Chapter VIII
Priority Arguments

In this chapter some standard results of classical recursion theory are lifted to every Σ_1 admissible ordinal α by techniques hinted at in earlier chapters. Since the classical proofs use the Σ_2 admissibility of $L(\omega)$, the proofs to come may be regarded as instances of the austere art of making Σ_1 admissibility do the work of Σ_2. The initial technique relies strongly on the L-ness of L, the ability of L to support downward Skolem–Löwenheim arguments. The later technique depends on combinatoric consequences of Σ_1 admissibility. Its dynamic nature makes it applicable to Σ_1 admissible structures for which hull-collapsing arguments fail.

1. α-Finite Injury via α^*

In this section and the next it will be shown that there exist α-recursively enumerable sets A and B such that neither is α-recursive in the other. The method extends that applied in Section 5.VII, to construct a hyperregular, non-α-recursive, α-recursively enumerable set. The injury sets become more complex. Back in Chapter VII each negative requirement was injured at most once for the sake of each positive requirement of higher priority. Such simplicity is rare in the present chapter. Consequently the α-recursive projection of α into α^*, which arranges that each negative requirement be opposed by less than-α^* positive requirements of higher priority, does not always compel the injury sets to be α-finite. In Section 2 below, when $\alpha^* = \alpha$ and there is a greatest α-cardinal, it will be necessary to project α downward by means of a carefully chosen Σ_2^α function.

1.1 Strategy. Define $\{p^{-1}\varepsilon\}^B$ for each $\varepsilon < \alpha^*$ as in the beginning of the proof of Theorem 5.5.VII. Thus $A \leq_{w\alpha} B$ iff $A = \{p^{-1}\varepsilon\}^B$ for some $\varepsilon < \alpha^*$. The requirements on A and B are as follows.

Requirement 2ε: If $\{p^{-1}\varepsilon\}^B$ is a total function, then $A \neq \{p^{-1}\varepsilon\}^B$.

Requirement $2\varepsilon + 1$: Same as req 2ε with A and B interchanged.

Let $\{Z_\varepsilon \mid \varepsilon < \alpha^*\}$ be a collection of simultaneously α-recursive, pairwise disjoint, unbounded subsets of α. The strategy for satisfying req 2ε consists of finding an $x \in Z_\varepsilon$ such that:

(1) $$\text{if } \{p^{-1}\varepsilon\}^B(x) = 0, \text{ then } x \in A.$$

(Recall that $A(x)$, the characteristic function of A, is 1 when $x \in A$.) As in the proof of Theorem 5.5.VII, $\{p^{-1}\varepsilon\}^B(x)$ is approximated at stage σ of the construction of A and B by $\{p^{-1}\varepsilon\}_\sigma^{B<\sigma}(x)$, an α-recursive function of ε, σ and x. $B^{<\sigma}$ is that part of B enumerated prior to stage σ.

Suppose $\{p^{-1}\varepsilon\}_\sigma^{B<\sigma}(x) = 0$ and $x \notin A^{<\sigma}$ for some $x \in Z_\varepsilon$. Then at stage σ an attempt to satisfy req 2ε can be made by adding x to A and making a commitment to preserve the computation of $\{p^{-1}\varepsilon\}_\sigma^{B<\sigma}(x)$ through all future stages. The commitment reduces to: exclude all the elements of some α-finite K from B, where K is the set of negative facts about $B^{<\sigma}$ used in the computation of $\{p^{-1}\varepsilon\}_\sigma^{B<\sigma}(x)$. If the attempt is made at stage σ, and if the resulting commitment is honored at all future stages, then $x \in A$, $\{p^{-1}\varepsilon\}^B(x) = 0$ and req 2ε is satisfied.

The commitment is broken the first time some member of K is added to B for the sake of some req $2\varepsilon_0 + 1$ at some stage $\tau > \sigma$. In that event req 2ε is said to be *injured* at stage τ for the sake of req $2\varepsilon_0 + 1$. Let

$$I_{2\varepsilon} = \{\tau \mid \text{req } 2\varepsilon \text{ is injured at stage } \tau\}.$$

$I_{2\varepsilon}$ is called an injury set. Define $I_{2\varepsilon+1}$ similarly. The injury sets are simultaneously α-recursively enumerable. Once it is seen that all injury sets are α-finite, it is not difficult to show all requirements are met.

Two devices are employed to limit the extent of injury sets. The first assigns priorities. Requirement x has higher priority than requirement y if $x < y$. The priority method consists of allowing requirement y to be injured for the sake of requirement x only if x has higher priority than y. Thus req 2ε can be injured by adding some element to B for the sake of req $2\varepsilon_0 + 1$ only when $2\varepsilon_0 + 1 < 2\varepsilon$. The second device enhances the first. It indexes the requirements so as to form the shortest possible list compatible with a Σ_1^α construction. The shorter the list, the smaller the set of requirements of higher priority than a given requirement. In this section the requirements are indexed by ordinals less than α^* with the aid of p, an α-recursive projection of α into α^*. In the next section they are indexed by ordinals less than a certain ordinal below α^* with the aid of a Σ_2^α projection.

1.2 Construction of A and B. At each stage attention is paid to only one requirement, but every requirement is paid attention unboundedly often. Let σ be a stage at which attention is paid to req 2ε.

First suppose an attempt was made to satisfy req 2ε prior to stage σ and no subsequent injury was inflicted on req 2ε prior to stage σ. To elaborate: there was a stage $\sigma_0 < \sigma$ at which some element of Z_ε was put in A and a commitment was made to keep some α-finite K from touching B; that commitment has been honored up to now, in short, $K \subseteq cB^{<\sigma}$. There is clearly no need to make a new attempt at stage σ to satisfy req 2ε. Let $A^\sigma = A^{<\sigma}$ and $B^\sigma = B^{<\sigma}$.

Now suppose every attempt to satisfy req 2ε made prior to stage σ was injured prior to stage σ. Let $m_B(\sigma, \varepsilon)$ be the supremum of all y such that prior to stage σ, a commitment was made to keep y out of A for the sake of req $2\varepsilon_0 + 1$ for some $\varepsilon_0 < \varepsilon$. Then any $y > m_B(\sigma, \varepsilon)$ can be added to A at stage σ without injuring any requirement of higher priority than req 2ε. Define

$$w(\sigma, \varepsilon) = \mu y[y > m_B(\sigma, \varepsilon) \quad \& \quad y \in Z_\varepsilon - A^{<\sigma}].$$

(1) If $\{p^{-1}\varepsilon\}_\sigma^{B<\sigma}(w(\sigma, \varepsilon)) = 0$, then

$$A^\sigma = A^{<\sigma} \cup \{w(\sigma, \varepsilon)\} \quad \text{and} \quad B^\sigma = B^{<\sigma};$$

in addition a commitment is made to keep all of K out of B, where K is an α-finite set of negative membership facts about $B^{<\sigma}$ needed for the computation of (1). If (1) is false, then $A^\sigma = A^{<\sigma}$ and $B^\sigma = B^{<\sigma}$.

Req $2\varepsilon + 1$ is handled similarly. End of construction.

The next lemma captures the so-called *α-finite injury method*. The analogous Friedberg–Muchnik lemma says that the n-th requirement is injured less than 2^n times.

1.3 Lemma. *Suppose β is an infinite α-cardinal and $\delta < \beta$. Then injury set I_δ is α-finite and has α-cardinality $< \beta$.*

Proof. Let $\beta_0 \leq \beta$ be a regular α-cardinal such that $\delta < \beta_0$. For the sake of an induction, assume for $\gamma < \delta$,

$$I_\gamma \text{ is } \alpha\text{-finite and has } \alpha\text{-cardinality} < \beta_0.$$

the I_γ's are simultaneously α-recursively enumerable, and so by Lemma 2.3.VII,

$$\cup \{I_\gamma | \gamma < \delta\} \text{ is } \alpha\text{-finite and has } \alpha\text{-cardinality} < \beta_0.$$

Now define

$$A_\gamma = \{\sigma | \text{an attempt is made at stage } \sigma \text{ to satisfy req. } \gamma\}.$$

Observe that A_γ and I_γ are *interlaced*: between any two elements of either set lies a member of the other. Hence if either set is α-finite, then so is the other, and in addition their α-cardinalities differ by at most 1. Consequently

$$\text{(1)} \qquad \cup \{A_\gamma | \gamma < \delta\} \text{ is } \alpha\text{-finite and has } \alpha\text{-cardinality} < \beta_0.$$

I_δ is a subset of $\cup \{A_\gamma | \gamma < \delta\}$, and so can be viewed as an α-recursively enumerable subset of some ordinal less than β_0. Since $\beta_0 \leq \alpha^*$, I_δ is α-finite. $\square$

Theorem 1.4 solves half of Post's problem. The other half is in the next section.

1.4 Theorem. *Suppose $\alpha^* < \alpha$, or $\alpha^* = \alpha$ and there is no greatest α-cardinal. Then there exist α-recursively enumerable sets A and B such that $A \not\leq_{w\alpha} B$ and $B \not\leq_{w\alpha} A$.*

Proof. Let A and B be the sets enumerated in subsection 1.2. Assume $\{p\varepsilon^{-1}\}^B$ is total with the intent of showing $A \neq \{p\varepsilon^{-1}\}^B$. According to formula (1) of the proof of Lemma 1.3, there exists a σ_0 such that for $\delta \leq 2\varepsilon$, all attempts to satisfy req 2ε occur prior to stage σ_0.

Case 1: there was an attempt at some stage σ to satisfy req 2ε, and the associated commitment was honored at all subsequent stages. Then

$$\{p^{-1}\varepsilon\}_\sigma^{B<\sigma}(w(\sigma, \varepsilon)) = 1 = \{p^{-1}\varepsilon\}^B(w(\sigma, \varepsilon))$$

and $A^\sigma(w(\sigma, \varepsilon)) = 0 = A(w(\sigma, \varepsilon))$.

Case 2: otherwise. Let $\sigma \geq \sigma_1 \geq \sigma_0$ and suppose σ and σ_1 are stages at which attention is paid to req 2ε. Then

$$m_B(\sigma, \varepsilon) = m_B(\sigma_1, \varepsilon) \quad \text{and} \quad w_B(\sigma, \varepsilon) = w_B(\sigma_1, \varepsilon) \notin A^{<\sigma}.$$

No attempt is made to satisfy req 2ε at stage σ, so

$$\{p^{-1}\varepsilon\}_\sigma^{B<\sigma}(w_\beta(\sigma, \varepsilon)) \neq 1.$$

Since $\{p^{-1}\varepsilon\}^B$ is total, it follows that

$$\{p^{-1}\varepsilon\}^B(w_B(\sigma_1, \varepsilon)) \neq 1.$$

Thus $w_B(\sigma_1, \varepsilon)$ is a witness to the inequality of A and $\{p^{-1}\varepsilon\}^B$. $\square$

The technology of the next section is more than is needed to solve Post's problem. Exercise 1.5 describes a much less technical solution. Sections 1 and 2 are intended to introduce ideas needed for further results such as the uniform solution of subsection 3.7.

1.5 Exercise. Assume $\alpha^* = \alpha$. Construct A and B as in subsection 1.2 with one additional proviso: suppose req x is injured at stage σ for the sake of req y; then req x can be injured at stage $\tau > \sigma$ for the sake of req z only if $z < y$. Show each injury set is finite. Show A and B are α-recursively enumerable, $A \not\leq_{w\alpha} B$, and $B \not\leq_{w\alpha} A$.

