Chapter 1
Further Topics in Computable Analysis

Introduction

This chapter starts where Chapter 0 leaves off, but involves a transition to more difficult topics. It begins with differentiation and the classes C^n, $1 \leqslant n \leqslant \infty$, and progresses through the computable theory of analytic functions. It concludes with a general theorem on "translation invariant operators" which subsumes several of our previous results. We continue to use the "Chapter 0" notion of computability for continuous functions.

Naively, one might suppose that since the derivative, the operations of complex analysis, etc. are given by formulas, they should be computable. Of course, as we know, this is not necessarily the case. In fact, it is not always easy to guess what the results should be. Consider, for example, differentiation, the topic of Section 1. If a computable function possesses a continuous derivative, is the derivative necessarily computable? The answer turns out to be no. This raises the question: are there regularity conditions which we can impose on the initial computable function (e.g. being C^n for some fixed n, or C^∞) which insure the computability of the derivative. Here the answer is yes. The problem is to find the right conditions.

Theorem 1, essentially due to Myhill [1971], asserts that there exists a computable C^1 function whose derivative is not computable. Theorem 2, due to the authors [1983a], asserts that, on the other hand, if the initial function is C^2, then the derivative is computable. Thus the cutoff point must lie somewhere between C^1 and C^2. Where is it located? We give a slight strengthening of Myhill's example, which shows that the function f can be twice differentiable (but not continuously so) and still give a noncomputable f'. Hence the cutoff is pinned between "twice differentiable" (Theorem 1) and "twice continuously differentiable" (Theorem 2).

A curious result which emerges from Theorems 1 and 2 is the following: There exists a function on [0, 1] which is effectively continuous at each point of [0, 1] but not effectively uniformly continuous on [0, 1] (Corollary 2b). Thus the classical theorem that a continuous function on a compact set is uniformly continuous does not effectivize.

An immediate consequence of Theorem 2 (Corollary 2a) asserts that if a computable function f is C^∞, then each derivative $f^{(n)}$ is computable. This raises the question: Is the sequence of derivatives $\{f^{(n)}\}$ computable, effectively in n? The answer is no, as we show in Theorem 3 (cf. Pour-El, Richards [1983a]).

We turn now to Section 2, on analytic functions. Let f be a computable function which is analytic (but with no computability assumptions on its derivatives). Here, in contrast to the C^∞ case, everything effectivizes—at least for compact domains. In particular, the sequence of derivatives $\{f^{(n)}\}$ and the sequence of Taylor coefficients are computable. Surprisingly enough, even analytic continuation is computable. These results are spelled out in Proposition 1.

In view of the rather pervasive effectiveness cited above, the following result seems a bit surprising. There exists an entire function which is computable on every compact disk, but not computable over the whole complex plane (Theorem 4).

Section 3 presents the Effective Modulus Lemma (Theorem 5), a technical result which is useful for the creation of counterexamples. We use it to produce a continuous function f which is sequentially computable—i.e. f maps every computable sequence $\{x_n\}$ onto a computable sequence $\{f(x_n)\}$—but such that f is not computable (Theorem 6). A second application, involving the wave equation, is cited without proof (cf. Pour-El, Richards [1981]).

The chapter closes with Section 4, translation invariant operators. The section begins with a detailed account of translation invariance. Many of the standard operators of analysis and physics are translation invariant. We prove a theorem about these operators (Theorem 7) which has several applications. In particular, Theorem 2 from Section 1 is an immediate corollary of Theorem 7. So too are the extensions of Theorem 2 to partial derivatives. A deeper application of Theorem 7 involves weak solutions of the wave equation (Theorem 8).

1. C^n Functions, $1 \leqslant n \leqslant \infty$

Here we present the key results about the computability theory of derivatives, as outlined in the introduction to this chapter.

Note. We recall that a function f, of one or several variables, is said to be C^n if all derivatives of order $\leqslant n$ exist and are continuous. We say that f is C^∞ if f is C^n for all n.

As noted above, our first theorem is essentially due to Myhill [1971].

Theorem 1 (Noncomputability of the derivative for a computable C^1 function). *There exists a computable function f on* $[0, 1]$ *such that f is C^1, but the derivative f' is not computable. Furthermore, the function f can be chosen to be twice differentiable* (*but not twice continuously differentiable*).

Proof. We begin by describing the derivative $f'(x)$, and then obtain $f(x)$ by integration:

$$f(x) = \int_0^x f'(u)\, du.$$

The derivative $f'(x)$ will be a superposition of countably many "pulses", and we start by taking a canonical C^∞ pulse function $\varphi(x)$:

$$\varphi(x) = \begin{cases} e^{-x^2/(1-x^2)} & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geqslant 1. \end{cases}$$

Then $\varphi \in C^\infty$, φ has support on $[-1, 1]$, and $\varphi(0) = 1$. The sequence of derivatives $\{\varphi^{(n)}(x)\}$ is computable, effectively in n.

Let $a\colon \mathbb{N} \to \mathbb{N}$ be a one-to-one recursive function listing a recursively enumerable nonrecursive set A. We can assume that $0 \notin A$. Define the n-th pulse $\varphi_n(x)$ by:

$$\varphi_n(x) = \varphi[2^{(n+a(n)+2)} \cdot (x - 2^{-a(n)})].$$

Then $\varphi_n(x)$ is a pulse of height 1 and half width $2^{-(n+a(n)+2)}$, centered on the point $x = 2^{-a(n)}$.

We now give a (noneffective) description of the derivative $f'(x)$:

$$f'(x) = \sum_{k=0}^{\infty} 4^{-a(k)} \varphi_k(x).$$

We observe that the sequence of partial sums

$$f_n'(x) = \sum_{k=0}^{n} 4^{-a(k)} \varphi_k(x)$$

is computable, effectively in n, since the sequences $\{4^{-a(n)}\}$ and $\{\varphi_n(x)\}$ are both computable. Moreover, $f_n'(x) \to f'(x)$ uniformly (although not necessarily effectively) as $n \to \infty$, since $a(n)$ gives a one-to-one listing of the set A, and hence the series converges uniformly by comparison with the series $\sum 4^{-a}$. Thus the limit function $f'(x)$ is continuous.

We assert that $f'(x)$ is also differentiable (although its derivative is discontinuous). For the individual pluses φ_n are C^∞ and their supports approach the point $x = 0$. Hence $f'(x)$ is differentiable except perhaps at $x = 0$.

We now show that near $x = 0$ the graph of $f'(x)$ is bounded between the two parabolas $y = \pm 4x^2$. As a first step, we observe that the pulses $4^{-a(n)}\varphi_n(x)$, centered at the points $x = 2^{-a(n)}$, have heights $y = 4^{-a(n)} = x^2$ when $x = 2^{-a(n)}$. However, the pulses have finite width, and we cannot just consider the central point $x = 2^{-a(n)}$. But the half-width of the pulse is $\leqslant (1/2) \cdot 2^{-a(n)}$ (with room to spare), and so for *any* point x in support(φ_n), $x \geqslant (1/2) \cdot 2^{-a(n)}$. Thus, for *any* such x, $2x \geqslant 2^{-a(n)}$, and $4x^2 \geqslant 4^{-a(n)} =$ amplitude of n-th pulse.

Since $f'(x)$ is bounded between the two parabolas $y = \pm 4x^2$, $f''(0)$ exists and is zero. Hence $f'(x)$ is differentiable at all points.

To complete the proof, we must show that $f'(x)$ is not computable, but that its antiderivative $f(x)$ is computable. We begin with $f'(x)$. As a first step, we observe that the pulses φ_n in the series for f' have disjoint supports. For the half-width of

φ_n is $2^{-(n+a(n)+2)} \leqslant 2^{-a(n)-2}$, whereas the pulse-center $x = 2^{-a(n)}$ differs from its nearest possible neighbor $x = 2^{-a(n)-1}$ by a distance of $2^{-a(n)-1}$.

Recall that each pulse φ_n, centered at $x = 2^{-a(n)}$, has height 1 and is multiplied by $4^{-a(n)}$. Hence the value of $f'(x)$ at the point $x = 2^{-a}$ is given by:

$$f'(2^{-a}) = \begin{cases} 4^{-a} & \text{if } a = a(n) \text{ for some } n, \\ 0 & \text{otherwise.} \end{cases}$$

That is, $f'(2^{-a}) = 4^{-a}$ if $a \in A$, and $f'(2^{-a}) = 0$ if $a \notin A$.

