Chapter IV

Models of Fragments of Arithmetic

Introduction. The present chapter is devoted to the study of models of arithmetic, i.e. structures M for the language L of arithmetic such that $M \models T$, where T is either PA or a fragment of PA. Models are useful as means for showing unprovability of a formula φ in T (by exhibiting a model M for $T + \neg\varphi$) as well as for showing provability (by proving that $(T + \neg\varphi)$ has no model). In particular, we shall prove some conservation results by model-theoretical methods.

Model theory of arithmetic is a rather broad field and we shall not try to be exhaustive; our aim will be to present selected typical results and techniques. Section 1 (Some basic constructions) makes the reader familiar with non-standard models and some techniques of their construction; the main results are hierarchy results (concerning theories $I\Sigma_n$, $B\Sigma_n$, $P\Sigma_n$), theorems concerning existence of elementary extensions and the Paris-Friedman conservation result. Section 2 (Cuts in models of arithmetic with top) introduces and studies cuts (roughly: initial segments of models having no greatest element) and various kinds of cuts: in particular, k-extendable cuts and their relationship to models of $B\Sigma_{k+1}$. Furthermore, the section contains a proof of the theorem saying that each non-standard model of $I\Sigma_1$ is isomorphic to a proper cut of itself. Section 3 (Provably recursive functions and the method of indicators) presents two characterizations of a $I\Sigma_k$-provably recursive function; a corollary is the fact that $I\Sigma_1$-provably recursive functions coincide with primitive recursive functions. We rely heavily on the method of indicators, a formalization of ordinals in $I\Sigma_1$, k-extendable (or k-restrainable) cuts and the notion of α-large sets. Finally, there is a short Sect. 4, dealing with a formalization of some parts of model theory of fragments; we formalize in $I\Sigma_1$. In particular, we give in $I\Sigma_1$ a model-theoretical proof of Paris-Friedman's conservation theorem and get various corollaries, among them the fact that $I\Sigma_{k+1}$ proves the consistency of $B\Sigma^{\bullet}_{k+1}$.

Our main omission is the absence of the theory of recursively saturated models. We apologize for this and refer to bibliographical remarks for references. But we hope that the present chapter will give the reader satisfactory insight into the world of models of arithmetic and their possible uses.

1. Some Basic Constructions

(a) Preliminaries

1.1 Introduction. Our starting point is the *standard* model N, consisting of all natural numbers endowed with the usual (standard) successor, addition, multiplication, zero, equality and the less-than-or-equal-to relation. Since isomorphic models differ in an inessential manner, each model isomorphic to N is also called standard. We shall be interested in non-standard models. We shall investigate only *countable* models; thus in saying "model" we always mean "countable model".

We shall often use Gödel's famous completeness theorem 0.12 and also the method of skolemization 0.14. We shall also make use of the fact that Matiyasevič's theorem is provable in $I\Sigma_1$ (cf. I.1.59). We shall construct new models as extensions of a given model (e.g. of N) or as submodels of a given (non-standard) model. In this section we shall deal with *definable ultrapower* as a method for construction of an extension of a given model and with a few methods of construction of *submodels* using *definable elements.* Definable ultrapower of N is one of the simplest examples of a non-standard model (and corresponds – mutatis mutandis – to the first historical construction of a non-standard model of PA due to Skolem). This is elaborated on in subsection (b). The methods of constructing submodels of a non-standard model are powerful enough to produce, for each theory T from the hierarchy of theories $I\Sigma_k$, $B\Sigma_{k+1}$, $P\Sigma_{k+1}$, a model of T which is not a model of any stronger theory in the hierarchy. For details see subsection (d). In (e) we generalize the construction of definable ultrapower and characterize models of $B\Sigma_{k+1}$ by the existence of some proper extension; as a by-product we obtain new models of some theories in our hierarchy that are not models of stronger theories. Subsection (f) contains two proofs of the theorem of Friedman and Paris saying that $B\Sigma_{k+1}$ is Π_{k+2}-conservative over $I\Sigma_k$ ($k \geq 0$). Thus the reader finds here some techniques for construction of models, examples of models showing that the hierarchy of fragments is proper (and non-linear) and the above-mentioned conservation result. Most results of this section are due – mutatis mutandis – to Skolem, Gaifman, Friedman, Paris, Kirby, Wilkie, Lessan, Kaye and Dimitracopulos; see bibliographical notes for details.

1.2 Conventions. M, K, I, J vary over models for L; N is reserved for the standard model. We use the same letter for a model and its domain if there is no danger of misunderstanding. A model M has its zero 0_M, successor S_M, addition $+_M$, multiplication $*_M$, inequality $\leq_M$; we deal with models having absolute equality. Sometimes we shall omit the index M in $+_M$ etc. If $t(x_1, \ldots, x_n)$ is a term L containing the variables displayed then t_M is the corresponding mapping of M^n into M, i.e., for each $a_1, \ldots, a_n \in M$,

$t_M(a_1, \ldots, a_n)$ is the value of t for $a_1, \ldots, a_n$. We use Tarski's definition of satisfaction; if φ is closed then $M \models \varphi$ means that φ is true in M. If $\varphi(x_1, \ldots, x_n)$ has the free variables displayed then $M \models \varphi(a_1, \ldots, a_n)$ means that φ is satisfied by $a_1, \ldots, a_n$ in M (which in turn is a condition concerning sequences of elements of M having the value a_i in the place corresponding to x_i). $M \subseteq K$ means that M is a submodel of K, i.e. the domain of M is a subset of the domain of K, $0_M = 0_K$ and S_M, $+_M$, $*_M$, $\leq_M$ are restrictions of S_K, $+_K$, $*_K$, $\leq_K$ respectively.

1.3 Some Definitions and Facts. (1) If $M \models I_{\text{open}}$ then $+_M$ and $*_M$ are associative and commutative, satisfy the distributive law for multiplication w.r.t. addition, $\leq_M$ is a discrete linear order with the least element O_M, S_M associates to each element its bigger neighbour, addition is monotone and so is multiplication by a non-zero element; etc., cf. Chap. I, Sect. 1.

(2) Let $K, M \models I_{\text{open}}$. K is an *end-extension* of M (M is a *cut* in K, notation $M \subseteq_e K$) if for each $a \in M$ and each $b \in K - M$ we have $K \models a \leq b$, i.e. each element of M $\leq_K$-precedes each element of $K - M$.

(3) Each $M \models I_{\text{open}}$ is an end-extension of (a copy of) the standard model: Let $I = \{a \in M \mid (\exists n \in N)(M \models a = \overline{n})\}$; then clearly I is a cut in M (cf. I.1.6 (4)) and is isomorphic to N. Evidently, M is non-standard iff $I \neq M$, i.e. iff there is an $a \in M$ such that for each $n \in N$, $M \models a \neq \overline{n}$.

(4) If $M, K \models I_{\text{open}}$, $M \subseteq_e K$, $\varphi \in \Sigma_0$ and $a, \ldots, b \in M$ then $M \models \varphi(a, \ldots, b)$ iff $K \models \varphi(a, \ldots, b)$ (obvious). Consequently, if $\varphi \in \Sigma_1$ and $a, \ldots, b \in M$ then $M \models \varphi(a, \ldots, b)$ implies $K \models \varphi(a, \ldots, b)$.

(5) In particular, the set of all L-formulas is defined in N by a formula $\mathit{Form}^\bullet(x)$ which is Δ_1 in $I\Sigma_1$. Thus φ is a formula iff $N \models \mathit{Form}^\bullet(\overline{\varphi})$ iff $M \models \mathit{Form}^\bullet(\overline{\varphi})$ for each $M \models I\Sigma_1$.

(6) We introduce the notion of a Σ_n-definable subset of a model $M \models I\Sigma_0$; $A \subseteq M$ is *parametrically Σ_n-definable in M* iff there is a Σ_n-formula φ and an element $c \in M$ such that $A = \{a \in M \mid M \models \varphi(a, c)\}$. Similarly for non-parametric definability and for definable subsets of M^n.