2. *α-Finite Injury and Tameness*

In this section it is assumed that $\alpha^* = \alpha$ and there exists a greatest α-cardinal, call it $\aleph$. The solution to Post's problem given in the previous section now fails, because α^* is no longer the limit of regular α-cardinals, and consequently Lemma 1.3 is not applicable to all $\delta < \alpha^*$. One way around the difficulty is to project α into something potentially smaller than α^*. The new mode of projection will be Σ_2^α rather than Σ_1^α, and the range of the projection will be some ordinal multiple of $\aleph$. Lemma 1.3 will apply to each block of requirements of length $\aleph$, and a further idea will be involved to control the injuries within a short union of blocks. The arguments of this section rely strongly on the fine structure of L and are as concrete as possible. Lemma 2.7 provides rare bounds on α-finite injury sets. In succeeding sections the method is dealt with more abstractly so as to be applicable to a wide range of Σ_1 admissible sets.

2.1 Tame Σ_2 Maps. The notion of tameness was invented by M. Lerman to clarify the proof of Theorem 2.6. It has many applications. Let $f\colon \alpha \to \alpha$ be Σ_2^α. f has an α-recursive approximation that will not surprise students of classical recursion theory. Since $f \in \Sigma_2^\alpha$, there is a $D \in \Delta_0^\alpha$ such that

$$f(x) = y \leftrightarrow L(\alpha) \models (\mathrm{Ev})(u)D(u, v, \underline{x}, \underline{y})$$

for all $x, y < \alpha$. Define α-recursive w and g by:

$$w(\sigma, x) = \mu w_{w<\sigma}(u)_{u<\sigma} D(u, (w)_0, x, (w)_1); \tag{1}$$

$$g(\sigma, x) = (w(\sigma, x))_1. \tag{2}$$

2.2 Proposition. *$f \in \Sigma_2^\alpha$ iff there exists an α-recursive g such that $f(x) = \lim_\sigma g(\sigma, x)$.*

Proof. Suppose $f(x) = \lim_\sigma g(\sigma, x)$. Then

$$f(x) = y \leftrightarrow (E\sigma)(\tau)_{\tau \geq \sigma}[g(\sigma, x) = y].$$

Now suppose f is Σ_2^α. Define $w(\sigma, x)$ and $g(\sigma, x)$ as in (1)–(2) of subsection 2.1. Fix x and let w be the least $\langle v, y \rangle$ such that $(u)D(u, v, x, y)$ holds in $L(\alpha)$. Then $(w)_1 = f(x)$. Define

$$h(z) \simeq \mu u \sim D(u, (z)_0, x, (z)_1).$$

h is partial α-recursive and defined for all $z < w$. Let

$$\tau = \sup\{h(z) \mid z < w\}.$$

$\tau < \alpha$ thanks to the Σ_1 admissibility of $L(\alpha)$. For all $\sigma > \tau$, $w(\sigma, x) = w$ and $g(\sigma, x) = f(x)$. $\square$

Let $f\colon \delta \to \alpha$ for some $\delta \leq \alpha$. f is said to be *tame* Σ_2^α if there exists an α-recursive g such that

$$(\gamma)_{\gamma<\delta}(E\tau)(\sigma)_{\sigma>\tau}(x)_{x<\gamma}[g(\sigma, x) = f(x)].$$

Thus $\lim_\sigma g(\sigma, x) = f(x)$, and so $f \in \Sigma_2^\alpha$ as in the first part of the proof of Proposition 2.2. The tameness of f refers to the way g approximates f on proper initial segments of the domain of f. Let g_σ denote $\lambda x | g(\sigma, x)$. Then

$$(\gamma)_{\gamma<\delta}(E\tau)(\sigma)_{\sigma>\tau}[g_\sigma \restriction \gamma = f \restriction \gamma].$$

A Σ_2^α function need not be tame Σ_2^α (cf. Exercise 2.14).

The *tame* Σ_2^α *projectum* of α, denoted by $t\sigma p2(\alpha)$, is

$$\mu\gamma(\mathrm{Ef})[f \in \text{tame}\, \Sigma_2^\alpha \;\&\; f \text{ maps } \gamma \text{ onto } \alpha].$$

Warning: it can happen that $\alpha > t\sigma 2p(\alpha)$ and $t\sigma 2p(\alpha)$ is not an α-cardinal.

2.3 Lemma (Lerman). *Let $g(\sigma, x)$ be an α-recursive function such that $\lim_\sigma g(\sigma, x)$ exists for all x. Suppose $\lim_\sigma g(\sigma, x)$ maps α one-one into δ. Assume: $(\gamma)_{\gamma<\delta}(E\tau)(y)_{y<\gamma}$ either*

$$(x)(\sigma)_{\sigma>\tau}[g(\sigma,x) \neq y] \quad \textit{or} \quad (\mathrm{Ex})(\sigma)_{\sigma>\tau}[g(\sigma,x) = y].$$

Then $t\sigma 2p(\alpha) \leq \delta$.

Proof. The map $\lim_\sigma g(\sigma, x)$ is collapsed and inverted. Let $h(\sigma, y)$ be the least member of

$$\{g(\sigma, w) | w \leq \sigma\} - \{h(\sigma, z) | z < y\}$$

if there is one, and zero otherwise. Let δ_0 be the ordertype of the range of $\lambda x | \lim_\sigma g(\sigma, x)$. Then for each $y < \delta_0$, $\lim_\sigma h(\sigma, y)$ exists and is equal to y-th smallest member of the range of $\lambda x | \lim_\sigma g(\sigma, x)$. For $y < \delta_0$, define

$$k(\sigma, y) = \mu x_{x \leq \sigma}[g(\sigma, x) = h(\sigma, y)].$$

Then $k(\sigma, y)$ is α-recursive, and $\lim_\sigma k(\sigma, y)$ is a tame Σ_2^α map of δ_0 onto α. $\square$

2.4 Corollary. $t\sigma 2p(\alpha) \leq \alpha^*$.

Proof. Let $g: \alpha \to \alpha^*$ be one-one, into, and α-recursive. Define $g(\sigma, x) = g(x)$ for all $\sigma, x < \alpha$. $g(\sigma, x)$ satisfies the hypotheses of Lemma 2.3 ($\delta = \alpha^*$) with the aid of Proposition 2.1.VII. $\square$

The Σ_2^α *cofinality of* α, denoted by $\sigma 2\mathrm{cf}(\alpha)$, is

$$\mu\beta(\mathrm{Ef})[f \in \Sigma_2^\alpha \quad \& \quad f \text{ is strictly increasing} \quad \& \quad \mathrm{dom} f = \beta \quad \& \quad \sup \text{ range } f = \alpha].$$

$\sigma 2\mathrm{cf}(\alpha)$ measures the failure of $L(\alpha)$ to be Σ_2 admissible. Thus $L(\alpha)$ is Σ_2 admissible iff $\sigma 2\mathrm{cf}(\alpha) = \alpha$.

The *greatest cardinal of* α, denoted by $\mathrm{gc}(\alpha)$, is the greatest α-cardinal, if there is one, and α otherwise.

2.5 Lemma. If $t\sigma 2p(\alpha) > \mathrm{gc}(\alpha)$, *then*

$$t\sigma 2p(\alpha) = \mathrm{gc}(\alpha) \cdot \sigma 2\mathrm{cf}(\alpha).$$

Proof. Let $t\sigma 2p(\alpha) = \mathrm{gc}(\alpha) \cdot \gamma + \lambda$, where $0 < \gamma \leq \alpha$ and $\lambda < \mathrm{gc}(\alpha)$; and let f be a tame Σ_2^α map from $t\sigma 2p(\alpha)$ onto α. Suppose $\lambda > 0$. Then $f[\mathrm{gc}(\alpha) \cdot \gamma]$ is α-finite by the tameness of f. Let t be a one-one, α-recursive map of $\alpha - f[\mathrm{gc}(\alpha) \cdot \gamma]$ onto α. Then f_0, defined by

$$f_0(x) = t(f(\mathrm{gc}(\alpha) \cdot \gamma + x)),$$

is a tame Σ_2^α map from λ onto α. But that is impossible because $\lambda < t\sigma 2p(\alpha)$.

Thus $t\sigma 2p(\alpha) = \mathrm{gc}(\alpha) \cdot \gamma$. Suppose γ is not a limit ordinal. Then $t\sigma 2p(\alpha) = \mathrm{gc}(\alpha) \cdot (\gamma - 1) + \mathrm{gc}(\alpha)$, and the argument of the previous paragraph can be repeated with $\mathrm{gc}(\alpha)$ in place of λ to obtain a tame Σ_2^α map from $\mathrm{gc}(\alpha)$ onto α. An impossibility since $\mathrm{gc}(\alpha) < t\sigma 2p(\alpha)$.

Thus $t\sigma 2p(\alpha) = \mathrm{gc}(\alpha) \cdot \gamma$ for some limit γ. For each $\delta < \gamma$, let

$$h(\delta) = \sup\{f(x) \mid x < \mathrm{gc}(\alpha) \cdot \delta\} + \delta.$$

The tameness of f implies $h(\delta) < \alpha$. h is a strictly increasing Σ_2^α map from γ into an unbounded subset of α. Hence $\sigma 2\mathrm{cf}(\alpha) \leq \gamma$.

It remains only to construct a tame Σ_2^α map j from $\mathrm{gc}(\alpha) \cdot \sigma 2\mathrm{cf}(\alpha)$ onto α. Let k be a strictly increasing Σ_2^α map from $\sigma 2\mathrm{cf}(\alpha)$ onto an unbounded subset of α. Assume $k(0) = 0$. The range of k divides α into blocks. For each $\delta < \sigma 2\mathrm{cf}(\alpha)$, the δ-th block is $[k(\delta), k(\delta + 1))$. Let z_δ be the least α-finite map of $\mathrm{gc}(\alpha)$ onto the δ-th block ("least" is defined by Proposition 1.8.VII). Define

$$j(x) = z_\delta(x - (\mathrm{gc}(\alpha) \cdot \delta)) \quad \text{if} \quad \mathrm{gc}(\alpha) \cdot \delta \leq x < \mathrm{gc}(\alpha)$$

for all $x < \mathrm{gc}(\alpha) \cdot \sigma 2\mathrm{cf}(\alpha)$. j is tame Σ_2^α because k is. k is tame because every Σ_2^α function with domain $\leq \sigma 2\mathrm{cf}(\alpha)$ is tame (cf. Exercise 2.15). $\square$

The ideas behind Lemma 2.5 are helpful when studying Σ_1 admissible structures that are not L-like. The present section concludes with the original solution to

Post's problem for Σ_1 admissible ordinals. It relies on a tame Σ_2 phenomenon derived from stability properties in L.

2.6 Theorem (Sacks & Simpson 1972). *There exist two α-recursively enumerable sets such that neither is weakly α-recursive in the other (Post's problem).*

Proof. By Theorem 1.4 it is safe to assume $\alpha^* = \alpha$ and there is a greatest α-cardinal, call it $\aleph$. By Lemma 2.6.VII, α is the limit of α-stable ordinals. Let

$$0 = \delta_0 < \aleph = < \delta_1 < \delta_2 < \cdots < \delta_\gamma < \cdots (\gamma < \lambda_0)$$

be a listing of all α-stable ordinals beyond $\aleph$. A tame Σ_2^α map f from $\aleph \cdot \lambda_0$ onto α will be defined in a moment. (It is possible that $t\sigma 2p(\alpha) < \aleph \cdot \lambda_0$.) The α-stable ordinals divide α into λ_0 blocks. Block γ is $[\delta_\gamma, \delta_{\gamma+1})$. Let h_γ be the least α-finite map of $\aleph$ onto block γ. Define

$$f(x) = h_\gamma(x - (\aleph \cdot \gamma)) \quad \text{if} \quad \aleph \cdot \gamma \leq x < \aleph \cdot (\gamma + 1).$$

f maps $\aleph \cdot \lambda_0$ onto α.