The sequence of values $\{f'(2^{-a})\}$, $a = 0, 1, 2, \ldots$, is not a computable sequence of reals. For suppose it were. Then there would exist a computable double sequence of *rationals* $\{r_{ak}\}$ such that $|r_{ak} - f'(2^{-a})| \leqslant 2^{-k}$ for all k and a (Definition 5a, Section 2, Chapter 0). From this we could derive the computable rational sequence $\{r'_a\}$ given by $r'_a = r_{a,2a+2}$, and we would have $|r'_a - f'(2^{-a})| \leqslant (1/4)\cdot 4^{-a}$ for all a. Since $\{r'_a\}$ is a computable rational sequence, exact comparisons involving $\{r'_a\}$ can be made effectively. This would give a decision procedure for the set A: $a \in A$ if $r'_a \geqslant (1/2)4^{-a}$, and $a \notin A$ otherwise. Since A is not recursive, this is a contradiction.

Now we recall (Definition A in Chapter 0) that a computable function must be sequentially computable. Since $\{2^{-a}\}$ is a computable sequence of reals which is mapped by f' onto a noncomputable sequence $\{f'(2^{-a})\}$, the function f' is not computable.

To show that the antiderivative $f(x)$ is computable, we reason as follows. Since the n-th pulse $4^{-a(n)}\varphi_n(x)$ has height $\leqslant 4^{-a(n)}$, and since this pulse has half-width $2^{-(n+a(n)+2)}$, its antiderivative

$$\Phi_n(x) = \int_0^x 4^{-a(n)}\varphi_n(u)\, du \leqslant (4^{-a(n)})(2\cdot 2^{-(n+a(n)+2)}) \leqslant 2^{-n}.$$

Hence the series for $f(x)$,

$$f(x) = \sum_{k=0}^{\infty} \Phi_k(x),$$

converges uniformly and effectively by comparison with $\sum 2^{-k}$. Furthermore the sequence of summands $\{\Phi_k(x)\}$ is computable. Hence by Theorem 4 in Chapter 0, f is computable. This proves Theorem 1. □

In the opposite direction, we have:

Theorem 2 (Computability of the derivative for computable C^2 functions). *Let $[a, b]$ be an interval with computable endpoints. Let f be a computable function on $[a, b]$ which is C^2. Then the derivative f' is computable. Moreover, the hypothesis that f is C^2 could be replaced by the weaker hypothesis that f' is effectively uniformly continuous.*

Before proving Theorem 2 we give two corollaries.

Corollary 2a (Computability of each derivative of a computable C^∞ function). *If f is computable on $[a, b]$ and f is C^∞—i.e. infinitely differentiable—then the n-th derivative $f^{(n)}$ is computable for each fixed n.*

Proof. This is an immediate consequence of Theorem 2. □

For the next corollary, we need a definition of "pointwise effective continuity". As expected, a function g is *effectively continuous at a point* x if there is a recursive function $d: \mathbb{N} \to \mathbb{N}$ such that $|y - x| \leqslant 1/d(N)$ implies $|g(y) - g(x)| \leqslant 2^{-N}$. A function which is effectively continuous at each point of its domain will be called, for simplicity, "effectively continuous".

Corollary 2b (Effective continuity does not imply effective uniform continuity). *There exists a function g on $[0, 1]$ which is effectively continuous at each point $x \in [0, 1]$, but which is not effectively uniformly continuous.*

Proof. We combine the results of Theorems 1 and 2. By Theorem 1, there exists a computable function f on $[0, 1]$ whose derivative f' is itself differentiable but not computable. Let $g = f'$.

By Theorem 2, g is not effectively uniformly continuous. For this theorem asserts that if $g = f'$ were effectively uniformly continuous, then it would be computable, which it is not.

On the other hand, since g is differentiable, it is effectively continuous at each point x. To see this: Fix x. Let $c = g'(x)$. Then $\lim_{y \to x} [(g(y) - g(x))/(y - x)] = c$. Let M be an integer with $M \geqslant |c| + 1$. Then for all points y sufficiently close to x, $|g(y) - g(x)| \leqslant M|y - x|$. (The points y which are not close to x are irrelevant.) Thus to achieve $|g(y) - g(x)| \leqslant 2^{-N}$, it suffices to take $|y - x| \leqslant 1/M \cdot 2^N$, so that an effective modulus of continuity is given by $d(N) = M \cdot 2^N$. □

Proof of Theorem 2. First we verify that if f is C^2, then f' is effectively uniformly continuous. For if f'' is continuous, then there is some integer M with $|f''(x)| \leqslant M$ for all $x \in [a, b]$. Then, by the Mean Value Theorem, f' satisfies

$$|f'(x) - f'(y)| \leqslant M \cdot |x - y|$$

for all $x, y \in [a, b]$. Hence f' is effectively uniformly continuous.

From here on, we will merely assume that f' is effectively uniformly continuous. This means (cf. Definition A in Chapter 0) that there is a recursive function $d(N)$ such that, for all $x, y \in [a, b]$, $|x - y| \leqslant 1/d(N)$ implies $|f'(x) - f'(y)| \leqslant 2^{-N}$. We can assume that $d(N)$ is a strictly increasing function.

The assertion (∗) below is a technicality, but it must be dealt with. The problem, of course, is to keep y_{kN} within the interval $[a, b]$, since we cannot decide effectively which of the boundary points a, b is closer to x_k.

(∗) *Let $\{x_k\}$ be a computable sequence of real numbers in $[a, b]$. Then there exists an integer N_0 and a computable double sequence $\{y_{kN}\}$, $y_{kN} \in [a, b]$, such that for each k* *and* $N \geqslant N_0$,

$$\textit{either} \quad y_{kN} = x_k + \frac{1}{d(N)} \quad \text{or} \quad y_{kN} = x_k - \frac{1}{d(N)}.$$

Proof of (∗). First we find suitable rational approximations to the computable reals a and b, and to the computable real sequence $\{x_k\}$. For a and b, we merely observe that there exist rational numbers A and B with $|a - A| \leqslant (b - a)/1000$ and $|b - B| \leqslant (b - a)/1000$. For $\{x_k\}$, we readily deduce from Definition 5a, Section 2, Chapter 0, that there exists a computable sequence of rationals $\{X_k\}$ with $|X_k - x_k| \leqslant (b - a)/1000$ for all k. We also choose N_0 so that $N \geqslant N_0$ implies $1/d(N) \leqslant (b - a)/1000$. Now, working with the computable *rational* sequence $\{X_k\}$, we set $y_{kN} = x_k + (1/d(N))$ if X_k is closer to A than to B, and set $y_{kN} = x_k - (1/d(N))$ otherwise. This is an effective procedure which satisfies all of the conditions set down in (∗). □

Now we come to the body of the proof. We want to show that f' is computable, and by Definition A in Chapter 0, this means that f' is (i) sequentially computable and (ii) effectively uniformly continuous. We have (ii) by hypothesis. Now, as we recall, (i) means that if $\{x_k\}$ is a computable sequence, then $\{f'(x_k)\}$ is computable. To compute $\{f'(x_k)\}$, we proceed as follows:

Take $\{y_{kN}\}$ as in (∗) above. Recall that the points y_{kN} are approximations to the x_k, lying either above or below x_k and spaced an exact distance $1/d(N)$ away from x_k. Now consider the difference quotients

$$D_{kN} = \frac{f(y_{kN}) - f(x_k)}{y_{kN} - x_k}.$$

Since the function f itself is computable, $\{D_{kN}\}$ is computable, effectively in k and N. Now by the mean value theorem,

$$D_{kN} = f'(\xi)$$

for some $\xi = \xi_{kN}$ between x_k and y_{kN}. (It makes no difference here whether we can compute ξ effectively or not. It suffices that ξ exists.) By definition of y_{kN}, the distance $|y_{kN} - x_k| = 1/d(N)$, and hence $|\xi - x_k| < 1/d(N)$. Finally, by definition of the modulus of continuity $d(N)$, $|\xi - x_k| < 1/d(N)$ implies $|f'(\xi) - f'(x_k)| \leqslant 1/2^N$. Hence

$$|D_{kN} - f'(x_k)| = |f'(\xi) - f'(x_k)| \leqslant 1/2^N.$$

Thus the computable sequence $\{D_{kN}\}$ converges to $\{f'(x_k)\}$ as $N \to \infty$, effectively in N and k. Hence $\{f'(x_k)\}$ is computable, as desired. This proves Theorem 2. □

