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Characteristic Varieties of Highest Weight Modules and Primitive Quotients

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Dedicated to Professor Ichiro Satake on his 60th birthday

§ 0. Introduction

0.1. We discuss about the characteristic varieties of certain modules over the enveloping algebra of a semisimple Lie algebra, such as highest weight modules and primitive quotients.

Let G be a connected semisimple algebraic group over the complex number field $\boldsymbol{C}$, $\mathfrak{g}$ its Lie algebra and $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. Let X be the flag variety of G and $\mathscr{D}_X$ the sheaf of linear (algebraic) differential operators on X. The natural action of G on X induces an algebra homomorphism $U(\mathfrak{g}) \to \Gamma(X, \mathscr{D}_X)$. Hence for a $U(\mathfrak{g})$-module M we have a $\mathscr{D}_X$-module $\mathscr{D}_X \otimes_{U(\mathfrak{g})} M$.

For a finitely generated $U(\mathfrak{g})$-module M (resp. a coherent $\mathscr{D}_X$-module $\mathscr{M}$) the associated variety $V(M)$ (resp. the characteristic variety $\mathrm{Ch}(\mathscr{M})$) is a subvariety of the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ (resp. a subvariety of the cotangent bundle T^*X). For simplicity we sometimes write $\mathrm{Ch}(M)$ instead of $\mathrm{Ch}(\mathscr{D}_X \otimes_{U(\mathfrak{g})} M)$ for a finitely generated $U(\mathfrak{g})$-module M and call it the characteristic variety of M.

We hope to determine the associated varieties and the characteristic varieties of the irreducible highest weight modules L with trivial central character and the quotients $U(\mathfrak{g})/I$, where I is a primitive ideal of $U(\mathfrak{g})$ with trivial central character. For a finitely generated $U(\mathfrak{g})$-module M with trivial central character $V(M)$ is determined from $\mathrm{Ch}(M)$ (Borho-Brylinski, see Proposition 1.2 below). Hence our problems are the following:

Problem 0.1. *Determine* $\mathrm{Ch}(L)$ *for irreducible highest weight modules* L *with trivial central character.*

Problem 0.2. *Determine* $\mathrm{Ch}(U(\mathfrak{g})/I)$ *for primitive ideals* I *of* $U(\mathfrak{g})$

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with trivial central character.

It follows from Propositions 1.2 and 1.9 below that $\mathrm{Ch}(U(\mathfrak{g})/I)$ is determined from $\mathrm{Ch}(L)$ for some L (Borho-Brylinski).

If G is of type A_n, we have explicit conjectures for the above problems which are known to be true for $n \leqq 5$ (see Section 3.2 below).

0.2. One of the purposes of this paper is to determine $\mathrm{Ch}(L)$ and $\mathrm{Ch}(U(\mathfrak{g})/I)$ when the rank of G is not greater than three using several known facts concerning them. There was a conjecture that $\mathrm{Ch}(U(\mathfrak{g})/I)$ is always irreducible ([BoB2]), but our calculation shows that there exist counter examples for B_3 and C_3.

We also propose a modification of the above conjecture (Conjecture 3.4 below). It is a more refined form of the following:

Conjecture 0 3. Let $\mathscr{X}_0$ be the set of the primitive ideals of $U(\mathfrak{g})$ with trivial central character.

(i) If $\mathrm{Ch}(U(\mathfrak{g})/I_1)=\mathrm{Ch}(U(\mathfrak{g})/I_2)$ for $I_1, I_2 \in \mathscr{X}_0$, then $I_1=I_2$.

(ii) There exists a natural one-to-one correspondence:

$$\mathscr{X}_0 \simeq \coprod_{O \in \mathrm{Nilp}_s} \mathrm{Irr}(O \cap \mathfrak{n}), \tag{0.4}$$

where Nilp_s is the set of special nilpotent orbits in the sence of Lusztig (see Section 2.3), $\mathfrak{n}$ is the Lie algebra of the unipotent radical of a Borel subgroup and $\mathrm{Irr}(O \cap \mathfrak{n})$ is the set of the irreducible components of $O \cap \mathfrak{n}$.

It is known that a subset $\mathscr{X}_{0,O}$ of $\mathscr{X}_0$ is determined for each $O \in \mathrm{Nilp}_s$ and we have:

$$\#(\mathscr{X}_{0,O}) = \#(\mathrm{Irr}(O \cap \mathfrak{n}) \qquad (O \in \mathrm{Nilp}_s)$$

$$\mathscr{X}_0 = \coprod_{O \in \mathrm{Nilp}_s} \mathscr{X}_{0,O}.$$

Our conjecture states that there exists a *natural* one-to-one correspondence:

$$\mathscr{X}_{0,O} \simeq \mathrm{Irr}(O \cap \mathfrak{n}) \tag{0.5}$$

for $O \in \mathrm{Nilp}_s$.

Conjecture 3.4, which tells more about the expected one-to-one correspondence (0.5), is true for G of type A_n ([BoB2]) and for G of rank $\leqq 3$ (Section 4). It is also true for $O \in \mathrm{Nilp}_s$ satisfying the condition (3.6) below (Proposition 3.5). Although (3.6) sometimes fails, it holds for many O's (e.g. for all $O \in \mathrm{Nilp}_s$ in E_6).

0.3. The contents of this paper are as follows. In Section 1 we recall some known results concerning characteristic varieties. In Section 2 some relations between characteristic varieties and Weyl group representations are stated. These two sections may be considered as a survey of the results of Joseph, Borho-Brylinski, $\cdots$ concerning the characteristic varieties. In Section 3 the existence of $I \in \mathscr{X}_0$ such that $\mathrm{Ch}(U(\mathfrak{g})/I)$ is not irreducible is explained from the representation theory of the Hecke algebra. We also propose a conjecture and prove this for some cases. Section 4 is devoted to the calculations for the cases when the rank of G is not greater than three. In Appendix we give proofs of some facts concerning the Springer representations, which are probably well-known to the experts.

0.4. We use the following notation for $\mathscr{D}$-modules. For a non-singular algebraic variety Y over $\boldsymbol{C}$ we denote by $\mathscr{D}_Y$ the sheaf of the linear algebraic differential operators. When $f\colon Y \to V$ is a morphism of non-singular varieties, an $(f^{-1}\mathscr{D}_V, \mathscr{D}_Y)$-bimodule $\mathscr{D}_{V \leftarrow Y}$ and a $(\mathscr{D}_Y, f^{-1}\mathscr{D}_V)$-bimodule $\mathscr{D}_{Y \to V}$ are defined as usual. We set:

$$\mathscr{H}^j f^*(\mathscr{M}) = \mathscr{H}^j\left(\mathscr{D}_{Y \to V} \overset{L}{\underset{f^{-1}\mathscr{D}_V}{\otimes}} f^{-1}\mathscr{M}\right)$$

for a $\mathscr{D}_V$-module (or a complex of $\mathscr{D}$-modules) $\mathscr{M}$. When f is smooth and $\mathscr{M}$ is a $\mathscr{D}_V$-module, we have: $\mathscr{H}^j f^*(\mathscr{M}) = 0$ for $j \neq 0$ and we write $f^*\mathscr{M}$ instead of $\mathscr{H}^0 f^*(\mathscr{M})$. For a $\mathscr{D}_Y$-module (or a complex of $\mathscr{D}_Y$-modules) $\mathscr{N}$ we set:

$$\int_f^j \mathscr{N} = \mathscr{H}^j\left(\boldsymbol{R}f_*\left(\mathscr{D}_{V \leftarrow Y} \overset{L}{\underset{\mathscr{D}_Y}{\otimes}} \mathscr{N}\right)\right).$$

§ 1. Characteristic varieties

1.1. Associated varieties and characteristic varieties.

Let $U_i(\mathfrak{g})$ be the subspace of $U(\mathfrak{g})$ consisting of the elements of order $\leqq i$. Then the associated graded algebra $\mathrm{Gr}\,U(\mathfrak{g}) := \bigoplus_i (U_i(\mathfrak{g})/U_{i-1}(\mathfrak{g}))$ is naturally isomorphic to the symmetric algebra $S(\mathfrak{g})$. Let M be a finitely generated $U(\mathfrak{g})$-module. An increasing filtration $\{M_j\}_{j \in \boldsymbol{Z}}$ of M consisting of finite-dimensional subspaces is called a good filtration if the following conditions are satisfied:

(F1) $M_j = \{0\}$ for a sufficiently small j,

(F2) $M = \bigcup_{j \in \boldsymbol{Z}} M_j$,

(F3) $U_i(\mathfrak{g}) M_j \subset M_{i+j}$,

(F4) $U_i(\mathfrak{g}) M_j = M_{i+j}$ for a sufficiently large j.

If $\{M_j\}_{j\in\boldsymbol{Z}}$ is a good filtration of a finitely generated $U(\mathfrak{g})$-module, the associated graded module $\mathrm{Gr}M:=\bigoplus_{j\in\boldsymbol{Z}}(M_j/M_{j-1})$ is a finitely generated $S(\mathfrak{g})$-module. Since $S(\mathfrak{g})$ is naturally identified with the ring $\boldsymbol{C}[\mathfrak{g}^*]$ of polynomial functions on $\mathfrak{g}^*$, $\mathrm{Gr}M$ determines a cohernet sheaf $\mathrm{gr}M$ on $\mathfrak{g}^*$. The associated variety $V(M)$ of M is defined to be the support of $\mathrm{gr}M$, which is known to be independent of the choice of a good filtration.

Example. Let M be a finitely generated $U(\mathfrak{g})$-module.

(i) If M is finite-dimensional, then $V(M)=\{0\}$.

(ii) If $M=U(\mathfrak{g})/I$ for a left ideal I, then $V(M)$ is the zero set of the ideal:

$$\mathrm{Gr}I:=\bigoplus_{j\in\boldsymbol{Z}}(((I\cap U_j(\mathfrak{g}))+U_{j-1}(\mathfrak{g}))/U_{j-1}(\mathfrak{g}))$$

of $\mathrm{Gr}U(\mathfrak{g})=S(\mathfrak{g})=\boldsymbol{C}[\mathfrak{g}^*]$.

(iii) If M has a central character, then $V(M)$ is a subvariety of the nilpotent variety $\mathcal{N}=\{x\in\mathfrak{g}\,|\,x\text{: nilpotent}\}$. Here we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the Killing form.

For a non-singular algebraic variety V over $\boldsymbol{C}$ we denote by $\mathcal{D}_{V,i}$ the subsheaf of $\mathcal{D}_V$ consisting of differential operators of order $\leqq i$. Let $\mathcal{M}$ be a coherent $\mathcal{D}_V$-module. An increasing filtration $\{\mathcal{M}_j\}_{j\in\boldsymbol{Z}}$ of $\mathcal{M}$ consisting of coherent $\mathcal{O}_V$-submodules is called a good filtration if the conditions, which are obtained by replacing M_j and $U_i(\mathfrak{g})$ by $\mathcal{M}_j$ and $\mathcal{D}_{V,i}$ respectively in (F1)–(F4), are satisfied. If $\{\mathcal{M}_j\}_{j\in\boldsymbol{Z}}$ is a good filtration of $\mathcal{M}$, the associated graded module $\mathrm{Gr}\mathcal{M}$ is a coherent $\mathrm{Gr}\mathcal{D}_V$-module. Let $p\colon T^*X\to X$ be the cotangent bundle. Since $\mathrm{Gr}\mathcal{D}_V$ is naturally isomorphic to $p_*(\mathcal{O}_{T^*X})$, $\mathrm{Gr}\mathcal{M}$ determines a coherent $\mathcal{O}_{T^*V}$-module $\mathrm{gr}\mathcal{M}$. The characteristic variety $\mathrm{Ch}(\mathcal{M})$ of $\mathcal{M}$ is defined to be the support of $\mathrm{gr}\mathcal{M}$, which is known to be independent of the choice of a good filtration. It is known that $\mathrm{Ch}(\mathcal{M})$ is an involutive subvariety of T^*V in the sence of the symplectic geometry (Sato-Kawai-Kashiwara). Especially, any irreducible component of $\mathrm{Ch}(\mathcal{M})$ has dimension $\geqq \dim V$.

Example. Let $\mathcal{M}$ be a coherent $\mathcal{D}_V$-module.

(i) If $\mathcal{M}$ is coherent as an $\mathcal{O}_V$-module, then $\mathrm{Ch}(\mathcal{M})$ is the zero section of T^*X.

(ii) If $\mathcal{M}=\mathcal{D}_V/I$, then $\mathrm{Ch}(\mathcal{M})$ is the zero set of $\mathrm{Gr}I$ ($\subset p_*\mathcal{O}_{T^*V}$).

(iii) If Y is a non-singular closed subvariety of V and $\mathcal{M}=\mathcal{B}_{Y|V}$ $(:=\mathcal{H}_Y^{\operatorname{codim} Y}(\mathcal{O}_V))$, then $\mathrm{Ch}(\mathcal{M})$ is the conormal bundle T_Y^*V.

1.2. The Beilinson-Bernstein category equivalence.

Recall that G is a connected semisimple algebraic group over $\boldsymbol{C}$ with Lie algebra $\mathfrak{g}$ and X is the flag manifold.

For a $U(\mathfrak{g})$-module M we have a $\mathscr{D}_X$-module $\mathscr{D}_X \otimes_{U(\mathfrak{g})} M$ as noted in Introduction. This is a kind of localization.

Proposition 1.1 ([BeB]). *The localization stated above gives an equivalence between the abelian category of finitely generated $U(\mathfrak{g})$-modules with trivial central character and that of coherent $\mathscr{D}_X$-modules.*

Let $\gamma : T^*X \to \mathfrak{g}^*$ ($\simeq \mathfrak{g}$) be the natural map induced by the action of G on X (moment map).