Guessing tamely at f proceeds as follows. At stage σ let

$$\aleph = \delta_1^\sigma < \delta_2^\sigma < \cdots < \delta_\gamma^\sigma < \cdots (\gamma < \lambda_0^\sigma)$$

be a listing of all σ-stable ordinals beyond $\aleph$. (β is σ-stable if $\beta < \sigma$ and $L(\beta) \prec_1 L(\sigma)$.) If β is σ-stable and $\sigma \geq$ some α-stable ordinal $\geq \beta$, then β is α-stable. Hence

$$\delta_\gamma^\sigma = \delta_\gamma \quad \text{for all} \quad \sigma \geq \delta_{\gamma+1}.$$

Thus δ_γ is a tame Σ_2^α function of γ.

Let h_γ^σ be the least α-finite map of $\aleph$ onto $[\delta_\gamma^\sigma, \delta_{\gamma+1}^\sigma)$ in $L(\sigma)$, if there is one; otherwise $h_\gamma^\sigma = 0$. Then $h_\gamma^\sigma = h_\gamma$ for all $\sigma \geq \delta_{\gamma+2}$. Let

$$g(\sigma, x) = h_\gamma^\sigma(x - (\aleph \cdot \gamma)) \quad \text{if} \quad \aleph \cdot \gamma \leq x < (\gamma + 1)$$

for all $x < \aleph \cdot \lambda_0^\sigma$. Then f is tame Σ_2^α via the α-recursive approximation g. In essence f is tame because a correct guess of one α-stable ordinal implies a correct guess of all lesser α-stable ordinals.

The construction of A and B proceeds as in subsections 1.1–1.2 save that requirements are indexed by ordinals less than $\aleph \cdot \lambda_0$. For each $\varepsilon < \aleph \cdot \lambda_0$, req 2ε is: if $\{f\varepsilon\}^B$ is total, then $A \neq \{f\varepsilon\}^B$. req $2\varepsilon + 1$ is similar with A and B exchanged. Stage σ unfolds as it did in subsection 1.2 with $\{p^{-1}\varepsilon\}_\sigma^{B^{<\sigma}}$ replaced by $\{g(\sigma, \varepsilon)\}_\sigma^{B^{<\sigma}}$. Consequently there is a new reason for making repeated attempts to satisfy req 2ε. Reduction procedure $\{g(\sigma, \varepsilon)\}$ varies with σ. The resulting disturbance dies down quickly. By the Σ_2 tameness of f via g,

$$(x)_{x<\aleph\cdot\lambda_0}(E\tau)(\sigma)_{\sigma\geq\tau}(\varepsilon)_{\varepsilon<x}[g(\sigma,\varepsilon)=f(\varepsilon)].$$

For $\varepsilon<\aleph\cdot\lambda_0$ and $i<2$, define injury set

$$I_\varepsilon^i=\{\sigma\,|\,\mathrm{req}(2\varepsilon+i)\text{ is injured at stage }\sigma\}.$$

The next lemma should be compared with Lemma 1.3.

2.7 Lemma. $I^i_{\aleph\cdot\gamma+\beta}\in L(\delta_{\gamma+2})\quad(\gamma<\lambda_0\ \ \&\ \ \beta<\aleph)$.

Proof. Req$(2\varepsilon+i)$ is said to be *active* at stage σ if it is injured at stage σ, or if an attempt to satisfy it is made at stage σ, or if

$$(\tau)_{\tau<\sigma}(E\eta)_{\tau\leq\eta<\sigma}[g(\eta,\varepsilon)\neq g(\sigma,\varepsilon)].$$

By induction on γ the following is proved: all activity occasional by req $2(\aleph\cdot\gamma+\beta)+i$ takes place before stage τ for some $\tau<\delta_{\gamma+2}$.

Req$(2\varepsilon+i)$ is said to be in block y if

$$\aleph\cdot y\leq\varepsilon<\aleph\cdot(y+1).$$

Fix γ. By induction all activity with respect to requirements in block y, for all $y<\gamma$, takes place before stage $\delta_{\gamma+1}$. $h_\gamma\in L(\delta_{\gamma+2})$, so there is a $\tau_0<\delta_{\gamma+2}$ such that

$$(\sigma)_{\sigma\geq\tau_0}(\varepsilon)_{\varepsilon<\aleph\cdot(\gamma+1)}[g(\sigma,\varepsilon)=f(\varepsilon)].$$

Now an induction on β shows that req $z(\aleph\cdot\gamma+\beta)+i$ is inactive after stage τ for some $\tau<\delta_{\gamma+2}$. The induction on β proceeds in the same fashion as the induction on δ in the proof of Lemma 1.3. After stage τ_0, injuries to, and attempts to satisfy, req $2(\aleph\cdot\gamma+\beta)+i$ are interlaced. As in the proof 1.3, Lemma 2.3.VII is applied to show $I^i_{\aleph\cdot\gamma+\beta}$ is α-finite and has α-cardinality less than $\aleph$. In short, the argument of 1.3 works within block β. Let

$$\tau_0\leq\sigma_0<\sigma_1<\ldots<\sigma_i<\ldots(i<\rho)$$

be a listing of $I^i_{\aleph\cdot\gamma+\beta}-\tau_0$. $\{\sigma_i|i<\rho\}$ is an α-recursively enumerable set defined by a Σ_1^α formula whose parameters are i, γ, β, τ_0 and $\aleph$. ($\aleph$ is the only parameter needed for the enumeration of A and B.) Since $\rho<\aleph<\delta_{\gamma+2}$, the α-stability of $\delta_{\gamma+2}$ pins down the σ_i's. Suppose $\{\sigma_i|i<j\}\subseteq L(\delta_{\gamma+2})$. Proceed by induction on j. If $j\leq\rho$, then $\{\sigma_i|i<j\}\in L(\delta_{\gamma+2})$, because $\delta_{\gamma+2}$ is Σ_1 admissible. Hence $\{\sigma_i|i<\rho\}\in L(\delta_{\gamma+2})$.

The proof of Theorem 2.6 is completed as in the proof of Theorem 1.4. □

2.8–2.15 Exercises

2.8. Assume $A, B \subseteq \alpha$. $L[B, \alpha]$ was defined in subsection 3.5.VII. $L[B, \alpha]^+$ is the least $C \supseteq L[B, \alpha]$ such that $\langle C, B\rangle$ is Σ_1 admissible. A is said to be α-computable from B (in symbols $A \leq_{\alpha c} B$) iff A is Δ_1 over $\langle L[B, \alpha]^+, B\rangle$. Suppose B is α-recursively enumerable and hyperregular. Show

$$A \leq_{\alpha c} B \leftrightarrow A \leq_\alpha B.$$

2.9. Show there exist α-recursively enumerable sets A and B such that $A \not\leq_{\alpha c} B$ and $B \not\leq_{\alpha c} A$.

2.10. Show that $\sigma 2\mathrm{cf}(\alpha)$ equals

$$\mu\beta(\mathrm{E}f)[f \in \Sigma_2^\alpha \ \& \ \mathrm{dom}\, f = \beta \ \& \ \sup \mathrm{range}\, f = \alpha].$$

(In other words the clause "f is strictly increasing" can be dropped from the definition of $\sigma 2\mathrm{cf}(\alpha)$.)

2.11. Show that $t\sigma 2p(\alpha)$ equals

$$\mu\gamma(\mathrm{E}f)\left[f \in \mathrm{tame}\, \Sigma_2^\alpha \ \& \ f : \gamma \xrightarrow[\text{onto}]{\text{1-1}} \alpha\right].$$

2.12. Find an α such that $t\sigma 2p(\alpha) < \alpha$ and $t\sigma 2p(\alpha)$ is not an α-cardinal.

2.13. Find an α such that $t\sigma 2p(\alpha) < \aleph \cdot \lambda_0$, where $\aleph$ and λ_0 are as in the proof of Theorem 2.6.

2.14. Show α is Σ_2 admissible iff every Σ_2^α function is tame Σ_2^α.

2.15. Suppose $f \in \Sigma_2^\alpha$ and $\mathrm{dom} f \leq \sigma 2\mathrm{cf}(\alpha)$. Show f is tame Σ_2^α.

3. *Dynamic Versus Fine-Structure*

An argument in higher recursion theory, particularly an argument about "recursively enumerable" sets, is said to be fine-structure in character if it relies on the collapsing (or condensation) method associated with L. The proof of Theorem 2.6 is such an argument, because the stable ordinals δ_γ ($\gamma < \lambda_0$) owe their existence to Lemma 2.6.VII, whose proof makes explicit use of Mostowski's collapsing map. A dynamic argument relies on combinatoric reasoning about cofinalities and projecta. It may form hulls, but it does not collapse them. The regular sets theorem, 4.2.VII, is proved dynamically. A less obvious example is Theorem 5.3.VII, the existence of a non-α-recursive, hyperregular set. It makes use of Lemma 2.3.VII and Proposition 2.1.VII, both proved dynamically. Both 2.3 and 2.1 have a dependence on admissibility. If α is not Σ_1 admissible, then 2.3 can fail, but 2.1 remains true by a fine-structure argument.

Dynamic methods possess the force needed to operate outside L and are in harmony with classical recursion theory. Fine structure techniques are more delicate. They keep careful count of injuries to requirements. Shore's density theorem, Chapter IX, is a powerful combination of both approaches.

The purpose of this section is to enlarge the repertoire of dynamic methods.

3.1 Lerman's Tame Σ_2 Approach. The fine-structure aspects of the proof of Theorem 2.6 (Post's problem) can be eliminated systematically as follows. (The proof of 2.6 employed a tame Σ_2^α map from $\aleph \cdot \lambda_0$ onto α when $\alpha^* = \alpha$ and $\aleph = \mathrm{gc}(\alpha) < \alpha$.) Requirements are indexed by ordinals less than $t\sigma 2p(\alpha)$.

The treatment of requirements breaks into two cases.

(i) $t\sigma 2p(\alpha) \leq \mathrm{gc}(\alpha)(\leq \alpha)$.
(ii) $t\sigma 2p(\alpha) = \mathrm{gc}(\alpha) \cdot \sigma 2\mathrm{cf}(\alpha)$.

By Lemma 2.5 either (i) or (ii) holds for all α. If (i) holds, then ε is less than some α-cardinal. If (ii) holds, then ε sits in block γ, that is,

$$\mathrm{gc}(\alpha) \cdot \gamma \leq \varepsilon < \mathrm{gc}(\alpha) \cdot (\gamma + 1),$$

and can be viewed as less than some α-cardinal modulo $\mathrm{gc}(\alpha)$.

The reasoning behind Lemma 1.3 applies to both cases. Fix ε and choose τ so that the α-recursive approximation of $f \upharpoonright (\varepsilon + 1)$ is correct from stage τ onward. Assume I_x, the x-th injury set, is α-finite for all $x < \varepsilon$. If (i) holds, then I_ε is α-finite, as in 1.3, by the combinatoric lemma (2.3.VII). Note the use of Proposition 2.1.VII in the proof of 1.3.

Suppose (ii) holds. Let

$$J_\delta = \cup \{I_x | \mathrm{gc}(\alpha) \cdot \delta \leq x < \mathrm{gc}(\alpha) \cdot (\delta + 1)\}.$$

Assume J_δ is α-finite for all $\delta < \gamma$. Since J_δ is a Σ_2^α function of δ, and $\gamma < \sigma 2\mathrm{cf}(\alpha)$, it follows that $\cup \{J_\delta | \delta < \gamma\}$ is α-finite. Hence there is a stage $\sigma_\gamma (\geq \tau)$ after which all activity with respect to requirements in block δ, for all $\delta < \gamma$, ceases. Then requirements in $[\mathrm{gc}(\alpha) \cdot \gamma, \varepsilon)$ can be handled the same way as requirements in $[0, \varepsilon)$ were handled in case (i). In short: $\gamma < \sigma 2\mathrm{cf}(\alpha)$ implied a bound on activity in the first γ blocks; and the combinatoric lemma implies a bound on activity up to any point within a block, since the length of a block is an α-cardinal.