Theorem 3 (Noncomputability of the sequence of n-th derivatives). *There exists a computable function f on $[-1, 1]$ which is C^∞, but for which the sequence of derivatives*

$\{f^{(n)}(x)\}$ *is not computable. In addition, f can be chosen so that the sequence of values* $\{|f^{(n)}(0)|\}$ *is not bounded by any recursive function of n.*

Proof. Take a one-to-one recursive function $a\colon \mathbb{N} \to \mathbb{N}$ which generates a recursively enumerable nonrecursive set A. We assume that $0, 1 \notin A$. Let $w(n)$ be the "waiting time"

$$w(n) = \max\{m\colon a(m) \leqslant n\}$$

defined in Chapter 0, Section 1. As we saw in Chapter 0, $w(n)$ is not bounded above by any recursive function. (Otherwise, as we recall, there would be an effective procedure for deciding whether or not an arbitrary integer belongs to A.)

We shall construct a computable C^∞ function f such that $|f^{(n)}(0)| \geqslant w(n)$ whenever $w(n) \geqslant n$. The exceptional cases where $w(n) < n$ are of no interest, since we know that $w(n)$ is not bounded above by any recursive function, and clearly this failure to be recursively bounded cannot involve those values n for which $w(n) < n$.

We define R_n recursively by setting $R_0 = 1$,

$$R_n = 2nR_{n-1}^n.$$

(Although we do not need to know the exact size of R_n, we observe that $2^{n!} \leqslant R_n \leqslant 2^{n^n}$ for $n \geqslant 1$.)

We now construct the desired function f. Let

$$f(x) = \sum_{k=0}^{\infty} R_k^{1-a(k)} \cdot \cos\left(R_k x + \frac{\pi}{4}\right), \qquad x \in [-1, 1].$$

By Theorem 4 of Chapter 0, f is computable, since the terms of the series are computable (effectively in k), and the series is effectively uniformly convergent (being dominated by R_k^{-1} since $0, 1 \notin A$ and thus $a(k) \geqslant 2$).

For fixed n, the series for the n-th derivative is also effectively uniformly convergent: since the series is

$$f^{(n)}(x) = \sum_{k=0}^{\infty} R_k^{n+1-a(k)} \begin{Bmatrix} \pm\cos \\ \pm\sin \end{Bmatrix} \left(R_k x + \frac{\pi}{4}\right),$$

and $a(k) \leqslant n + 1$ for only finitely many k. Thus, again by Theorem 4 of Chapter 0, $f^{(n)}(x)$ exists and is a computable function. In particular, since $f^{(n)}$ exists for all n, f is C^∞.

We now show that $|f^{(n)}(0)| \geqslant w(n)$ for $w(n) \geqslant n \geqslant 3$. Take such an n, and for convenience set

$$m = w(n).$$

In the series for $f^{(n)}(x)$, the m-th term (which turns out to dominate all of the others) is

$$R_m^{n+1-a(m)} \cdot \{\pm\sin \text{ or } \cos(\ )\}.$$

Hence $|(m\text{-th term})(x = 0)| = R_m^{n+1-a(m)}\sqrt{1/2}$, since the sin or cos is evaluated at $\pi/4$. Since $a(m) \leqslant n$ (by definition $m = w(n)$ is the last k for which $a(k) \leqslant n$), we obtain

$$|(m\text{-th term})(x = 0)| \geqslant R_m\sqrt{1/2}.$$

We must show that the sum of all other terms in the series for $f^{(n)}(x)$ is smaller than the m-th term. Consider first the previous terms, involving $k < m$. These terms

$$R_k^{n+1-a(k)} \cdot \{\pm\sin \text{ or } \cos(\ )\}, \qquad k < m,$$

are dominated by R_k^n (since $a(k) \geqslant 2$), and $R_k^n \leqslant R_k^m$ (here we use the fact that $m = w(n) \geqslant n$). Furthermore, there are m such terms. Since $R_m = 2mR_{m-1}^m$, the sum of these previous terms (i.e. the terms with $k < m$) has absolute value at most $R_m/2$. Since $\sqrt{1/2} - (1/2) > 1/10$,

$$|m\text{-th term}| - \left|\sum_{k=0}^{m-1} (k\text{-th term})\right| > R_m/10.$$

Now consider the terms with $k > m$. These give $a(k) \geqslant n + 1$ (again since $m = w(n)$ is the last k for which $a(k) \leqslant n$). For at most one value of k is $a(k) = n + 1$. Otherwise $a(k) > n + 1$, and the exponent $n + 1 - a(k)$ in

$$R_k^{n+1-a(k)} \cdot \{\pm\sin \text{ or } \cos(\ )\}$$

is negative. We bound the possible term with $a(k) = n + 1$ by $R_k^0 = 1$, and the sum of the other terms (with $k > m$) by $\sum_{k=1}^{\infty} R_k^{-1} \leqslant 1$ (since $R_k \geqslant 2^k$). Thus the effect of all terms with $k > m$ is dominated by $1 + 1 = 2$. Hence for $f^{(n)}(0)$, which is the sum of *all* the terms (with $k = m$, $k < m$, and $k > m$):

$$|f^{(n)}(0)| \geqslant (R_m/10) - 2.$$

Now for $m = w(n) \geqslant n \geqslant 3$,

$$(R_m/10) - 2 \geqslant m.$$

(In fact, since $R_m \geqslant 2^{m!}$, this inequality is absurdly weak.) Combining the last two displayed inequalities, we have, for $w(n) \geqslant n \geqslant 3$:

$$|f^{(n)}(0)| \geqslant m = w(n),$$

as desired. This proves Theorem 3. □

The final remarks in this section all relate to Theorem 2. As we recall, this theorem gives conditions under which the derivative *is* computable (i.e. if f is computable and C^2, then f' is computable).

Partial derivatives. Theorem 2 extends in an obvious way to partial derivatives. Thus if $f(x, y)$ is computable and C^2 on a computable rectangle in $\mathbb{R}^2$, then $\partial f/\partial x$ and $\partial f/\partial y$ are computable. If $f(x, y)$ is computable and C^∞ on this rectangle, then each partial derivative

$$\frac{\partial^{m+n} f}{\partial x^m \, \partial y^n}$$

is computable. Similar results hold for $\mathbb{R}^q$, $q > 2$.

In Section 4 we give a general result (Theorem 7) from which Theorem 2, its extension to partial derivatives, and numerous other results are immediate corollaries.

It is natural to ask whether Theorem 2 extends (A) to noncompact domains, or (B) to sequences of functions. The answer is "no" in both cases, as the following two remarks show.

Remark A (The noncompact case). Theorem 2 breaks down if the domain $[a, b]$ is replaced by $\mathbb{R}$. For there exists a C^∞ function f on $\mathbb{R}$ which is computable on $\mathbb{R}$ (in the sense of Definition A″ of Chapter 0), but such that $f'(x)$ is not computable on $\mathbb{R}$.

Here is the construction of f. Start with the C^∞ function φ used in the proof of Theorem 1 above. Let $\psi(x) = \varphi[x - (1/2)]$, so that $\psi'(0) = c > 0$. Set $\psi_k(x) = (1/k) \cdot \psi(k^2 x)$, so that ψ_k has amplitude $1/k$ but ψ_k' has amplitude $=$ Const $\cdot\, k$. As previously, let $a\colon \mathbb{N} \to \mathbb{N}$ be a recursive function generating a recursively enumerable nonrecursive set A in a one-to-one manner. Let

$$f(x) = \sum_{k=2}^{\infty} \psi_k(x - a(k)).$$

Then it is easy to verify that f is C^∞ and computable on $\mathbb{R}$.