Proposition 1.2 ([BoB2]). *For a finitely generated $U(\mathfrak{g})$-module M with trivial central character we have:*

$$V(M) = \gamma(\mathrm{Ch}(M)),$$

where $\mathrm{Ch}(M)$ *stands for* $\mathrm{Ch}(\mathscr{D}_X \otimes_{U(\mathfrak{g})} M)$.

1.3. Characteristic varieties of highest weight modules.

We fix a Borel subgroup B of G and a maximal torus H of G contained in B. The Lie algebras of B and H are denoted by $\mathfrak{b}$ and $\mathfrak{h}$, respectively. Let $\rho \in \mathfrak{h}^*$ be the half of the sum of the positive roots. Here the ordering on the root system is chosen so that the weights of $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ is positive. For an element w of the Weyl group W let M_w be the Verma module with highest weight $-w\rho - \rho$ and L_w its simple quotient. Let $\mathcal{O}_0$ be the abelian category of finitely generated $U(\mathfrak{g})$-module with trivial central character so that the action of $\mathfrak{b}$ lifts to the locally finite algebraic action of B. M_w and L_w are objects of $\mathcal{O}_0$ and the Grothendieck group $K(\mathcal{O}_0)$ has two bases $\{[M_w]\}_{w \in W}$ and $\{[L_w]\}_{w \in W}$.

Let M be an object of $\mathcal{O}_0$. It is easily seen that $V(M)$ is a B-stable subvariety of $\mathfrak{n}$. Hence we have $\mathrm{Ch}(M) \subset \gamma^{-1}(\mathfrak{n}) = \coprod_{w \in W} T^*_{X_w}X = \bigcup_{w \in W} \overline{T^*_{X_w}X}$ (see Section 1.2). Here X_w is the Schubert cell BwB/B. Hence any irreducible component of $\mathrm{Ch}(M)$ for $M \in \mathcal{O}_0$ is of the form $\overline{T^*_{X_w}X}$ for some $w \in W$. Taking into account the multiplicity of $\mathrm{gr}(\mathscr{D}_X \otimes_{U(\mathfrak{g})} M)$ at each irreducible component of $\mathrm{Ch}(M)$, the characteristic cycle $\mathbf{Ch}(M) \in \bigoplus_{w \in W} \boldsymbol{Z}_{\geq 0}[\overline{T^*_{X_w}X}]$ of M is defined. By the additivity of $\mathbf{Ch}$ we have a $\boldsymbol{Z}$-linear map

$$\mathbf{Ch}\colon K(\mathcal{O}_0) \longrightarrow \bigoplus_{w \in W} \boldsymbol{Z}[\overline{T^*_{X_w}X}].$$

We define $m(y, w) \in \boldsymbol{Z}_{\geq 0}$ for $y, w \in W$ by $\mathbf{Ch}(L_w) = \sum_{y \in W} m(y, w)[\overline{T^*_{X_y}X}]$. Setting $\Sigma(w) = \{y \in W \mid m(y, w) > 0\}$ we have $\mathrm{Ch}(L_w) = \bigcup_{y \in \Sigma(w)} \overline{T^*_{X_y}X}$.

It follows from Proposition 1.1 that the category $\mathcal{O}_0$ is equivalent to the category $\mathscr{A}(X, B)$ consisting of coherent $\mathscr{D}_X$-modules with B-action.

By the above arguments objects of $\mathscr{A}(X, B)$ are holonomic (actually they are regular holonomic ([BK], [BeB])).

Set $\partial X_w = \bar{X}_w - X_w$. By an easy calculation we see that $\mathscr{D}_X \otimes_{U(\mathfrak{g})} M_w | X - \partial X_w$ coincides with $\mathscr{B}_{X_w | X - \partial X_w} = \mathscr{H}^{d-l(w)}_{X_w}(\mathscr{O}_{X - \partial X_w})$. Here $d = \dim X$ and $l(w)$ is the length of w. Hence it is easily shown by induction on $l(w)$ that $\mathscr{D}_X \otimes_{U(\mathfrak{g})} L_w$ coincides with the minimal extension ${}^{\pi}\mathscr{B}_{X_w | X - \partial X_w}$ as a holonomic system (in the algebraic category).

We denote the Bruhat ordering on W by $\leqq$ ($y \leqq w$ if and only if $X_y \subset \bar{X}_w$) and the set of simple reflections in W by S. Set $\mathscr{L}(w) = \{s \in S \mid sw < w\}$ and $\mathscr{R}(w) = \{s \in S \mid ws < w\}$.

Lemma 1.3. (i) *$w \in \Sigma(w)$ and $m(w, w) = 1$.*

(ii) *If $y \in \Sigma(w)$, then $y \leqq w$.*

(iii) *If $y < w$ and X_y is contained in the non-singular part of $\bar{X}_w$, then $m(y, w) = 0$. Especially if w is the longest element of the subgroup of W generated by a subset of S, then $\Sigma(w) = \{w\}$.*

(iv) *If $y \in \Sigma(w)$, then $\mathscr{L}(y) \supset \mathscr{L}(w)$.*

Proof. The statements (i), (ii), (iii) follow from the fact that $\mathscr{D}_X \otimes_{U(\mathfrak{g})} L_w \simeq {}^{\pi}\mathscr{B}_{X_w | X - \partial X_w}$. The statement (iv) follows from the following. Let P be the parabolic subgroup of G containing B whose Levi part has the Weyl group $\langle \mathscr{L}(w) \rangle$. Then the decomposition of X into P-orbits gives a Whitney stratification of X, and X_y is open in PyB/B if and only if $\mathscr{L}(y) \supset \mathscr{L}(w)$.

1.4. Symmetry.

Let $\mathscr{A}(X \times X, G)$ be the abelian category consisting of coherent $\mathscr{D}_{X \times X}$-modules with (diagonal) G-actions. We define $i: X \to X \times X$ by $i(gB) = (eB, gB)$.

Lemma 1.4. (i) *For $\mathscr{M} \in \mathscr{A}(X \times X, G)$ we have $(\mathscr{H}^j i^*)(M) = 0$ for $j \neq 0$.*

(ii) *The functor $\mathscr{H}^0 i^*: \mathscr{A}(X \times X, G) \to \mathscr{A}(X, B)$ gives an equivalence of the abelian categories.*

Sketch of the proof. This follows from the following observation. The orbit decomposition of $X \times X$ under the diagonal action of G is given by $X \times X = \coprod_{w \in W} D_w$ ($D_w = G(eB, wB)$). Regarding $X \times X$ as a fiber bundle over $X = G/B$ via the projection onto the first factor, we have $X \times X \simeq G \times^B X$ and $D_w \simeq G \times^B X_w$.

Let $\mathscr{L}_w$ and $\mathscr{M}_w$ be the objects of $\mathscr{A}(X \times X, G)$ so that $\mathscr{H}^0 i^*(\mathscr{L}_w) = \mathscr{D}_X \otimes_{U(\mathfrak{g})} L_w$ and $\mathscr{H}^0 i^*(\mathscr{M}_w) = \mathscr{D}_X \otimes_{U(\mathfrak{g})} M_w$. Then we have $\mathscr{L}_w \simeq {}^{\pi}\mathscr{B}_{D_w | X \times X - \partial D_w}$. Set $Z_w = \overline{T^*_{D_w}(X \times X)}$. It is easily seen that $\mathrm{Ch}(M) \subset$

$\bigcup_{w\in W} Z_w$ for $\mathcal{M} \in \mathcal{A}(X\times X, G)$. As in Section 1.3 we have a $\boldsymbol{Z}$-linear map $\mathbf{Ch}$: $K(\mathcal{A}(X\times X, G)) \to \bigoplus_{w\in W} \boldsymbol{Z}[Z_w]$. The following holds by the same reason as the one for Lemma 4.1.

Lemma 1.5. *The diagram*:

$$\begin{array}{ccc} K(\mathcal{A}(X\times X, G)) & \xrightarrow{\mathbf{Ch}} & \bigoplus_{w\in W} \boldsymbol{Z}[Z_w] \\ \downarrow & & \downarrow \\ K(\mathcal{O}_0) & \xrightarrow{\mathbf{Ch}} & \bigoplus_{w\in W} \boldsymbol{Z}[\overline{T^*_{X_w}X}] \end{array}$$

commutes. Here the left vertical arrow is the isomorphism induced by $\mathcal{A}(X\times X, G)\simeq\mathcal{A}(X, B)\simeq\mathcal{O}_0$ *and the right vertical arrow is the* $\boldsymbol{Z}$*-linear map given by* $[Z_w]\mapsto[\overline{T^*_{X_w}X}]$. *Especially, we have* $\mathbf{Ch}(\mathcal{L}_w)=\sum_{y\in W} m(y, w)[Z_y]$.

We define j: $X\times X\to X\times X$ by $j(g_1B, g_2B)=(g_2B, g_1B)$. It is a G-equivariant isomorphism such that $j(D_w)=D_{w^{-1}}$. Hence $j^*(\mathcal{L}_w)=\mathcal{L}_{w^{-1}}$. It is easily seen that the diagram:

$$\begin{array}{ccccc} K(\mathcal{A}(X\times X, G)) & \xrightarrow{\mathbf{Ch}} & \bigoplus_{w\in W} \boldsymbol{Z}[Z_w] & \ni & [Z_w] \\ {\scriptstyle j^*}\downarrow & & \downarrow & & \downarrow \\ K(\mathcal{A}(X\times X, G)) & \xrightarrow{\mathbf{Ch}} & \bigoplus_{w\in W} \boldsymbol{Z}[Z_w] & \ni & [Z_{w^{-1}}] \end{array}$$

commutes. Hence we have the following.

Lemma 1.6. $m(y, w)=m(y^{-1}, w^{-1})$. *Especially* $\Sigma(w)=(\Sigma(w^{-1}))^{-1}$. *Hence if* $y \in \Sigma(w)$, *then* $\mathcal{R}(y)\supset\mathcal{R}(w)$.

1.5. Cells in the Weyl groups ([J1], [KL1]).

We define integers $a(y, w)$ ($y, w \in W$) by $[L_w]=\sum_{y\in W} a(y, w)[M_y]$ (in $K(\mathcal{O}_0)$). By the Kazhdan-Lusztig conjecture ([KL1]) which is proved in [BK] and [BeB] these integers are computable (at least in principle). Set $\boldsymbol{a}(w)=\sum_{y\in W} a(y, w)y \in \boldsymbol{Q}[W]$. Note that $\{\boldsymbol{a}(w)\}_{w\in W}$ is a basis of $\boldsymbol{Q}[W]$. A subspace of $\boldsymbol{Q}[W]$ is said to be $\boldsymbol{a}$-basal if it is spanned by a subset of $\{\boldsymbol{a}(w)\}_{w\in W}$. We denote by $\bar{V}^L_w$ the minimal $\boldsymbol{a}$-basal subspace which contains $\boldsymbol{a}(w)$ and is invariant under the *left* action of W. Define a preorder $\underset{L}{\geqq}$ and an equivalence relation $\underset{L}{\sim}$ on W by:

$$w\underset{L}{\geqq} y \text{ if and only if } \bar{V}^L_w\supset\bar{V}^L_y,$$

$$w\underset{L}{\sim} y \text{ if and only if } \bar{V}^L_w=\bar{V}^L_y.$$

Equivalence classes of $\underset{L}{\sim}$ are called left cells and the left cell containing w

is denoted by $\mathscr{C}_w^L$. The representation of W on the space $V_w^L := \bar{V}_w^L/$(sum of $\bar{V}_y^L$'s which are properly contained in $\bar{V}_w^L$) is called the left cell representation attached to the left cell $\mathscr{C}_w^L$. Replacing "the *left* action of W" above by "the *right* action of W" (resp. "the *two-sided* action of $W \times W$"). we have the similar notions $\bar{V}_w^R$, $\geqq_R$, $\sim_R$, $\mathscr{C}_w^R$, V_w^R (resp. $\bar{V}_w^{LR}$, $\geqq_{LR}$, $\sim_{LR}$, $\mathscr{C}_w^{LR}$, V_w^{LR}).

By the arguments as in Section 1.4 (or by the Kazhdan-Lusztig conjecture) we see that $a(y, w) = a(y^{-1}, w^{-1})$ for $y, w \in W$. Hence $w \geqq_L y$ if and only if $w^{-1} \geqq_R y^{-1}$, and $w \sim_L y$ if and only if $w^{-1} \sim_R y^{-1}$. Let $W^\wedge$ be the set of the irreducible representations of W over $\boldsymbol{Q}$. Since any irreducible representation of W over $\boldsymbol{Q}$ is absolutely irreducible, we have $\boldsymbol{Q}[W] \simeq \bigoplus_{\sigma \in W^\wedge} (\sigma \otimes \sigma)$ as a $W \times W$-module and hence $w \sim_{LR} w^{-1}$ for any $w \in W$.

1.6. Characteristic varieties of primitive quotients.

We denote by $\mathscr{X}_0$ the set of the primitive ideals of $U(\mathfrak{g})$ with trivial central character, that is,

$$\mathscr{X}_0 = \{\mathrm{Ann}(M) \mid M\text{: irreducible } U(\mathfrak{g})\text{-module with trivial central character}\},$$

where $\mathrm{Ann}(M) = \{u \in U(\mathfrak{g}) \mid u \cdot M = \{0\}\}$. Set $I_w = \mathrm{Ann}(L_w) \in \mathscr{X}_0$.

Proposition 1.7 ([D]). $\mathscr{X}_0 = \{I_w \mid w \in W\}$.

Proposition 1.8 ([J1], [V]). *$I_w \subset I_y$ if and only if $w \geqq_L y$, and hence $I_w = I_y$ if and only if $w \sim_L y$.*

Therefore we have $\mathscr{X}_0 \simeq W/\sim_L$.

Since I_w is a two-sided ideal of $U(\mathfrak{g})$, $\mathrm{Ch}(U(\mathfrak{g})/I_w)$ is a G-invariant closed subvariety of T^*X. We identify T^*X with $G \times^B \mathfrak{n}$ via the Killing form.