It still has to be shown that J_γ is α-finite. Let I'_x be the set of stages after σ_γ at which req x is injured. It suffices to show

$$\cup \{I'_x | \mathrm{gc}(\alpha) \cdot \gamma \leq x < \mathrm{gc}(\alpha) \cdot (\gamma + 1)\} \tag{1}$$

is α-finite. The combinatoric lemma implies (by induction on x in block γ) that each I'_x is α-finite and has α-cardinality less than $\mathrm{gc}(\alpha)$. The simultaneous enumeration of the I'_x's gives rise to a partial α-recursive, 1-1 map of (1) into $\mathrm{gc}(\alpha) \cdot \mathrm{gc}(\alpha)$. If b is the β-th member of I'_x to be enumerated, then b is mapped to $\langle \beta, x - \mathrm{gc}(\alpha) \rangle$. Suppose (1) is not α-finite. Then there is an α-recursive, one-one map from α into (1), and from (1) into $\mathrm{gc}(\alpha)\cdot\mathrm{gc}(\alpha)$. Hence $\alpha^* \leq \mathrm{gc}(\alpha)$, and so $t\sigma 2p(\alpha) \leq \mathrm{gc}(\alpha)$ by Corollary 2.4. But that contradicts the hypothesis of case (ii). □

3.2 Shore's Blocking Method. The notion of block is carried a step further. For Shore there is no conflict between requirements in the same block. For example, the requirements in a given block might all be of the form $A \neq \{\varepsilon\}^B$. The success of the method turns on a lemma concerning cofinalities. Define $\sigma 2\mathrm{cf}^{\alpha}(\delta)$, the Σ_2^{α} cofinality of δ, to be

$$\mu\beta(\mathrm{Eh})[h \in \Sigma_2^{\alpha} \;\&\; h \text{ is strictly increasing} \;\&\; \mathrm{dom}\, h = \beta \;\&\; \sup \text{ range } h = \delta].$$

Thus $\sigma 2\mathrm{cf}^{\alpha}(\alpha) = \sigma 2\mathrm{cf}(\alpha)$.

3.3 Lemma (Shore). $\sigma 2\mathrm{cf}^{\alpha}(\alpha^*) = \sigma 2\mathrm{cf}(\alpha)$.

Proof. Let $f\colon \alpha \to \alpha^*$ be α-recursive, one-one and into. Suppose g is a strictly increasing Σ_2^{α} map from β into α^* with range unbounded in α^*. Then there exists a strictly increasing Σ_2^{α} map h from β into α with unbounded range. A rough approximation of h is $f^{-1}g$. The details are as follows. Define

$$h(x) = \sup\{f^{-1}(y) \mid y < g(x) \;\&\; y \in \text{range } f\} + x.$$

The "$+x$" insures that h is strictly increasing. Proposition 2.1.VII implies h is Σ_2^{α}; for each $x < \beta$ there exists α-finite sets w_1 and w_2 such that

$$w_1 \cup w_2 = g(x) \;\&\; w_1 \subseteq \text{range} f \;\&\; w_2 \cap \text{range} f = \phi.$$

Thus $h(x) = z$ iff

(1) $$(\mathrm{Ew})(\mathrm{Ew}_1)(\mathrm{Ew}_2)[w = g(x) \;\&\; w = w_1 \cup w_2 \;\&\; w \subseteq \text{range } f \;\&\; w_2 \cap \text{range} f = \phi \;\&\; z = \sup\{f^{-1}(y) \mid y \in w_1\} + x].$$

(1) is easily seen to be Σ_2^{α} save for perhaps one detail. The formula, $w \subseteq \mathrm{rg} f$, rendered as

(2) $$(u)_{u \in w}(\mathrm{Ex})(f(x) = u),$$

appears to be Π_2^{α}, but is in fact Σ_1^{α}. Let $P(x, u, v)$ be a Δ_0^{α} formula such that

$$f(x) = u \leftrightarrow (\mathrm{Ev})P(x, u, v).$$

Then (2) is equivalent to

$$(\mathrm{Eq})(u)_{u \in w}(\mathrm{Ex})_{x \in q}(\mathrm{Ev})_{v \in q}P(x, u, v).$$

Thus the Σ_1 admissibility of $L(\alpha)$ is needed to show h is Σ_2^α. Hence $\sigma 2\text{cf}(\alpha) \leq \sigma 2\text{cf}^\alpha(\alpha^*)$.

Now suppose a strictly increasing Σ_2^α map h from $\sigma 2\text{cf}(\alpha)$ into α is given. A good approximation of the desired cofinality map g from $\sigma 2\text{cf}(\alpha)$ into α^* is fh. Clearly fh is Σ_2^α. The range of fh is unbounded in α^*; otherwise range $fh \subseteq \delta$ for some $\delta < \alpha^*$, and then range h is bounded by $\sup f^{-1}[\delta] < \alpha$. The only difficulty is that fh may not be strictly increasing. Define

$$\beta = \mu\gamma(\mathrm{E}t)[t \in \Sigma_2^\alpha \ \& \ \operatorname{dom} t = \gamma \ \& \ \sup \operatorname{range} t = \alpha].$$

Let t be a Σ_2^α map from β into α with unbounded range. ft is Σ_2^α and its range is unbounded in α^*. Define β_0 to be

$$\mu\gamma(\mathrm{E}p)[p \in \Sigma_2^\alpha \ \& \ \operatorname{dom} p = \gamma \ \& \ \sup \operatorname{range} = \alpha^*].$$

then $\beta_0 \leq \beta \leq \sigma 2\text{cf}(\alpha)$. Let g_0 be a Σ_2^α map from β_0 into α^* with range unbounded in α^*. To show $\sigma 2\text{cf}^\alpha(\alpha^*) \leq \sigma 2\text{cf}(\alpha)$, it suffices to transform g_0 into a strictly increasing function g with the same domain $(=\beta_0)$.

For $x < \beta_0$, let

$$g(x) = \sup\{g_0(y) \mid y \leq x\} + x.$$

To see g is Σ_2^α, let $k(\alpha, x)$ be an α-recursive function such that

$$g_0(x) = \lim_\sigma k(\sigma, x) \quad (x < \beta_0).$$

k exists by Proposition 2.2. Let

$$m(x) = \mu\sigma(\tau)_{\tau \geq \sigma}[k(\sigma, x) = k(\tau, x)].$$

m is Σ_2^α, since $m(x) = \sigma$ iff

$$\text{(3a)} \qquad (\tau)_{\tau \geq \sigma}[k(\sigma, x) = k(\tau, x)]$$

$$\text{(3b)} \qquad \& \ (\rho)_{\rho < \sigma}(E\gamma)_{\rho < \gamma < \sigma}[k(\rho, x) \neq k(\gamma, x)],$$

Hence for each $x < \beta_0 \leq \beta$,

$$\{m(y) \mid y \leq x\}$$

is bounded below α. In other words, for each $x < \beta_0$

$$\text{(4)} \qquad (E\sigma)(\tau)_{\tau \geq \sigma}(y)_{y \leq x}[k(\sigma, y) = k(\tau, y)].$$

(4) is Σ_2^α and serves as the principal part of a Σ_2^α definition of g. Note that $\sup\{g_0(y) \mid y \leq x\}$ equals $\sup\{k(\sigma, y) \mid y \leq x\}$ for all sufficiently large σ. □

Note well the use of Σ_1 admissibility in the above proof to show a formula is Σ_2^α.

3.4 Blocking. As in subsection 1.1 the requirements for Post's problem are indexed by ordinals less than α^*. Let g be a strictly increasing, Σ_2^α map from $\sigma 2\mathrm{cf}(\alpha^*)$ into α^* with range unbounded in α^*. Shore's blocking method uses g to distribute requirements as follows. p is an α-recursive, 1-1 map of α into α^*.

Block 2δ: all requirements of the form $A \neq \{p^{-1}\varepsilon\}^B$, where $g(\delta) \leq \varepsilon < g(\delta + 1)$.
Block $2\delta + 1$: all requirements of the form $B \neq \{p^{-1}\varepsilon\}^A$, $g(\delta) \leq \varepsilon < g(\delta + 1)$.

The procedure for trying to meet requirements is the one established in subsection 1.1. Hence there is no conflict between requirements in the same Shore block. Block γ is said to have a higher priority than block ρ if $\gamma < \rho$. Thus a requirement in block 2δ can be injured only by an attempt to satisfy a requirement in block $2\eta + 1$ for some $\eta < \delta$. In order to proceed effectively at stage σ, the Σ_2^α cofinality map $g(\delta)$ is replaced by an α-recursive approximation $g(\sigma, \delta)$ supplied by Proposition 2.2.

Define $\lim_{\tau<\sigma} g(\tau,x)$ to be z iff

$$(E\gamma)_{\gamma<\sigma}(\tau)_{\gamma\leq\tau<\sigma}[g(\tau,x) = z].$$

For each $\rho < \sigma 2\mathrm{cf}^\alpha(\alpha^*)$, let J_ρ be the set of all σ such that:

(i) an attempt is made to satisfy, or an injury occurs to, a requirement in $\cup\{\text{block}\, x \mid x \leq \rho\}$ at stage σ; or

(ii) $g(\sigma,x) \neq \lim_{\tau<\sigma} g(\tau,x)$ for some $x \leq \rho$.

As in Section 1 it suffices to see J_ρ is α-finite.

Assume J_x is α-finite for $x < \rho$. Then J_x is a Σ_2^α function of x below ρ. Since $\rho < \sigma 2\mathrm{cf}^\alpha(\alpha^*)$, Lemma 3.3 implies

$$\sup\{J_x \mid x < \rho\} = \sigma_0 < \alpha.$$

Choose $\sigma_1 > \sigma_0$ so that $(\tau)_{\tau\geq\sigma_1}[g(\tau,\rho) = g(\sigma_1,\rho)]$. Let $J_\rho^1 = J_\rho - \sigma_1$. Suppose $\sigma \in J_\rho^1$. The *only* activity at stage σ with respect to block δ is an attempt to meet one requirement in that block. (Recall that the procedure for meeting requirements established in subsection 1.1 allows an attempt on at most one requirement at each stage.) Each requirement in block δ will be attempted at at most one stage in J_ρ^1. Thus J_ρ^1 is in α-recursive, 1-1 correspondence with an α-recursively enumerable subset of block ρ. That subset is α-finite, since block ρ is shorter than α^*. Hence J_ρ^1 is α-finite.

3.5 Post's Problem for Admissible Sets. Let A be a Σ_1 admissible set as defined in Section 1.VII. Recall that $B \subseteq A$ is said to be A-recursively enumerable if B is Σ_1^A and A-recursive if A is Δ_1^A. Also z is said to be A-finite if $z \in A$. Many, but not all, of the notions of α-recursion theory extend to A. For example, regularity can only mean: $B \subseteq A$ is regular iff $(B \cap z) \in A$ for all $z \in A$. On the other hand, the notion of Σ_1 projectum does not seem to make sense for all Σ_1 admisible A's. If A is $(Ad(2^\omega))$, the least Σ_1 admissible set with 2^ω as an element, then there does not appear to be any useful way of defining the Σ_1 projectum of A.

To formulate Post's problem for all A, all that is needed is an extension of "α-recursive in" to "A-recursive in". Suppose $B, C \subseteq A$. B is A-recursive in C, symbolically $B \leq_A C$, if there exist partial A-recursive functions ϕ and ψ such that

$$x \subseteq B \leftrightarrow (\mathrm{Ey})(\mathrm{Ez})[\phi(x,y,z) = 0 \quad \& \quad y \subseteq C \quad \& \quad z \subseteq A - C],$$

$$x \subseteq A - B \leftrightarrow (\mathrm{Ey})(\mathrm{Ez})[\psi(x,y,z) = 0 \quad \& \quad y \subseteq C \quad \& \quad z \subseteq A - C],$$

for all $x \in A$. (y and z range over A.) Clearly $\leq_A$ is reflexive and transitive. Two sets have the same A-degree if each is A-recursive in the other. The notion of "weakly A-recursive in" ($\leq_{wA}$) is defined by substituting "$x \in$" for "$x \subseteq$" in the definition of $\leq_A$.