(7) Recall that in I.1.73–74 we defined satisfaction for Σ_n-formulas in $I\Sigma_1$ and in I.1.81 we defined Σ_n-definable sets and their codes in $I\Sigma_1$; we have predicates $\mathit{Code}_{\Sigma,n}$ and $\in_{\Sigma,n}$. The relation to definable subsets of models is as follows:

1.4 Claim. Let $M \models I\Sigma_1$ and $A \subseteq M$. The following is equivalent:

(1) A is parametrically Σ_n-definable in M;

(2) there is a $c \in M$ such that $M \models$ (c is a Σ_n-formula$^\bullet$ with one variable) and $A = \{a \in M \mid M \models \mathit{Sat}_{\Sigma,n}(c, \langle a \rangle)\}$

(3) there is a $d \in M$ such that $M \models \Sigma_n^\bullet\text{-set}^\bullet(d)$ and $A = \{a \in M \mid M \models a \in_{\Sigma,n} d\}$. (Cf. I.1.78)

Proof. (1) $\Rightarrow$ (2). Let $A = \{a \mid M \models \varphi(a,p)\}$; $M \models \varphi(a,p)$ iff $M \models Sat_{\Sigma,n}(\overline{\varphi}, \langle a,p\rangle)$ iff $M \models Sat_{\Sigma,n}(c, \langle a\rangle)$ where c results from $\overline{\varphi}$ by substituting the p-th numeral for the second variable).

(2) $\Rightarrow$ (3). Let $A = \{a \mid M \models Sat_{\Sigma,n}(c, \langle a\rangle)\}$, then $M \models \Sigma_n^\bullet\text{-set}^\bullet(c)$ and $A = \{a \mid a \in_{\Sigma,n} c\}$.

(3) $\Rightarrow$ (1). If $A = \{a \in M \mid M \models a \in_{\Sigma,n} d\}$ then A is Σ_n-definable in M since $\in_{\Sigma,n}$ is a Σ_n-formula. □

Example of a non-standard model. The existence of a non-standard-model follows trivially from compactness and completeness: let $Th(N)$ be the set of all sentences true in N. Let α be a new constant and put $T' = T \cup \{\overline{n} \leq \alpha \mid n \in N\}$. T' is consistent by compactness and therefore has a model M; M is non-standard since if c is the meaning of α then $M \models c \geq \overline{n}$ for each $n \in N$. In the next subsection we shall *construct* a non-standard model in an apparently more transparent way.

(b) Definable Ultrapower of the Standard Model

1.6 Construction. Let F be the set of all definable mappings of N into N (as particular subsets of N^2). Let D be the set of all definable subsets of N. Clearly, D is a Boolean algebra, i.e. is closed under meet, union and complement. D is countable; let U be a non-trivial ultrafilter in D, i.e. we have the following:

$$A \subseteq B \text{ and } A \in U \text{ implies } B \in U\,.$$
$$A, B \in U \text{ implies } A \cap B \in U\,,$$
$$\text{for each } A \in D, \text{ either } A \in U \text{ or } (N - A) \in U\,,$$
$$A \in U \text{ implies that } A \text{ is infinite}\,.$$

(Hint: let $A_0, A_1, A_2, \ldots$ be the sequence of all (countably many) definable subsets of N; define $D_0 = N$ and for each n, let D_n be $D_0 \cap \cdots \cap D_{n-1} \cap A_n$ if this set is infinite, otherwise let $D_n = D_0 \cap \cdots \cap D_{n-1} \cap (N - A_n)$. Let $A \in D$ iff $(\exists n)(D_n \subseteq A)$.)

Let, for $f, g \in F$, $f =_U g$ mean $\{i \in N \mid N \models f(i) = g(i)\} \in U$. (Observe that the last set is in D.) It is easy to show that $=_U$ is an equivalence, i.e. is reflexive, symmetric and transitive. Let $[f]_U$ be the equivalence class of f, i.e. $[f]_n = \{g \mid f =_U g\}$. Let $N^* = \{[f]_U \mid f \in F\}$. Put $f +_F g = h$ iff, for each $i \in N$, $f(i) + g(i) = h(i)$, similarly for $S, *$. Let $f \leq_U g$ mean $\{i \in N \mid f(i) \leq g(i)\} \in U$. Show that $=_U$ is a congruence w.r.t. the operations S_F, $+_F$, $*_F$; this means that we may define $[f]_U +_{N^*} [g]_U = [f +_F g]_U$ and similarly for S. Doing this we endow N^* with a structure for L (we put $0_N = k_0$ where $k_0(i) = 0$ for each i). This structure is denoted by N^* (or $N^*(U)$) and is called the *definable ultrapower* of N (given by U).

1.7 Theorem. (1) N^* is a structure for L elementarily equivalent to N; thus N^* is a model of PA.

(2) The mapping associating with each $n \in N$ the element $[k_n]_U$ (where $k_n(i) = n$ for each i) is the unique isomorphic embedding of N onto a cut in N^*.

(3) N^* is non-standard; e.g. the diagonal d (such that $d(i) = i$ for each i) is a non-standard element.

The rest of the subsection consists of the proof of this theorem.

1.8 Lemma. (1) For each term $t(x_1, \ldots, x_n)$ of L and all $f_1, \ldots, f_n \in F$, let $t_F(f_1, \ldots, f_n)$ be the function $f(i) = t_N(f_1(i), \ldots, f_n(i))$. Then $t_{N^*}([f_1]_U, \ldots, [f_n]_U) = [t_F(f_1, \ldots, f_n)]_U$.

(2) For any terms $t(x_1, \ldots, x_n)$, $s(x_1, \ldots, x_n)$ and $f_1, \ldots, f_n \in F$,

$$N^* \models t(f_1, \ldots, f_n) = s(f_1, \ldots, f_n)$$

iff

$$\{i \in N \mid N \models t(f_1(i), \ldots, f_n(i)) = s(f_1(i), \ldots, f_n(i))\} \in U\,.$$

The same replacing $=$ with $\leq$.

Proof. (1) by induction on the complexity of t. (2) follows immediately from (1). □

1.9 Łoś's Lemma. For each L-formula $\varphi(x_1, \ldots, x_n)$ and any $f_1, \ldots, f_n \in F$,

$$N^* \models \varphi([f_1]_U, \ldots, [f_n]_U) \text{ iff } \{i \in N \mid N \models \varphi(f_1(i), \ldots, f_n(i))\} \in U\,.$$

Proof by induction on the complexity of φ; we elaborate the induction step for quantification. Let φ be $(\exists y)\psi(x_1, \ldots, x_n, y)$ and assume the assertion for ψ. If $N^* \models (\exists y)\psi([f_1]_U, \ldots, y)$ then for some g, $N^* \models \varphi([f_1]_U, \ldots, [f_n]_U, [g]_U)$, thus

$$\{i \mid N \models \varphi(f_1(i), \ldots, g(i))\} \in U\,,$$

which implies $\{i \mid N \models (\exists u)\varphi(f_1(i), \ldots, f_n(i), u)\} \in U$. Conversely, let $A = \{i \mid N \models (\exists u)\varphi(f_1(i), \ldots, f_n(i), u)\} \in U$. Define g as follows:

$$\text{for } i \in A,\ g(i) = \min_j N \models \varphi(f_1(i), \ldots, f_n(i), j)\,;$$
$$\text{for } i \notin A,\ g(i) = 0\,.$$

Since $f_1, \ldots, f_n$ are definable, one easily sees that A is definable and so is g. Clearly,

$$A = \{i \mid N \models \varphi(f_1(i), \ldots, f_n(i), g(i)\} \in U\,,$$

thus $N^* \models \psi([f_1]_U, \ldots, [f_n]_U, [g]_U)$ and $N^* \models (\exists u)\psi([f_1]_U, \ldots, u)$. □

1.10 Corollary. 1.7 (1) follows.

1.11 Lemma. For each n, let $k_n(i) = n$ for each i; then k_n is the value of the n-th numeral $\overline{n}$ in N.

Proof. Clearly, $[k_0]_U$ is the least element in N^* and $n < m$ implies $N^* \models [k_n]_U < [k_m]_U$. It is sufficient to show that if $N^* \models [f]_U \leq [k_n]_U$ then for some $m < n$, $[f]_U = [k_m]_U$. Let $A_m = \{i \mid f(i) = m\}$; clearly, $\{i \mid f(i) < n\} = A_0 \cup \cdots \cup A_n \in U$ and therefore for an $m < n$ we have $A_m \in U$. Thus $f =_U k_m$. Thus 1.7 (2) is proved. □

1.12 Lemma. The diagonal is non-standard.

Proof. Clearly, for each n, $[d]_U \neq [k_n]_U$ since the set $\{i \mid d(i) = n\}$ is a one-element set. This completes the proof of 1.7. □

1.13 Remark. In subsection (e) we shall generalize this construction in two ways: first, we shall start with a possibly non-standard model M instead of N and, second, we shall only assume that M is a model of a fragment of PA.