Proposition 1.9 ([BoB2]). $\mathrm{Ch}(U(\mathfrak{g})/I_w) = G \times^B V(L_{w^{-1}})$ *for* $w \in W$.

Corollary 1.10 ([J3], [BoB2]). *If $w \geqq_R y$, then $V(L_w) \supset V(L_y)$. Hence if $w \sim_R y$, then $V(L_w) = V(L_y)$.*

§ 2. Weyl group representations

2.1. The Springer representations.

We denote by Nilp the set of nilpotent orbits in $\mathfrak{g}$. Let $O \in$ Nilp and $x \in O$. The set of the irreducible local systems over $\boldsymbol{Q}$ on O with G-actions is denoted by $\mathscr{S}_O$. Set $A(x) = Z_G(x)/Z_G(X)^0$, where $Z_G(x)$ is the centralizer of x. Then there exists a natural one-to-one correspondence:

$$\mathscr{S}_0 \simeq A(x)^\wedge \qquad (\xi \leftrightarrow \chi_\xi),$$

where $A(x)^\wedge$ is the set of the irreducible representations of $A(x)$ over $\boldsymbol{Q}$. It is known that any irreducible representation of $A(x)$ over $\boldsymbol{Q}$ is absolutely irreducible.

Set $d=\dim X$ and $d_O=d-(\dim O)/2$. The variety

$$X^x := \{gB \in X \mid x \in g\cdot\mathfrak{n}\}$$

has pure dimension d_O ([Spa]). An action of the Weyl group W on the top homology group $H_{2d_O}(X^x)$ $(:=\mathrm{Hom}_{\boldsymbol{Q}}(H_c^{2d_O}(X^x, \boldsymbol{Q}), \boldsymbol{Q}))$, which commutes with the natural action of $A(x)$, is defined (the Springer representation, see [Spr]).

Let $H_{2d_O}(X^x) \simeq \bigoplus_{\xi\in\mathscr{S}_O}(\tau_{(O,\xi)}\otimes\chi_\xi)$ be the decomposition as a $W\times A(x)$-module. Set $\mathscr{S}'_O=\{\xi \in \mathscr{S}_O \mid \tau_{(O,\xi)}\neq 0\}$. Then it is known that $\tau_{(O,\xi)}$ is irreducible for $\xi \in \mathscr{S}'_O$ and we have:

$$\coprod_{O\in\mathrm{Nilp}} \mathscr{S}'_O \xrightarrow{\sim} W^\wedge \qquad (\mathscr{S}'_O \ni \xi \mapsto \tau_{(O,\xi)} \in W^\wedge).$$

We denote $\tau_{(O,\mathcal{Q}_O)}$ $(=H_{2d_O}(X^x)^{A(x)})$, which is clearly non-zero, by $\mathrm{Sp}(O)$.

For a pure-dimensional variety V we denote the set of its irreducible components by $\mathrm{Irr}(V)$. The variety $D=\{(y, gB) \mid y \in g\cdot\mathfrak{n}\}$ is isomorphic to $G\times^B(O\cap\mathfrak{n})$ as a fiber bundle over $X=G/B$ and is isomorphic to $G\times^{Z_G(x)}X^x$ as a fiber bundle over $O=G/Z_G(x)$. Since $\mathrm{Irr}(\overline{O\cap\mathfrak{n}})\simeq \mathrm{Irr}(O\cap\mathfrak{n})\simeq\mathrm{Irr}(D)$, we have a surjection:

$$h\colon \mathrm{Irr}(X^x)\longrightarrow\mathrm{Irr}(\overline{O\cap\mathfrak{n}}) \tag{2.1}$$

and the inverse image of each element of $\mathrm{Irr}(\overline{O\cap\mathfrak{n}})$ under h is an $A(x)$-orbit. Hence we have $\mathrm{Irr}(X^x)/A(x)\simeq\mathrm{Irr}(\overline{O\cap\mathfrak{n}})$. For each $Y \in \mathrm{Irr}(\overline{O\cap\mathfrak{n}})$ set:

$$q_Y=\sum_{C\in h^{-1}(Y)}[C] \in H_{2d_O}(X^x)^{A(x)}=\mathrm{Sp}(O).$$

Then $\{q_Y \mid Y \in \mathrm{Irr}(\overline{O\cap\mathfrak{n}})\}$ is a basis of $\mathrm{Sp}(O)$, especially we have $\dim\mathrm{Sp}(O)=\sharp(\mathrm{Irr}(\overline{O\cap\mathfrak{n}}))$.

Let $\mathfrak{h}^*_{\boldsymbol{Q}}$ be the $\boldsymbol{Q}$-form of $\mathfrak{h}^*$ spanned by the roots. The Weyl group W acts on $S^m(\mathfrak{h}^*_{\boldsymbol{Q}})$, the space of symmetric m-th tensors, for each $m \in \boldsymbol{Z}_{\geq 0}$. By a theorem of Borho-MacPherson [BM] we have:

$$\dim\mathrm{Hom}_W(\mathrm{Sp}(O), S^m(\mathfrak{h}^*_{\boldsymbol{Q}}))=\begin{cases}0 & (m<d_O)\\ 1 & (m=d_O).\end{cases}$$

Let σ_O be the unique W-submodule of $S^{d_O}(\mathfrak{h}_Q^*)$ which is isomorphic to $\mathrm{Sp}(O)$.

For an H-invariant subvariety Y of $\mathfrak{n}$ Joseph [J3] defined a polynomial $p_Y \in S^{d-\dim Y}(\mathfrak{h}_Q^*)$ as follows. Let $I(Y)$ be the defining ideal of Y. Since Y is H-invariant, $\mathfrak{h}$ acts on the coordinate algebra $M=C[\mathfrak{n}]/I(Y)$ locally semisimply with weights in $-\sum_{\alpha\in\Delta^+} Z_{\geqq 0}\alpha$. Here Δ^+ denotes the set of positive roots. For $h\in\mathfrak{h}$ with $\alpha(h)\in Z_{<0}$ for any $\alpha\in\Delta^+$, set $M_i^h=\{m\in M \mid h\cdot m=im\}$. Then p_Y is defined by:

$$\sum_{i=0}^{k} \dim M_i^h=(-1)^d p_Y(h)/(\prod_{\alpha\in\Delta^+}\alpha(h))k^{\dim Y}+O(k^{\dim Y-1}).$$

If $Y\in\mathrm{Irr}(O\cap\mathfrak{n})$ for $O\in\mathrm{Nilp}$, then $\dim Y=(\dim O)/2$ and hence $d-\dim Y=d_O$.

Proposition 2.2 ([H], see also [BBM]). *The set $\{p_Y \mid Y\in\mathrm{Irr}(\overline{O\cap\mathfrak{n}})\}$ is a basis of σ_O, and an isomorphism from* $\mathrm{Sp}(O)$ *to σ_O of W-modules is given by $q_Y\mapsto p_Y$.*

2.2. Joseph's Goldie rank polynomials.

We denote Joseph's Goldie rank polynomial ([J2]) corresponding to L_w by p_w, that is,

$$p_w=\sum_{y\in W} a(y, w)(-y^{-1}\rho)^{d_w}\in S^{d_w}(\mathfrak{h}_Q^*),$$

where d_w is the least non-negative integer so that the right hand side is non-zero.

Proposition 2.3 ([J2]). (i) *$Qp_w=Qp_y$ if and only if $w\sim_L y$.*

(ii) *If $w\sim_{LR} y$, then $d_w=d_y$.*

(iii) *The space $\sigma(w):=\sum_{y\in\mathscr{C}_w^{LR}} Qp_y$ is W-invariant and is irreducible as a W-module.*

(iv) $\sigma(w)=\bigoplus_{y\in\mathscr{C}_w^{LR}/\sim_L} Qp_y$.

(v) $\dim \mathrm{Hom}_W(\sigma(w), S^m(\mathfrak{h}_Q^*))=\begin{cases}0 & (m<d_w)\\ 1 & (m=d_w).\end{cases}$

(vi) *$\sigma(w)=\sigma(y)$ if and only if $w\sim_{LR} y$.*

Hence an injection $W/\sim_{LR}\to W^\wedge$ is determined by $w\to\sigma(w)$. The image $W_s^\wedge$ of this map coincides with the set of the special representations defined by Lusztig ([BV1, 2]).

2.3. Irreducibility of $V(U(\mathfrak{g})/I_w)$.

A closed subvariety of $\mathfrak{n}$ which is an irreducible component of $\overline{O\cap\mathfrak{n}}$

for some $O \in \mathrm{Nilp}$ is called an orbital variety (associated to O). Set $Y^r(w)=\overline{B\cdot(\mathfrak{n}\cap w\mathfrak{n})}$ and $Y^l(w)=Y^r(w^{-1})$. We define $O_w^{lr} \in \mathrm{Nilp}$ by $\overline{O_w^{lr}}=\overline{G\cdot(\mathfrak{n}\cap w\mathfrak{n})}$. Then $Y^l(w)$ and $Y^r(w)$ are orbital varieties associated to O_w^{lr}. Furthermore any orbital variety coincides with $Y^r(w)$ for some $w \in W$ ([St]).

Since $V(L_w)=\gamma(\mathrm{Ch}(L_w))=\bigcup_{y\in\Sigma(w)}\gamma(\overline{T^*_{X_y}X})=\bigcup_{y\in\Sigma(w)}Y^r(y)$, each irreducible component of $V(L_w)$ is an orbital variety.

Considering the character of L_w as an $\mathfrak{h}$-module, we have $p_{w^{-1}}=\sum_Y c_Y p_Y$ for some $c_Y>0$, where Y runs through the irreducible components of $V(L_w)$ with maximal dimension ([J3]). On the other hand by the pure-dimensionality theorem of Kashiwara-Gabber $V(L_w)$ is pure-dimensional. Hence we have the following.

Proposition 2.4 ([J3]). $p_{w^{-1}}=\sum_{Y\in\mathrm{Irr}(V(L_w))}c_Y p_Y$ *for some* $c_Y>0$.

By Propositions 2.2, 2.3 and 2.4 we see that $W_s^\wedge$ is contained in the image of $\mathrm{Sp}\colon \mathrm{Nilp}\to W^\wedge$. Nilpotent orbits which belong to $\mathrm{Nilp}_s=\{O\in\mathrm{Nilp}\mid \mathrm{Sp}(O)\in W_s^\wedge\}$ are called special nilpotent orbits. A bijection:

$$W/\underset{LR}{\sim}\longrightarrow \mathrm{Nilp}_s \quad (w\to O_w^{LR})$$

is determined by $\sigma_{O_w^{LR}}=\sigma(w)$. By the above arguments we have:

Proposition 2.5. $\mathrm{Irr}(V(L_w))\subset \mathrm{Irr}(\overline{O_w^{LR}\cap\mathfrak{n}})$ *for* $w\in W$.

Hence by Propositions 1.2, 1.9 and 2.5 we have:

Proposition 2.6 (Borho-Brylinski, Joseph).

$V(U(\mathfrak{g})/I_w)=\overline{O_w^{LR}}$ *for* $w\in W$.

The irreducibility of $V(U(\mathfrak{g})/I_w)$ was conjectured by Borho and was proved by Borho-Brylinski [BoB1] using case-by-case method. Later the unified proof indicated above was obtained using results of several people. Joseph ([J4]) has given a different proof without using Proposition 1.9.

The follownig is clear from Propositions 1.2 and 2.5.

Lemma 2.7. $V(L_w)=\bigcup_{y\in\Sigma(w)}Y^r(y)=\bigcup_{\substack{y\in\Sigma(w)\\ O_y^{lr}=O_w^{LR}}}Y^r(y)$ *for* $w\in W$.

2.4. The Springer representations and the Steinberg cells.

The variety $Z=\bigcup_{w\in W}Z_w$ $(Z_w=\overline{T^*_{D_w}(X\times X)})$ has pure dimension $2d$ $(d=\dim X)$. Kazhdan-Lusztig [KL2] defined a $W\times W$-module structure on $H_{4d}(Z)$ $(=\mathrm{Hom}_{\boldsymbol{Q}}(H_c^{4d}(Z,\boldsymbol{Q}),\boldsymbol{Q})=\bigoplus_{w\in W}\boldsymbol{Q}[Z_w])$ and showed that an isomorphism $f\colon H_{4d}(Z)\to\boldsymbol{Q}[W]$ of $W\times W$-modules is given by $f([Z_e])=e$. Hence setting $\boldsymbol{b}(w)=f([Z_w])$, we have a basis $\{\boldsymbol{b}(w)\}_{w\in W}$ of $\boldsymbol{Q}[W]$.

Proposition 2.8 ([KL2]). *Let $s \in S$ and $w \in W$.*

(i) *If $sw<w$, then $s\boldsymbol{b}(w)=-\boldsymbol{b}(w)$.*

(ii) *If $sw>w$, then $s\boldsymbol{b}(w)=\boldsymbol{b}(w)+\boldsymbol{b}(sw)+\sum_{sy<y<sw}\delta_s(y, w)\boldsymbol{b}(y)$, where $\delta_s(y, w)$ are certain non-negative integers.*

A subspace of $\boldsymbol{Q}[W]$ is called a $\boldsymbol{b}$-basal subspace if it is spanned by a subset of $\{\boldsymbol{b}(w)\}_{w\in W}$. Using $\boldsymbol{b}$-basal subspaces instead of $\boldsymbol{a}$-basal subspaces in the definitions in Section 1.6 we have the notions corresponding to $\bar{V}^L_w$, $\geqq_L$, $\cdots$, $\mathscr{C}^{LR}_w$. They will be denoted by $\bar{V}^l_w$, $\geqq_l$, $\cdots$, $\mathscr{C}^{lr}_w$ replacing L and R by l and r, respectively.