One formulation of Post's problem for A is: do there exist A-recursively enumerable sets B and C such that $B \not\leq_A C$ and $C \not\leq B$? Another formulation that asks for more: $B \not\leq_{wA} C$ and $C \not\leq_{wA} B$? Still another asks for less: does there exist an A-recursively enumerable set B such that B is neither A-recursive nor complete? (B is complete if every A-recursively enumerable set is A-recursive in B.) Simpson showed that HC, the set of all hereditarily countable sets, yields a negative answer to the third formulation above, if the axiom of determinateness is assumed. Later Harrington showed, as a theorem of ZF, that there exists a countable Σ_1 admissible set for which the third formulation has a negative answer. His proof can be found in Chong 1984.

On the other hand, Simpson 1974b and Stoltenberg–Hansen 1977 showed that the second (and strongest) formulation has a positive answer if A can be suitably well-ordered. The condition to be considered here is: A is *effectively well-orderable*, that is, there exists a one-one A-recursive map of A onto ord(A), the least ordinal not in A. If A is effectively well-orderable, then the notion of Σ_1 projectum is meaningful for A, and the dynamic solutions of Post's problem given above via Lerman's tame Σ_2 projectum (subsection 3.1) or Shore's blocking method (subsection 3.2) apply to A. The fine structure approach (Theorem 2.6) fails in A, because it is based on collapsing arguments that need initial segments of L to succeed. The next proposition is an aid to understanding why dynamic methods succeed in A when A is effectively well-orderable.

3.6 Proposition. *Assume A is a Σ_1 admissible set. Let $\alpha = \mathrm{ord}(A)$. Then (i) iff (ii).*

(*i*) *A is effectively well-orderable.*
(*ii*) *There exists a Δ_1^A $B \subseteq \alpha$ such that $A = L[B, \alpha]$.*

Proof. $L[B,\alpha]$ was defined in subsection 3.5.VII. If (ii) holds, then the natural enumeration of $L(\alpha)$, described in subsection 1.7.VII, extends to one of $L[B,\alpha]$, and yields an effective well-ordering of A.

Suppose (i) holds. Let $f\colon A \to \alpha$ be a one-one, onto, Σ_1^A map. B will encode all sets in A by relations on ordinals. Let $x \in A$, and define $tc(x)$, the transitive closure of x, to be the least transitive $y \supseteq x$. (Note that $tc(x)$, as a function of x, is Σ_1^A.) Define

$$\delta \,\varepsilon_f\, \gamma \text{ by } f^{-1}(\delta) \in f^{-1}(\gamma).$$

then $\langle tc(x), \in \rangle \approx \langle f[tc(x)], \varepsilon_f \rangle$. Let

$$r = \langle f[x], f[tc(x)],\ \varepsilon_f \cap (f[tc(x)])^2 \rangle.$$

r is an effective code for x in that the passage from r to x can be accomplished inside $L(\alpha, r)$. Note that the collection of all such r is A-recursive. r can be further coded as a set of ordinals.

Thus it will be enough for B to encode all sets of ordinals in A. A-recursive functions $g, h\colon \alpha \to \alpha$ are defined by recursion. Let f_0 be a one-one, A-recursive map of α onto $A \cap 2^\alpha$. Define

$$h(\delta) = (\sup_{\gamma < \delta} h(\gamma)) + \sup{}^+ f_0(\delta),$$

$$g(h(\delta) + x) = \begin{matrix} 0 & \text{if} & x \in f_0(\delta) \\ 1 & \text{if} & x \notin f_0(\delta) \end{matrix} \qquad (x < \sup{}^+ f_0(\delta)).$$

Each $y \in A \cap 2^\alpha$ can be recovered from a sufficiently long initial segment of h and g. Since h, $g \subseteq \alpha^2$, they can be encoded by a $B \subseteq \alpha$. $\square$

3.7 Theorem. *Let A be an effectively well-orderable, Σ_1 admissible set. Then there exist two A-recursively enumerable sets such that neither is weakly A-recursive in the other.*

Proof. By Shore's blocking method. Define

$$\alpha_A^* = \mu\gamma(\mathrm{E}g)\left[\, g \in \Sigma_1^A \ \&\ g\colon \alpha \xrightarrow[\text{into}]{\text{1-1}} \gamma \,\right].$$

α_A^* is the Σ_1 projectum of A. The proof of Proposition 2.1.VII was entirely dynamic in nature and so applies to α_A^* . Suppose $X \subseteq \delta \subseteq \alpha_A^*$ and X is A-recursively enumerable. Let f be a one-one, A-recursive map of A onto $\mathrm{ord}(A)$. With the aid of f, X becomes the range of a partial, one-one, A-recursive map g with domain equal to an initial segment of $\mathrm{ord}(A)$. The domain of g cannot be $\mathrm{ord}(A)$ because $\delta \subseteq \alpha_A^*$. Thus the domain of g, and hence X, is α-finite.

Similarly the proof of Proposition 2.2 shows that each Σ_2^A function from $\mathrm{ord}(A)$ into $\mathrm{ord}(A)$ is the limit of an A-recursive approximation. And the proof of Lemma 3.3 becomes a proof of

$$\sigma 2\mathrm{cf}^A(\alpha_A^*) = \sigma 2\mathrm{cf}^A(\mathrm{ord}(A)).$$

The arguments of subsection 3.4 transfer to A straightforwardly. The two "incomparable" sets obtained are subsets of α. □

3.8 Uniform Solutions to Post's Problem. Call a Σ_1^α subset of $L(\alpha)$ *lightface* if its Σ_1 definition has all its parameters in ω, and *boldface* otherwise. Thus the set of ordinals less than α^* in $L(\alpha)$ is boldface, but not lightface, Σ_1^α, because its definition needs α^* as a parameter. All the solutions of Post's problem given prior to this section yield "incomparable" sets that are boldface Σ_1^α. The parameters needed arise out of various definable projecta and confinalities. For example, the blocking argument of subsection 3.4 needs the parameters occurring in the Σ_1^α definition of p and the Σ_2^α definition of g. The solution given below avoids all such parameters and is uniform in α.

3.8 Theorem (R. Shore 1974). *There exist integers m and n such that for all α: the m-th and n-th lightface Σ_1^α sets have the property that neither is weakly α-recursive in the other.*

Proof. Curiously the uniform construction, viewed locally, is similar to that given in Section 1 when $\alpha^* < \alpha$. The idea is to divide α into intervals that are independent in the sense that requirements in different intervals do not conflict. The top of each interval is to be a Σ_1 admissible ordinal with strictly smaller Σ_1 projectum. The α-stable ordinals are Σ_1 admissible and, according to Lemma 2.7, define somewhat independent intervals.

Let $\{t(\sigma,\delta) \mid 0 < \delta < \sigma_0\}$ be a list in ascending order of the σ-stable ordinals. Set $t(\sigma,0) = \omega$ and $t(\sigma,\sigma_0) = \sigma$.

Define $t_\delta = t(\alpha,\delta)$. Thus t_δ is the δ-th α-stable ordinal. $t(\sigma,\delta) = t_\delta$ if $\sigma \geq t_\delta$, and $t(\sigma,\delta)$ is a lightface α-recursive function of σ and δ. Hence t_δ is a lightface, tame Σ_2^α function of δ.

Let h be the universal, partial α-recursive function central to the proof of Lemma 2.6.VII. Recall that h is lightface Σ_1^α, uniformly in α, and that $h[\gamma]$ is a Σ_1 substructure, and an initial segment, of $L(\alpha)$ for every infinite $\gamma < \alpha$. It follows that

$$(0) \qquad \text{(i) } h[t_\delta + 1] = L(t_{\delta+1}) \quad \text{and} \quad \text{(ii) } t_{\delta+1}^* \leq t_\delta$$

for all $\delta < \sigma_0$. (ii) is obtained from (i) by inverting h.

Since $t_{\delta+1}^* < t_{\delta+1} (\delta < \alpha_0)$, the local strategy of the uniform construction can be modeled on that of subsection 1.1. Suppose

$$t_\delta \leq \rho < \sigma < t_{\delta+1}.$$

At stage σ the intention is to consider only requirements of the form $A \neq \{\rho\}^B$ (req 2ρ) or $B \neq \{\rho\}^A$ (req $2\rho + 1$). The priorities are governed by f_σ, a partial, one-one $\Sigma_1^{t_{\delta+1}}$ map from $t_{\delta+1}^*$ onto $t_{\delta+1}$ defined below. The local strategy yields the desired global result: req ρ is satisfied prior to stage $t_{\delta+1}$, and that will be so thanks to the α-stability of $t_{\delta+1}$. Of course at stage σ the local strategy can only be guessed at. The guesses must converge properly, and above all, must be lightface α-recursive uniformly in α.

Define $f_\delta = hg_\delta$, where g_δ is the first member of $L(t_{\delta+1})$ to be a one-one map of $t^*_{\delta+1}$ onto t_δ. $h \upharpoonright t_\delta$ is $\Sigma_1^{t_{\delta+1}}$ because $t_{\delta+1}$ is stable. Thus f_δ is a partial $\Sigma_1^{t_{\delta+1}}$ function from $t^*_{\delta+1}$ onto $t_{\delta+1}$. It is safe to assume f_δ is one-one (cf. Exercise 1.16.VII). The only non-trivial parameter in the Σ_1 definition of f_δ is g_δ. g^σ_δ is the best guess in $L(\sigma)$ at a one-one map of the σ-cardinality of $t(\sigma,\delta)$ onto $t(\sigma,\delta)$. (Recall that for any Σ_1 admissible γ, if $\gamma^* < \gamma$, then γ^* is the greatest γ-cardinal.) As σ approaches $t_{\delta+1}$

$$t(\sigma,\delta) = t_\delta,$$

$$\sigma\text{-card}(t_\delta) = t^*_{\delta+1},$$

and $g^\sigma_\delta = g_\delta$. Let h^σ be the result of restricting the Σ_1^α definition of h to $L(\sigma)$. Then h^σ is lightface Σ_1^σ.

Define $f^\sigma_\delta = h^\sigma g^\sigma_\delta$. Then

(1) $$z < t^*_{\delta+1} \rightarrow \lim_{\sigma \to t_{\delta+1}} f^\sigma_\delta \upharpoonright z = f_\delta \upharpoonright z.$$

Suppose $\omega \leq \rho < \sigma$. Let $t_{m(\sigma,\rho)}$ be the greatest σ-stable ordinal $\leq \rho$. Then

(2) $$t_\delta \leq \rho < t_{\delta+1} \rightarrow \lim_{\sigma \to t_{\delta+1}} t_{m(\sigma,\rho)} = t_\delta.$$

(Remember that $\lim\limits_{\sigma \to y} x_\sigma = x$ means $(E\tau)_{\tau < y}(\sigma)_{\tau \leq \sigma < y}(x_\sigma = x)$.)

The uniform solution.

Stage $\sigma < \omega$. Identical with the Friedberg–Muchnik solution to Post's problem.

Stage $\sigma \geq \omega$. Suppose $\rho < \sigma$ and an attempt was made at stage $\tau < \sigma$ to satisfy req ρ. That attempt, specifically the negative requirement imposed by that attempt on A or B, is now *discarded* if

$$t_{m(\tau,\rho)} > t_{m(\sigma,\rho)}.$$

If $r < \sigma$, then r belongs to a unique block of the form $[t(\sigma,\delta), t(\sigma,\delta+1))$. The local *priority* assigned to req r at stage σ is $(f^\sigma_\delta)^{-1}(r)$. As usual req x has *higher priority* than req y if the priority assigned to x is less than that assigned to y.