(c) On Submodels and Cuts

This subsection is somewhat technical; we collect here several facts useful further on. They concern three things; first we formulate very useful principles of overspill and underspill, second we introduce the notion of a k-elementary submodel and prove some lemmas about this notion and third, for $M \subseteq K$ we define $\sup_K(M) = \{a \in K \mid (\exists b \in M) M \models a \leq b\}$ and prove some facts about this notion.

1.14 Definition. Let $M \models I_{\text{open}}$. A set $I \subseteq M$ is a *cut* (notation: $I \subseteq_e M$) if it is closed under successor (in the sense of M) and contains with each a each $b \in M$ that is less than a, i.e.

$$(\forall a \in I)[S_M(a) \in I \,\&\, (\forall b \in M)(M \models b < a \rightarrow b \in I]\,.$$

A cut $I \subseteq M$ is *proper* if $I \neq M$.

1.15 Observation. Let $M \models I_{\text{open}}$. $M \models I\Sigma_k$ iff no proper cut is $\Sigma_0(\Sigma_k)$-definable.

Proof. (ii) $\Rightarrow$ (i) is evident; and (i) $\Rightarrow$ (ii) follows from the fact that $I\Sigma_k$ implies $I\Sigma_0(\Sigma_k)$ (see I.2.4). □

1.16 Corollary. Let $M \models I\Sigma_k$. Let $\varphi(x) \in \Sigma_0(\Sigma_k)$ and let $I \subseteq M$ be a proper cut. Then

(i) (overspill) if for each $a \in I$, $M \models \varphi(a)$ then for some $b \in M - I$, $M \models (\forall x \le b)\varphi(x)$;
(ii) (underspill) if for each $a \in M - I$, $M \models \varphi(a)$ then for some $b \in I$, $M \models (\forall x > b)\varphi(x)$.

Proof. If for each $b \in M - I$, $M \models (\exists x < b)\neg\varphi(x)$ then $I = \{a \in M \mid M \models (\forall x < a)\varphi(x)\}$; thus I is $\Sigma_0(\Sigma_k)$-definable. Similarly for (ii). □

1.17 Definition. Let $M \subseteq K$. M is a *k-elementary* substructure of K (notation: $M \prec_k K\}$ if for each $\varphi(x_1, \ldots, x_n) \in \Sigma_k$ and each $a_1, \ldots, a_n \in M$, $M \models \varphi(a_1, \ldots, a_n)$ iff $K \models \varphi(a_1, \ldots a_n)$. M is an *elementary* substructure of K if $M \prec_k K$ for each k.

1.18 Remark and Notation. Clearly if $M \subseteq_e K$ (i.e. K is an end-extension of M) then $M \prec_0 K$. See below for a stronger result. We write $M \prec_{e,k} K$ for $M \subseteq_e K$ & $M \prec_k K$.

1.19 Lemma. Let $k \ge 1$, and $M, K \models I_{\text{open}}$, $M \subseteq K$. Then $M \prec_k K$ iff, for each Π_{k-1}-formula $\pi(x, \mathbf{y})$ and each $\mathbf{b} \in M$ such that $K \models (\exists x)\pi(x, \mathbf{b})$, there is an $e \in M$ such that $K \models \pi(a, \mathbf{b})$.

Proof. The implication $\Rightarrow$ is obvious; let us prove the converse. Assume the condition and prove $M \prec_i K$ by induction on $i \le k$. For $i = 0$ use induction on the complexity of the formula in question. □

1.20 Lemma. Let $M, K \models I\Sigma_1$ and $M \subseteq K$. Then $M \prec_0 K$.

Proof. Let $\varphi(x)$ be Σ_0; then both φ and $\neg\varphi$ are Σ_1. By Matiyasevič's theorem (provable in $I\Sigma_1$, cf. I.3.25) there are pure existential formulas $\sigma_1(x)$ and $\sigma_2(x)$ such that $I\Sigma_1 \vdash \varphi(x) \equiv \sigma_1(x)$ and $I\Sigma_1 \vdash \neg\varphi(x) \equiv \sigma_2(x)$. Let $a \in M$ and assume $M \models \varphi(x)$. Then $M \models \sigma_1(a)$, thus $K \models \sigma_1(a)$ and $K \models \varphi(a)$. Similarly for $\neg\varphi$. □

1.21 Remark. (1) Note that $I\Sigma_0(exp)$ proves Matiyasevič's theorem as well (cf. I.3.26); thus $I\Sigma_1$ may be replaced by $I\Sigma_0(exp)$ in the previous theorem.

(2) Note that for $k \ge 1$, $B\Sigma_k$ and $P\Sigma_k$ are axiomatized by Π_{k+2}-formulas (see Chap. I, Sect. 2 and subsection (d)) below. Furthermore observe that $M \prec_k K$ implies that M satisfies each Π_{k+1} sentence true in K. Thus if $M \prec_{k+1} K$ and K is a model of $I\Sigma_k$, $B\Sigma_k$, $P\Sigma_k$ respectively, then so is M. (Furthermore, $I\Sigma_0$ is Π_1-axiomatized; thus $M \prec_0 K \models I\Sigma_0$ implies $M \models I\Sigma_0$.)

1.22 Lemma. Let $M \prec_{e,k} K \models I\Sigma_k$. If $M \neq K$ then $M \models B\Sigma_{k+1}$.

Proof. Clearly, $M \models I\Sigma_0$. Let $\pi(x,y,z) \in \Pi_k$, $a,b \in M$, $M \models (\forall x < a)(\exists y)\pi(x,y,b)$. For each $c \in K - M$, $K \models (\forall x < a)(\exists y < c)\pi(x,y,b)$. By underspill, there is a $c \in M$ such that $K \models (\forall x < a)(\exists y < c)\pi(x,y,b)$; thus $M \models (\exists t)(\forall x < a)(\exists y < t)\pi(x,y,b)$. □

1.23 Definition. Let $M \subseteq K$; put $\sup_K(M) = \{a \in K \mid (\exists b \in M)(K \models a < b)\}$. (The cut in K determined by M.)

1.24 Theorem. Let $k \geq 1$, $M \models B\Sigma_k$ and $M \subseteq K \models I\Sigma_1$. Then $M \prec_{k+1} \sup_K(M)$.