Lemma 2.9 ([KL2]). *Let $O \in$ Nilp. The subspace $\bigoplus_{O^{lr}_w\subset\bar{O}}\boldsymbol{Q}\boldsymbol{b}(w)$ of $\boldsymbol{Q}[W]$ is $W\times W$-invariant. If $x \in O$, then we have:*

$$\Big(\bigoplus_{O^{lr}_w\subset\bar{O}}\boldsymbol{Q}\boldsymbol{b}(w)\Big)\Big/\Big(\bigoplus_{O^{lr}_w\subset\bar{O}-O}\boldsymbol{Q}\boldsymbol{b}(w)\Big)\simeq(H_{2d_O}(X^x)\otimes H_{2d_O}(X^x))^{A(x)}$$
$$\simeq\bigoplus_{\xi\in\mathscr{S}'_O}(\tau_{(O,\xi)}\otimes\tau_{(O,\xi)}),$$

as $W\times W$-modules.

Similarly we have the following.

Lemma 2.10. *If Y is an orbital variety, the subspace $\bigoplus_{Y^l(w)\subset Y}\boldsymbol{Q}\boldsymbol{b}(y)$ (resp. $\bigoplus_{Y^r(w)\subset Y}\boldsymbol{Q}\boldsymbol{b}(y)$) of $\boldsymbol{Q}[W]$ is invariant under the left (resp. right) action of W. Hence if $w\geqq_l y$ (resp. $w\geqq_r y$), then $Y^l(w)\supset Y^l(y)$ (resp. $Y^l(w)\supset Y^l(y)$).*

The following Lemma was suggested to the author by Joseph. See Appendix for the proof.

Lemma 2.11. (i) *$w\sim_l y$ if and only if $Y^l(w)=Y^l(y)$.*

(ii) *$w\sim_r y$ if and only if $Y^r(w)=Y^r(y)$.*

(iii) *$w\sim_{lr} y$ if and only if $O^{lr}_w=O^{lr}_y$.*

2.5. $\{\boldsymbol{a}(w)\}_{w\in W}$ and $\{\boldsymbol{b}(w)\}_{w\in W}$.

Let p_{ij}: $X\times X\times X\to X\times X$ be the projection onto the (i,j)-factor. The Grothendieck group $K(\mathscr{A}(X\times X, G))$ is endowed with a ring structure via the product:

$$[\mathscr{M}_1]\cdot[\mathscr{M}_2]=\sum_i(-1)^i\Big[\int^i_{p_{13}}\Big(p^*_{12}\mathscr{M}_1\overset{L}{\underset{\mathcal{O}_{X\times X\times X}}{\otimes}}p^*_{23}\mathscr{M}_2\Big)\Big]$$

for $\mathscr{M}_1$, $\mathscr{M}_2 \in \mathscr{A}(X\times X, G)$ and the isomorphism $K(\mathscr{A}(X\times X, G))\simeq\boldsymbol{Z}[W]$ of rings is given by $[\mathscr{M}_w]\leftrightarrow w$. Especially $K(\mathscr{A}(X\times X, G))\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$ is a $W\times W$-module.

Proposition 2.12 ([KT], [T2]). *The Q-linear map*:

$$\mathbf{Ch}\colon K(\mathscr{A}(X\times X,G))\underset{Z}{\otimes}Q\to H_{4d}(Z)=\bigoplus_{w\in W}Q[Z_w]$$

is a homomorphism of $W\times W$-modules.

Since $\mathbf{Ch}(\mathscr{M}_e)=[Z_e]$, the above Proposition is equivalent to the commutativity of the diagram:

$$\begin{array}{ccc} K(\mathscr{A}(X\times X,G)) & \xrightarrow{\mathbf{Ch}} & H_{4d}(Z) \\ \downarrow & & \downarrow f \\ Z[W] & \hookrightarrow & Q[W]. \end{array}$$

It is also equivalent to the following.

Lemma 2.13. $\boldsymbol{a}(w)=\sum_{y\in W}m(y,w)\boldsymbol{b}(y)$ *for $w\in W$.*

Remark. Proposition 2.12 is generalized in [T3], where we consider the Hecke algebra instead of the Weyl group.

§ 3. Counter examples and conjectures

3.1. Counter examples and Hecke algebras.

Borho-Brylinski [BoB2] conjectured that the characteristic varieties $\mathrm{Ch}(U(\mathfrak{g})/I_w)$ of the primitive quotients $U(\mathfrak{g})/I_w$ are always irreducible. By Proposition 1.9 it is equivalent to the irreducibility of $V(L_{w^{-1}})$. It is also equivalent to the following statement (A) by Proposition 2.3.

(A) The two natural bases $\{p_Y\mid Y\in\mathrm{Irr}(\overline{O_w^{LR}\cap\mathfrak{n}})\}$ and $\{p_{y^{-1}}\mid y\in\mathscr{C}_w^{LR}/\sim_R\}$ of the special representation $\mathrm{Sp}(O_w^{LR})\simeq\sigma_{O_w^{LR}}=\sigma(w)$ coincide up to constant multiples.

Yet the calculations in Section 4 show that there exist counter examples to the conjecture above in B_3 and C_3. The existence of the counter examples can be explained from the representation theory of the Hecke algebra as follows.

Let $\mathscr{H}$ be the Hecke algebra of W. It is an algebra over the Laurent polynomial ring $Q[q^{1/2},q^{-1/2}]$ so that $\mathscr{H}\otimes_{Q[q^{1/2},q^{-1/2}]}Q\simeq Q[W]$ via the specialization: $q^{1/2}\mapsto 1$. We have the notion of W-graphs and to each W-graph a representation of $\mathscr{H}$ with a specified basis is associated ([KL1]).

Proposition 3.1. *For any special representation* $\mathrm{Sp}(O)$ ($O\in\mathrm{Nilp}_s$) *there exists a representation of $\mathscr{H}$ arising from a W-graph such that its specialization via*: $q^{1/2}\mapsto 1$ *is isomorphic to* $\mathrm{Sp}(O)$ *and the basis of* $\mathrm{Sp}(O)$

$(\simeq\sigma_O)$ comming from the W-graph coindides with $\{p_{y^{-1}} | y \in \mathscr{C}_w^{LR}/\sim_R\}$ $(O_w^{LR}=O)$ up to constant multiples.

Proof. Let τ be the (unique) two-sided cell representation containing $\mathrm{Sp}(O)\otimes\mathrm{Sp}(O)$ and $\tau=\bigoplus_i \tau_i$ be the decomposition into right cell representations. τ is a $W\times W$-module and each τ_i is a $1\times W$-submodule. Since $\deg(p_{y_1})<\deg(p_{y_2})$ for y_1, y_2 with $y_1 \geqq_{LR} y_2$ and $y_1 \gneqq_{LR} y_2$, the W-homomorphism $\boldsymbol{Q}[W]\to S^{d_O}(\mathfrak{h}_{\boldsymbol{Q}}^*)$ $(y\to(y\rho)^{d_O})$ induces a surjective W-homomorphism $r: \tau\to\sigma_O$ so that each $r(\tau_i)$ is spanned by some $p_{y^{-1}}$ (see Proposition 2.3). Here the actions of W on $\boldsymbol{Q}[W]$ and τ are induced from those of $W\times 1$. Consider the vector space $\mathrm{Hom}_W(\mathrm{Sp}(O), \tau)$. Here the W-module structure on τ is given by the action of $1\times W$. Then the action of $W\times 1$ on τ induces a W-module structure on $\mathrm{Hom}_W(\mathrm{Sp}(O), \tau)$ $(=\bigoplus_i \mathrm{Hom}_W(\mathrm{Sp}(O), \tau_i))$, which is clearly isomorphic to $\mathrm{Sp}(O)$. If x is a non-zero vector of $\mathrm{Sp}(O)$, the map F_x: $\mathrm{Hom}_W(\mathrm{Sp}(O), \tau)\to\sigma_O$ $(f\mapsto r(f(x)))$ is a non-zero intertwining operator and hence an isomorphism. By definition we have $F_x(\mathrm{Hom}_W(\mathrm{Sp}(O), \tau_i))\subset r(\tau_i)$. Therefore $\dim \mathrm{Hom}_W(\mathrm{Sp}(O), \tau_i)=1$ and $F_x(\mathrm{Hom}_W(\mathrm{Sp}(O), \tau_i))$ is spanned by some $p_{y^{-1}}$ for each i.

On the other hand by [Gy] there exists a representation of $\mathscr{H}$ arising from a W-graph such that its specialization via: $q^{1/2}\to 1$ is isomorphic to $\mathrm{Sp}(O)$ $(\simeq\mathrm{Hom}_W(\mathrm{Sp}(O), \tau))$ and each element of the basis of $\mathrm{Sp}(O)$ comming from the W-graph spans $\mathrm{Hom}_W(\mathrm{Sp}(O), \tau_i)$ for some i. Hence the lemma.

On the other hand Kazhdan-Lustztig found counter examples to the following hypothesis (B) in 1979 (unpublished).

(B) For any special representation $\mathrm{Sp}(O)$ $(O\in \mathrm{Nilp}_s)$ there exists a representation of $\mathscr{H}$ arising from a W-graph such that its specialization via: $q^{1/2}\to 1$ is isomorphic to $\mathrm{Sp}(O)$ and the basis of $\mathrm{Sp}(O)$ comming from the W-graph coincides with $\{q_Y \mid Y\in \mathrm{Irr}(\overline{O\cap \mathfrak{n}})\}$.

Hence by Proposition 3.1 counter examples to (B) due to Kazhdan and Lusztig imply the existence of counter examples to (A).

The author thanks Kazhdan and Lusztig for informing him of the above counter examples.

3.2. Conjectures.

We have the following conjectures for type A_n ([KT], [BoB2]).

Conjecture 3.2. If G is of type A_n, then $\mathrm{Ch}(L_w)=\overline{T_{X_w}^* X}$ for any $w\in W$.

Conjecture 3.3. If G is of type A_n, then $V(L_w)=Y^r(w)$ for any $w\in W$.

Since $\gamma(\overline{T^*_{X_w}X})=Y^r(w)$, Conjecture 3.2 implies Conjecture 3.3 by Proposition 1.2. By Lemma 2.13, Conjecture 3.2 is equivalent to a conjecture of Kazhdan-Lusztig ([KL2]) which says that if G is of type A_n, then $\boldsymbol{a}(w)=\boldsymbol{b}(w)$ for any $w \in W$.

In general $V(L_w)$ is not irreducible. We would like to propose the following.

Conjecture 3.4. Let O be a special nilpotent orbit and $\mathscr{C}$ the two-sided cell corresponding to O ($W/\sim_{LR}\simeq\mathrm{Nilp}_s$, Section 2.3). Then there exists a bijection from $\mathscr{C}/\sim_R$ to $\mathrm{Irr}(\overline{O\cap\mathfrak{n}})$ ($w\to Y_w$) and an ordering $\prec$ on the set $\mathrm{Irr}(\overline{O\cap\mathfrak{n}})$ such that $V(L_w)=Y_w\cup\tilde{Y}_w$. Here $\tilde{Y}_w$ is the union of some Y's in $\mathrm{Irr}(\overline{O\cap\mathfrak{n}})$ with $Y\prec Y_w$.

The above conjecture is equivalent to the following.

(A′) For any special representation the matrix describing the relation of the two natural bases is triangular.

This is proved for type A_n ([BoB2]; Section 6.10).

3.3. A partial result.

We define Φ: $W^\wedge\to\mathrm{Nilp}_s$ and Ψ: $W^\wedge\to\mathrm{Nilp}$ as follows. For $\sigma\in W^\wedge$ if $\sigma\otimes\sigma\subset V_w^{LR}$ (resp. $\sigma\otimes\sigma\subset V_w^{lr}$), then $\Phi(\sigma)=O_w^{LR}$ ($\Psi(\sigma)=O_w^{lr}$). Note that $\overline{O_w^{lr}}\subset\overline{O_w^{LR}}$. Indeed $\overline{O_w^{LR}}=\overline{G\cdot V(L_w)}\supset\overline{G\cdot Y^r(w)}=\overline{O_w^{lr}}$.

Proposition 3.5. *Let* $O\in\mathrm{Nilp}_s$. *If the condition*:

$$\{\tau\in W^\wedge\,|\,\Phi(\tau)\subset\bar{O}\}=\{\tau\in W^\wedge\,|\,\Psi(\tau)\subset\bar{O}\} \tag{3.6}$$

holds, then Conjecture 3.4 *holds for O*.

Proof. As $W\times W$-modules $\sum_{O_w^{LR}\subset\bar{O}}\boldsymbol{Q}\boldsymbol{a}(w)\simeq\sum_{\Phi(\tau)\subset\bar{O}}(\tau\otimes\tau)$ and $\sum_{O_w^{lr}\subset\bar{O}}\boldsymbol{Q}\boldsymbol{b}(w)\simeq\sum_{\Psi(\tau)\subset\bar{O}}(\tau\otimes\tau)$. Hence by our assumption we have $\sum_{O_w^{LR}\subset\bar{O}}\boldsymbol{Q}\boldsymbol{a}(w)=\sum_{O_w^{lr}\subset\bar{O}}\boldsymbol{Q}\boldsymbol{b}(w)$. Since $\boldsymbol{a}(w)\in\boldsymbol{b}(w)+\sum_{y<w}\boldsymbol{Z}_{\geqq0}\boldsymbol{b}(y)$, $O_w^{LR}\subset\bar{O}$ if and only if $O_w^{lr}\subset\bar{O}$. Noting that $O_w^{lr}\subset\overline{O_w^{LR}}$ we have the following.

(3.7) If $O_w^{lr}=O$, then $O_w^{LR}=O$ especially $Y^r(w)\in\mathrm{Irr}(V(L_w))$.