We now show that $f'(x)$ is not computable. This is done by showing that the sequence $\{f'(n)\}$ is not computable. Let $k \geqslant 2$. Then

$$f'(n) = \begin{cases} k\varphi'(-1/2) & \text{if } n = a(k), \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $\{f'(n)\}$ were computable. Then, since $\varphi'(-1/2)$ is computable, the sequence $\{\mu_n\}$ where

$$\mu_n = \frac{f'(n)}{\varphi'(-1/2)}$$

would also be computable. Hence A would be recursive. For $n \in A$ if and only if one of the following holds: $n = a(0)$ or $n = a(1)$ or $\mu_n \geqslant 2$. Thus we have a contradiction. $\square$

It is interesting to note that the sequence $\{f'(n)\}$ grows faster than any recursive function. For, except for a finite number of values,

$$\max_{p \leqslant n} f'(p) = \max_{a(k) \leqslant n} (k\varphi'(-1/2)) = \varphi'(-1/2)w(n),$$

where $w(n) = \max\{k\colon a(k) \leqslant n\}$ is the function defined in the Waiting Lemma of Chapter 0, Section 1. As we showed there, $w(n)$ grows faster than any recursive function of n.

Remark B (Sequences of functions). Theorem 2 also breaks down for sequences of functions. For there exists a computable sequence $\{f_k\}$ on $[0, 1]$, such that each f_k is C^∞, but the sequence of derivatives $\{f_k'\}$ is not computable.

To construct this example, simply take the function f from Remark A and set

$$f_k(x) = f(x + k), \qquad 0 \leqslant x \leqslant 1. \qquad \square$$

A final note. The explanation of the pathological behavior exhibited in Remarks A and B above is that, in both cases, there is a noncompact domain. In Remark A the domain is $\mathbb{R}$; whereas in Remark B the sequence $\{f_k\}$ can be viewed as a function on the noncompact set $[0, 1] \times \mathbb{N}$.

We will see further instances of these phenomena in the section which follows.

2. *Analytic Functions*

Here we deal with the computability theory of analytic functions of a complex variable. In what follows, the function f will be assumed to be analytic on some region Ω in the complex plane, and we will also assume that f is computable in the sense of Chapter 0 (details to follow). However, we do *not* assume that f is "computably analytic"—i.e. we do not assume that the derivatives or the power series expansion of f are computable. In fact, this assumption is redundant, as we show in the proposition below.

First some technicalities. In Chapter 0 we defined "computability" for a function f whose domain was a computable closed rectangle in $\mathbb{R}^q$—i.e. a rectangle with computable coordinates for its corners. To do complex analysis, one needs more general domains. For, in complex analysis, we are usually given a function f which is analytic on a connected open region Ω. The shape of Ω can be quite complicated, and frequently f will have no analytic extension beyond Ω. We want to describe "computability" on arbitrary compact subsets K of Ω. The natural approach, following Chapter 0, is to begin by defining computability on a closed region Δ which is a finite union of computable rectangles. Then any compact set $K \subseteq \Omega$ can be covered by such a region Δ, $K \subseteq \Delta \subseteq \Omega$. Thus:

Definition. (a) Let $\Delta \subseteq \mathbb{C}$ be a finite union of computable rectangles. A function $f: \Delta \to \mathbb{C}$ is called *computable* if f is computable on each of the constituent computable rectangles in Δ.

(b) A function f defined on a compact set $K \subseteq \mathbb{C}$ is called *computable* if f has a computable extension to a finite union of computable rectangles $\Delta \supseteq K$.

In the proposition below, we deal with all of the standard themes of elementary complex analysis, including analytic continuation.

Proposition 1 (Basic facts for analytic functions). *Let Ω be an open region in $\mathbb{C}$. Let $f: \Omega \to \mathbb{C}$ be analytic in Ω. Suppose f is computable on some computable rectangle D with nonempty interior, $D \subseteq \Omega$. Then*:

(a) *Effective analytic continuation—the function f, originally assumed to be computable only on D, is computable on any compact subset $K \subseteq \Omega$.*

(b) *The sequence of derivatives $\{f^{(n)}(z)\}$ is computable on K, effectively in n.*

(c) *For any computable point $a \in \Omega$, the sequence of Taylor coefficients of $f(z)$ about $z = a$ is computable.*

(d) *For any computable point $a \in \Omega$ and all computable reals $M \geqslant 0$, the Taylor series for f converges effectively and uniformly in any closed disk $\{|z - a| \leqslant M\} \subseteq \Omega$.*

Note. This result contrasts with Theorem 3 (for C^∞ functions) in the preceding section. In Theorem 3, each derivative $f^{(n)}(x)$ was computable, but the sequence of derivatives was not. Here the entire sequence of derivatives is computable. Of course, this corresponds to well known distinctions between the properties of C^∞ functions and analytic functions.

We observe that part (a), about the computability of analytic continuation, applies only when the continuation is to a compact region K. For continuations to the entire complex plane, the result fails, as Theorem 4 below shows.

Proof. Since any compact set $K \subseteq \Omega$ can be covered by a finite union of computable rectangles $\Delta \subseteq \Omega$, it suffices to prove the proposition for Δ. Thus we replace the arbitrary compact set K by a set of the special form Δ.

The proof proceeds in several stages. Initially we know only that f is computable on the rectangle D. We first prove parts (b) and (c) for a smaller rectangle D' contained in the interior of D. Then, using this information, we prove parts (d) and (a) in general. Then we come back and prove parts (b) and (c) for the larger region Δ, $\Delta \supseteq K$.

Let γ be the boundary of the rectangle D, where the curve γ is taken in the positive sense. Let D' be any computable rectangle in the interior of D (and hence inside of γ). As already noted, we first prove (b) and (c) of the proposition for the special case of the rectangle D'.

For part (b) on D'. By the Cauchy integral formula, for points $z \in D'$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w - z)^{n+1}} \, dw.$$

Now γ (= boundary of D) consists of a finite number of vertical or horizontal line segments with computable coordinates for their endpoints. Hence, by Corollary 6b in Chapter 0, the integration can be carried out effectively, and $\{f^{(n)}(z)\}$ is computable on D', effectively in n.

For part (c) on D'. It follows immediately that, for any computable point $a \in D'$, the sequence of Taylor coefficients $\{f^{(n)}(a)/n!\}$ is computable.

For part (d) in general. Here assume that we have *any* computable point $a \in \Omega$ for which it is known that the sequence of Taylor coefficients $\{f^{(n)}(a)/n!\}$ is computable.

We wish to show that the Taylor series for f converges effectively and uniformly in any disk $\{|z - a| \leqslant M\} \subseteq \Omega$. Take a larger closed disk $\{|z - a| \leqslant M + \varepsilon\}$, still contained inside the region Ω. Now the standard estimates found in any complex analysis text (e.g. Ahlfors [1953]) are already effective. Namely, let

$$f(z) = f(a) + \cdots + \frac{f^{(n)}(a)}{n!}(z - a)^n + R_{n+1}(z),$$

where

$$R_{n+1}(z) = \frac{1}{2\pi i} \int_{|w-a|=M+\varepsilon} \frac{f(w)(z - a)^{n+1}}{(w - a)^{n+1}(w - z)} \, dw.$$

Then for $|z - a| \leqslant M$, the "error" $R_{n+1}(z)$ satisfies

$$|R_{n+1}(z)| \leqslant \text{Const} \frac{M}{\varepsilon} \left(\frac{M}{M + \varepsilon} \right)^n,$$

where

$$\text{Const} = \max \{|f(w)|: |w - a| \leqslant M + \varepsilon\}.$$

Thus the "Const" depends on f, M, and ε, but it does not depend on z or n. This gives effective uniform convergence of $R_{n+1}(z)$ to zero as $n \to \infty$, and hence proves part (d).

Now we turn to part (a)—effective analytic continuation. Take a computable point $a \in D'$ and a closed disk $\{|z - a| \leqslant M\}$ as above. The Taylor series for f about $z = a$ is computable by part (c), and this series is effectively uniformly convergent on the disk $\{|z - a| \leqslant M\}$ by part (d). Hence, by Theorem 4 in Chapter 0, f has a computable analytic continuation to this disk. Now, starting with D' and using a finite chain of overlapping closed disks, we can cover any compact region $\Delta \subseteq \Omega$. Hence by part (d), already proved, f has a computable analytic continuation to Δ. This proves part (a).

