mid x_0, y_0; t) \sim \frac{1}{2\pi t}$$

which, as we have seen, yields Carleman's result (9.3). To get Weyl's theorem we integrate (9.13) over S_ϵ obtaining

$$\sum_{m,n=1}^{\infty} e^{-[(m^2+n^2)\pi^2/2\epsilon^2]t} \leqq \iint_{S_\epsilon} P(x_0, y_0 \mid x_0, y_0; t)\, dx_0 dy_0. \tag{9.16}$$

If we cover Ω with a net of squares S_ϵ and apply (9.16) to each of the squares we obtain by adding

$$\left[\frac{|\Omega|}{\epsilon^2}\right] \sum_{m,n=1}^{\infty} e^{-[(m^2+n^2)\pi^2/2\epsilon^2]t} \leqq \iint_{\Sigma S_\epsilon} P(x_0, y_0; t)\, dx_0 dy_0 \leqq \iint_{\Omega} P(x_0, y_0 \mid x_0, y_0; t)\, dx_0 dy_0, \tag{9.17}$$

where $[x]$ denotes, as usual, the greatest integer in x. From (9.12) it follows that

$$\iint_{\Omega} P(x_0, y_0 \mid x_0, y_0; t)\, dx_0 dy_0 \leqq \frac{|\Omega|}{2\pi t}$$

and by combining it with (9.17) we obtain

$$\left[\frac{|\Omega|}{\epsilon^2}\right] \sum_{m,n=1}^{\infty} e^{-[(m^2+n^2)\pi^2/2\epsilon^2]t} \leqq \iint_{\Sigma S_\epsilon} P(x_0, y_0 \mid x_0, y_0; t)\, dx_0 dy_0 \leqq \frac{|\Omega|}{2\pi t}. \tag{9.18}$$

An elementary computation yields

$$\lim_{t \to 0} t \sum_{m,n=1}^{\infty} e^{-[(m^2+n^2)\pi^2/2\epsilon^2]t} = \frac{\epsilon^2}{2\pi}$$

and hence from (9.18)

$$\frac{\epsilon^2}{2\pi}\left[\frac{|\Omega|}{\epsilon^2}\right] \leqq \liminf_{t \to 0} t \iint_{\Omega} P(x_0, y_0 | x_0, y_0; t)\, dx_0 dy_0$$
$$\leqq \limsup_{t \to 0} t \iint_{\Omega} P(x_0, y_0 | x_0, y_0; t)\, dx_0 dy_0 \leqq \frac{|\Omega|}{2\pi}.$$

Letting $\epsilon \to 0$ we have

$$\lim_{t \to 0} t \iint_{\Omega} P(x_0, y_0 | x_0, y_0; t)\, dx_0 dy_0 = \frac{|\Omega|}{2\pi}$$

and this, as we have seen, implies Weyl's theorem (9.2). Thus a rigorous derivation hinges on the inequalities (9.12) and (9.13). These inequalities are trivial consequences of the probabilistic interpretation of $P(x_0, y_0 | x, y; t)$. The rigorous justification of this probabilistic interpretation is, however, not entirely trivial. In particular, one must make some assumptions about the boundary Γ.[4] The inequalities (9.12) and (9.13) can also be derived (as was pointed out to me by Dr. G. A. Hunt), without appealing to probabilistic notions, within the framework of the classical diffusion theory.

Our approach has a drawback. If, for instance, instead of the boundary condition $u = 0$ on Γ we consider the boundary condition

$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma, \tag{9.19}$$

the heuristic argument leads again to (9.2) and (9.3). To establish inequalities analogous to (9.12) and (9.13) (they are reversed in this case) one can again appeal to the theory of Brownian motion except that the 'absorbing barrier' Γ must now be replaced by the 'reflecting barrier.' Unfortunately, no rigorous treatment of reflecting barriers seems to be available. It should, however, be possible to prove the desired inequalities directly from the differential equation.

It must also be emphasized that our approach is closely related to that of Minakshisundaram [16], [17] (see also [18]). In two, most important, respects they are identical inasmuch as both use the diffusion equation and a Tauberian argument. The difference is that we use the inequalities (9.12) and (9.13) whereas Minakshisundaram uses different estimates. Also, Minakshisundaram [17] goes much farther obtaining highly elegant analogies between the eigenvalue problem and Riemann's approach to the problem of the distribution of primes.