Proof. Put $I = \sup_K(M)$. Clearly $I \prec_0 K$ (since $I \subseteq_e K$), and $M \prec_0 K$ by 1.20. Thus $M \prec_0 I$. We prove $M \prec_1 I$. Let $(\forall x)\varphi(x,y)$ be Π_1 and assume $M \models (\forall x)\varphi(x,a)$. Let $c \in I$, take a $c' \in M$, $I \models c < c'$. Then $M \models (\forall x < c')\varphi(x,a)$, thus $I \models (\forall x < c')\varphi(x,a)$, thus $I \models \varphi(c,a)$. We have proved $I \models (\forall x)\varphi(x,a)$. Now let $k \geq 1$, i.e. $k+1 > 1$; assume the theorem for k, let $M \models B\Sigma_{k+1}$, $M \subseteq K \models I\Sigma_1$, let Φ be a Π_{k+1}-formula $(\forall x)(\exists y)\pi(x,y,z)$ and let $M \models (\forall x)(\exists y)\pi(x,y,a)$. By induction, we assume $M \prec_k I$. Let c, c' be as above; since $M \models B\Sigma_{k+1}$, there is a $d \in M$ such that $M \models (\forall x < c')\pi(x,(d)_x,a)$. Thus $K \models (\forall x < c')\pi(x,(d)_y,a)$ and $K \models (\exists y)\pi(x,y,a)$. We have proved $K \models (\forall x)(\exists y)\pi(x,y,a)$. □

1.25 Remark. (1) In the next subsection we shall show that the above theorem is the best possible: for each $k > 1$, there are $M \models B\Sigma_k$ and $K \models I\Sigma_1$ such that for $I = \sup_K(M)$ M is not a $(k+2)$-elementary substructure of I. (See 1.51)

(2) For PA we get the following Corollary: If $M, K \models PA$, $M \subseteq K$ and $I = \sup_K(M)$, then $M \prec I$ (M is an elementary substructure of I).

(d) Models for the Hierarchy

In this section we shall show that the hierarchy of theories $I\Sigma_k$, $B\Sigma_k$ is proper; we shall also show the exact place of theories $P\Sigma_k$ investigated in Chap. I, Sect. 2.

Recall that $P\Sigma_k$ is $I\Sigma_1$ plus the schema saying that each Σ_k-definable partial function has arbitrarily long finite approximations, i.e. if F is such a function then for each x there is a segment s of length x such that, for each $i \leq x-1$ and each $u < (s)_i$, u from $dom(F)$, we have $F(u) < (s)_{i+1}$. Recall the positive results we have:

1.26 Theorem. (1) $PA = \bigcup_n I\Sigma_n = \bigcup_n B\Sigma_n = \bigcup_n P\Sigma_n$

(2) for $n \geq 0$, $I\Sigma_{n+1}$ implies $B\Sigma_{n+1}$ and $B\Sigma_{n+1}$ implies $I\Sigma_n$;

(3) for $n \geq 1$, $I\Sigma_{n+1}$ implies $P\Sigma_n$ and $P\Sigma_n$ implies $I\Sigma_n$;

(4) $P\Sigma_0$ is equivalent to $P\Sigma_1$ and $B\Sigma_0$ is equivalent to $B\Sigma_1$.

Recall also the following fact (more or less explicit in Chap. I, Sect. 2, see also the preceding subsection).

1.27 Theorem. (1) $I\Sigma_0$ is a Π_1-theory.

(2) For $n \geq 1$, each of $I\Sigma_n$, $B\Sigma_n$, $P\Sigma_n$ is a Π_{n+2}-theory (i.e. it has a system of axioms consisting only of Π_{n+2} formulas).

Remark. The proof is an easy exercise; note that for $n > 1$, $B\Sigma_n$ is equivalent to $I\Sigma_1$ plus

$$(\ldots)(\forall a)[(\forall x < a)(\exists y)\varphi(x, y) \to (\exists s)(\forall x < a)\varphi(x, (s)_x)]$$

(cf. I.2.24).

1.28. Thus the relative strength of the theories in question may be visualized by the following diagram where $T \to S$ means: T is at least as strong as S (cf. I.2.26):

$$\begin{tikzcd}
& & & I\Sigma_3 \arrow[dl] \arrow[dr] & & \\
& & P\Sigma_2 \arrow[dr] & & B\Sigma_3 \arrow[dl] & \\
& & & I\Sigma_2 \arrow[dl] \arrow[dr] & & \\
P\Sigma_0 & \equiv & P\Sigma_1 \arrow[dr] & & B\Sigma_2 \arrow[dl] & \\
& & & I\Sigma_1 \arrow[d] & & \\
& & & B\Sigma_1 \arrow[d] & \equiv & B\Sigma_0 \\
& & & I\Sigma_0 & &
\end{tikzcd}$$

We shall show that the diagram remains valid if the arrow is understood as meaning "is strictly stronger than". In more details:

1.29 Theorem. (1) For $n \geq 1$, $I\Sigma_{n+1}$ is strictly stronger than either of $B\Sigma_{n+1}$ and $P\Sigma_n$; both $B\Sigma_{n+1}$ and $P\Sigma_n$ are strictly stronger than $I\Sigma_n$.

Furthermore, $I\Sigma_1$ is strictly stronger than $B\Sigma_1$ and $B\Sigma_1$ is strictly stronger than $I\Sigma_0$.

(2) For n$\geq$ 1, $P\Sigma_n$ and $B\Sigma_{n+1}$ are incomparable, i.e. neither of them is a subtheory of the other.

(3) For $n \geq 1$, $P\Sigma_n$ & $B\Sigma_{n+1}$ does not imply $I\Sigma_{n+1}$.

1.30. To prove this theorem we define three constructions of submodels of a given non-standard model; all are given by some definable elements. Note that an element $a \in M$ is *Σ_n-definable* if there is a Σ_n-formula $\varphi(x)$ such that a is the unique element of M satisfying φ in M. More generally, a is *Σ_n-definable from* $\mathbf{b}$ in M (where $\mathbf{b}$ is a tuple of elements of M) if there is a Σ_n-formula $\varphi(x,y)$ such that a is the unique $c \in M$ such that $M \models \varphi(c,\mathbf{b})$. If $X \subseteq M$ then a is called *Σ_n-definable from* X in M if it is Σ_n-definable from a tuple of elements of X in M.

Clearly, each standard element of an $M \models I_{\text{open}}$ (or even $M \models Q$) is Σ_0-definable (n is definable by $x = \overline{n}$). Our starting point is the following.

1.31 Observation. There is a model $M \models PA$ containing non-standard Σ_0-definable elements.

Proof. By Gödel's first incompleteness theorem, there is a Π_1-formula $(\forall x)\varphi(x)$ unprovable in *PA*; thus $PA+(\exists x)\neg\varphi(x)$ has a model M. Let a be the least element of M satisfying $\neg\varphi$; then a is defined by $\neg\varphi(x)$ & $(\forall y < x)\varphi(x)$ which is Σ_0. □

1.32 Definition. Let $M \models I\Sigma_1$ and $k \geq 0$.

$$K^k(M) = \{a \in M \mid a \text{ is } \Sigma_k\text{-definable in } M\}\,,$$
$$I^k(M) = \sup_M(K^k(M)) =$$
$$\{a \in M \mid a \text{ is majorized by a } \Sigma_k\text{-definable element of } M\}\,.$$

If $X \subseteq M$ then $K^k(M,X)$, $I^k(M,X)$ are defined as $K^k(M)$, $I^k(M)$ but replacing "definable" by "definable from X".

$H^k(M) = \bigcup_{n=0} H^k_n(M)$ where $H^k_0(M)$ is $I^k(M)$ and $H^k_{n+1}(M)$ is $I^k(M, H^k_n(M))$.

1.33 Theorem. (1) Let $k > 0$ and $M \models I\Sigma_k$. Then
(i) $K^k(M) \prec_k M$, thus $K^k(M) \models Th_{\Pi,k+1}(M)$
(ii) $I^k(M) \prec_{k-1} M$ and $I^k(M) \models Th_{\Pi,k+1}(M)$
(iii) $H^k(M) \prec_k M$, thus $H^k(M) \models Th_{\Pi,k+1}(M)$.

Evidently $I^k(M) \subseteq_e M$ and $H^k(M) \subseteq_e M$.

(2) If $N \neq K^k(M)$ (i.e. if $K^k(M)$ is non-standard) then

(i) $K^k(M)$ is a model of $I\Sigma_{k-1}$ and $\neg B\Sigma_k$; if $k > 1$ then also $K^k(M) \models P\Sigma_{k-1}$.