Now, as promised, we extend parts (b) and (c) from the small rectangle D' to any finite union of computable rectangles $\Delta \subseteq \Omega$. Let $\Delta^* \subseteq \Omega$ be another finite union of computable rectangles, such that each rectangle in Δ lies in the *interior* of some rectangle in Δ^*. By part (a), already proved, f has a computable analytic continuation to Δ^*. But then our previous proofs of (b) and (c) (for the rectangle D' inside of D) extend mutatis mutandis to all of the rectangles which make up Δ. This completes the proof of Proposition 1. $\square$

We have seen that on compact domains, everything goes as one would expect. However, for noncompact domains, the results are a little more startling. The following theorem is due to Caldwell and Pour-El [1975].

Theorem 4 (An entire function which is computable on every compact disk but not computable over the whole plane). *There exists an entire function f which is computable on any compact domain in $\mathbb{C}$, but such that f is not computable over the whole complex plane. Furthermore, f can be chosen so that, as $x \to \infty$ along the positive real axis, $f(x)$ grows more rapidly than any computable function of x.*

Note. In fact we will have that the sequence of values $f(0)$, $f(1)$, $f(2)$, ... is not bounded by any recursive function.

Proof. Let $a\colon \mathbb{N} \to \mathbb{N}$ be a one-to-one recursive function enumerating a recursively enumerable non recursive set A. We can assume that $0 \notin A$. Now we define f by

$$f(z) = \sum_{m=0}^{\infty} \frac{z^m}{a(m)^m}.$$

Clearly the sequence of Taylor coefficients $\{1/a(m)^m\}$ is computable. Furthermore, the series is effectively uniformly convergent on any compact disk $\{|z| \leqslant M\}$, where M is an integer >0. To see this, we observe that there are only finitely many values of m with $a(m) \leqslant M$. For all other m, $a(m) \geqslant M + 1$, and since $|z| \leqslant M$, the series is dominated by

$$\sum \left(\frac{M}{M+1}\right)^m.$$

Of course, this dominating series is effectively convergent.

Thus on the disk $\{|z| \leqslant M\}$, f is the effective uniform limit of a computable sequence of functions (the partial sums of its Taylor series), and so by Theorem 4 of Chapter 0, f is computable.

[It should be noted that the above construction is not effective in M. For, as $M \to \infty$, there is no effective way of telling how many m's there are with $a(m) \leqslant M$.]

We now show that the sequence of values $f(0), f(1), f(2), \ldots$ is not bounded above by any recursive function. Suppose otherwise, that there is a recursive function g with

$$f(n) \leqslant g(n) \qquad \text{for all } n.$$

Now we recall the "waiting time"

$$w(n) = \max\{m: a(m) \leqslant n\}$$

defined in Section 1 of Chapter 0. As we showed in Chapter 0, $w(n)$ is not bounded above by any recursive function. Now we will show that $w(n) \leqslant f(2n) \leqslant g(2n)$, and thus derive a contradiction.

To show that $w(n) \leqslant f(2n)$. We observe that all of the coefficients in the Taylor series for f are positive, and hence $f(2n) >$ any single term in its Taylor series expansion. We use the term with $m = w(n)$. Then, since by definition $a(m) \leqslant n$,

$$f(2n) > \left(\frac{2n}{a(m)}\right)^m \geqslant \left(\frac{2n}{n}\right)^m = 2^m = 2^{w(n)} > w(n).$$

Hence $w(n) < f(2n) \leqslant g(2n)$, giving a recursive upper bound $g(2n)$ for $w(n)$, and thus giving the desired contradiction. This proves Theorem 4. □

We conclude this section with another counterexample. In Proposition 1 we proved the effectiveness of analytic continuation to compact domains. (Theorem 4 shows that this breaks down for noncompact domains.) Even for compact domains, however, the result fails if we consider *sequences* of analytic functions. The failure for sequences is connected with the fact that analytic continuation is not "well posed" in the sense of Hadamard. That is, analytic functions which are "small" on a compact domain D_1 can grow arbitrarily rapidly when continued to a larger compact domain D_2. This the following example shows.

Example (Failure of effective analytic continuation for sequences of functions). There exists a sequence $\{f_k\}$ of entire functions which is computable on the disk $D_1 = \{|z| \leqslant 1\}$ but not computable on $D_2 = \{|z| \leqslant 2\}$.

Since this is similar to several of our other examples, we shall be very terse. Let $a: \mathbb{N} \to \mathbb{N}$ give a one-to-one recursive listing of a recursively enumerable non recursive set A. Delete the value $a(0)$ from A. Define

$$f_{kn}(z) = \begin{cases} \dfrac{z^m}{m} & \text{if } k = a(m) \text{ for some } m,\ 1 \leqslant m \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\{f_{kn}\}$ is computable. Let

$$f_k(z) = \begin{cases} \dfrac{z^m}{m} & \text{if } k = a(m) \text{ for some } m \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

As displayed, $\{f_k\}$ is not computable. But on D_1, f_{kn} converges effectively and uniformly to f_k as $n \to \infty$. Namely when $z \in D_1$, $|f_{kn}(z) - f_k(z)| \leqslant 1/n$. Thus $\{f_k\}$ is computable on D_1.

On the other hand, when we enlarge the domain to D_2, the values $\{f_k(2)\}$ increase faster than any recursive function of k. To see this: We recall that $f_k(z) = z^m/m$ if $k = a(m)$ for some $m \geqslant 1$, and $f_k(z) = 0$ otherwise. Set $z = 2$. We observe that $2^m/m \geqslant m$ for all $m \geqslant 4$. So, with finitely many exceptions,

$$\max_{j \leqslant k} f_j(2) \geqslant \max \{m: a(m) \leqslant k\} = w(k),$$

where $w(k)$ is the function defined in the Waiting Lemma. By that lemma, $w(k)$ and hence $f_j(2)$, $j \leqslant k$, are not dominated by any recursive function. □

Real-analytic functions. Part (b) of Proposition 1 above (computability of the sequence of derivatives) extends in a natural way to real analytic functions. For details, see Pour/El, Richards [1983a].

3. The Effective Modulus Lemma and Some of Its Consequences

Our first result, the Effective Modulus Lemma, is useful for the construction of counterexamples. We give this result and then give two applications of it.

Before we state the Effective Modulus Lemma, a brief introduction seems in order. Recall from Chapter 0, Sections 1 and 2, that if $\{r_m\}$ is a computable monotone sequence of reals which converges noneffectively to a limit α, then α is not computable. Examples of this type are used several times in this book. However, sometimes one needs a sharper version of this construction. The Effective Modulus Lemma provides such a sharpening. In this lemma, we have $\{r_m\}$ and α as above (so that, in particular, α is a noncomputable real), but we also have more. Namely, for any computable real sequence $\{\gamma_k\}$, there exists a recursive function $d: \mathbb{N} \to \mathbb{N}$ such that $|\gamma_k - \alpha| \geqslant 1/d(k)$ for all k. The striking thing here is that, although the moduli $|\gamma_k - \alpha|$ are effectively bounded away from zero, there is no effective way to determine the *signs* of the numbers $(\gamma_k - \alpha)$. Indeed, if there were, then α would be computable—for to compute α, we would merely take for $\{\gamma_k\}$ any recursive enumeration of the rationals.

Both of the applications given below appear to require the full strength of the Effective Modulus Lemma. The first application gives an example of a continuous function f which is sequentially computable—i.e. $\{f(x_n)\}$ is computable whenever $\{x_n\}$ is—but such that f is not computable. This example has an interesting connection with the history of recursive analysis. Originally, Banach and Mazur defined a function f to be "computable" if f was sequentially computable (condition (i) in Definition A of Chapter 0). Later it was realized that this definition was too broad, and condition (ii) of Definition A—effective uniform continuity—was added.

In Theorem 6 below, we give an example of a *continuous* function f which satisfies the Banach-Mazur condition (i), but which is not "computable" in the modern sense, since it fails to satisfy condition (ii).

Our second application, which we cite without proof (cf. Pour-El, Richards [1981]), shows that all of the above mentioned phenomena can occur for solutions of the wave equation of mathematical physics. Again, the proof of this assertion depends on the Effective Modulus Lemma.