The probabilistic approach has the advantage of a strong intuitive appeal making the theorems (9.2) and (9.3) almost obvious. It seems almost incredible that these beautiful theorems are consequences of the crude principle of 'not feeling' the boundary as $t \to 0$ and that their depth seems to be hidden in inequalities like (9.12) and (9.13) which, on physical grounds, are so immediate.

[4] A discussion of this problem is being prepared by Dr. M. Rosenblatt and the present writer. A brief discussion can be found in the recent book of P. Lévy [15, pp. 259–261].

The applicability of the heuristic principle is not limited to the classical case discussed above.

We have seen in section 8 that for a Cauchy process $x(\tau)$ $[x(0) = 0]$

$$Pr\{\underset{0\leq\tau\leq t}{\text{l.u.b.}} |x(\tau)| < a\} = \frac{1}{a}\sum_{1}^{\infty} e^{-\pi t/2a\lambda_j} g_j(0) \int_{-a}^{a} g_j\left(\frac{x}{a}\right) dx,$$

where the λ_j's are the eigenvalues and $g_j(x)$ the normalized eigenfunctions of the kernel

$$\frac{1}{4}\log\frac{1 - xy + \sqrt{(1-x^2)(1-y^2)}}{1 - xy - \sqrt{(1-x^2)(1-y^2)}}. \tag{9.20}$$

By a slight modification of the argument which led to this result we can obtain the following: Let $P(x_0|x;t)dx$, $-a < x_0, x < a$ be the probability that a 'Cauchy particle' starting from x_0 will be at x (within dx) at time t without having left the strip $(-a, a)$ in the meantime. Then

$$P(x_0|x;t) = \frac{1}{a}\sum_{1}^{\infty} e^{-\pi t/2a\lambda_j} g_j\left(\frac{x_0}{a}\right) g_j\left(\frac{x}{a}\right). \tag{9.21}$$

By the principle of 'not feeling' the boundary we get

$$P(x_0|x;t) \sim \frac{1}{\pi}\frac{t}{t^2 + (x-x_0)^2}, \qquad t \to 0,$$

and by setting $x = x_0$ and integrating from $-a$ to a

$$\sum_{1}^{\infty} e^{-\pi t/2a\lambda_j} \sim \frac{2a}{\pi t}, \qquad t \to 0,$$

setting $a = \pi/2$ and applying the Hardy-Littlewood Tauberian theorem we get

$$\sum_{1/\lambda_j<\lambda} 1 \sim \lambda, \qquad \lambda \to \infty,$$

or in other words

$$\lambda_n \sim \frac{1}{n}. \tag{9.22}$$

Unfortunately we are unable to prove this result rigorously even though the analogues of (9.12) and (9.13) are trivially true in this case. The reason for this is that we do not have in this case the analogue of the *explicit* formula for P_ϵ.

10. Connections with potential theory

Connections between the Dirichlet problem and random walk were known for a long time [19], [20], [21]. Considerable progress was made recently by Kakutani [22], [23].

In this section we shall be concerned with a particular aspect of this subject. Let Ω be a bounded region in the three dimensional Euclidean space whose volume $|\Omega|$ is different from 0.

Let $r(t) = [x(t), y(t), z(t)]$ be the three dimensional Brownian motion [that is,

$x(t)$, $y(t)$, $z(t)$ are independent Wiener processes] and y a point in space. Let

$$V(r) = \begin{cases} 1, & r \in \Omega \\ 0, & r \notin \Omega \end{cases} \tag{10.1}$$

and consider the random variable

$$T \equiv T_\Omega(y) = \int_0^\infty V[y + r(\tau)]\, d\tau. \tag{10.2}$$

It is clear that T represents the total time which a Brownian particle starting from y spends in Ω. It is easily verified that

$$E\{T\} = \frac{1}{2\pi}\int_\Omega \frac{1}{|r-y|}\, dr < \infty$$

and consequently T is finite with probability 1. (Here as in the sequel $|r - y|$ denotes the distance between r and y). It is not too difficult to calculate the moments of T and one obtains

$$\mu_k = E\{T^k\} = \frac{k!}{(2\pi)^k}\int_\Omega \cdots \int_\Omega \frac{1}{|r_1 - y|}\,\frac{1}{|r_2 - r_1|}\cdots\frac{1}{|r_k - r_{k-1}|}\, dr_1 \ldots dr_k. \tag{10.3}$$

Consider now the integral equation

$$\frac{1}{2\pi}\int_\Omega \frac{1}{|r-\varrho|}\,\varphi(\varrho)\, d\varrho = \lambda\varphi(r) \tag{10.4}$$

and note that the kernel is L^2 and positive definite. Denote by $\lambda_1, \lambda_2, \ldots$ its eigenvalues and by $\varphi_1(\varrho), \varphi_2(\varrho), \ldots$ the corresponding normalized eigenfunctions.