We define a sequence:

$$\phi=J_0\subset J_1\subset J_2\subset\cdots\subset\mathrm{Irr}(\overline{O\cap\mathfrak{n}})$$

of subsets of $\mathrm{Irr}(\overline{O\cap\mathfrak{n}})$ inductively as follows. When J_{t-1} is already defined, then

$$J_t=\{Y^r(w)\,|\,O_w^{lr}=O,\ V(L_w)\subset Y^r(w)\cup(\bigcup_{Y\in J_{t-1}}Y)\}.$$

We prove the following.

(3.8) If $J_{t-1} \subsetneqq \mathrm{Irr}(\overline{O \cap \mathfrak{n}})$, then $J_{t-1} \subsetneqq J_t$.

The set $\mathscr{S} = \{w \in W \mid O_w^{lr} = O,\ Y^r(w) \notin J_{t-1}\}$ is non-empty by the assumption. Let w be the minimal element of $\mathscr{S}$ with respect to the Bruhat ordering. It is sufficient to show that $Y^r(w) \in J_t$. By Lemma 2.7 and (3.7) we have $V(L_w) = Y^r(w) \cup (\bigcup_y Y^r(y))$, where y is running through elements of $\Sigma(w) - \{w\}$ such that $O_y^{lr} = O$. Assume that $V(L_w) \not\subset Y^r(w) \cup (\bigcup_{y \in J_{t-1}} Y)$. Then there exists some $y < w$ so that $O_y^{lr} = O$ and $Y^r(y) \notin J_{t-1}$. Hence $y \in \mathscr{S}$ and this contradicts with the minimality of w. Therefore we have $V(L_w) \subset Y^r(w) \cup (\bigcup_{Y \in J_{t-1}} Y)$ and (3.8) is proved.

We define an ordering $\prec$ on $\mathrm{Irr}(\overline{O \cap \mathfrak{n}})$ so that $Y' \prec Y$ for $Y \in J_t - J_{t-1}$ and $Y' \in J_u - J_{u-1}$ with $t > u$. For each $Y \in \mathrm{Irr}(\overline{O \cap \mathfrak{n}})$ choose $w_Y \in \mathscr{C}$ so that if $Y \in J_u - J_{u-1}$, then $Y^r(w_Y) = Y$ and $V(L_{w_Y}) \subset Y \cup (\bigcup_{Y' \in J_{u-1}} Y')$. Then the set $\{w_Y \mid Y \in \mathrm{Irr}(\overline{O \cap \mathfrak{n}})\}$ is a set of complete representatives of $\mathscr{C}/\sim_R$. Indeed $w_Y \not\sim_R w_{Y'}$ for $Y \neq Y'$ because $V(L_{w_Y}) \neq V(L_{w_{Y'}})$ (Corollary 1.10), and $\sharp(\mathscr{C}/\sim_R) = \dim \mathrm{Sp}(O) = \sharp(\mathrm{Irr}(\overline{O \cap \mathfrak{n}}))$ (Section 2). It is clear that the above ordering $\prec$ and the bijection $\mathscr{C}/\sim_R \ni w_Y \to Y \in \mathrm{Irr}(\overline{O \cap \mathfrak{n}})$ satisfy the required properties and Proposition is proved.

For $O \in \mathrm{Nilp}_s$ we have $\{\tau \in W^\wedge \mid \Phi(\tau) \subset \bar{O}\} \subset \{\tau \in W^\wedge \mid \Psi(\tau) \subset \bar{O}\}$ ([KT]). But the opposite inclusion does not hold in general.

Since Φ and Ψ are described in [BV1, 2], [Sh1, 2] and [ALS] explicitly, and since the closure relations of the nilpotent orbits are already known (well-known for classical types and the exceptional types are treated in [Sh3] and [M]), we can check the condition (3.6). For example since (3.6) holds for any $O \in \mathrm{Nilp}_s$ in E_6, Conjecture 3.4 is true for E_6. In F_4 (3.6) holds for nine special nilpotent orbits among eleven but fails for the remaining two.

§ 4. Explicit calculations for low rank cases

4.1. $V(L_w)$ for C_2, C_3.
Set

$$T_n = \begin{bmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{bmatrix} \in M_n(\boldsymbol{C}), \qquad J_n = \begin{bmatrix} 0 & T_n \\ -T_n & 0 \end{bmatrix} \in M_{2n}(\boldsymbol{C}),$$

$\mathfrak{g} = \{x \in M_{2n}(\boldsymbol{C}) \mid {}^t x J_n + J_n x = 0\}$,

$\mathfrak{b} = \{$upper triangular matrix in $\mathfrak{g}\}$,

$\mathfrak{n} = \{x \in \mathfrak{b}$ with diagonal entries $= 0\}$.

Then $\mathfrak{g}$ is a simple Lie algebra of type C_n, $\mathfrak{b}$ is its Borel subalgebra and $\mathfrak{n}=[\mathfrak{b},\mathfrak{b}]$. The Weyl group W is identified with the set of permutations w of $2n$ letters $\{\pm 1, \pm 2, \cdots, \pm n\}$ so that $w(-i)=-w(i)$ for any i. When $w(i)=a_i$ for $i=1, \cdots, n$, we write $w=(a_1, \cdots, a_n)$. Set $s_1=(-1, 2, 3, \cdots, n)$, $s_2=(2, 1, 3, 4, \cdots, n)$, $s_3=(1, 3, 2, 4, \cdots, n)$, $\cdots$, $s_n=(1, 2, \cdots, n-2, n, n-1)$. Then $S=\{s_1, s_2, \cdots, s_n\}$ is the set of simple reflections.

For a positive integer m let P_m be the set of the partitions of m, that is, $P_m=\{(1^{n_1}2^{n_2}\cdots m^{n_m}) \mid \sum_{i=1}^m in_i=m\}$. For $\sigma=(1^{n_1}2^{n_2}\cdots) \in P_m$ we denote by C_σ by the set of nilpotent matrices in $M_m(\boldsymbol{C})$ whose Jordan normal forms have exactly n_i Jordan blocks of size i for each i.

Set $P_{2n}^C=\{(1^{n_1}2^{n_2}\cdots) \in P_{2n} \mid n_i$ is even for i odd$\}$. As is well-known, for $\sigma \in P_{2n}$ $C_\sigma \cap \mathfrak{g} \neq \phi$ if and only if $\sigma \in P_{2n}^C$, and for each $\sigma \in P_{2n}^C$ $O_\sigma=C_\sigma \cap \mathfrak{g}$ is a single nilpotent orbit in $\mathfrak{g}$. Hence we have $\mathrm{Nilp} \simeq P_{2n}^C$. It is known that $\bar{O}_\sigma=\bar{C}_\sigma \cap \mathfrak{g}$ and hence $\overline{O_\sigma \cap \mathfrak{n}}=\bar{O}_\sigma \cap \mathfrak{n}=\bar{C}_\sigma \cap \mathfrak{n}$. For each $\sigma \in P_m$ a family $\{f_i^\sigma \mid i \in I_\sigma\}$ of polynomial functions on $M_m(\boldsymbol{C})$, whose zero set coincides with $\bar{C}_\sigma$, is explicitly constructed in [T1]. Here I_σ is some indexing set. Hence we have $\overline{O_\sigma \cap \mathfrak{n}}=\{x \in \mathfrak{n} \mid f_i^\sigma(x)=0$ for any $i \in I_\sigma\}$. Using this we calculate the explicit forms of the orbital varieties in C_2 and C_3. We identify Nilp with P_{2n}^C and the orbital varieties associated to the nilpotent orbit corresponding to $(1^{n_1}2^{n_2}\cdots) \in P_{2n}^C$ will be denoted by $(1^{n_1}2^{n_2}\cdots)_1$, $(1^{n_1}2^{n_2}\cdots)_2, \cdots$.

(i) C_2

$\mathrm{Nilp}=\{(4), (2^2), (1^2 2), (1^4)\}$,

$$\mathfrak{n}=\left\{\begin{bmatrix} 0 & a & c & b \\ 0 & 0 & d & c \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| \ a, b, c, d \in \boldsymbol{C}\right\},$$

$(4)_1=\mathfrak{n}$, $(2^2)_1=\{a=0\}$, $(2^2)_2=\{d=0\}$, $(1^2 2)_1=\{a=c^2-bd=0\}$, $(1^4)_1=\{0\}$.

(ii) C_3

$\mathrm{Nilp}=\{(6), (24), (1^2 4), (3^2) (2^3), (1^2 2^2), (1^4 2), (1^6)\}$,

$$\mathfrak{n}=\left\{\begin{bmatrix} 0 & a & b & f & e & d \\ 0 & 0 & c & h & g & e \\ 0 & 0 & 0 & k & h & f \\ 0 & 0 & 0 & 0 & -c & -b \\ 0 & 0 & 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \middle| \ a, b, \cdots, k \in \boldsymbol{C}\right\},$$

$(6)_1=\mathfrak{n}$,

$(24)_1=\{a=0\}$, $(24)_2=\{c=0\}$, $(24)_3=\{k=0\}$,

$$(1^2 4)_1=\left\{a=\det\begin{bmatrix} b & f & e & d \\ c & h & g & e \\ 0 & k & h & f \\ 0 & 0 & -c & -b \end{bmatrix}=0\right\},\ (1^2 4)_2=\{c=h^2-gk=0\},$$

$(3^2)_1=\{c=a^2g+2abh+b^2k=0\}$, $(3^2)_2=\{a=k=0\}$,

$(3^2)_3=\{k=ag+2bh-2cf=0\}$,

$(2^3)_1=\{a=b=c=0\}$, $(2^3)_2=\{a=k=bh-cf=0\}$,

$(2^3)_3=\{c=h^2-gk=ag+bh=ah+bk=0\}$,

$(1^2 2^2)_1=\{a=b=c=f^2g+h^2d+e^2k-gkd-2ehf=0\}$,

$(1^2 2^2)_2=\{c=k=h=g=0\}$,

$$(1^2 2^2)_3=\{a=k=bh-cf=f^2g+h^2d-2ehf=b^2g+c^2d-2bce$$
$$=bgf+chd-cef-beh=0\},$$

$$(1^4 2)_1=\{a=b=c=fg-eh=fh-ke=fe-hd=f^2-kd=e^2-gd$$
$$=h^2-gk=0\},$$

$(1^6)_1=\{0\}$.

Including relations of orbital varieties are given in Figure 1 and Figure 2. Since $Y^l(w)$ (resp. $Y^r(w)$) is the minimal orbital variety including $\mathfrak{n}\cap w^{-1}(\mathfrak{n})$ (resp. $\mathfrak{n}\cap w(\mathfrak{n})$), we can calculate them easily (Table 1 and Table 3).

Next we give the right cells and the two-sided cells.

(i) C_2

Set $s=s_1$, $t=s_2$, $\mathscr{C}_1=\{e\}$, $\mathscr{C}_{21}=\{t, ts, tst\}$, $\mathscr{C}_{22}=\{s, st, sts\}$, $\mathscr{C}_3=\{stst\}$ and $\mathscr{C}_2=\mathscr{C}_{21}\cup\mathscr{C}_{22}$. Then, $\mathscr{C}_1$, $\mathscr{C}_{21}$, $\mathscr{C}_{22}$, $\mathscr{C}_3$ are right cells and $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$ are two-sided cells.

(ii) C_3

We use the numbering of the elements of W given in Table 3.
Set

$\mathscr{C}_1=\{1\}$,

$\mathscr{C}_{21}=\{5, 9, 11, 33, 34\}$, $\mathscr{C}_{22}=\{3, 13, 17, 18, 25\}$, $\mathscr{C}_{23}=\{2, 19, 20, 29\}$,

$\mathscr{C}_{31}=\{26, 43, 44\}$, $\mathscr{C}_{32}=\{10, 35, 36\}$, $\mathscr{C}_{33}=\{30, 47, 48\}$,

$\mathscr{C}_{41}=\{27, 41, 42\}$, $\mathscr{C}_{42}=\{31, 45, 46\}$, $\mathscr{C}_{43}=\{15, 37, 38\}$,

$\mathscr{C}_{51}=\{7, 21, 22, 28\}$, $\mathscr{C}_{52}=\{4, 14, 16, 39, 40\}$, $\mathscr{C}_{53}=\{6, 12, 23, 24, 32\}$,

$\mathscr{C}_6=\{8\}$.

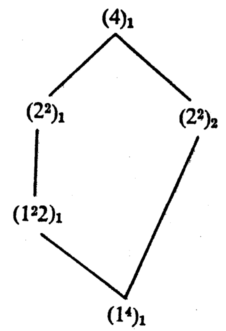


Figure 1.1 Including relation of orbital varieties for C_2.

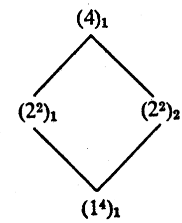


Figure 1.2 Including relation of orbital varieties associated to special nilpotent orbits for C_2.

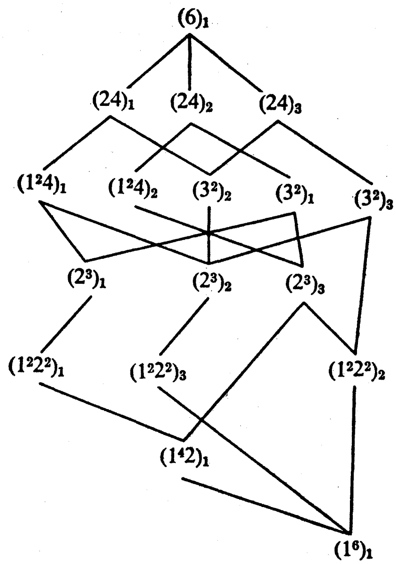


Figure 2.1 Including relation of orbital varieties for C_3.

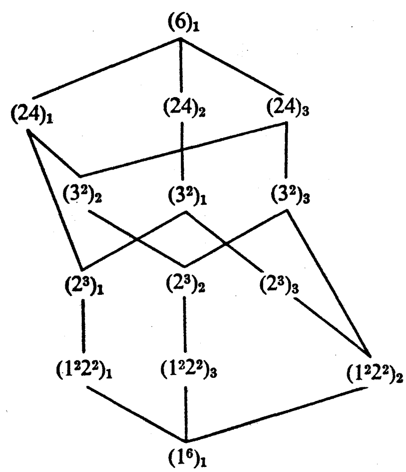


Figure 2.2 Including relation of orbital varieties associated to special nilpotent orbits for C_3.