Theorem 5 (Effective Modulus Lemma). *There exists a computable sequence of rational numbers* $\{r_m\}$ *such that:*

(1) *the sequence* $\{r_m\}$ *is strictly increasing,* $0 < r_m < 1$, *and* $\{r_m\}$ *converges to a noncomputable real number* α.

(2) *the differences* $(r_m - r_{m-1})$ *do not approach zero effectively.*

(3) *for any computable sequence of reals* $\{\gamma_k\}$, *there exist recursive functions* $d(k)$ *and* $e(k)$ *such that*

$$|\gamma_k - r_m| \geqslant 1/d(k) \qquad \text{for } m \geqslant e(k).$$

[*In particular,* $|\gamma_k - \alpha| \geqslant 1/d(k)$ *for all* k.]

Proof. We begin with a pair of recursively inseparable sets of natural numbers A and B; as we recall, this means that the sets A and B are recursively enumerable and disjoint, and there is no recursive set C with $A \subseteq C$ and $B \cap C = \emptyset$. Let $a\colon \mathbb{N} \to \mathbb{N}$ and $b\colon \mathbb{N} \to \mathbb{N}$ be one-to-one recursive functions listing the sets A and B respectively. We assume that $0 \notin A$.

For the time being, we put aside the set B and work with A. We set

$$r_m = \frac{5}{9} + \sum_{n=0}^{m} 10^{-a(n)}, \qquad \alpha = \frac{5}{9} + \sum_{n=0}^{\infty} 10^{-a(n)}.$$

Thus the decimal expansion of α is a sequence of 5's and 6's, with a 6 in the s-th place if and only if $s \in A$. (Since $0 \notin A$, $0 < \alpha < 1$.)

Here we recall briefly some facts which were worked out in detail in Sections 1 and 2 of Chapter 0. The above construction already gives parts (1) and (2) of Theorem 5. Begin with (2): $r_m - r_{m-1} = 10^{-a(m)}$, which cannot approach zero effectively, else we would have an effective test for membership in the set A, and A would be recursive. Now (2) implies (1)—for, since $\alpha - r_m \geqslant r_{m+1} - r_m$, (2) shows that r_m cannot approach α effectively. We know from Chapter 0 that this forces α to be a noncomputable real.

Now we come to part (3). Take any computable sequence of reals $\{\gamma_k\}$. We must construct the recursive functions $d(k)$ and $e(k)$ promised in (3). To do this, we will use the decimal expansions of the numbers γ_k. However, there is the difficulty that the determination of these expansions may not be effective, because of the ambiguity between decimals ending in 000... and those ending in 999.... To circumvent this difficulty, we use "finite decimal approximations" to the γ_k. These are constructed as follows:

Since $\{\gamma_k\}$ is computable, there exists a computable double sequence of rationals $\{R_{kq}\}$ such that $|R_{kq} - \gamma_k| \leqslant 10^{-(q+2)}$ (cf. Chapter 0, Section 2). Of course, as we have often noted, exact comparisons between rationals can be made effectively. We define the (q-th decimal for γ_k) to be that decimal of length $q + 1$ which most closely approximates R_{kq}—in case of ties we take the smaller one. Then $|(q\text{-th decimal for } \gamma_k) - R_{kq}| \leqslant (1/2)10^{-(q+1)}$ and $|R_{kq} - \gamma_k| \leqslant (1/10)10^{-(q+1)}$, so that

$$|(q\text{-th decimal for } \gamma_k) - \gamma_k| \leqslant 10^{-(q+1)}.$$

Now suppose we write out this decimal:

$$(q\text{-th decimal for } \gamma_k) = N_{q0} \cdot N_{q1} N_{q2} \ldots N_{q,q+1}.$$

Then the "digits" $N_{qs} = N_{qs}(k)$ (where $s \leqslant q + 1$) are recursive in q, s, and k. This is the desired effective sequence of finite decimal approximations to γ_k.

Now we come to the heart of the proof. It is here that we use the other set B in our recursively inseparable pair. We shall give the construction for a particular γ_k, but in a manner which is clearly effective in k. Here is the construction:

List the sets A and B in turn, using the recursive functions a and b. Stop when an integer $s \in A \cup B$ occurs such that either:

a) $s \in A$ and $N_{ss} \neq 6$, $N_{s,s+1} \neq 0$ or 9, or
b) $s \in B$ and $N_{ss} = 6$, $N_{s,s+1} \neq 0$ or 9, or
c) $s \in A \cup B$ and $N_{s,s+1} = 0$ or 9.

This process eventually halts. To prove this, suppose that (c) never occurs. Let C denote the set of integers $\{s: N_{ss} = 6\}$. Since $N_{ss}(k)$ is recursive, C is recursive, effectively in k. Hence, since the sets A and B are recursively inseparable, we cannot have $A \subseteq C$ and $B \cap C = \emptyset$. If $A \nsubseteq C$ then we have (a): for there is some $s \in A$ with $s \notin C$, which means that $s \in A$ and $N_{ss} \neq 6$, and—since we have ruled out (c)—this is precisely what we need for (a). Similarly, if $B \cap C \neq \emptyset$ then we have (b). This covers all cases; for we have shown that if (c) fails, then we eventually arrive at either (a) or (b)—i.e. the process halts. Furthermore, the process can be carried out for all of the γ_k, by using an effective procedure which returns to each k infinitely often.

Now we define the functions $d(k)$ and $e(k)$. Let s be the first occurrence of a value $s = a(n)$ or $s = b(n)$ for which (a), (b), or (c) holds. Then set

$$d(k) = 10^{s+1}, \qquad e(k) = n.$$

We must show that $|\gamma_k - r_m| \geqslant 1/d(k)$ for $m \geqslant e(k)$.

First we examine the case where the above process terminates in (c). Consider the true (i.e. exact) decimal expansion for γ_k. By (c), the $(s + 1)st$ decimal digit for γ_k is 8, 9, 0, or 1 (allowing for errors in the decimal approximation). By contrast, the $(s + 1)st$ digit for r_m is 5 or 6. This gives $|\gamma_k - r_m| \geqslant 10^{-(s+1)} = 1/d(k)$, as desired.

Now we turn to the case where the process terminates in (a) or (b). This will be a standing assumption throughout the remainder of the proof.

Recall that the s-th decimal digit for α is a 6 if $s \in A$ and a 5 if $s \notin A$. Furthermore, since $a(n)$ gives a one-to-one listing of A, and since $A \cap B = \emptyset$, the s-th decimal digit for both r_m and α is determined as soon as some $a(n)$ or $b(n)$ equals s (the s-th digit is 6 if $s \in A$, 5 if $s \in B$). By definition of $e(k)$, this occurs as soon as $m \geqslant e(k)$. Thus, for $m \geqslant e(k)$, the s-th digit for r_m coincides with that for α.

Now suppose that the process terminates in (a). Then $s \in A$, so that the s-th digit for r_m is 6. However, N_{ss} = the s-th digit for γ_k is *not* 6. Finally, $N_{s,s+1} \neq 0$ or 9, which guarantees that N_{ss} is the "true" s-th decimal digit for γ_k (i.e. that N_{ss} is not "off by 1" due to an error in approximation). Hence $|\gamma_k - r_m| \geqslant 10^{-(s+1)} = 1/d(k)$, as desired.

The case (b) is handled similarly, and this completes the proof of Theorem 5. □

Now as promised above, we give two results which depend on the Effective Modulus Lemma.

Theorem 6 (A continuous function which is sequentially computable but not computable). *There exists a continuous function f on $[0, 1]$ such that f is sequentially computable—i.e. f maps computable sequences $\{x_n\}$ onto computable sequences $\{f(x_n)\}$—but f is not computable. Furthermore, f can be chosen to be differentiable (but not continuously differentiable).*

Proof. We use the results and notation of the Effective Modulus Lemma (Theorem 5 above). Let $\{r_m\}$ be as in the lemma, and set

$$a_m = r_m - r_{m-1},$$

$$b_m = \min(a_m, a_{m-1}), \quad m \geqslant 2.$$

Since $\{r_m\}$ is strictly increasing, $b_m > 0$. Finally, we use a computable C^∞ pulse function $\varphi(x)$ with support on $[-1, 1]$ as in the proof of Theorem 1 in Section 1. Now we define f by:

$$f(x) = \sum_{m=2}^{\infty} a_m^2 \cdot \varphi[2^m b_m^{-1}(x - r_{m-1})].$$

Since $a_m = r_m - r_{m-1}$ and $\{r_m\}$ converges, $\sum a_m$ converges, and since $0 < a_m < 1$, $\sum a_m^2$ converges (although not effectively). Hence the series for f is uniformly convergent (again not effectively), and f is continuous.