(ii) $I^k(M)$ is a model of $B\Sigma_k$ and $\neg I\Sigma_k$; if $k > 1$ then also $I^k(M) \models \neg P\Sigma_{k-1}$.

(iii) $H^k(M)$ is a model of $B\Sigma_{k+1}$ and $\neg P\Sigma_k$; thus $H^k(M) \models \neg I\Sigma_{k+1}$.

Remark. Assertion (1) and the positive parts of assertion (2) relativize, i.e. $K^k(M)$ may be replaced by $K^k(M, X)$ etc.; but trivially, the negative parts do not, e.g. if $X = M$ then $K^k(M, X) = M$ and $M \models I\Sigma_k$ by assumption; thus $M \models B\Sigma_k$.

1.34 Corollary. (1) If $M \models PA$ and $K^\infty(M)$ means the substructure of all definable elements then $K^\infty(M) \prec M$.

(2) Theorem 1.29 follows.

(3) *PA* is not finitely axiomatizable, since each finite subtheory of *PA* is a subtheory of some $I\Sigma_k$. (Cf. 3.2.24)

The rest of the subsection contains a proof of Theorem 1.33 and a proof of the fact that Theorem 1.24 in the previous subsection cannot be strengthened.

1.35 Definition (only for this section). A Σ_k-formula $\varphi(x)$ (possibly containing other free variables as parameters) is *special* if $I\Sigma_0 \vdash \varphi(x)\ \&\ \varphi(y) :\rightarrow x = y$, i.e. in each model of $I\Sigma_0$, at most one object satisfies φ.

1.36 Lemma. (1) An element $a \in M$ is Σ_k-definable in $M \models I\Sigma_k$ iff it is defined by a special Σ_k-formula.

(2) More generally, if $X \subseteq M$ is Σ_k-definable and non-empty in $M \models I\Sigma_k$ then there is a special Σ_k-formula defining an element of X.

Proof. Let $\varphi \in \Sigma_k$ define $X \subseteq M$. If $k = 0$ then $\min X$ has a special Σ_0 definition. Assume $k > 0$ and let $\varphi(x)$ be $(\exists y)\psi(x, y)$; let $\chi(x)$ be $(\exists y, z)\psi(x, y)\ \&\ z = \langle x, y\rangle\ \&\ (\forall u < z)(\neg\psi((u)_0, (u)_1))$. Then χ is special and clearly defines an element of X (not necessarily the least element). $\square$

1.37 Lemma. If $M \models I\Sigma_k$ ($k \geq 1$) is non-standard then Σ_k-definable elements are not cofinal (i.e. there is an $a \in M$ such that no $b \in M$, $b >_M a$ is Σ_k-definable). This generalizes to Σ_k-definability from a proper cut.

Proof. Formalize the notion of a special definition in $I\Sigma_1$. In M, let $y = F(x)$ if x is a special$^\bullet$ $\Sigma_k^\bullet$-formula$^\bullet$ and y satisfies x (w.r.t. $Sat_{\Sigma,k}$). Clearly, F is a Σ_k-definable partial function. If a is non-standard then, by $S\Sigma_k$, there is a t such that $(\forall x < a)(x \in dom\, F :\rightarrow F(x) < t)$. If b is Σ_k-defined in M by a special Σ_k-formula φ then $M \models b = F(\overline{\varphi})\ \&\ \overline{\varphi} < a$, thus $M \models b < t$. $\square$

1.38 Convention. In the following lemmas 1.39–1.44 we assume $k \geq 1$, $M \models I\Sigma_k$, $X \subseteq M$.

1.39 Lemma. $K^k(M) \prec_k M$.
(The same for $K^k(M,X)$.)

Proof. Use lemma 1.19: let $M \models (\exists y)\pi(a,y)$, $\pi \in \Pi_{k-1}$, $a \in K^k$, a defined by $\theta \in \Sigma_k$; the least y such that $\pi(a,y)$ is defined in M by

$$(\exists u)(\theta(u) \,\&\, \pi(u,y) \,\&\, (\forall y' < y)\neg\pi(u,y')),$$

the last formula is Σ_k in $I\Sigma_k$ (even in $B\Sigma_{k-1}$) and such a y exists by L_{k-1}. Thus the least y such that $\pi(a,y)$ is in M. □

1.40 Corollary. $K^k(M,X)$ satisfies $I\Sigma_{k-1}$; if $k \geq 2$ then it also satisfies $P\Sigma_{k-1}$.

1.41 Lemma. If $N \neq K^k(M)$ then $K^k(M) \models \neg B\Sigma_k$.

Proof. Assume $K^k \models B\Sigma_k$. In K^k, each element is Σ_k-definable (by any special Σ_k-formula defining this element in M). Thus for $s,t \in K^k - N$, $K^k \models s < t$, we have (after an obvious formalization, see above)

$$K^k \models (\forall x < t)(\exists e < s)(e \text{ is a special } \Sigma_k^\bullet\text{-definition of } x).$$

By $B\Sigma_k$ the last formula is equivalent to a Σ_k-formula and consequently it is true in M. The restriction of the relation "e is a special Σ_k-definition of x" to $s \times t$ is coded (by $I\Sigma_k$) and yields a finite set which is a one-one mapping from $(< s)$ onto $(< t)$, which contradicts the pigeon-hole principle for finite sets, provable in $I\Sigma_1$. This completes the proof. □

1.42 Lemma. $I^k(M,X) \prec_{k-1} M$.

Proof. Trivial for $k=1$ since $I^k(M,X) \subseteq_e M$. Suppose $k \geq 2$, let $\varphi \in \Pi_{k-2}$ and $M \models (\exists x)\varphi(x,a)$ where $a \in I^k$. (Write I^k instead of $I^k(M,X)$.) We find a $d \in K^k$ such that $M \models (\exists x < d)\varphi(x,a)$; and since such an x is necessarily in I^k, the lemma follows by 1.19. Thus let $c \in K^k$ be such that $M \models a < c$; in M, take $d = \max_{y \leq c} \min_x \varphi(x,y)$. In more detail, for $y \leq c$ put $F(y) = \min_x \varphi(x,y)$ if there is such an x. Clearly, F is $\Sigma_0(\Pi_{n-2})$-defined and therefore coded; let $d = \max(\mathit{range}(F))$. Then d is defined as follows:

$$\begin{aligned} x = d \equiv\; & (\exists y \leq c)(\varphi(x,y) \,\&\, (\forall y' < y)\neg\varphi(x,y')) \\ & \&\, (\forall y \leq c)(\forall z)(\varphi(z,y) \rightarrow (\exists z' \leq x)\varphi(z',y)). \end{aligned}$$

Write this as $x = d \equiv \alpha(x, c)$; clearly α is Σ_k (even Π_{k-1}) in M; if $\gamma(y)$ is a Σ_k-definition of c, d is Σ_k-defined by $(\exists y)(\overline{\gamma}(y)\ \&\ \alpha(x, y))$. $\square$

1.43 Lemma. $I^k(M, X) \models Th_{\Pi,k+1}(M)$, i.e. each Π_{k+2} sentence true in M is true in $I^k(M, X)$.

Proof. Let π be Π_{k-1} and let $M \models (\forall x)(\exists y)\pi(x, y)$. Take an arbitrary $a \in K^k$. Then $M \models (\forall x < a)(\exists y)\pi(x, y)$ and, by $B\Sigma_k$, $M \models (\exists t)(\forall x < a)(\exists y < t)$ $\pi(x, y)$. We show that the least such t_0 is in K. Indeed t_0 is $\Sigma_0(\Pi_{k-1})$ in a, thus Σ_k-definable (without parameters) in M, hence $t_0 \in K$. Thus $I^k \models (\forall x < a)(\exists y < t_0)\pi(x, y)$ (thanks to $I^k \prec_{k-1} M$) and since a was an arbitrary element of K we get $I^k \models (\forall x)(\exists y)\pi(x, y)$. $\square$

1.44 Lemma. (1) $I^k(M, X) \models B\Sigma_k$.

(2) If $k \geq 2$ then $I^k(M, X) \models P\Sigma_{k-1}$.