The above fourteen subsets of W are right cells, and there are six two-sided cells, namely $\mathscr{C}_1$, $\mathscr{C}_2=\mathscr{C}_{21}\cup\mathscr{C}_{22}\cup\mathscr{C}_{23}$, $\mathscr{C}_3=\mathscr{C}_{31}\cup\mathscr{C}_{32}\cup\mathscr{C}_{33}$, $\mathscr{C}_4=\mathscr{C}_{41}\cup\mathscr{C}_{42}\cup\mathscr{C}_{43}$, $\mathscr{C}_5=\mathscr{C}_{51}\cup\mathscr{C}_{52}\cup\mathscr{C}_{53}$ and $\mathscr{C}_6$.

The ordering on the set of right cells induced by $\geqq_R$ is given in Figure 5 and Figure 7.

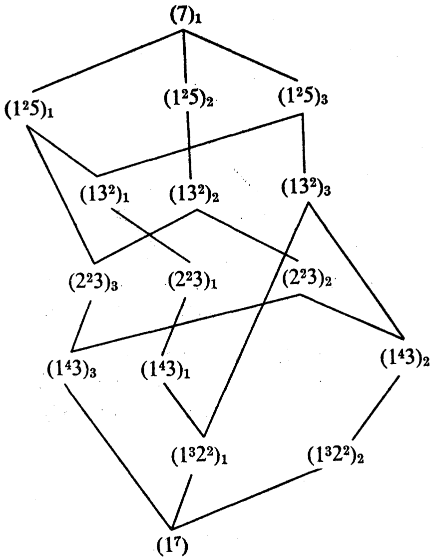


Figure 3.1 Including relation of orbital varieties for B_3.

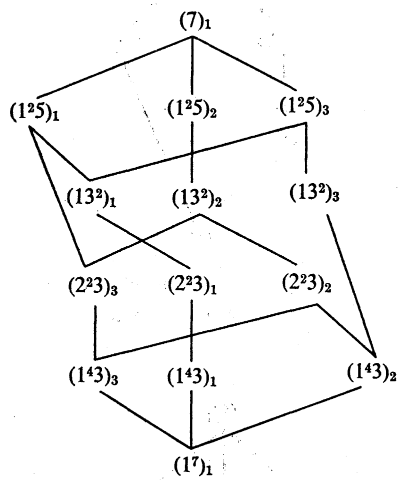


Figure 3.2 Including relation of orbital varieties associated to special nilpotent orbits for B_3.

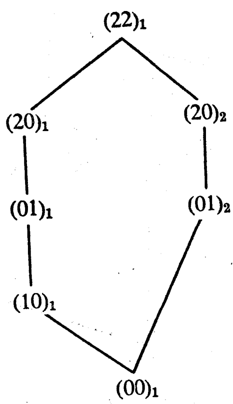


Figure 4.1 Including relation of orbital varieties for G_2.

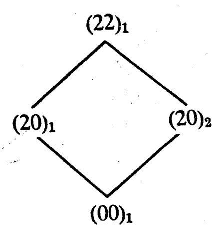


Figure 4.2 Including relation of orbital varieties associated to special nilpotent orbits for G_2.

Under the above preparation, we determine $V(L_w)$.

(i) C_2

When $w=e$, s, t or $stst$, $\overline{X}_w$ is non-singular. Indeed such w is the longest element of a parabolic subgroup of W. Hence $V(L_w)=Y^r(w)$

Table 1 C_2

w	$Y^l(w)$	$Y^r(w)$	right cell	$V(L)_w$	$a(w)$
e	$(4)_1$	$(4)_1$	$\mathscr{C}_1$	$(4)_1$	$b(e)$
s	$(2^2)_2$	$(2^2)_2$	$\mathscr{C}_{22}$	$(2^2)_2$	$b(s)$
t	$(2^2)_1$	$(2^2)_1$	$\mathscr{C}_{21}$	$(2^2)_1$	$b(t)$
st	$(2^2)_1$	$(2^2)_2$	$\mathscr{C}_{22}$	$(2^2)_2$	$b(st)$
ts	$(2^2)_2$	$(2^2)_1$	$\mathscr{C}_{21}$	$(2^2)_1$	$b(ts)$
sts	$(2^2)_2$	$(2^2)_2$	$\mathscr{C}_{22}$	$(2^2)_2$	$b(sts)$
tst	$(1^22)_1$	$(1^22)_1$	$\mathscr{C}_{21}$	$(2^2)_1$	$b(tst)+b(t)$
$stst$	$(1^4)_1$	$(1^4)_1$	$\mathscr{C}_3$	$(1^4)_1$	$b(stst)$

Table 2 G_2

w	$Y^l(w)$	$Y^r(w)$	right cell	$V(L_w)$	$a(w)$
e	$(22)_1$	$(22)_1$	$\mathscr{C}_1$	$(22)_1$	$b(e)$
s	$(20)_2$	$(20)_2$	$\mathscr{C}_{22}$	$(20)_2$	$b(s)$
t	$(20)_1$	$(20)_1$	$\mathscr{C}_{21}$	$(20)_1$	$b(t)$
st	$(20)_1$	$(20)_2$	$\mathscr{C}_{22}$	$(20)_2$	$b(st)$
ts	$(20)_2$	$(20)_1$	$\mathscr{C}_{21}$	$(20)_1$	$b(ts)$
sts	$(20)_2$	$(20)_2$	$\mathscr{C}_{22}$	$(20)_2$	$b(sts)$
tst	$(01)_1$	$(01)_1$	$\mathscr{C}_{21}$	$(20)_1$	$b(tst)+2b(t)$
$stst$	$(01)_1$	$(01)_2$	$\mathscr{C}_{22}$	$(20)_2$	$b(stst)+b(st)$
$tsts$	$(01)_2$	$(01)_1$	$\mathscr{C}_{21}$	$(20)_1$	$b(tsts)+b(ts)$
$ststs$	$(01)_2$	$(01)_2$	$\mathscr{C}_{22}$	$(20)_2$	$b(ststs)+b(s)$
$tstst$	$(10)_1$	$(10)_1$	$\mathscr{C}_{21}$	$(20)_1$	$b(tstst)+b(tst)+b(t)$
$ststst$	$(00)_1$	$(00)_1$	$\mathscr{C}_3$	$(00)_1$	$b(ststst)$

(Lemma 1.3, Lemma 2.7). Hence by Corollary 1.10 we have:

$$V(L_w)=\begin{cases}(4)_1 & w\in\mathscr{C}_1\\ (2^2)_1 & w\in\mathscr{C}_{21}\\ (2^2)_2 & w\in\mathscr{C}_{22}\\ (1^4)_1 & w\in\mathscr{C}_3.\end{cases}$$

(ii) C_3

When $w=1, 9, 17, 2, 10, 41, 4$ or 8, $V(L_w)=Y^r(w)$ since such w is the longest element of a parabolic subgroup of W. Hence $V(L_w)=(6)_1$ for $w\in\mathscr{C}_1$, $V(L_w)=(24)_1$ for $w\in\mathscr{C}_{21}$, $V(L_w)=(24)_2$ for $w\in\mathscr{C}_{22}$, $V(L_w)=(24)_3$ for $w\in\mathscr{C}_{23}$, $V(L_w)=(3^2)_2$ for $w\in\mathscr{C}_{32}$, $V(L_w)=(2^3)_1$ for $w\in\mathscr{C}_{41}$, $V(L_w)=(1^22^2)_2$ for $w\in\mathscr{C}_{52}$ and $V(L_w)=(1^6)_1$ for $w\in\mathscr{C}_6$. Since $V(L_w)\supset Y^r(w)$ in general, $V(L_w)\supset(3^2)_1$ for $w\in\mathscr{C}_{31}$, $V(L_w)\supset(3^2)_3$ for $w\in\mathscr{C}_{33}$, $V(L_w)\supset(2^3)_2$ for $w\in\mathscr{C}_{42}$, $V(L_w)\supset(2^3)_3$ for $w\in\mathscr{C}_{43}$, $V(L_w)\supset(1^22^2)_1$ for $w\in\mathscr{C}_{51}$ and $V(L_w)\supset(1^22^2)_3$ for $w\in\mathscr{C}_{53}$. For $w\in\mathscr{C}_{31}$ $(3^2)_1$ is an irreducible component of $V(L_w)$

Table 3 C_3, B_3 ([k] stands for $b(k)$ for $k=1, \cdots, 48$)

No.	w	reduced expression	w^{-1}	right cell	C_3			B_3		
					$Y^r(w)$	$V(L_w)$	$\mathbf{a}(w)$	$Y^r(w)$	$V(L_w)$	$\mathbf{a}(w)$
1	$(1\ \ 2\ \ 3)$		1	$\mathscr{C}_1$	$(6)_1$	$(6)_1$	[1]	$(7)_1$	$(7)_1$	[1]
2	$(-1\ \ 2\ \ 3)$	1	2	$\mathscr{C}_{23}$	$(24)_3$	$(24)_3$	[2]	$(1^25)_3$	$(1^25)_3$	[2]
3	$(1\ -2\ \ 3)$	212	3	$\mathscr{C}_{22}$	$(1^24)_2$	$(24)_2$	[3]+[17]	$(1^25)_2$	$(1^25)_2$	[3]
4	$(-1\ -2\ \ 3)$	2121	4	$\mathscr{C}_{52}$	$(1^22^2)_2$	$(1^22^2)_2$	[4]	$(1^43)_2$	$(1^43)_2$	[4]
5	$(1\ \ 2\ -3)$	32123	5	$\mathscr{C}_{21}$	$(1^24)_1$	$(24)_1$	[5]+[9]+[41]	$(1^25)_1$	$(1^25)_1$	[5]
6	$(-1\ \ 2\ -3)$	321213	6	$\mathscr{C}_{53}$	$(1^22^2)_3$	$(1^22^2)_3$	[6]	$(1^43)_1$	$(1^43)_1$	[6]
7	$(1\ -2\ -3)$	32123212	7	$\mathscr{C}_{51}$	$(1^42)_1$	$(1^22^2)_1$	[7]+[21]	$(1^43)_3$	$(1^43)_3$	[7]
8	$(-1\ -2\ -3)$	132123212	8	$\mathscr{C}_6$	$(1^6)_1$	$(1^6)_1$	[8]	$(1^7)_1$	$(1^7)_1$	[8]
9	$(1\ \ 3\ \ 2)$	3	9	$\mathscr{C}_{21}$	$(24)_1$	$(24)_1$	[9]	$(1^25)_1$	$(1^25)_1$	[9]
10	$(-1\ \ 3\ \ 2)$	13	10	$\mathscr{C}_{32}$	$(3^2)_2$	$(3^2)_2$	[10]	$(13^2)_1$	$(13^2)_1$	[10]
11	$(1\ -3\ \ 2)$	3212	13	$\mathscr{C}_{21}$	$(1^24)_1$	$(24)_1$	[11]+[33]	$(1^25)_1$	$(1^25)_1$	[11]
12	$(-1\ -3\ \ 2)$	32121	14	$\mathscr{C}_{53}$	$(1^22^2)_3$	$(1^22^2)_3$	[12]	$(1^43)_1$	$(1^43)_1$	[12]
13	$(1\ \ 3\ -2)$	2123	11	$\mathscr{C}_{22}$	$(1^24)_2$	$(24)_2$	[13]+[25]	$(1^25)_2$	$(1^25)_2$	[13]
14	$(-1\ \ 3\ -2)$	21213	12	$\mathscr{C}_{52}$	$(1^22^2)_2$	$(1^22^2)_2$	[14]	$(1^43)_2$	$(1^43)_2$	[14]
15	$(1\ -3\ -2)$	2123212	15	$\mathscr{C}_{43}$	$(2^3)_3$	$(2^3)_3 \cup (2^3)_1$	[15]+[27]+[37]+[41]	$(2^23)_2$	$(2^23)_2$	[15]
16	$(-1\ -2\ -3)$	21232121	16	$\mathscr{C}_{52}$	$(1^22^2)_2$	$(1^22^2)_2$	[16]	$(1^32^2)_2$	$(1^43)_2$	[16]+[4]
17	$(2\ \ 1\ \ 3)$	2	17	$\mathscr{C}_{22}$	$(24)_2$	$(24)_2$	[17]	$(1^25)_2$	$(1^25)_2$	[17]
18	$(-2\ \ 1\ \ 3)$	21	19	$\mathscr{C}_{22}$	$(24)_2$	$(24)_2$	[18]	$(1^25)_2$	$(1^25)_2$	[18]
19	$(2\ -1\ \ 3)$	12	18	$\mathscr{C}_{23}$	$(24)_3$	$(24)_3$	[19]	$(1^25)_3$	$(1^25)_3$	[19]
20	$(-2\ -1\ \ 3)$	121	20	$\mathscr{C}_{23}$	$(24)_3$	$(24)_3$	[20]	$(13^2)_3$	$(1^25)_3$	[20]+[2]
21	$(2\ \ 1\ -3)$	321323	21	$\mathscr{C}_{51}$	$(1^22^2)_1$	$(1^22^2)_1$	[21]	$(1^43)_3$	$(1^43)_3$	[21]
22	$(-2\ \ 1\ -3)$	3212321	23	$\mathscr{C}_{51}$	$(1^22^2)_1$	$(1^22^2)_1$	[22]	$(1^43)_3$	$(1^43)_3$	[22]
23	$(2\ -1\ -3)$	3212132	22	$\mathscr{C}_{53}$	$(1^22^2)_3$	$(1^22^2)_3$	[23]	$(1^43)_1$	$(1^43)_1$	[23]
24	$(-2\ -1\ -3)$	32123121	24	$\mathscr{C}_{53}$	$(1^22^2)_3$	$(1^22^2)_3$	[24]	$(1^32^2)_1$	$(1^43)_1$	[24]+[6]