In terms of these eigenvalues and eigenfunctions the moments can be expressed simply by the formulas

$$\frac{\mu_k}{k!} = \frac{1}{2\pi}\sum_{j=1}^{\infty}\lambda_j^{k-1}\int_\Omega \varphi_j(r)\, dr \int_\Omega \frac{\varphi_j(\varrho)}{|\varrho - y|}\, d\varrho, \qquad k \geqq 1. \tag{10.5}$$

If $y \in \Omega$ formula (10.5) assumes the simpler form

$$\frac{\mu_k}{k!} = \sum_{j=1}^{\infty}\lambda_j^k \varphi_j(y)\int_\Omega \varphi_j(r)\, dr, \qquad k \geqq 1. \tag{10.6}$$

Let now $u > 0$. It is easily verified that

$$\begin{aligned} E\{e^{-uT}\} &= \sum_1^\infty \frac{(-u)^k}{k!}\mu_k \\ &= 1 - \frac{u}{2\pi}\sum_{j=1}^\infty \frac{1}{1+\lambda_j u}\int_\Omega \varphi_j(r)\, dr \int_\Omega \frac{\varphi_j(\varrho)}{|\varrho - y|}\, d\varrho. \end{aligned} \tag{10.7}$$

For $y \notin \Omega$ the series on the right is a uniformly convergent series of harmonic functions and hence $E\{e^{-uT}\}$ is a harmonic function of y. If $u \to \infty$ (increasingly)

$$\lim_{u \uparrow \infty} E\{e^{-uT}\} = Pr\{T = 0\} = U(y)$$

and hence $Pr\{T = 0\} = U(y)$ being a limit of a decreasing sequence of harmonic functions is itself harmonic. Suppose now that P is a point on the boundary of Ω which is regular in the sense of Poincaré, that is, one can find a sphere through P

which lies totally inside Ω. Let $\boldsymbol{a}$ be the center of this sphere and ϵ its radius. Let now

$$V_P(\vec{r}) = \begin{cases} 1, & |r-a| < \epsilon, \\ 0, & |r-a| > \epsilon, \end{cases}$$

note that

$$V_P(r) \leqq V(r)$$

and consequently

$$\lim_{u\uparrow\infty} E\left\{ e^{-u\int_0^\infty V[r(\tau)+y]d\tau} \right\} \leqq \lim_{u\uparrow\infty} E\left\{ e^{-u\int_0^\infty V_P[r(\tau)+y]d\tau} \right\}.$$

Thus

$$U(y) \leqq \lim_{u\uparrow\infty} E\left\{ e^{-u\int_0^\infty V_P[r(\tau)+y]d\tau} \right\} = U_P(y).$$

It is not difficult to show that

$$U_P(y) = 1 - \frac{\epsilon}{|y-a|}$$

and thus

$$U(y) \to 0 \quad \text{as} \quad y \to P.$$

If every point on the boundary of Ω is regular in the sense of Poincaré, $U(y)$ is the harmonic function which vanishes at the boundary of Ω and is 1 at infinity. From (10.7) we obtain ($\delta = 1/u$)

$$(10.8) \qquad U(y) = 1 - \lim_{\delta\downarrow 0} \frac{1}{2\pi} \sum_{j=1}^{\infty} \frac{1}{\delta+\lambda_j} \int_\Omega \varphi_j(r)\, dr \int_\Omega \frac{\varphi_j(\varrho)}{|\varrho - y|}\, d\varrho.$$

If $y \in \Omega$ one obtains

$$(10.9) \qquad 1 = \lim_{\delta\downarrow 0} \sum_{j=1}^{\infty} \frac{\lambda_j}{\delta+\lambda_j} \int_\Omega \varphi_j(r)\, dr \varphi_j(y).$$

This is a consequence of the *continuity* of $r(\tau)$ inasmuch as a Brownian motion starting from $y \in \Omega$ must spend some time in Ω and consequently $Pr\{T = 0\} = 0$.