(3) If X is a proper cut in M or if $X = \emptyset$ and M is non-standard then $I^k(M, X) \not\models I\Sigma_k$, thus $I^k(M, X) \neq M$.

Proof. (1) follows by Lemma 1.22 for $I^k \neq M$ and is trivial for $I^k = M$. (2) follows by 1.33 since $P\Sigma_{k-1}$ is Π_{k+1}. (3) follows by 1.37. $\square$

1.45 Lemma. If $k \geq 1$, and $M \models B\Sigma_{k+1}$ then (1) $H^k(M) \models B\Sigma_{k+1}$ and (2) $H^k(M) \prec_k M$.

Proof. (1) follows by Lemma 1.22 for $H^k \neq M$ and is trivial for $H^k = M$ (since $M \models B\Sigma_{k+1}$). (2) is clear since $H^k(M)$ is the sum of the increasing chain of models $K^k(H^k_n(M))$ and each such model is $\prec_k M$. $\square$

1.46 Remark. (1) Observe that the assumption $M \models B\Sigma_{k+1}$ may be replaced by $M \models I\Sigma_k$ and $H^k(M) \neq M$.

(2) Recall the notion "a finite increasing sequence is an approximation of F" (where F is a Σ_k-defined partial function); note that this is a Π_k notion. Thus (1) $I\Sigma_k$ proves that if there is an approximation s of F such that $lh(s) = x$ then there is a least such s; (2) for each n, $I\Sigma_k$ proves that there is an approximation s of F with $lh(s) = n$. (See I.2.21).

1.47 Definition ($I\Sigma_1$). From here on out, let F be the function associating with each pair (z, x) (where z is a special parametrical Σ_k definition and x is a parameter) the only y defined by z from x if there is such a y, and undefined otherwise.

(2) A finite increasing sequence s is said to *capture* x if x is less than or equal to the last member of s.

1.48 Lemma. Let $M \models I\Sigma_k$ and $a \in M$; $a \in H^k(M)$ iff, for some standard n, the least approximation s of F with $lh(s) = n$ captures a.

Proof. Since $H_k(M) \models I\Sigma_k$, for each standard n there is an $s_n \in H_k(M)$ such that, in $H_k(M)$, s_n is the least approximation of F such that $lh(s_n) = n$. Since $H_k(M) \prec_k M$, s_n has the same property in M; this proves that if a is captured by such an s_n then $a \in H^k(M)$.

Conversely, let us prove by induction on i that each element of H_i^k is captured by some s_n. This is clear for $i = 0$; assume the assertion for i and consider $(i+1)$. It suffices to show that an element of $K^k(H_i^k)$, i.e. an element a Σ_k-definable from an element c of H_i^k, is captured. Let $\psi \in \Sigma_k$ and ψ define a from c in M; observe that ψ, being standard, is in H_i^k and hence $(\overline{\psi}, c) \in H_i^k$. By the induction assumption, let s_n capture $\langle\overline{\psi}, c\rangle$; then clearly, s_{n+1} captures a. □

1.49 Corollary. Let $M \models I\Sigma_k$ be non-standard and let $H^k(M) \neq N$. Then $H^k(M)$ is not a model of $P\Sigma_k$. Thus $H^k(M)$ is not a model of $I\Sigma_{k+1}$.

Proof. In $H^k(M)$, the function F defined in 1.37 has no approximation of non-standard lenght. □

1.50 Summary. We check that Theorem 1.33 has been proved. (1) (i) by 1.39, (2) (i) by 1.40 and 1.41; (1) (ii) by 1.42 and 1.43; (2) (ii) by 1.44; (1) (iii) by 1.45, (2) (iii) by 1.45 and 1.49.

*

The last task of the present subsection is to show that the result of 1.24 is optimal.

1.51 Theorem. For each $k \geq 1$ there are $M \subseteq K$ such that $M \models I\Sigma_k$, $K \models I\Sigma_{k+1}$ and M is *not* a $(k+2)$-elementary substructure of $\sup_K(M)$.

Proof. Take a $K \models I\Sigma_{k+1}$ containing non-standard Σ_0-definable elements, and let $M = K^{k+1}(K)$; thus $M \models I\Sigma_k \;\&\; \neg B\Sigma_{k+1}$ (cf. 1.33). Let $I = \sup_K(M) = I^{k+1}(K)$. Note that in M, each element is Σ_{k+1}-definable; in other words, for each non-standard t,

$$M \models (\forall x)(\exists y < t)(y\,\text{special}^\bullet \text{ and } Sat_{\Sigma,k+1}(y,x)).$$

Assuming $M \prec_{k+2} I$, the last formula would be true in I for each non-standard t (since M is cofinal in I), thus in I each element would be Σ_{k+1}-definable. But by 1.33, $I \models B\Sigma_{k+1}$ and the following lemma gives a contradiction. □

1.52 Lemma. If $k \geq 1$ and $M \models B\Sigma_{k+1}$ is non-standard then some elements of M are not Σ_{k+1}-definable. More generally, assume $a \in M$ and $t \in M - N$; then under the above assumption, there are elements $b \in M - N$, such that $M \models b \leq t$ and b is not Σ_{k+1}-definable from a.

Proof. Assume that each $b <_M t$ is Σ_{k+1}-definable from a; thus $M \models (\forall x < t+1)(\exists y < t)(y \text{ special}^\bullet \text{ and } Sat_{\Sigma,k+1}(y,x,a))$.

This can be written as

$$M \models (\forall x < t+1)(\exists y < t)(\exists u)\pi(x,y,u,a)$$

where $\pi \in \Pi_k$. Using $M \models B\Sigma_{k+1}$ we can bound the quantifier $(\exists u)$; thus for some q,

$$M \models (\forall x < t+1)(\exists y < t)(\exists u < q)\pi(x,y,u,a).$$

By $I\Sigma_k$, there is a $d \in M$ coding the relation on $(t+1) \times t$ defined by $(\exists u < q)\pi(x,y,u,a)$. From d we may construct a d' which is, in M, a finite mapping associating to each $x \leq t$ its least special Σ_{k+1}-definition from a. Since $M \models$ (d' is one-one) we get a contradiction with the pigeon hole principle for finite functions (provable in $I\Sigma_1$). □

(e) Elementary End Extensions

1.53 Theorem. Let $M \models I\Sigma_0$ (countable!).

(1) For $k \geq 1$, the following are equivalent:
  (1i) $M \models B\Sigma_{k+1}$
  (1ii) M has a proper $(k+1)$-elementary end extension $K \models I\Sigma_0$
(2) If M has a proper 1-elementary end extension $K \models I\Sigma_0$ then $M \models B\Sigma_2$.
(3) If $M \models I\Sigma_0$ then [M has a proper elementary end extension iff $M \models PA$].

1.54 Remark. The theorem is proved in this subsection. In (1) observe that (1ii) evidently implies that the K in question is a model of $I\Sigma_{k-1}$ (since this theory is Π_{k+1}). We don't know if we may always have $K \models B\Sigma_k$; but if $k \geq 2$ we may have $K \models P\Sigma_{k-1}$.

The reader may show as an exercise that (1ii) *cannot* be strengthened to $K \models I\Sigma_k$ (this would imply $M \models I\Sigma_{k+1}$). But comparing this theorem with 1.22, we may ask if 1.22 can be converted, i.e. if (1i) implies that M has a proper k-elementary end extension to a model of $I\Sigma_k$. The answer is negative (personal communication by R. Kaye).

1.55 (*Proof of* (2)). Assume $M \models I\Sigma_0$, $M \prec_{e,1} K \models I\Sigma_0$; we prove $B\Pi_1$. Let $\pi(x,y) \equiv (\forall z)\varphi(x,y,z)$ where φ is Σ_0 and assume $M \models (\forall x < a)(\exists y)\pi(x,y)$ (we disregard parameters). Let $t \in K - M$; then $K \models (\forall x < a)(\exists y < t)\pi(x,y)$

– here we use $M \prec_1 K$. For each $c \in K - M$, $K \models (\forall x < a)(\exists y < t)(\forall z < c)\varphi(x, y, z)$.