Req r needs *attention* at stage σ if:

(i) every attempt to satisfy req r prior to stage σ was injured prior to stage σ, or discarded prior to or at stage σ; and

(ii) there is an opportunity to satisfy req r at stage σ that does not threaten injury to any req of higher priority in the same block as r, or to any requirement in any block below that of r.

Go to the lowest block in which there is a requirement that needs attention. In that block go to the highest priority requirement that needs attention, and take the least opportunity to satisfy it.

End of uniform solution.

Fix $\delta < \alpha_0$ and $z < t^*_{\delta+1}$ in order to show by induction that J^δ_z (the set of all σ such that an injury to, or attempt to satisfy, req $f_\delta(z)$ occurs at stage σ) is $t_{\delta+1}$-finite. As in (1) and (2) there is a $\sigma_1 \in [t_\delta, t_{\delta+1})$ such that for all $\sigma \geq \sigma_1$:

$$t(\sigma, \delta) = t_\delta,$$

$$f^\sigma_\delta \restriction (z+1) = f_\delta \restriction (z+1)$$

$$t(\sigma, \delta+1) > \sup f_\delta[z+1].$$

Then for all $x \leq z$ at stage $\sigma \geq \sigma_1$:

(3) no prior attempt to satisfy req $f_\delta(x)$ is discarded;

(4) req $f_\delta(x)$ is injured only for the sake of $f_\delta(y)(y < x)$.

(4) follows from the induction hypothesis that $J^\gamma_v \in L(t_{\gamma+1})$; for all $\gamma < \delta$ and $v < t^*_{\gamma+1}$.

By induction the J^δ_x's $(x < z)$ are simultaneously $t_{\delta+1}$-recursively enumerable, and each is $t_{\delta+1}$-finite. The combinatoric lemma (2.3.VII) implies $\cup \{J^\delta_x \mid x < z\}$ is $t_{\delta+1}$-finite. After stage σ_1 and after $U\{J^\delta_x \mid x < z\}$ is enumerated, there is at most one attempt to satisfy req z. Hence J^δ_z is α-finite.

Fix $r \in [t_\delta, t_{\delta+1})$ to see req r is met. $r = f_\delta(z)$ for some $z < t^*_{\delta+1}$. Assume req r is of the form $A \neq \{r\}^B$. Choose $\sigma_2 \in (\sigma_1, t_{\delta+1})$ so that

$$\cup \{J^\delta_x \mid x \leq z\} \subseteq \sigma_2.$$

σ_1 is as above. Any attempt to satisfy req r involves a witness w and the inequation $A(w) \neq \{r\}^B(w)$. w is drawn from a witness set Z^δ_r. The witness sets are pairwise disjoint, unbounded in α, and simultaneously lightface α-recursive. The α-stability of $t_{\delta+1}$ implies that $Z^\delta_r - t_\delta$ is unbounded in $t_{\delta+1}$. Let w_0 be a member of $Z^\delta_r - t_\delta$ not put in A prior to stage σ_2. Then w_0 is never put in A, i.e. $A(w_0) = 0$. Suppose $\{r\}^B(w_0.) = 0$. Then there is a $\sigma < \alpha$ such that

(5) $$\sigma > \sigma_2 \;\&\; A^{<\sigma}(w_0) = 0 \;\&\; \{r\}^{B^{<\sigma}}_\sigma(w_0) = 0.$$

Since A and B are lightface Σ^α_1, there is a $\sigma < t_{\delta+1}$ that satisfies (5). No requirement in any block below block δ, or in block δ and of higher priority than r, receives attention at stage σ. If r does not need attention at stage σ, then it was satisfied at an earlier stage by an attempt not yet discarded or injured, and hence never to be discarded or injured. If r does need attention at stage σ, then it will be satisfied at stage σ and remain so forever. $\square$

The proof of Theorem 3.8 was dynamic in nature save for one fine-structure fact:

(6) $$h[t_\delta + 1] = L(t_{\delta+1}).$$

The proof of (6) is a collapsing argument. Nonetheless the proof of Theorem 3.8 can be adjusted so as to avoid (6) (cf. Exercise 3.10). The result is an entirely dynamic, lightface solution to Post's problem that is uniform with respect to a wide class of structures.

3.9–3.12 Exercises

3.9. Suppose $\alpha^* < \alpha$. Show there exists a lightface, tame Σ_2^α map with range α and domain less than α.

3.10. Let A be a Σ_1 admissible set. Suppose A is effectively wellorderable via a lightface one-one, A-recursive map of A onto ord(A). Show Post's problem for A has a lightface solution.

3.11. (R. Shore 1974). Suppose $L(\alpha)$ is Σ_n admissible, that is, it satisfies Σ_n^α replacement. Find sets A_i $(i \leq 1)$ such that A_i is Σ_n^α, but is not Δ_n^α, over $\langle L[A_{1-i}, \alpha], \varepsilon, A_{1-i} \rangle$.

3.12. Let A be a Σ_1 admissible structure of the form $\langle L[B,\alpha], \varepsilon, B \rangle$. Solve Post's problem for A.

4. Σ_1 *Doing the Work of* Σ_2

Chapter VIII ends on the same note with which it began. Certain Σ_2 constructions that occur in classical recursion theory can, after some modification, be carried out for every Σ_1 admissible ordinal. A Σ_2 construction is one that succeeds by appeal to Σ_2 replacement. Although the Friedberg–Muchnick (F–M) construction is a Σ_1 recursion, Σ_2 replacement is needed to check that every requirement is met. $\sigma(e)$, the stage by which the e-th requirement is met, is a Σ_2 function of e; and the proof that the e-th requirement is met begins with finding a bound on $\{\sigma(c) | c < e\}$. Thus the proof that the F–M construction works can be lifted, with no conceptual change, from $L(\omega)$ to every Σ_2 admissible $L(\alpha)$.

The previous sections of this chapter studied various methods of lifting the F–M construction to every Σ_1 admissible ordinal. All the methods had in common the idea of projecting α downward to an ordinal with combinatoric properties reminiscent of Σ_2 replacement. The present section focuses on Lerman's tame Σ_2 approach and uses it to lift tame Σ_2 recursions, in a systematic fashion, from ω to α. The simplest example of a tame Σ_2 recursion in classical recursion theory is the construction of a 1-generic Δ_2 set. After the combinatoric facts about tame Σ_2^α recursion are established, they will be applied to obtain a 1-generic Δ_2^α set, and finally to prove Simpson's inversion theorem for the α-jump.

4.1 Tame Σ_2 Recursion. Let R be an α-recursively enumerable predicate, and S an α-recursive function. A *tame Σ_2^α recursion* is defined by

$$f(\gamma) = \begin{matrix} \mu y R(f \upharpoonright \gamma, \gamma, y) & \text{if} \quad (\mathrm{E}y) R(f \upharpoonright \gamma, \gamma, y) \\ S(f \upharpoonright \gamma) & \text{otherwise.} \end{matrix}$$

If α is Σ_2 admissible, then there exists a unique $\Sigma_2^\alpha f$ from α into α satisfying the above recursion equations. If α is merely Σ_1 admissible, then the recursion may break down at some δ such that $(f \upharpoonright \delta) \notin L(\alpha)$. The first such δ is at least $t\sigma 2p(\alpha)$ according to Lemma 4.2.

A tame Σ_2^α recursion arises out of attempts to meet requirements indexed by ordinals less than α. The requirements are simultaneously α-recursively enumerable sets. A typical member of a requirement is an ordered pair of disjoint α-finite sets. $f(\gamma)$ might be the least extension of $f \upharpoonright \gamma$ that satisfies requirement γ. Since the desired extension may not exist, f is Σ_2 in character rather than Σ_1.

4.2 Lemma. *The equations for a tame Σ_2^α recursion define a unique tame Σ_2^α f from $t\sigma 2p(\alpha)$ into α.*

Proof. The idea is to construe f as the limit of an α-recursive sequence f_σ. $f(\gamma)$ is approximated at the beginning of stage σ by $f_{<\sigma}(\gamma)$, and at the end of stage σ by $f_\sigma(\gamma)$. Convergence of f_σ to f is assured by a system of priorities. At stage σ the best guess for $f(\gamma_1)$ may be inconsistent with the best guess for $f(\gamma_2)$. If $\gamma_1 < \gamma_2$ then γ_1 is given preference. Thus a guess made for $f(\gamma_2)$ at stage σ may be discarded at a later stage for the sake of a new guess for $f(\gamma_1)$. Recall that

$$\lim_{\tau \to \sigma} k(\tau) = z \text{ means } (E\rho)_{\rho<\sigma}(\tau)_{\rho \le \tau < \sigma}[k(\tau) = z].$$

Define (by induction on γ):

$$f_{<\sigma}(\gamma) = \begin{cases} \lim_{\tau \to \sigma} f_\tau(\gamma) & \text{if the limit exists,} \\ S(f_{<\sigma} \upharpoonright \gamma) & \text{otherwise.} \end{cases}$$

γ needs attention at stage σ if:

$$L(\sigma) \models (\text{Ey}) R(f_{<\sigma} \upharpoonright \gamma, \gamma, y)]; \text{ and}$$

(1)
$$f_{<\sigma}(\gamma) \neq \mu y[L(\sigma) \models R(f_{<\sigma} \upharpoonright \gamma, \gamma, y)].$$

Let γ^σ be the least γ that needs attention at stage σ, and y_0 the least y alluded to in (1) when $\gamma = \gamma^\sigma$. Define

$$f_\sigma(\gamma) = \begin{cases} f_{<\sigma}(\gamma) & \text{if } \gamma < \gamma^\sigma \\ y_0 & \text{if } \gamma = \gamma^\sigma \\ S(f_\sigma \upharpoonright \gamma) & \text{if } \gamma > \gamma^\sigma. \end{cases}$$

Let I_γ be $\{\sigma \mid \gamma \ge \gamma^\sigma\}$. It need only be shown that I_γ is α-finite for all $\gamma < t\sigma 2p(\alpha)$. It then follows from the definition of f_σ that f_σ converges tamely to some f with domain $t\sigma 2p(\alpha)$. Then f, by induction on γ, is a solution of the tame Σ_2^α recursion equation.

Fix γ to see I_γ is α-finite. The argument splits into two cases, as it did in Lerman's tame Σ_1 approach to Post's problem in subsection 3.1.

Case 1: $t\sigma 2p(\alpha) \leq \mathrm{gc}(\alpha) \leq \alpha$. Let κ be a regular α-cardinal such that $\gamma < \kappa$. Assume I_x $(x < \gamma)$ is α-finite and of α-cardinality less than κ. Then the combinatoric lemma (2.3.VII) implies $\cup \{I_x | x < \gamma\}$ is α-finite and of α-cardinality less than κ. It follows from the manner in which $I_\gamma - \cup\{I_x | x < \gamma\}$ is interlaced with $\cup\{I_x | x < \gamma\}$ that I_γ is α-finite with α-cardinality less than κ. The interlacing effect is similar to the one discussed in the proof of Lemma 1.3 with one change. Finite sequences of elements of $I_\gamma - \cup\{I_x | x < \gamma\}$, rather than single elements, are separated by elements of $\cup\{I_x | x < \gamma\}$. The change is caused by the insistence that $f(\gamma)$ be the *least* y that satisfies R.

Case 2: $t\sigma 2p(\alpha) = \mathrm{gc}(\alpha)\cdot\sigma 2\mathrm{cf}(\alpha)$. Hence

$$\gamma = \mathrm{gc}(\alpha)\cdot\delta + w$$

for some $\delta < \sigma 2\mathrm{cf}(\alpha)$ and $w < \mathrm{gc}(\alpha)$. Assume

$$J_z = \cup\{I_x | \mathrm{gc}(\alpha)\cdot z \leq x < \mathrm{gc}(\alpha)\cdot(z+1)\}$$

is α-finite for each $z < \delta$. Then J_z is a Σ_2^α function of z $(z < \delta)$, and so $\cup\{J_z | z < \delta\}$ is α-finite, since $\delta < \sigma 2\mathrm{cf}(\alpha)$. Let its supremum be σ_2. The argument of Case 1 shows, for each $x \in (\mathrm{gc}(\alpha)\cdot\delta,\ \mathrm{gc}(\alpha)\cdot(\delta+1))$, that $I_x - \sigma_2$ is α-finite and of α-cardinality less than some regular α-cardinal. Hence I_γ is α-finite.