It is worth emphasizing that the purely analytic fact (10.9) emerges here as a consequence of the measure theoretical fact that almost all sample functions $r(\tau)$ are continuous at $\tau = 0$. If y is on the boundary of Ω we still have

$$(10.10) \qquad Pr\{T=0\} = 1 - \lim_{\delta\downarrow 0} \sum_{j=1}^{\infty} \frac{\lambda_j}{\delta+\lambda_j} \int_\Omega \varphi_j(r)\, dr \varphi_j(y)$$

we cannot however assert that $Pr\{T = 0\} = 0$. It seems natural to conjecture[5] that if y is a regular boundary point (in the sense used in potential theory) $Pr\{T = 0\} = 0$ but if y is a singular point $Pr\{T = 0\} > 0$. The equivalent way of stating this conjecture is to say that a necessary and sufficient condition for a boundary point y to be regular is that

$$1 = \lim_{\delta\downarrow 0} \sum_{j=1}^{\infty} \frac{\lambda_j}{\delta+\lambda_j} \int_\Omega \varphi_j(r)\, dr \varphi_j(y).$$

Consideration of this section enables us also to find the distribution function of T.

[5] This conjecture has, in the meantime, been proved by A. Dvoretzky.

In fact, denoting this distribution function by $\sigma(\beta)$ we can rewrite (10.7) in the form

$$\int_0^\infty e^{-u\beta} d\sigma(\beta) = 1 - u \sum_{j=1}^\infty \frac{a_j}{1+\lambda_j u}, \tag{10.11}$$

where

$$a_j = \frac{1}{2\pi} \int_\Omega \varphi_j(r)\, dr \int_\Omega \frac{\varphi_j(\varrho)}{|\varrho - y|}\, d\varrho .$$

Now, the inversion can be performed by the following simple argument suggested by H. Pollard.

Dividing by u we get

$$\frac{1}{u}\int_0^\infty e^{-u\beta} d\sigma(\beta) = \int_0^\infty e^{-u\beta}\sigma(\beta)\, d\beta = \frac{1}{u} - \sum_{j=1}^\infty \frac{a_j}{1+\lambda_j u}.$$

We can now invert [24, p. 334] obtaining

$$\sigma(\beta) = 1 - \sum_{j=1}^\infty \frac{a_j}{\lambda_j} e^{-\beta/\lambda_j}$$

or

$$1 - \sigma(\beta) = \sum_{j=1}^\infty \frac{a_j}{\lambda_j} e^{-\beta/\lambda_j}.$$

In particular,

$$Pr\{T > \beta\} \sim C e^{-\beta/\lambda_1}, \qquad \beta \to \infty . \tag{10.12}$$

That the probability (10.12) goes down exponentially was conjectured by Erdös. It should be recalled that λ_1 is the largest eigenvalue of the integral equation (10.4).

The theory of this section can be extended to dimensions higher than 3. The principal difference is that the kernels of the analogues of (10.4) are no longer L^2. This difficulty can, however, be circumvented. There is no analogous theory in one and two dimensions.

11. An application to the integral equation (10.4)

As a final illustration of how probability methods can be used in deriving purely analytic results we shall consider the problem of the distribution of eigenvalues of the integral equation (10.4).

Let A be sphere contained in Ω and consider

$$\nu_k = \int_0^\infty E\left\{\left(\int_0^t V[y + r(\tau)]\, d\tau\right)^k,\ y + r(t) \in A\right\} dt, \tag{11.1}$$

the symbol $E\{X, y + r(t) \in A\}$ denoting the integral of X over the portion of the sample space in which $y + r(t) \in A$. It is not difficult to verify that

$$\nu_k = \frac{k!}{(2\pi)^{k+1}} \int_A \int_\Omega \cdots \int_\Omega \frac{1}{|r_1 - y|} \cdots \frac{1}{|r_k - r_{k-1}|} \frac{1}{|r - r_k|}\, dr_1 \ldots dr_k dr,\ k \geq 0,$$

and hence for $y \in \Omega$

$$\frac{\nu_k}{k!} = \sum_{j=1}^\infty \lambda_j^{k+1} \varphi_j(y) \int_A \varphi_j(r)\, dr, \qquad k \geq 0. \tag{11.2}$$

Thus

$$\int_0^\infty E\left\{e^{-u\int_0^t V[y+r(\tau)]d\tau},\ y+r(t)\in A\right\}dt=\sum_{j=1}^\infty \frac{\lambda_j}{1+\lambda_j u}\varphi_j(y)\int_A \varphi_j(r)\,dr.$$

Inverting with respect to u, we obtain

$$(11.3)\quad \int_0^\infty Pr\left\{\int_0^t V[y+r(\tau)]\,d\tau>\beta,\ y+r(t)\in A\right\}dt$$
$$=\sum_{j=1}^\infty e^{-\beta/\lambda_j}\lambda_j\varphi_j(y)\int_A \varphi_j(r)\,dr.$$