Fix a and c; by underspill, there is a $t \in M$ such that the last formula is true in K. Thus, $M \models (\forall x < a)(\exists y < t)\pi(x, y, z)$. □

1.56 (*Proof of* (1ii) $\Rightarrow$ (1i).) Let $k \geq 1$, $M \models I\Sigma_0$, $M \prec_{e,k+1} K \models I\Sigma_0$. By 1.22 and by induction on k, we know that $M \models B\Sigma_k$ and $K \models I\Sigma_{k-1}$. We prove $M \models B\Pi_k$. Let $\pi(x, y) \equiv (\forall z)\varphi(x, y, z)$ where φ is Σ_{k-1}. Assume $M \models (\forall x < a)(\exists y)\pi(x, y)$. For a $t \in K - M$, we have

$$K \models (\forall x < a)(\exists y < t)(\forall z)\varphi(x, y, z).$$

Thus

$$K \models (\forall x < a)(\forall u)(\exists y < t)(\forall z < u)\varphi(x, y, z),$$
$$K \models (\exists t)(\forall x < a)(\forall u)(\exists y < t)(\forall z < u)\varphi(x, y, z);$$

since $K \models B\Sigma_{k-1}$ the last sentence is Σ_{k+1} and therefore holds in M.

$$M \models (\exists t)(\forall x < a)(\forall u)(\exists y < t)(\forall z < u)\varphi(x, y, z).$$

Since $M \models B\Sigma_k$ we may use $R\Sigma_{k-1}$ (regularity) and get

$$M \models (\exists t)(\forall x < a)(\exists y < t)(Cu)(\forall z < u)\varphi(x, y, z)$$

where (Cu) is "for infinitely many u" (cf. I.2.20); thus

$$M \models (\exists t)(\forall x < a)(\exists y < t)(\forall z)\varphi(x, y, z),$$ □

1.57 (*Proof of* (1i) $\Rightarrow$ (1ii). We generalize and refine the construction of the definable ultrapower described in subsection (b) (1.6–1.12). First, we start with an $M \models B\Sigma_{k+1}$, not just by N. Second, we take algebra D of (parametrically) Δ_k-definable subsets of M and construct an ultrafilter U on D. To be able to prove that in the corresponding model constants form an initial segment (cf. 1.11), U must have the following property: each parametrically Δ_k-definable partition of M into a M-finite number of sets intersects U. In other words, if f is a Δ_k-definable mapping of M into $(< a)$ where $a \in M$ then for some $b <_M a$, $f^{-1}(b) \in U$. Due to countability, we can arrange all such F's (satisfying the "if"-part above) into a sequence $f_0, f_1, \ldots$ and define: $X_0 = M$; and if X_n has been defined and is unbounded in M, put $f' = f_n \restriction X_n$. Since $M \models B\Sigma_{k+1}$, it satisfies $\omega \to \omega'_{<\omega}(\Delta_k, \Delta_k)$ and therefore there is a b such that $f'^{-1}(b)$ is unbounded; we put $X_{n+1} = f'^{-1}(b)$. Then we put, for $A \in D$, $A \in U$ iff $(\exists n)(X_n \subseteq A)$.

Let F be the set of all parametrically Δ_k-definable mappings of M into itself. Define $f =_U y$, $[f]_U$ as in 1.6 and let $K = \{[f]_U \mid f \in F\}$. Define $\leq_K$, $+_K$, $*_K$, etc. as in 1.6. We have the following

Claim 1. Łoś's theorem holds for Σ_k-formulas, i.e. for each such formula $\varphi(x_1,\ldots,x_n)$ and $f_1,\ldots,f_n \in F$, $K \models \varphi([f]_U,\ldots,[f_n]_U)$ iff there is an $X \in U$ such that

$$(\forall a \in X)(M \models \varphi(f_1(a),\ldots,f_n(a)))\,.$$

To prove this we first prove a subclaim: if φ is a Boolean combination of Σ_{k-1}-formulas and f_i are as above then

$$K \models \varphi([f_1]_U,\ldots,[f_n]_U) \text{ iff } \{a \in M \mid M \models \varphi(f_1(a),\ldots,f_n(a))\} \in U\,.$$

The subclaim is proved by induction on the complexity of φ (cf. 1.8–1.9 observing that the sets in question are Δ_k). Then consider $\varphi(x_1,\ldots,x_n) \equiv (\exists y)\psi(x,y)$ where ψ is Π_{k-1}.

First assume $K \models (\exists y)\psi([f_1]_U,\ldots,y)$; thus there is a g such that $K \models \psi([f_1]_U,\ldots,[g]_U)$. By the subclaim, $\{a \in M \mid M \models \varphi(f_1(a),\ldots,g(a))\} \in U$. Conversely, let for each $a \in X \in U$, $M \models (\exists y)\psi(f_1(a),\ldots,y)$. Then define

$$g(a) = \begin{cases} \min_y M \models \psi(f_1(a),\ldots,y) & \text{for } a \in X \\ 0 & \text{for } a \in M - X\,. \end{cases}$$

Clearly, g is Δ_k-defined and total and $K \models \psi([f_1]_U,\ldots,[g]_U)$. This proves Claim 1.

Claim 2. The mapping associating with each $a \in M$ the constant function k_a on M with range $(k_a) = \{a\}$ is a $(k+1)$-elementary embedding of M onto a proper initial segment of K.

The proof of the fact that this is an embedding of M onto a proper initial segment of K is fully analogous to 1.11–12 (due to our careful choice of U). Furthermore, it is immediate from Claim 1 that for each $\varphi \in \Sigma_k$, $M \models \varphi(a,\ldots,b)$ iff $K \models \varphi([k_a]_U,\ldots[k_b]_U)$. Now let $\varphi \in \Sigma_{k+1}$, $\varphi(x_1,\ldots,y)$ and let $K \models \varphi([k_a]_U,\ldots)$, i.e. for some g, $K \models \psi([k_a]_U,\ldots,[g]_U)$. Then for some $c \in M$, $M \models \psi(a,\ldots,g(c))$ since otherwise we would get $K \models \neg\psi([k_a]_U,\ldots,[g]_U)$ from Claim 1. This proves Claim 2.

Since $k \geq 1$ and after the trivial identification of M with its isomorphic image in K, $M \prec_{k+1} K$, we get $K \models I\Sigma_0$ and (1ii) is proved. □

1.58 (*Proof of* 1.53 (3).) The implication $\Rightarrow$ follows by 1.55 (1). Conversely, if $M \models PA$ we may modify the construction of the definable K by investigating all definable subsets and construct U such that it intersects each definable partition of M into M-finitely many subsets. Then evidently $M \prec K$ and thus $K \models PA$. □

(f) A Conservation Result

The main aim of this subsection is to prove the following:

1.59 Theorem. For each $k \geq 0$, $B\Sigma_{k+1}$ is a Π_{k+2}-conservative extension of $I\Sigma_k$.

The theorem is due to Paris and Friedman, independently. We shall present two proofs. The first is simpler (due to Kaye; apparently this was the original unpublished Friedman's proof) and the second brings additional information. The first proof relies on the following:

1.60 Lemma. For each $k \geq 0$, if $I\Sigma_k + \psi(a)$ is consistent, where ψ is Π_{k+1} and a is a constant, then there is a model $M \models I\Sigma_k + \psi(a)$ which has a proper k-elementary cut containing a.

Clearly, the lemma implies the theorem, using 1.22: if $I\Sigma_k + (\exists x)\psi(x)$ is consistent then the cut I above is a model of $B\Sigma_{k+1} + \psi(a)$. The second proof uses the following theorem, which is of independent interest.

1.61 Theorem. For each $k \geq 0$, each model M of $I\Sigma_k$ has a cofinal $(k+1)$-elementary extension K to a model of $B\Sigma_{k+1}$.