Table 3 (Continued)

No.	w	reduced expression	w^{-1}	right cell	C_3			B_3		
					$Y^r(w)$	$V(L_w)$	$a(w)$	$Y^r(w)$	$V(L_w)$	$a(w)$
25	(2 3 1)	23	33	$\mathscr{C}_{22}$	$(24)_2$	$(24)_2$	[25]	$(1^25)_2$	$(1^25)_2$	[25]
26	(−2 3 1)	213	35	$\mathscr{C}_{31}$	$(3^2)_1$	$(3^2)_1$	[26]	$(13^2)_2$	$(13^2)_2$	[26]
27	(2 −3 1)	23212	37	$\mathscr{C}_{41}$	$(2^3)_1$	$(2^3)_1$	[27]+[41]	$(2^23)_3$	$(2^23)_3$	[27]
28	(−2 −3 1)	232121	39	$\mathscr{C}_{51}$	$(1^22^2)_1$	$(1^22^2)_1$	[28]	$(1^43)_3$	$(1^43)_3$	[28]
29	(2 3 −1)	123	34	$\mathscr{C}_{23}$	$(24)_3$	$(24)_3$	[29]	$(1^25)_3$	$(1^25)_3$	[29]
30	(−2 3 −1)	1213	36	$\mathscr{C}_{33}$	$(3^2)_3$	$(3^2)_3$	[30]	$(13^2)_3$	$(13^2)_3 \cup (13^2)_1$	[30]+[10]
31	(2 −3 −1)	123212	38	$\mathscr{C}_{42}$	$(2^3)_2$	$(2^3)_2$	[31]+[45]	$(2^23)_1$	$(2^23)_1$	[31]
32	(−2 −3 −1)	1232121	40	$\mathscr{C}_{53}$	$(1^22^2)_3$	$(1^22^2)_3$	[32]	$(1^32^2)_1$	$(1^43)_1$	[32]+[12]
33	(3 1 2)	32	25	$\mathscr{C}_{21}$	$(24)_1$	$(24)_1$	[33]	$(1^25)_1$	$(1^25)_1$	[33]
34	(−3 1 2)	321	29	$\mathscr{C}_{21}$	$(24)_1$	$(24)_1$	[34]	$(1^25)_1$	$(1^25)_1$	[34]
35	(3 −1 2)	132	26	$\mathscr{C}_{32}$	$(3^2)_2$	$(3^2)_2$	[35]	$(13^2)_1$	$(13^2)_1$	[35]
36	(−3 −1 2)	3121	30	$\mathscr{C}_{32}$	$(3^2)_2$	$(3^2)_2$	[36]	$(13^2)_1$	$(13^2)_1$	[36]+[10]
37	(3 2 −1)	21323	27	$\mathscr{C}_{43}$	$(2^3)_3$	$(2^3)_3 \cup (2^3)_1$	[37]+[41]	$(2^23)_2$	$(2^23)_2$	[37]
38	(−3 1 −2)	212321	31	$\mathscr{C}_{43}$	$(2^3)_3$	$(2^3)_3 \cup (2^3)_1$	[38]+[42]	$(2^23)_2$	$(2^23)_2$	[38]
39	(3 −1 −2)	212132	28	$\mathscr{C}_{52}$	$(1^22)_2$	$(1^22^2)_2$	[39]	$(1^43)_2$	$(1^43)_2$	[39]
40	(−3 −1 −2)	2123121	32	$\mathscr{C}_{52}$	$(1^22^2)_2$	$(1^22^2)_2$	[40]	$(1^32^2)_2$	$(1^43)_2$	[40]+[14]
41	(3 2 1)	323	41	$\mathscr{C}_{41}$	$(2^3)_1$	$(2^3)_1$	[41]	$(2^23)_3$	$(2^23)_3$	[41]
42	(−3 2 1)	2321	45	$\mathscr{C}_{41}$	$(2^3)_1$	$(2^3)_1$	[42]	$(2^23)_3$	$(2^23)_3$	[42]
43	(3 −2 1)	2132	43	$\mathscr{C}_{31}$	$(3^2)_1$	$(3^2)_1$	[43]	$(13^2)_2$	$(13^2)_2$	[43]
44	(−3 −2 1)	23121	47	$\mathscr{C}_{31}$	$(3^2)_1$	$(3^2)_1$	[44]	$(13^2)_2$	$(13^2)_2$	[44]+[26]
45	(3 2 −1)	1323	42	$\mathscr{C}_{42}$	$(2^3)_2$	$(2^3)_2$	[45]	$(2^23)_1$	$(2^23)_1$	[45]
46	(−3 2 −1)	12321	46	$\mathscr{C}_{42}$	$(2^3)_2$	$(2^3)_2$	[46]	$(2^23)_1$	$(2^23)_1$	[46]
47	(3 −2 −1)	12132	44	$\mathscr{C}_{33}$	$(3^2)_3$	$(3^2)_3$	[47]	$(13^2)_3$	$(13^2)_3 \cup (13^2)_1$	[47]+[35]
48	(−3 −2 −1)	123121	48	$\mathscr{C}_{33}$	$(3^2)_3$	$(3^2)_3$	[48]	$(13^2)_3$	$(13^2)_3 \cup (13^2)_1$	[48]+[4] +[36]+[30] +[10]+[20]

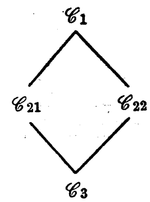


Figure 5 Ordering of right cells for C_2.

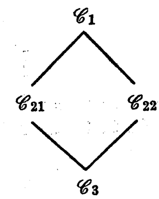


Figure 6 Ordering of right cells for G_2.

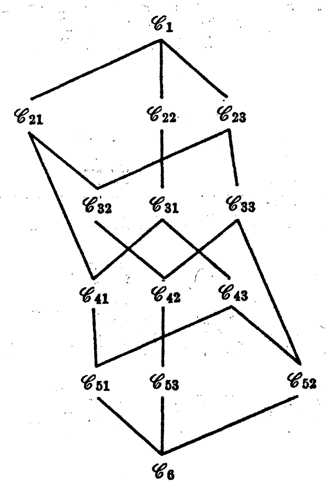


Figure 7 Ordering of right cells for B_3, C_3.

and if there is an irreducible component different from $(3^2)_1$, it must be $(3^2)_2$ or $(3^2)_3$. Since $\mathscr{C}_{22} \geqq_R \mathscr{C}_{31}$, $V(L_w) \subset V(L_y) = (24)_2$ for $w \in \mathscr{C}_{31}$ and $y \in \mathscr{C}_{22}$. Hence by Figure 2 $V(L_w) = (3^2)_1$ for $w \in \mathscr{C}_{31}$. We will repeat the above arguments. Comparing $\mathscr{C}_{42}$ to $\mathscr{C}_{32}$ $V(L_w) = (2^3)_2$ for $w \in \mathscr{C}_{42}$. Comparing $\mathscr{C}_{51}$ to $\mathscr{C}_{41}$ $V(L_w) = (1^2 2^2)_1$ for $w \in \mathscr{C}_{51}$. Comparing $\mathscr{C}_{52}$ to $\mathscr{C}_{31}$ and $\mathscr{C}_{23}$ $V(L_w) = (1^2 2^2)_2$ for $w \in \mathscr{C}_{52}$. Comparing $\mathscr{C}_{53}$ to $\mathscr{C}_{42}$ $V(L_w) = (1^2 2^2)_3$ for $\mathscr{C}_{53}$. Comparing $\mathscr{C}_{43}$ to $\mathscr{C}_{31}$ and $\mathscr{C}_{51}$ $V(L_w) = (2^3)_3 \cup (2^3)_1$ for $w \in \mathscr{C}_{43}$. Comparing $\mathscr{C}_{33}$ to $\mathscr{C}_{23}$ $V(L_w) = (3^2)_3$ or $(3^2)_3 \cup (3^2)_2$ for $w \in \mathscr{C}_{33}$. Let $w = 30$ $(\in \mathscr{C}_{33})$. If $V(L_w) = (3^2)_3 \cup (3^2)_2$, there exists some $y \in \Sigma(w)$ such that $Y^r(y) = (3^2)_2$. The only element $y \in W$ so that $y \leqq w$ and $Y^r(y) = (3^2)_2$ is 10. But by a direct calculation we see that X_y is contained in the non-singular part of $\bar{X}_w$ $(y=10, w=30)$ and hence $y = 10 \notin \Sigma(w)$. Thus $V(L_w) = (3^2)_3$ for $w \in \mathscr{C}_{33}$. $V(L_w)$ is determined for any $w \in W$ (Table 3).

4.2. $V(L_w)$ for B_3.
Set

$$K=\begin{bmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{bmatrix} \in M_7(C).$$

Then $\mathfrak{g}=\{x \in M_7(C) \mid {}^t xK+Kx=0\}$ is a simple Lie algebra of type B_3, $\mathfrak{b}=$ {triangular matrix in $\mathfrak{g}$} is a Borel subalgebra and we have:

$$\mathfrak{n}=[\mathfrak{b},\mathfrak{b}]=\left\{\left.\begin{bmatrix} 0 & a & b & g & e & d & 0 \\ 0 & 0 & c & h & f & 0 & -d \\ 0 & 0 & 0 & k & 0 & -f & -e \\ 0 & 0 & 0 & 0 & -k & -h & -g \\ 0 & 0 & 0 & 0 & 0 & -c & -b \\ 0 & 0 & 0 & 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right| a, \cdots, k \in C\right\}.$$

The Weyl group W, the right cells and two-sided cells are the same as that of C_3. The set Nilp of nilpotent orbits is identified with the set of partitions of seven in which each even integer appears even times, that is, Nilp $=\{(7), (1^25), (13^2), (2^23), (1^43), (1^32^2), (1^7)\}$. The orbital varieties are given as follows.

$(7)_1=\mathfrak{n}$,

$(1^25)_1=\{a=0\}$, $(1^25)_2=\{c=0\}$, $(1^25)_3=\{k=0\}$,

$(13^2)_1=\{a=k=0\}$, $(13^2)_2=\{c=ah+bk=0\}$,

$(13^2)_3=\{k=h^2+2cf=0\}$,

$(2^23)_1=\{a=k=(bf-ce)^2+2(fg-eh)(bh-cg)=0\}$,

$(2^23)_2=\{c=ah+bk=2k^2d+2fkg-2ehk-af^2$
$=2hfg+2hkd-2eh^2+bf^2=0\}$,

$(2^23)_3=\{a=b=c=0\}$,

$(1^43)_1=\{a=k=bh-gc=bf-ec=gf-eh=0\}$,

$(1^43)_2=\{c=k=h=f=0\}$,

$(1^43)_3=\{a=b=c=eh-dk-gf=0\}$,

$(1^32^2)_1=\{a=k=bh-gc=bf-ec=gf-eh=hg+2bf=g^2+2be$
$=h^2+2cf=0\}$,

$(1^32^2)_2=\{c=k=h=f=g^2+2ad+2be=0\}$,

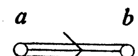
$(1^7)_1=\{0\}$.

We can calculate $V(L_w)$ as in the case of C_3 (Table 3).

Remark. Equations describing the orbital varieties for C_3 and B_3 given in Sections 4.1 and 4.2 define reduced schemes.

4.3. $V(L_w)$ for G_2.

Let $\mathfrak{g}$ be the simple Lie algebra of type G_2. When the weighted Dynkin diagram corresponding to a nilpotent orbit is given by:



$(a, b=0$ or 1 or $2)$,

this nilpotent orbit is denoted by (ab). Then we have Nilp$=\{(22), (20), (01), (10), (00)\}$. Let Δ^+ be the set of positive roots. We denote the simple roots by α and β. Here α is a long root and β is a short root. For $\gamma \in \Delta^+$ let $\mathfrak{g}_\gamma$ be the root space corresponding to γ and U_γ the subgroup of G corresponding to $\mathfrak{g}_\gamma$. The subgroup of G corresponding to $\mathfrak{n}$ is denoted by U. Then the orbital varieties are described as follows.

$(22)_1=\mathfrak{n}$,

$(20)_1=\bigoplus_{\gamma\in\Delta^+-\{\beta\}}\mathfrak{g}_\gamma$, $\qquad (20)_2=\bigoplus_{\gamma\in\Delta^+-\{\alpha\}}\mathfrak{g}_\gamma$,

$(01)_1=(\mathrm{Ad}\,(U_\beta)(\mathfrak{g}_\alpha+\mathfrak{g}_{\alpha+\beta}))+\mathfrak{g}_{2\alpha+3\beta}$, $\qquad (01)_2=\mathrm{Ad}\,(U)\mathfrak{g}_\beta$,

$(10)_1=(\mathrm{Ad}\,(U_\beta)\mathfrak{g}_\alpha)+\mathfrak{g}_{2\alpha+3\beta}$,

$(00)_1=\{0\}$.

Let s and t be the reflections relative to α and β, respectively. The four subsets $\mathscr{C}_1=\{e\}$, $\mathscr{C}_{21}=\{t, ts, tst, tsts, tstst\}$, $\mathscr{C}_{22}=\{s, st\ sts, stst, ststs\}$, $\mathscr{C}_3=\{ststst\}$ of W are right cells and the three subsets $\mathscr{C}_1$, $\mathscr{C}_2=\mathscr{C}_{21}\cup\mathscr{C}_{22}$, $\mathscr{C}_3$ are two-sided cells.

We can calculated $V(L_w)$ as in the case of C_2 (Table 2).

4.4. $\mathbf{Ch}(L_w)$.

We show the calculations for G_2 only. The results for C_2, C_3, B_3 are given in Table 1 and Table 3.