Let now A be the sphere with center y and radius δ. It can be shown by dividing both sides of (11.3) by $|A| = 4/3\pi\delta^3$ and letting $\delta\to 0$ that

$$\frac{1}{(2\pi)^{3/2}}\int_0^\infty \frac{1}{t^{3/2}}Pr\left\{\int_0^t V[y+r(\tau)]\,d\tau>\beta\,|\,r(t)=0\right\}dt=\sum_{j=1}^\infty e^{-\beta/\lambda_j}\lambda_j\varphi_j^2(y).$$

Noticing that

$$Pr\left\{\int_0^t V[y+r(\tau)]\,d\tau>\beta\,|\,r(t)=0\right\}=0$$

for $t<\beta$ we obtain

$$(11.4)\quad \frac{1}{(2\pi)^{3/2}}\int_\beta^\infty \frac{1}{t^{3/2}}Pr\left\{\int_0^t V[y+r(\tau)]\,d\tau>\beta\,|\,r(t)=0\right\}dt$$
$$=\sum_{j=1}^\infty e^{-\beta/\lambda_j}\lambda_j\varphi_j^2(y).$$

Setting $t=\beta\xi$ and $\tau=\beta\eta$ we get

$$\frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{\beta}}\int_1^\infty \frac{1}{\xi^{3/2}}Pr\left\{\int_0^\xi V[y+r(\beta\eta)]\,d\eta>1\,|\,r(\beta\xi)=0\right\}d\xi$$
$$=\sum_{j=1}^\infty e^{-\beta/\lambda_j}\lambda_j\varphi_j^2(y).$$

As $\beta\to 0$ we see that

$$(11.5)\qquad Pr\left\{\int_0^\xi V[y+r(\beta\eta)]\,d\eta>1\,|\,r(\beta\xi)=0\right\}\to 1,$$

because the Brownian particle starting from $y\in\Omega$ is extremely unlikely to leave Ω in the time interval $(0, \beta\xi)$ (this can be made quite rigorous). Thus

$$\sum_{j=1}^\infty e^{-\beta/\lambda_j}\lambda_j\varphi_j^2(y)\sim\frac{2}{(2\pi)^{3/2}}\frac{1}{\sqrt{\beta}},\qquad \beta\to 0,$$

and by the Hardy-Littlewood Tauberian theorem

$$(11.6)\qquad \sum_{1/\lambda_j<\lambda}\lambda_j\varphi_j^2(y)\sim\frac{\sqrt{2}}{\pi^2}\sqrt{\lambda},\qquad \lambda\to\infty.$$

By a slightly more complicated argument we also get

$$(11.7)\qquad \sum_{1/\lambda_j<\lambda}\lambda_j\sim\frac{\sqrt{2}}{\pi^2}|\Omega|\sqrt{\lambda},\qquad \lambda\to\infty.$$

It should be mentioned that the integral equation (10.4), in case the boundary S of Ω is sufficiently regular, is equivalent to the differential equation

$$\frac{1}{2}\Delta\varphi + \frac{1}{\lambda}\varphi = 0 \tag{11.8}$$

and the 'boundary condition'

$$\int_S \left\{ \varphi(\varrho) \frac{\partial}{\partial n}\left[\frac{1}{|r-\varrho|}\right] - \frac{\partial\varphi(\varrho)}{\partial n}\frac{1}{|r-\varrho|}\right\} d\sigma = 0 \tag{11.9}$$

for *every* $r \in \Omega$ ($r \notin S$).

The heuristic principle of 'not feeling' the boundary section 9 would yield the asymptotic formula

$$\sum_{1/\lambda_j<\lambda} 1 \sim \frac{\sqrt{2}\,|\Omega|}{3\pi^2}\lambda^{3/2}, \qquad \lambda\to\infty, \tag{11.10}$$

and this can indeed be obtained from (11.7).