To complete case 2 it must be shown that J_δ is α-finite. Consider the simultaneous α-recursive enumeration of $\{I_x - \sigma_2 | x \in [\mathrm{gc}(\alpha)\cdot\delta,\ \mathrm{gc}(\alpha)\cdot(\delta+1))\}$. It gives rise to a one-one, α-recursive map of $J_\delta - \sigma_2$ into $\mathrm{gc}(\alpha)\cdot\mathrm{gc}(\alpha)$. When the ρ-th entry in the enumeration of $I_x - \sigma_2$ appears, it is mapped to $\langle x - \mathrm{gc}(\alpha)\cdot\delta, \rho\rangle$. $\rho < \mathrm{gc}(\alpha)$ because the α-cardinality of $I_x - \sigma_2$ is less than some regular α-cardinal. If $J_\sigma - \sigma_2$ is not α-finite, then there is one-one, α-recursive map of α onto $J_\delta - \sigma_2$. But then $\alpha^* \leq \mathrm{gc}(\alpha) < t\sigma 2p(\alpha)$, an impossibility by Corollary 2.4.VII. $\square$

4.3 Reduced Tame Σ_2 Recursion. The equations for a tame Σ_2 recursion were formulated with the idea that $f(\gamma)$ would be defined so as to satisfy requirement γ in some construction involving α requirements. Since a tame Σ_2 recursion may break down long before α is reached, it is necessary to re-index requirements. Let t be a tame Σ_2^α map from $t\sigma 2p(\alpha)$ onto α. The equations for tame Σ_2 recursion, *reduced* by t, are

$$f(\gamma) = \begin{array}{ll} \mu y\, R(f \upharpoonright \gamma, t(\gamma), y) & \text{if} \quad (\mathrm{E}y)\, R(f \upharpoonright \gamma, t(\gamma), y) \\ S(f \upharpoonright y) & \text{otherwise.} \end{array}$$

The intention now is that $f(\gamma)$ be defined to satisfy requirement $t(\gamma)$. Thus there will be time to satisfy all requirements if the recursion does not break down before stage $t\sigma 2p(\alpha)$. And it does not, according to the next result.

4.4 Theorem. *Let t be a tame Σ_2^α map from $t\sigma 2p(\alpha)$ onto α. Then the equations for a tame Σ_2^α recursion, reduced by t, define a unique function f from $t\sigma 2p(\alpha)$ into α. Furthermore f is tame Σ_2^α.*

Proof. Same as that of Lemma 4.2. The presence of t makes very little difference. At stage σ, $t(\gamma)$ is guessed at by $t(\sigma, \gamma)$, an α-recursive function that converges tamely to $t(\gamma)$. Each of the arguments of 4.3 is altered in the same fashion. First wait for $t(\sigma, \gamma)$ to settle down on the appropriate proper initial segment of $t\sigma 2p(\sigma)$, and then proceed as in 4.2. $\square$

Theorem 4.4 is a precise interpretation of the phrase: "making Σ_1 do the work of Σ_2". It will be applied in the proof of Theorem 4.5 to obtain a 1-generic subset of α for every Σ_1 admissible α. More generally, and less precisely, if a Σ_2 construction is tame, then it can be executed using only Σ_1 admissibility.

4.5 1-Genericity. A set $A \subseteq \alpha$ is 1-generic if it is generic with respect to certain bounded Π_2^α sentences in the sense defined below. ("Bounded" means the universal quantifier is bounded.)

$p, q, r \ldots$ are forcing conditions. A condition p is a pair (p^+, p^-) of disjoint α-finite sets. $p \geq q$ (p is extended by q) if $p^+ \subseteq q^+$ and $p^- \subseteq q^-$. $A \in p$ (A satisfies p) if $p^+ \subseteq A$ and $p^- \subseteq \alpha - A$. Define

$$\{\gamma\}^p(\delta) = x \quad \text{by} \quad (p^+, p^-, \delta, x) \in R_\gamma.$$

R_γ is the γ-th reduction procedure as in subsection 3.2.VII. Let $\uparrow$ mean undefined, and $\downarrow$ defined. Thus

$$\{\gamma\}^p(\delta)\downarrow \quad \text{if} \quad (\mathrm{E}x)[\{\gamma\}^p(\delta) = x].$$

Note that $\{\gamma\}^p(\delta)\downarrow$ is an α-recursively enumerable relation on γ, p and δ.

A is *1-generic* if for all γ, δ, $x < \alpha$:

$$[\{\gamma\}^A(\delta)\uparrow] \to (\mathrm{E}p)_{A \in p}(q)_{p \geq q}[\{\gamma\}^q(\delta)\uparrow]; \text{ and} \tag{1}$$

$$(\delta)_{\delta < x}[\{\gamma\}^A(\delta)\downarrow] \to (\mathrm{E}p)_{A \in p}(\delta)_{\delta < x}[\{\gamma\}^p(\delta)\downarrow]. \tag{2}$$

A set is said to be tame Δ_2^α if its characteristic function is tame Σ_2^α.

4.6 Theorem. *There exists a tame Δ_2^α, 1-generic set.*

Proof. Let S be an α-recursive function such that

$$S(s) = \left(\bigcup_{\delta < \beta} s_\delta^+, \bigcup_{\delta < \beta} s_\delta^- \right),$$

where s is an α-finite sequence (of length β) of forcing conditions. Let $R(s, \gamma, p)$ be the α-recursively enumerable relation

$$S(s) \geq p \quad \& \quad \{(\gamma)_0\}^p((\gamma)_1)\downarrow.$$

According to Theorem 4.4.VII there is a tame Σ_2^α function f with domain $t\sigma 2p(\alpha)$ that satisfies

$$f(\gamma) = \begin{matrix} \mu y R(f\restriction\gamma, t(\gamma), y) & \text{if} \quad (\mathrm{E}y)R(f\restriction\gamma, t(\gamma), y) \\ S(f\restriction\gamma) \text{ otherwise.} & \end{matrix}$$

f is a sequence of forcing conditions. $f(\delta_1) \geq f(\delta_2)$ when $\delta_1 \leq \delta_2$. Define

$$A = \cup\{f(\gamma)^+ \mid \gamma < t\sigma 2p(\alpha)\}.$$

Suppose $\{y\}^A(\delta)\uparrow$ with the intent of checking clause (1) of the definition of 1-genericity. Choose x so that $(t(x))_0 = y$ and $(t(x))_1 = \delta$, where t is as in subsection 4.3. Let $p = S(f\restriction x)$. Then $A \in p$ and there is no $q \leq p$ such that $\{y\}^q(\delta)\downarrow$. If there were such a q, then $f(x)$ would be such a q, $A \in f(x)$, and $\{y\}^A(\delta)$ would be defined.

The checking of clause (2) of the definition of 1-genericity is managed by a trick learned from Normann [1975]. It is based on standard manipulations with closed unbounded subsets of regular cardinals. Let f_σ be the α-recursive function in the proof of Theorem 4.4. Define

$$A^{<\sigma} = \cup\{f_{<\sigma}(\gamma)^+ \mid \gamma < t\sigma 2p(\alpha)\}.$$

Thus $A^{<\sigma}$ is the α-recursive approximation to A provided by the proof of Theorem 4.4 at the beginning of stage σ.

Assume $(\delta)_{\delta<x}[\{y\}^A(\delta)\downarrow]$. The "closed unbounded sets" trick is needed to show

$$(1) \qquad (\tau)(E\sigma)_{\sigma\geq\tau}(L(\sigma) \models (\delta)_{\delta<x}[\{y\}^{A<\sigma}(\delta)\downarrow]).$$

Suppose for the moment that (1) holds. Let ε be an index such that for all $B \subseteq \alpha$,

$$\{\varepsilon\}^B(0)\downarrow \leftrightarrow (\mathrm{E}q)_{B\in q}(\delta)_{\delta<x}[\{y\}^q(\delta)\downarrow].$$

Choose y so that $(t(y))_0 = \varepsilon$ and $(t(y))_1 = 0$. Then (1), and the tame Σ_2 convergence of f_σ to f, imply $\{y\}^{f(y)}(\delta)\downarrow$ for all $\delta < x$. $A \in f(y)$, so clause (2) of the definition of 1-genericity holds.

It remains to prove (1). Let ρ be the α-cardinality of x, and m an α-finite map of ρ onto x. γ_m is an index such that

$$\{\gamma_m\}^B(z)\downarrow \leftrightarrow \{y\}^B(m(z))\downarrow$$

for all $B \subseteq \alpha$. Hence

$$(z)_{z<\rho}[\{\gamma_m\}^B(z)\downarrow] \leftrightarrow (\delta)_{\delta<x}[\{y\}^B(\delta)\downarrow].$$

Thus it suffices to prove (1) when x is an α-cardinal. Proving (1) by induction on the α-cardinality of x makes it safe to assume x is a regular α-cardinal. (Replace x by its cofinality.) According to Lemma 2.5 there are only two cases.

Case 1: $t\sigma 2p(\alpha) \leq \mathrm{gc}(\alpha) \leq \alpha$. By Exercise 4.14 there is a regular α-cardinal β and an α-finite sequence $y_\delta(\delta < x)$ such that

$$y_\delta < \beta, (t(y_\delta))_0 = \gamma \quad \text{and} \quad (t(y_\delta))_1 = \delta.$$

Define

$$k(\sigma, \delta) = \mu\tau_{\tau \geq \sigma}[\{\gamma\}^{f_\tau(y_\delta)}(\delta)\downarrow].$$

$k(\sigma, \delta)$ is defined for all $\sigma < \alpha$ and $\delta < x$, since

$$\{\gamma\}^{f(y_\delta)}(\delta) \simeq \{\gamma\}^A(\delta) \text{ is defined.}$$

Let v be an α-finite map of β onto x such that $v^{-1}(\delta)$ is an unbounded subset of β for every $\delta < x$.

Fix τ in order to generate a σ that satisfies the matrix of (1). Let

$$\begin{aligned} \sigma_0 &= \tau, \\ \sigma_{w+1} &= k(\sigma_w, v(w)) \quad (w < \beta), \\ \sigma_\lambda &= \sup\{\sigma_w \mid w < \lambda\} \quad (\lambda \text{ a limit} \leq \beta). \end{aligned}$$

Fix δ and consider the behavior of $\{\gamma\}^{f_\sigma(y_\delta)}(\delta)$ as $\sigma \to \sigma_\beta$. The choice of v implies $\{\gamma\}^{f_\sigma(y_\delta)}(\delta)\downarrow$ unboundedly often as $\sigma \to \sigma_\beta$. Since $y_\delta < \beta$ and β is α-regular, $f_\sigma(y_\delta)$ suffers fewer than β changes as $\sigma \to \alpha$. (This point is discussed in Case 1 of the proof of Lemma 4.2. The argument there uses Lemma 2.3.VII to show $f_\sigma(\gamma)$ changes less than κ times, where κ is a regular α-cardinal and $\gamma < \kappa$.) Consequently $f_\sigma(y_\delta)$ changes less than β times as $\sigma \to \sigma_\beta$. Since the cofinality of σ_β is β, it follows that $f_\sigma(y_\delta)$ is constant for all sufficiently large σ as $\sigma \to \sigma_\beta$. That constant value is $f_{<\sigma_\beta}(y_\delta)$, and so $\{\gamma\}^{A^{<\sigma_\beta}}(\delta)\downarrow$.