Now the individual pulses in the series for f have

$$\text{amplitudes} = a_m^2 \quad (\text{since } \varphi(0) = 1),$$

$$\text{half-widths} = 2^{-m} b_m \leqslant 2^{-m},$$

and are centered at the points $x = r_{m-1}$. By definition of b_m (and since $m \geqslant 2$) these pulses do not overlap. Hence $f(x)$ is C^∞ except at the point $\alpha = \lim r_m$. But $f'(\alpha)$ exists and is zero, since the graph of $f(x)$ is squeezed between the two parabolas $y = \pm 4(x - \alpha)^2$, which are tangent to the x-axis at $x = \alpha$.

To see this: The amplitude of the m-th pulse is a_m^2, and the pulse is centered at $x = r_{m-1}$. However, the pulses have finite width, and we must consider how close the support of the m-th pulse comes to the limit point α. Now the half-width of the m-th pulse is $\leqslant (1/2)b_m \leqslant (1/2)a_m = (1/2)(r_m - r_{m-1}) < (1/2)(\alpha - r_{m-1})$. Thus the support of the m-th pulse reaches less than half of the way from r_{m-1} to α. Hence, for $x \in$ support of m-th pulse, $(\alpha - x) \geqslant (1/2)(\alpha - r_{m-1})$. Thus $4(\alpha - x)^2 \geqslant (\alpha - r_{m-1})^2 \geqslant (r_m - r_{m-1})^2 = a_m^2 =$ amplitude, as desired.

Thus $f'(\alpha) = 0$, and hence f is differentiable at all points.

We now show that f is not effectively uniformly continuous, and hence not computable. Recall that the pulses in the series for f have disjoint supports. The m-th pulse has amplitude $= a_m^2$ and half-width $\leqslant 2^{-m}$. Thus the half-widths approach zero effectively, and effective uniform continuity would force the amplitudes to do likewise. But by (2) in the Effective Modulus Lemma, we have that $a_m = r_m - r_{m-1}$ does not approach zero effectively, and so neither does a_m^2.

Finally, we show that f is sequentially computable, i.e. that $\{f(\gamma_n)\}$ is computable for any computable sequence of reals $\{\gamma_n\}$. It is here that we use part (3) of the Effective Modulus Lemma. We recall that (3) gives recursive functions $d(k)$ and $e(k)$ such that

$$|\gamma_k - r_m| \geqslant 1/d(k) \qquad \text{for } m \geqslant e(k).$$

This allows us to sum the series for $f(\gamma_k)$ in a finite number of steps, effectively in k. Namely let

$$M(k) = \max\,(d(k), e(k)).$$

Then to compute $f(\gamma_k)$, we simply compute the first $M(k)$ terms in the series for f; all other terms vanish at $x = \gamma_k$. To see this, we recall that the m-th pulse is centered on r_{m-1} and has half-width $\leqslant 2^{-m}$. When $m > M(k)$, then

$$2^{-m} < 2^{-d(k)} < 1/d(k);$$

but $m - 1 \geqslant e(k)$ and hence

$$|\gamma_k - r_{m-1}| \geqslant 1/d(k),$$

so that the support of the m-th pulse does not contain the point $x = \gamma_k$. This proves Theorem 6. □

Another result which depends on the Effective Modulus Lemma is the following. We omit its proof; it can be found in Pour-El, Richards [1981].

Theorem* (The wave equation with computable initial data but a noncomputable solution). *Consider the wave equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = 0,$$

with the initial conditions

$$u(x, y, z, 0) = f(x, y, z),$$
$$\frac{\partial u}{\partial t}(x, y, z, 0) = 0.$$

There exists a computable function $f(x, y, z)$ such that the unique solution $u(x, y, z, t)$ is continuous and sequentially computable on $\mathbb{R}^4$, but the solution $u(x, y, z, 1)$ at time $t = 1$ is not computable on $\mathbb{R}^3$.

It is worth remarking that the main point of Theorem*—the wave equation with computable initial data f can have a noncomputable continuous solution u—will be proved in Chapter 3, Section 5 below. The proof will be based on the First Main Theorem and will be quite short.

4. *Translation Invariant Operators*

This section is not essential for the rest of the book and could be omitted on first reading. It marks a transition between the concrete topics treated in Part I and the more general notions introduced in Parts II and III. In Part I we have mainly dealt with specific problems, and we have used one notion of computability—that introduced in Chapter 0. In parts II and III we will introduce an infinite class of "computability structures", of which the Chapter 0 notion becomes a special case. Furthermore, rather than treating problems one at a time, we shall seek general theorems which encompass a variety of applications.

This section harks back to Part I in that we are still dealing with the classical—Chapter 0—notion of computability. On the other hand, it reflects the spirit of Parts II and III in that we give a general theorem which has several applications. In order to state this theorem, we need the idea of a translation invariant operator.

We now give a brief introduction to translation invariant operators. In a technical sense, this introduction is unnecessary. Readers who prefer a pure definition/theorem/proof style of presentation can turn to the conditions (1)–(3) below, and omit the explanations which precede them.

The following is a heuristic discussion, designed to show that the formal conditions which we give below do embody what we mean by "translation invariance". Begin with translation itself. There are two notions of "translation" which are commonly encountered in real analysis. The first is "discrete translation": the translation of a function f through a displacement a, given by the mapping $f(x) \to f(x - a)$. The second (which could be called "continuous translation") is convolution, the mapping from f into $f * g$ given by the convolution formula

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - a)g(a)\, da.$$

Actually we shall need the q-dimensional analog, in which $x, a \in \mathbb{R}^q$. This is

$$(f * g)(x) = \int \cdots \int_{\mathbb{R}^q} f(x - a)g(a)\, da_1 \ldots da_q.$$

The two versions of "translation"—discrete translation and convolution—are closely related. For it is a small step from the single discrete translation $f(x) \to f(x - a)$ above to a finite linear combination of such translations. Namely, consider a finite sequence $f(x - a_1), \ldots, f(x - a_n)$ of translates of f, and together with these a sequence of coefficients $g_1, \ldots, g_n$. Then the corresponding linear combination is

$$\sum_{i=1}^{n} g_i \cdot f(x - a_i).$$

This is clearly analogous to a Riemann sum for the integral $(f * g)$ above. Since we are only doing heuristics, there is no need to pursue this analogy further. For our purposes it is enough to know that convolution is a natural extension of the idea of a discrete translation. We shall work with convolutions because, in most applications, they are easier to deal with.

Now thinking of convolution as a kind of "continuous translation", we define a *translation invariant operator* T to be one which commutes with convolution, i.e. such that:

$$T(f * g) = (Tf) * g.$$

[To show the analogy with the discrete case, recall that there the operation $f \to f * g$ is replaced by $f(x) \to f(x - a)$. Then, by analogy with the above, translation invariance means that $T[f(x - a)] = (Tf)(x - a)$, just as we would expect.]

Many of the standard operators of analysis and physics are translation invariant: among them the derivative d/dx, the partial derivatives $\partial/\partial x_i$, the Laplace operator, and the solution operators for the wave and heat equations.

For our purposes, we must consider translation invariant operators which satisfy two side-conditions. We now list all of the hypotheses which we will set for the operator T.

Let X and Y be vector spaces of real-valued functions on $\mathbb{R}^q$, and let $T: X \to Y$ be a linear operator such that:

(1) (Translation invariance.) T commutes with convolution, i.e. $T(f * g) = (Tf) * g$.

(2) (Compact support). If f has compact support, then so does Tf.