Set $\mu_j = 1/\lambda_j$ and note that

$$\sum_{1/\lambda_j<\lambda} 1 = \sum_{\mu_j<\lambda} 1 = \sum_{\mu_j<\lambda} \mu_j \frac{1}{\mu_j} = \int_0^\lambda \mu\, d\sigma(\mu),$$

where

$$\sigma(\mu) = \sum_{\mu_j<\mu} \frac{1}{\mu_j}.$$

Integration by parts gives

$$\sum_{1/\lambda_j<\lambda} 1 = \lambda\sigma(\lambda) - \int_0^\lambda \sigma(\mu)\, d\mu.$$

From (11.7)

$$\sigma(\lambda) \sim \frac{\sqrt{2}\,|\Omega|}{\pi^2}\sqrt{\lambda}, \qquad \lambda\to\infty,$$

and consequently

$$\sum_{1/\lambda\ <\lambda} 1 \sim \frac{\sqrt{2}}{3}\frac{|\Omega|}{\pi^2}\lambda^{3/2}, \qquad \lambda\to\infty.$$

REFERENCES

[1] M. Kac, "On distributions of certain Wiener functionals," *Trans. Amer. Math. Soc.*, Vol. 65 (1949), pp. 1–13.

[2] P. Erdös and M. Kac, "On certain limit theorems of the theory of probability," *Bull. Amer. Math. Soc.*, Vol. 52 (1946), pp. 292–302.

[3] E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second Order Differential Equations*, Oxford University Press, Oxford, 1946.

[4] M. D. Donsker and M. Kac, "A sampling method for determining the lowest eigenvalue and the principal eigenfunction of Schrödinger's equation," to appear in the *Jour. Res. Nat. Bureau of Standards*.

[5] A. Kolmogoroff, "Sulla determinazione empirica di una legge di distribuzione," *Inst. Ital. Attuari*, Vol. 4 (1933), pp. 1–11.

[6] H. Cramér, "On the composition of elementary errors, second paper, statistical applications," *Skand. Aktuar.* (1928), pp. 141–180.

[7] R. von Mises, *Wahrscheinlichkeitsrechnung und ihre Anwendung in der Statistik und theoretischen Physik*, Deuticke, Leipzig and Wien, 1931.
[8] N. V. Smirnoff, "Sur la distribution de ω^2 (criterium de M. von Mises)," *C. R. Acad. Sci. Paris*, Vol. 202 (1936), pp. 449–452.
[9] J. L. Doob, "Heuristic approach to the Kolmogorov-Smirnov theorems," *Annals of Math. Stat.*, Vol. 20 (1949), pp. 393–403.
[10] M. Kac, "On deviations between theoretical and empirical distributions," *Proc. Nat. Acad. Sci.*, Vol. 35 (1949), pp. 252–257.
[11] M. Kac and A. J. F. Siegert, "On the theory of noise in radio receivers with square law detectors," *Jour. Applied Physics*, Vol. 18 (1947), pp. 383–397.
[12] ———, "An explicit representation of stationary Gaussian processes," *Annals of Math. Stat.*, Vol. 18 (1947), pp. 438–442.
[13] E. S. Andersen, "On the number of positive sums of random variables," *Skand. Aktuar.*, Vol. 32 (1949), pp. 27–36.
[14] M. Kac and H. Pollard, "On the distribution of the maximum of partial sums of independent random variables," to appear in the *Can. Jour. Math.*
[15] P. Lévy, *Processus Stochastiques et Mouvement Brownien*, Gauthier-Villars, Paris, 1948.
[16] S. Minakshisundaram, "Notes on Fourier expansions II," *Jour. London Math. Soc.*, Vol. 20 (1945), pp. 148–153.
[17] ———, "A generalization of Epstein zeta function," *Can. Jour. Math.*, Vol. 1 (1949), pp. 320–327.
[18] S. Minakshisundaram and A. Pleijel, "Some properties of the eigenfunctions of the Laplace operator on Riemannian manifold," *Can. Jour. Math.*, Vol. 1 (1949), pp. 242–256.
[19] H. B. Philips and N. Wiener, "Nets and the Dirichlet problem," *Jour. Math. and Phys.*, Vol. 2 (1923), pp. 105–124.
[20] R. Courant, K. Friedrichs and H. Lewy, "Über die partiellen Differenzengleichungen der mathematischen Physik," *Math. Annalen*, Vol. 100 (1928), pp. 32–74.
[21] A. Khintchine, *Asymptotishe Gesetze der Wahrscheinlichkeitsrechnung*, Springer, Berlin, 1933; Chelsea, New York, 1948. A summary of Petrovsky's work can be found in this book.
[22] S. Kakutani, "Two-dimensional Brownian motion and harmonic functions," *Proc. Acad. Tokyo*, Vol. 20 (1944), pp. 706–714.
[23] ———, "Markoff process and the Dirichlet problem," *Proc. Acad. Tokyo*, Vol. 21 (1945), pp. 227–233.
[24] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1941.