Case 2: $t\sigma 2p(\alpha) = \mathrm{gc}(\alpha) \cdot \sigma 2\mathrm{cf}(\alpha)$. Similar to Case 1. The idea is to repeat the argument of Case 1 inside the block occupied by γ. Define σ_2 as in Case 2 of the proof of Lemma 4.2. Then consider only what happens after stage σ_2. For example, $f_\sigma(y_\delta)$ suffers fewer than β changes after stage σ_2 as $\sigma \to \alpha$. □

4.7 Proposition. *If A is 1-generic, then A is regular and hyperregular.*

Proof. Suppose $f \leq_{w\alpha} A$ and $x < \alpha$. By Lemma 5.2.VII it suffices to show $f \restriction x$ is α-finite. If f is $\{\gamma\}^A$, then there exists a p such that

$$A \in p \quad \text{and} \quad (\delta)_{\delta < x}[\{\gamma\}^p(\delta)\downarrow].$$

Then $f \restriction x$ is $\{\gamma\}^p \restriction x$. □

4.8 The α-Jump. Let $A \subseteq \alpha$. By analogy with the Turing jump of classical recursion theory, the α-jump of A should be a universal, α-recursively-enumerable-in-A set. To be precise, the α-jump of A should be a set B such that B is α-recursively enumerable in A, and such that $C \leq_\alpha B$ for every C α-recursively enumerable in A. Three candidates for the definition of "B is α-recursively enumerable in A" come to mind.

(i) There is a $\gamma < \alpha$ such that

$$B = \{\delta \mid \{\gamma\}^A(\delta)\downarrow\}.$$

(Recall that "$\downarrow$" means "defined".)

(ii) There is a $\gamma < \alpha$ such that for all $\delta < \alpha$,

$$K_\delta \subseteq B \leftrightarrow \{\gamma\}^A(\delta)\downarrow.$$

($\{K_\delta \mid \delta < \alpha\}$ is the strong enumeration of α-finite sets, cf. 3.1.VII.)

(iii) B is $\Sigma_1^{\alpha, A}$. That is, B is definable over $\langle L[A,\alpha], \varepsilon, A\rangle$ by means of a Σ_1 formula with parameters in $L[A,\alpha]$ with $x \in A$ regarded as a Δ_0 formula.

Clearly (ii) $\to$ (i) $\to$ (iii). ($x \to y$ means if B is enumerable by definition x, then it is enumerable by definition y.) If A is regular, then (iii) $\to$ (i). If A is regular and hyperregular, then all three notions agree by Lemma 5.2.VII. Hence if A is α-recursively enumerable and hyperregular, then all three notions coincide. There exists an α and an α-recursively enumerable A on which all three differ.

One objection to (iii) is its dependence on non-α-finite computations. If (iii) holds, then an element is added to B because some bounded, but not necessarily α-finite, set of membership statements is satisfied by A. (ii) and (i) are based on α-finite computations. (ii) has the virtue of symmetry over (i). If (ii) holds, then an α-finite set of positive facts about B follows from an α-finite set of facts about A. Symmetry considerations led to the rejection of $\leq_{w\alpha}$ in favor of $\leq_\alpha$. On the other hand there is in general no universal set as defined above in the sense of (ii).

Thus (i) is the preferred choice for the definition of relative α-recursive enumerability on which to build the definition of α-jump. Let

$$A' \text{ be } \{\langle\gamma,\delta\rangle \mid \{\gamma\}^A(\delta)\downarrow\}.$$

Then A' is α-recursively enumerable in A (in the sense of (i)). Suppose B is α-r.e. in A. Thus

$$B = \{\delta \mid \{\gamma_0\}^A(\delta)\downarrow\}.$$

Then

$$H \subseteq B \leftrightarrow \{\gamma_0\} \times H \subseteq A', \text{ and}$$

$$J \subseteq \alpha - B \leftrightarrow \{\gamma_0\} \times J \subseteq \alpha - A'.$$

Hence $B \leq_\alpha A'$. If $A_1 \leq_\alpha A_2$, then A_1' is α-recursively enumerable in A_2, consequently α-recursive in A_2'. Thus the α-jump is well defined on α-degrees. For any α-degree, d, let d', the *α-jump* of d, be the α-degree of D, where D is any set of degree d.

Define $A^{(n+1)}$ to be $(A^{(n)})'$, and $A^{(0)}$ to be A.

Warning: ϕ' has the same α-degree as some complete Σ_1^α set, but in general $\phi^{(2)}$ does not have the same α-degree as some complete Σ_2^α set. Thus the familiar connection between $\phi^{(n)}$ and Σ_n is broken.

4.9 Theorem. *If A is regular and hyperregular, then A' has the same α-degree as some regular set α-recursively enumerable in A.*

Proof. Since A is regular and hyperregular, the structure $\langle L[A, \alpha], \varepsilon, A\rangle$ is Σ_1 admissible (cf. Exercise 5.8.VII). In addition $L[A, \alpha]$ equals $L(\alpha)$. The proof of the regular sets theorem (4.2.VII) is entirely dynamic in nature, and consequently can be extended from $L(\alpha)$ to $\langle L[A, \alpha], \varepsilon, A\rangle$ without significant change. The extended version states: if $B \in \Sigma_1^{\alpha, A}$, then there is a $C \in \Sigma_1^{\alpha, A}$ such that

(i) C is regular in the sense of $L[A, \alpha]$, and
(ii) $C \equiv_{\alpha, A} B$.

(i) means $(C \cap x) \in L[A, \alpha]$ for all $x \in L[A, \alpha]$. In (ii), $\equiv_{\alpha, A}$ refers to reduction procedures that are $\Sigma_1^{\alpha, A}$ sets of 4-tuples $\langle H, J, \delta, \gamma\rangle$ from $L[A, \alpha]$. To visualize the extended proof, extend the natural enumeration of $L(\alpha)$ to one of $L[A, \alpha]$.

Since $L[A, \alpha] = L(\alpha)$, it follows that C is regular, $C \oplus A \equiv_\alpha B \oplus A$, and C is α-recursively enumerable in A. If B is A', then $C \oplus A$ is the desired regular set. $\square$

The Simpson jump theorem is a lifting of the Friedberg jump theorem of classical recursion theory. The latter states: $\phi' \leq d$ iff $c' = d$ for some c.

4.10 Theorem (Simpson 1974a). *(i) and (ii) are equivalent.*
(i) $\phi' \leq_\alpha D$ and D has the same α-degree as some regular set.
(ii) $C' \equiv_\alpha D$ for some regular, hyperregular C.

Proof. (ii) implies (i) by Theorem 4.9. Now assume D satisfies (i). C is constructed from D so that

$$(0) \qquad \text{(a)} \quad C' \leq_\alpha \phi', D \quad \text{and} \quad \text{(b) } D \leq_\alpha C, \phi'.$$

The idea is to make C 1-generic so that (a) holds, and to code D into C so that (b) holds. In addition, the 1-genericity of C implies C is regular and hyperregular by Proposition 4.7.

Note that 4.7 makes it safe to assume that all forcing conditions are equivalent to initial segments of characteristic functions. If p and q are forcing conditions then $p^\frown q$ is the condition that begins with p and continues with q.

$$(p^\frown q)(x) = \begin{matrix} p(x) & \text{if } x < lh(p) \\ q(x - lh(p)) & \text{if } lh(p) \leq x < lh(p) + lh(q). \end{matrix}$$

Assume D is regular. Let s be an α-finite sequence of length γ of forcing conditions. Let $S(s)$ be as in the proof of Theorem 4.6. Define

$$S_0(s) = S(s)^\frown D\restriction(t(\gamma)),$$
$$R_0(s, \gamma, q) \leftrightarrow S_0(s) \geq q \quad \& \quad \{(\gamma)_0\}^q((\gamma)_1)\downarrow.$$

Consider the reduced recursion equations

$$(1) \qquad f(\gamma) = \begin{matrix} \mu y R_0(f\restriction\gamma, t(\gamma), y) & \text{if} \quad (\mathrm{Ey})R_0(f\restriction\gamma, t(\gamma), y) \\ S_0(f\restriction\gamma) \quad \text{otherwise.} & \end{matrix}$$

t is a tame Σ_2^α map from $t\sigma 2p(\alpha)$ onto α.

Fix $z < t\sigma 2p(\alpha)$ in order to study the recursion defined by (1). As the recursion progresses through γ's less than z, the only information about D that is needed is

$$D_z = D\restriction\left(\sup_{\gamma<z} t(\gamma)\right).$$

The regularity of D and tameness of t imply D_z α-finite. Let $(1)_2$ be the result of replacing D by D_z in (1). By Theorem 4.4, $(1)_2$ has a tame Σ_2^α solution f_2 from $t\sigma 2p(\alpha)$ into α. The α-finite function $f_z\restriction z$ is a solution of (1) for $\gamma < z$. Hence

$$\cup\{f_z\restriction z \mid z < t\sigma 2p(\alpha)\}$$

is a solution of (1). In short, (1) does have a solution f, necessarily unique, from $t\sigma 2p(\alpha)$ into α. Define

$$C = \cup\{f(\gamma)^+ \mid \gamma < t\sigma 2p(\alpha)\}.$$

Initial segments of the characteristic function of C correspond to initial segments of f. The value of $f\restriction\gamma$ is determined by (1) from an α-finite set of facts about D, t and ϕ'. ϕ' is needed to decide when there is a y that satisfies R_0. R_0 was defined so that

$$\gamma \in C' \leftrightarrow (\mathrm{Ey})R_0(f\restriction\gamma, t(\gamma), y).$$

It follows that $C' \leq_\alpha \phi', D, t$. The Σ_2^α tameness of t implies $t \leq_\alpha \phi'$ (cf. Exercise 4.13). Thus (0) (a) is proved.

A simultaneous recursion on γ shows $\lambda\gamma\mid f\restriction\gamma$ and $\lambda\gamma\mid D\restriction t(\gamma)$ are weakly α-recursive in C, ϕ'. Suppose $f\restriction\gamma$ has been computed α-finitely from C, ϕ'. $I(f\restriction\gamma)^\frown D\restriction t(\gamma)$ is an initial segment of $f(\gamma)$, hence an initial segment of C. So $D\restriction b(\gamma)$ can be extracted from C, $f\restriction\gamma$, $t(\gamma)$. Then $f(\gamma)$ can be computed α-finitely from $f\restriction\gamma$, $D\restriction t(\gamma)$, ϕ'. Thus (0) (b) is proved. □

A jump theorem (Sacks 1963) of classical recursion theory that relates recursive enumerability and the Turing jump states: D is recursively enumerable in ϕ' and

$D \geq_\omega \phi'$ iff $D \equiv_\omega C'$ for some incomplete, recursively enumerable C. Maass 1977 has shown this statement holds for α (in place of ω) iff α is Σ_2 admissible.

4.11–4.15 Exercises

4.11. Use Shore's blocking method to prove Theorem 4.10.

4.12. Show a 1-generic set cannot be α-recursively enumerable.

4.13. Let $A \subseteq \alpha$. Show A is α-recursively enumerable in ϕ' iff A is Σ_2^α. Show A is tame Δ_2^α iff $A \leq_\alpha \phi'$ and A is regular.

4.14. Suppose $t\sigma 2p(\alpha) \leq \text{gc}(\alpha)$. Find a tame Σ_2^α function t from $t\sigma 2p(\alpha)$ onto α with the following properties. For each $\gamma < \alpha$ and each regular α-cardinal x, there exists a regular α-cardinal β and an α-finite sequence $y_\delta(\delta < x)$ such that

$$y_\delta < \beta,\ (t(y_\delta))_0 = \gamma \quad \text{and} \quad (t(y_\delta))_1 = \delta.$$

4.15. Provide the details of Case 2 of the proof of Theorem 4.6.