When $\mathbf{Ch}(L_w)=\sum_{y\in W} m(y, w)[\overline{T^*_{X_y}X}]$, we have $\boldsymbol{a}(w)=\sum_{y\in W} m(y, w)\,\boldsymbol{b}(y)$. For type G_2 it is shown that $\boldsymbol{a}(w)=\sum_{y\leqq w}(-1)^{l(w)+l(y)}y$. When $w=e$, s, t or $ststst$, $\bar{X}_w$ is non-singular because such w is the longest element of a parabolic subgroup of W. It is shown that $\bar{X}_{sts}$ is also non-singular (although the author checked this by himself, this should be known). Hence $\boldsymbol{a}(w)=\boldsymbol{b}(w)$ for $w=e$, s, t, sts, $ststst$ by Lemma 1.3. Furthermore by Lemma 1.3 and Lemma 1.6 $\boldsymbol{a}(st)=\boldsymbol{b}(st)$ and $\boldsymbol{a}(ts)=\boldsymbol{b}(ts)$. By Lemma 1.3, Lemma 1.6 and Lemma 2.7 we have:

$\boldsymbol{a}(tst)=\boldsymbol{b}(tst)+x_1\boldsymbol{b}(t),$
$\boldsymbol{a}(stst)=\boldsymbol{b}(stst)+x_2\boldsymbol{b}(st),$
$\boldsymbol{a}(tsts)=\boldsymbol{b}(tsts)+x_2\boldsymbol{b}(ts),$
$\boldsymbol{a}(ststs)=\boldsymbol{b}(ststs)+x_3\boldsymbol{b}(sts)+x_4\boldsymbol{b}(s),$
$\boldsymbol{a}(tstst)=\boldsymbol{b}(tstst)+x_5\boldsymbol{b}(tst)+x_6\boldsymbol{b}(t)$

with $x_1>0$, $x_2>0$, $x_3\geqq 0$, $x_4\geqq 0$, $x_3+x_4>0$, $x_5\geqq 0$, $x_6>0$. By Proposition 2.8 and Lemma 2.10 we have:

$$u\boldsymbol{b}(w)\in\boldsymbol{b}(w)+\boldsymbol{b}(uw)+\sum_{\substack{uy<y<uw\\ \Upsilon^l(y)\subset\Upsilon^l(w)}} \boldsymbol{Z}_{\geqq 0}\boldsymbol{b}(y),$$

for $u=s$ or t, $w\in W$ with $uw>w$. Hence by Figure 4 we have $x_1=2$, $x_2=1$, $x_3=0$, $x_4=1$, $x_5=1$, $x_6=1$.

Appendix. Some remarks on the Springer representation

A.1. The image of the moment map $\gamma\colon T^*X\to\mathfrak{g}^*\simeq\mathfrak{g}$ is the set $\mathcal{N}$ of nilpotent elements in $\mathfrak{g}$ and $\gamma\colon T^*X\to\mathcal{N}$ gives a resolution of the singularity of $\mathcal{N}$ (Springer). An action of the Weyl group W on $\boldsymbol{R}\gamma_!(\boldsymbol{Q}_{T^*X})$, which is an object of the derived category, is defined in [L] (see also [BM]). Hence for any locally closed subvariety D of $\mathcal{N}$ we have an action of W on the vector space $H_c^*(\gamma^{-1}(D),\boldsymbol{Q})=H_c^*(D,\boldsymbol{R}\gamma_!(\boldsymbol{Q}_{T^*X})\,|\,D)$ and its dual $H_*(\gamma^{-1}(D))$. Since $\gamma^{-1}(x)\simeq X^x$ $(:=\{gB\in X\,|\,g^{-1}x\in\mathfrak{n}),$ we have an actino of W on $H_*(X^x)$. Since the inverse image of $\mathcal{N}^a:=\{(x,-x)\,|\,x\in\mathcal{N}\}$ under $\gamma\times\gamma\colon T^*(X\times X)=T^*X\times T^*X\to\mathcal{N}\times\mathcal{N}$ coincides with Z (see Section 2.5), we have an action of $W\times W$ on $H_*(Z)$. The action of W (resp. $W\times W$) on $H_{2d_O}(X^x)$ (resp. $H_{4d}(Z)$) described above coincides with the one given in [KL2] ([H]). Here O is the nilpotent orbit containing x.

A.2. Proof of Lemma 2.9 and Lemma 2.10.

Identifying $\mathcal{N}^a$ with $\mathcal{N}$ and restricting $\gamma\times\gamma$ to Z we have a map $\rho\colon Z\to\mathcal{N}$. Let $O\in$ Nilp. Since $\rho^{-1}(\bar{O})$ $(=\bigcup_{O_w^{lr}\subset\bar{O}}Z_w)$ is Zariski closed in Z, we have the following exact sequence of $W\times W$-modules:

$$O\longrightarrow H_{4d}(\rho^{-1}(\bar{O}))\longrightarrow H_{4d}(Z)\longrightarrow H_{4d}(\rho^{-1}(\mathcal{N}-\bar{O}))\longrightarrow 0.$$

Hence $\bigoplus_{O_w^{lr}\subset\bar{O}}\boldsymbol{Q}\boldsymbol{b}(w)\,(\simeq\bigoplus_{O_w^{lr}\subset\bar{O}}\boldsymbol{Q}[Z_w]=H_{4d}(\rho^{-1}(\bar{O}))$ is a $W\times W$-submodule of $\boldsymbol{Q}[W]\,(\simeq H_{4d}(Z))$. Setting $\partial\bar{O}=\bar{O}-O$ we have also the following exact sequence of $W\times W$-modules:

$$0\longrightarrow H_{4d}(\rho^{-1}(\partial\bar{O}))\longrightarrow H_{4d}(\rho^{-1}(\bar{O}))\longrightarrow H_{4d}(\rho^{-1}(O))\longrightarrow 0.$$

Let $x\in O$. Regarding $\rho^{-1}(O)$ as a fiber bundle over $O\simeq G/Z_G(x)$ we have $\rho^{-1}(O)\simeq G\times^{Z_G(x)}(X^x\times X^x)$. Hence $H_{4d}(\rho^{-1}(O))\simeq(H_{2d_O}(X^x)\otimes H_{2d_O}(X^x))^{A(x)}$ and Lemma 2.9 is proved.

Consider the following commutative diagram:

$$\begin{array}{ccc} T^*X\times T^*X & \xrightarrow{\gamma\times 1} & \mathcal{N}\times T^*X \\ \bigcup\!\!\uparrow & & \bigcup\!\!\uparrow \\ Z & \xrightarrow{\quad f \quad} & (\gamma\times 1)(Z). \end{array}$$

Since $Z=(\gamma\times 1)^{-1}((\gamma\times 1)(Z))$, we have the action of W on $Rf_!(\boldsymbol{Q}_Z)$ and hence $H_{4d}(Z)$ $(=(H_c^{4d}((\gamma\times 1)(Z), Rf_!(\boldsymbol{Q}_Z)))^*)$ is a W-module. This action coincides with the restriction of the action of $W\times W$ on $H_{4d}(Z)$ (given in Section A.1) to $W\times 1$. Identifying $(\gamma\times 1)(Z)$ with $T^*X\simeq G\times^B\mathfrak{n}$ we have $f(Z_w)=G\times^B Y^l(w)$. Hence Lemma 2.10 is proved similarly to Lemma 2.9.

A.3. Proof of Lemma 2.11.

We first prove the statement (i), which says that $Y^l(w)=Y^l(y)$ if and only if $w\sim_l y$.

Let Y be an orbital variety associated to $O\in$ Nilp. We denote by δY the union of all the orbital varieties properly contained in Y. Set $Y^0=Y-\delta Y$. Since $\bigoplus_{Y^l(w)\subset Y}\boldsymbol{Q}\boldsymbol{b}(w)$ and $\bigoplus_{Y^l(w)\subset\delta Y}\boldsymbol{Q}\boldsymbol{b}(w)$ are $W\times 1$-invariant by Section A.2, $M:=(\bigoplus_{Y^l(w)\subset Y}\boldsymbol{Q}\boldsymbol{b}(w))/(\bigoplus_{Y^l(w)\subset\delta Y}\boldsymbol{Q}\boldsymbol{b}(w))$ is a W-module. For $w\in W$ with $Y^l(w)=Y$ we denote the image of $\boldsymbol{b}(w)$ in M by $\bar{\boldsymbol{b}}(w)$. Then we clearly have $M=\bigoplus_{Y^l(w)=Y}\boldsymbol{Q}\bar{\boldsymbol{b}}(w)$. It is enough to show the following statement:

(A1) If M_0 is a W-invariant subspace of M and is spanned by a subset of $\{\bar{\boldsymbol{b}}(w)\mid Y^l(w)=Y\}$, then $M_0=\{0\}$ or $M_0=M$.

Set $Z_{Y^0}=f^{-1}(G\times^B Y^0)$. Here we identify $(\gamma\times 1)(Z)$ with $G\times^B\mathfrak{n}$. Then by the arguments similar to that of Section A.2 we have $M\simeq H_{4d}(Z_{Y^0})$. Note that Z_{Y^0} is naturally isomorphic to $\{(x, g_1B, g_2B)\in O\times X\times X\mid x\in g_2Y^0\cap g_1\mathfrak{n}\}$. Hence if $x\in Y^0$, Z_{Y^0} is isomorphic to $G\times^{Z_G(x)}(X^x\times\tilde{X}^x)$ as a fiber bundle over $O\simeq G/Z_G(x)$. Here $\tilde{X}^x$ is the union of $C\in\operatorname{Irr}(X^x)$ such that $h(C)=Y$ (see (2.1)). Hence we have $M\simeq(H_{2d_O}(X^x)\otimes H_{2d_O}(\tilde{X}^x))^{A(x)}$ as a W-module, where W acts on $H_{2d_O}(X^x)$ as in Section A.1 and trivially on $H_{2d_O}(\tilde{X}^x)$. Fix $C_0\in\operatorname{Irr}(\tilde{X}^x)=h^{-1}(Y)$ and set $A(x, C_0)=\{z\in A(x)\mid z\cdot C_0=C_0\}$. Since $A(x)$ acts on $\operatorname{Irr}(\tilde{X}^x)$ transitively, we have the following.

(A2) $M\simeq H_{2d_O}(X^x)^{A(x,C_0)}$ as a W-module.

Let $\operatorname{Irr}(X^x)=I_1\coprod I_2\coprod\cdots\coprod I_k$ be the orbit decomposition under the action of $A(x, C_0)$. Then $\{\sum_{C\in I_j}[C]\mid j=1,\cdots,k\}$ is a basis of $H_{2d_O}(X^x)^{A(x,C_0)}$ and this corresponds to the basis $\{\bar{\boldsymbol{b}}(w)\mid Y^l(w)=Y\}$ of M via (A.2).

Since $H_{2d_O}(X^x)=\bigoplus_{\xi\in\mathscr{S}'_O}(\tau_{(O,\xi)}\otimes\xi)$ and $H_{2d_O}(X^x)^{A(x)}=\tau_{(O,1)}\otimes 1=\tau_{(O,1)}$, we have the projection $p\colon H_{2d_O}(X^x)\to H_{2d_O}(X^x)^{A(x)}$ of W-modules. Let V

be a non-zero W-invariant subspace of $H_{2d_O}(X^x)^{A(x,C_0)}$ spanned by a subset of $\{\sum_{C\in I_j}[C]\,|\,j=1,\cdots,k\}$. Since $H_{2d_O}(X^x)^{A(x)}$ is irreducible and $p(\sum_{C\in I_j}[C])=(\sum_{C\in I_j}\sum_{z\in A(x)}[z\cdot C])/\#(A(x))\neq 0$, we have $p(V)=H_{2d_O}(X^x)^{A(x)}$ and hence V contains $H_{2d_O}(X_x)^{A(x)}$. Let $\mathrm{Irr}(X^x)=J_1\coprod\cdots\coprod J_l$ be the orbit decomposition under the action of $A(x)$. Then we have $H_{2d_O}(X^x)^{A(x)}=\bigoplus_{r=1}^{l}Q(\sum_{C\in J_r}[C])$. Since each J_r is a union of some I_j's we have $V=H_{2d_O}(X^x)^{A(x,C_0)}$ by the assumption. (i) is proved.

The statement (ii) is equivalent to (i) by the symmetry.

We prove (iii). Assume that $O_w^{lr}=O_y^{lr}=O$ for $w, y\in W$. Since the map $\{z\in W\,|\,O_z^{lr}=O\}\to\mathrm{Irr}(\overline{O\cap\mathfrak{n}})\times\mathrm{Irr}(\overline{O\cap\mathfrak{n}})$ $(z\mapsto(Y^l(z),\ Y^r(z)))$ is surjective by [St], there exists some $z\in W$ such that $Y^l(z)=Y^l(w)$ and $Y^r(z)=Y^r(y)$. By (i) and (ii) we have $w\sim_l z\sim_r y$. Hence $w\sim_{lr}y$ and Lemma 2.11 is proved.

As a corollary to the proof we have the following.

Corollary A.3. *Let $w\in W$, $O=O_w^{lr}$, $Y=Y^l(w)$, $x\in O$ and $C\in h^{-1}(Y)$ (h: $\mathrm{Irr}(X^x)\to\mathrm{Irr}(\overline{O\cap\mathfrak{n}})$, see (2.1)). Set $A(x,C)=\{z\in A(x)\,|\,z\cdot C=C\}$. Then V_w^l is isomorphic to $H_{2d_O}(X^x)^{A(x,C)}$ as a W-module.*

Conjecture A.4.

(i) $w\geqq_l y$ if and only if $Y^l(w)\supset Y^l(y)$.

(ii) $w\geqq_r y$ if and only if $Y^r(w)\supset Y^r(y)$.

(iii) $w\geqq_{lr} y$ if and only if $\overline{O_w^{lr}}\supset\overline{O_y^{lr}}$.

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