(3) (Computability for smooth functions.) If $\{\varphi_k\}$ is a sequence of C^∞ functions with compact support on $\mathbb{R}^q$, and if the partial derivatives $\{\varphi_k^{(\alpha)}\}$ (α = multi-index) are computable effectively in k and α, then the sequence $\{T\varphi_k\}$ is computable.

Now we give a general result which holds for operators of this type. By way of preface, we recall the two conditions in the definition of a computable function on a compact domain (Definition A, Chapter 0):

(i) sequential computability;
(ii) effective uniform continuity.

These conditions are independent in the sense that, in general, neither implies the other. However, there are situations where (ii) implies (i), and this has numerous applications.

Theorem 7 (Translation invariant operators). *Let T be a linear operator which satisfies* (1)–(3) *above. Let $f\colon \mathbb{R}^q \to \mathbb{R}^1$ be a computable function with compact support. Suppose that f lies in the domain of T, and that Tf is effectively uniformly continuous. Then Tf is computable.*

[I.e. if Tf satisfies (ii) above, then Tf also satisfies (i).]

Proof. Since the groundwork has been carefully laid, the proof is quite easy. We begin by constructing a computable sequence $\{\varphi_k\}$ of C^∞ functions which form an "approximate identity" for convolution—i.e., as $k \to \infty$, the support of φ_k approaches zero effectively, while the integral of φ_k over $\mathbb{R}^q$ remains equal to 1. In detail:

Begin with any C^∞ function $\varphi \geqslant 0$ with support on the unit disk in $\mathbb{R}^q$, and such that the sequence of derivatives $\{\varphi^{(\alpha)}\}$ (α = multi-index) is computable. We can assume without loss of generality that

$$\int \cdots \int_{\mathbb{R}^q} \varphi \, dx_1 \ldots dx_q = 1.$$

Let

$$\varphi_k(x) = k^q \varphi(kx).$$

Then the support of φ_k is a disk of radius $1/k$ about the origin, and the integral of φ_k over $\mathbb{R}^q$ is equal to 1. Furthermore, $\{\varphi_k^{(\alpha)}\}$ is computable, effectively in k and α. Thus $\{\varphi_k\}$ is the desired computable "approximate identity".

The following fact about convolutions is well known:

(*) If $g\colon \mathbb{R}^q \to \mathbb{R}^1$ is uniformly continuous, and ε, $\delta > 0$ are chosen so that $|x - y| \leqslant \delta$ implies $|g(x) - g(y)| \leqslant \varepsilon$, then

$$1/k \leqslant \delta \quad \text{implies} \quad |(g * \varphi_k) - g| \leqslant \varepsilon.$$

Our proof consists in effectivizing this elementary fact, and at the same time using the conditions (1)–(3) to connect it to the properties of the operator T.

Since the operator T satisfies (2) and f has compact support, so does Tf.

By construction, $\{\varphi_k^{(\alpha)}\}$ is a computable multi-sequence. Since f is computable and integration is a computable process, $\{f * [\varphi_k^{(\alpha)}]\}$ is computable. Since differentiation commutes with convolution, $f * [\varphi_k^{(\alpha)}] = [f * \varphi_k]^{(\alpha)}$. Hence by (3), $\{T[f * \varphi_k]\}$ is a computable sequence.

Now we use (1)—translation invariance. This gives $T[f * \varphi_k] = [Tf] * \varphi_k$, and hence by the above $\{[Tf] * \varphi_k\}$ is a computable sequence. We will show that $[Tf] * \varphi_k$ converges effectively and uniformly to Tf, so that by Theorem 4 of Chapter 0, Tf is computable.

Here we use the assumption that Tf is effectively uniformly continuous. Thus there is a recursive function $d(N)$ such that $|x - y| \leqslant 1/d(N)$ implies $|Tf(x) - Tf(y)| \leqslant 2^{-N}$. Now the estimate (∗) above becomes effective, and we have

$$k \geqslant d(N) \quad \text{implies} \quad |([Tf] * \varphi_k) - Tf| \leqslant 2^{-N}.$$

Thus $[Tf] * \varphi_k$ approaches Tf effectively and uniformly. As we have already noted, this implies that Tf is computable. The proof of Theorem 7 is complete. □

As promised, we give two applications of Theorem 7. The first—relating to partial derivatives—is a generalization of Theorem 2 in Section 1. The second gives information about noncomputable solutions of the wave equation.

Partial derivatives. We now give conditions which ensure the computability of partial derivatives. We shall take pains to present the result in its most general form.

Let $\alpha = (\alpha_1, \ldots, \alpha_q)$ be a multi-index of order $|\alpha| = \alpha_1 + \cdots + \alpha_q$. Let D^α denote the partial differential operator

$$D^\alpha f = f^{(\alpha)} = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_q^{\alpha_q}}.$$

Let $f\colon \mathbb{R}^q \to \mathbb{R}^1$ be a computable function with compact support. Suppose that all partial derivatives of f of order $< |\alpha|$ are continuous (although not necessarily computable, and not necessarily effectively uniformly continuous). Suppose that the particular derivative $D^\alpha f$ is effectively uniformly continuous. Then $D^\alpha f$ is computable.

[Of course, as a corollary, if f is C^∞, then every partial derivative $D^\alpha f$ is computable.]

To prove this, we merely observe that the operator $T = D^\alpha$ satisfies conditions (1)–(3) above. Hence the result is an immediate consequence of Theorem 7.

Note. The general hypotheses above—in which only $D^\alpha f$ itself is assumed effectively uniformly continuous—can actually occur. Example: Let $f(x)$ be as in Theorem 1 above, so that $f'(x)$ is continuous but not effectively uniformly continuous.

Let $h(x, y) = f(x)$. Then $\partial h/\partial x = f'(x)$ is not effectively uniformly continuous, but $\partial^2 h/\partial x \partial y = 0$ is.

Of course this is a technicality. However, the general result for $D^\alpha f$ above—a trivial consequence of Theorem 7—would be much harder to prove by the elementary methods used in the proof of Theorem 2.

The wave equation. In Section 3 above, we remarked that the wave equation with computable initial data can have a noncomputable solution. In fact we can start with computable initial data at time $t = 0$ and obtain a noncomputable solution at time $t = 1$. (This is further discussed in Section 5 of Chapter 3.)

Here we consider one aspect of these noncomputable solutions—an aspect which is related to Theorem 7 above. The noncomputable solutions are always "weak solutions"—i.e. although continuous, they are not C^2 or even C^1. Indeed, if they were C^1, they would be effectively uniformly continuous. This is impossible, for we have:

Theorem 8 (Noncomputable solutions of the wave equation must be weak solutions). *Consider the wave equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = 0,$$

with the initial conditions $u(x, y, z, 0) = f(x, y, z)$, $\partial u/\partial t = 0$ at time $t = 0$. Suppose that f is computable and continuous with compact support. Suppose, however, that the solution $u(x, y, z, 1)$ is not computable. Then $u(x, y, z, 1)$ is not effectively uniformly continuous, and hence is not C^2 or even C^1.

Notes. As we will show in Chapter 3, such noncomputable solutions can be continuous. Thus the break-point lies between continuity and effective uniform continuity.

Finally a trivial note: of course the time $t = 1$ could be replaced by any computable time $t = t_0$.

Proof. Suppose that $u(x, y, z, 1)$ is effectively uniformly continuous. We will show that then $u(x, y, z, 1)$ is computable.

Let T be the solution operator which maps the initial data $f(x, y, z)$ onto the solution $u(x, y, z, 1)$. If we can show that T satisfies conditions (1)–(3) above, then the desired result will follow immediately from Theorem 7.

Now there is a well known formula (Kirchhoff's equation) for T: it is displayed in Chapter 3, Section 5. Using this formula, it is a routine matter to verify that T satisfies (1)–(3). However, as an illustration of technique the following seems more interesting:

Two of the key properties of T can be seen on "physical" grounds. That T is translation invariant—condition (1) above—follows from the fact that the wave equation is translation invariant. That T preserves compact supports—condition (2) above—follows from the fact that waves travel with a finite velocity. Finally (3)—computability for C^∞ functions—does require a glance at the formula in Chapter 3, Section 5: but only long enough to verify that the formula involves integrals and partial derivatives. The exact shape of the formula is irrelevant.

Since conditions (1)–(3) above are satisfied by T, the desired result follows at once from Theorem 7. □