This again implies 1.60: if $M \models \psi(a)$ where ψ is as above then $K \models \psi(a)$ since $M \prec_{k+1} K$. We have also the following

1.62 Corollary. For each $k \geq 1$, $(B\Sigma_{k+1} + P\Sigma_k)$ is a Π_{k+2}-conservative extension of $P\Sigma_k$.

Proof. Observe that if M and K are as above and $M \models P\Sigma_k$ then $K \models P\Sigma_k$: if F is a partial function Σ_k-definable in K then its restriction to M (i.e. $F \cap M \times M$) is Σ_k-definable in M; and if $c \in M$ and $M \models$ (s is an approximation of F with $lh(s) = c$) then the same is true in K (since $M \prec_{k+1} K$). This suffices because M is cofinal in K. □

The rest of the subsection elaborates proofs of 1.60 and 1.61.

1.63 (*Proof of* 1.60). First assume $k = 0$ and let $(I\Sigma_0 + \psi(a))$ be consistent where ψ is Π_1. Let c be a new constant; then the theory $T = (I\Sigma_0 + \psi(a) + \{a^n < c \mid n \in N\}$ is consistent (by compactness). Let $M \models T$ and let $I = a^N = \{b \in M \mid (\exists n \in N) M \models (b < a^n)\}$. Then I is a proper cut in M, $I \prec_0 M$, $I \models \psi(a)$ and $I \models B\Sigma_1$ by 1.22.

Now assume $k > 0$ and work in $I\Sigma_k$. Say that y majorizes witnesses of Σ_k-formulas beneath z with parameters beneath u (in symbols: $MWitn(y, z, u)$) if the following holds for each Σ_k formula $x \le z$ of the form $(\exists v_0)\pi(v_0, \dots, v_t)$ and each t-tuple $s < u$: if there is a w such that $Sat_{\Pi,k-1}(\pi, \langle w\rangle \frown s)$(a witness for the fact that s satisfies x) then such a w exists beneath y. Define $E(z, u)$ be the minimal y such that $MWitn(y, z, u)$. Observe that E is $\Sigma_0(\Sigma_k)$ and total (due to $S\Sigma_k$).

Now take the following theory T:

$$I\Sigma_k, \quad \psi(a_0), \quad \{a_{n+1} = E(n, a_n) \mid n \in N\}, \quad \{c > a_n \mid n \in N\}.$$

Clearly, T is consistent; let $M \models T$ and $I = \sup_n a_n = \{b \in M \mid (\exists n)(M \models b < a_n)\}$. Then $a \in I$, $I \neq M$, $I \prec_k M$ as desired. □

1.64 Remark. The reader may prove the following fact as an exercise: In the above proof, $I = H^k(M, (\le a_0))$. The rest of the subsection contains a proof of Theorem 1.61. (The proof differs from the original proof by Paris.)

1.65 (Construction). Recall that each theory T has a conservative extension $S = Sk(T)$ having the following *strong Skolem property:* for each formula $\varphi(y, \mathbf{x})$ of S, there is a function symbol F_φ in S such that $S \models (\exists y)\varphi(\mathbf{x}, y) :\rightarrow \varphi(\mathbf{x}, F_\varphi(\mathbf{x}))$. Moreover, each model $M \models T$ has an expansion to a model of S. (see 0.15.)

Let $M \models I\Sigma_k$ and let M^* be the expansion of M to a model of $Sk(I\Sigma_k)$. Let B be the Boolean algebra of all subsets of M definable in the language of $Sk(I\Sigma_k)$ (B is countable). Let U be a non-trivial ultrafilter on B containing all upper segments $(> b) = \{a \in M \mid M \models a > b\}$. Let K^* be the definable ultrapower of M^* given by U and let K be the reduct of K^* to the language of arithmetic. Identifying each $a \in M$ with $\overline{a} = [k_a]$ where $k_a : M \to \{a\}$ we get $M \subseteq K$. Finally, let $I = \sup_K(M) = \{[f] \in K \mid (\exists a \in M)(K \models [f] \le \overline{a})\}$. We show that I is the desired extension, i.e. $M \subseteq_{cf} I$, $M \prec_{k+1} I$, $I \models B\Sigma_{k+1}$.

1.66 Fact. (1) $M \prec K$ and M is not cofinal in K.
(2) $M \prec_{k+1} I$.
(3) $I \prec_k K$.

Proof. (1) $M \prec K$ follows from the fact that Łoś's theorem holds for *all* formulas of S:

$$K^* \models \varphi([f_1], \dots, [f_n]) \text{ iff } \{a \in M \mid M^* \models \varphi(f_1(a), \dots, f_n(a)\} \in U.$$

(The verification is left to the reader as an easy exercise.) In particular, for $a_1, \dots, a_n \in M$,

$$K^* \models \varphi(\overline{a}_1, \dots, \overline{a}_n) \text{ iff } M^* \models \varphi(a_1, \dots, a_n),$$

which means, after the identification of M with the isomorphic substructure of K^*, $M^* \prec K^*$. Consequently, $M \prec K$. M is not cofinal in K since the diagonal d defined by $d(a) = a$ satisfies $K \models d > \overline{a}$ for each $a \in M$ (by the choice of U).

(2) For $k \geq 1$, the assertion follows by 1.24. For $k = 0$ argue as follows: Let $\pi(x)$ be Π_1 and let $M \models \pi(a)$. Then $K \models \pi(a)$ (by $M \prec K$), hence $I \models \pi(a)$ (since $I \subseteq_e K$).

(3) The claim is trivial for $k = 0$. For $k > 0$ use the function $E(u, z)$ from 1.63 ($E(u, z)$ is the least upper bound for a witness of Σ_k-formulas $\leq u$ with parameters $\leq z$). Let $\mathbf{b} \in I$ and $K \models (\exists u)\varphi(u, \mathbf{b})$ where $\varphi \in \Pi_{k-1}$. Let $m \in N$ such that $\varphi < m$ (φ is standard). Furthermore, let $a \in M$ be bigger than $\mathbf{b}$; in M, let $e = E(m, a)$. Since $M \prec K$ we have $K \models e = E(m, a)$, i.e. $K \models (\exists u < e)\varphi(u, \mathbf{b})$. This means that there is a $c \in I$ such that $K \models \varphi(c, \mathbf{b})$. By 1.19, this implies $I \prec_k K$. □

1.67 Corollary. $I \models B\Sigma_{k+1}$, thus I is the desired extension of M. This follows by 1.22 since $I \neq K$ by 1.66 (1) (M is not cofinal in K). This completes the proof of 1.61.

2. Cuts in Models of Arithmetic with a Top

Cuts in models of Peano arithmetic have played an important role in the development of model theory of arithmetic: Paris and Kirby did pioneering work that continued by the indicator theory and a model-theoretic proof of the independence of the Paris-Harrington principle. Later Paris developed the theory of cuts in models of $I\Sigma_1$; this is the topic of our main interest. Important in Paris's work are cuts in models of arithmetic with top (TA'): they present a powerful device for the study of cuts in models of $I\Sigma_1$, which in turn, are very useful in the study of provably recursive functions. This and the next section are devoted to this.

(a) Arithmetic with a Top and Its Models

2.1. Recall the theory BA' – arithmetic without function symbols, introduced in Chap. I, Sect. 2 (e). Its language L' consists of the constant $\overline{0}$, the predicates $=$, $\leq$, S (binary), A, M (ternary) having the usual meaning; axioms assure that $\leq$ is a discrete linear order with the least element $\overline{0}$, S, A, M define partial functions having natural properties of successor, addition and multiplication and, in addition, axioms contain the induction schema for Σ'-formulas (bounded formulas of L').

TA' (arithmetic with top) results by adding the axiom "there is a largerst element" and $I\Sigma'_0$ results by adding "successor, addition and multiplication are total functions". The only reason for the present short subsection is to prove the following

2.2 Theorem. (1) If $M \models I\Sigma_0'$ and $a \in M$ then the substructure $M \upharpoonright a$ of M with the domain $(< a)$ is a model of TA'.

(2) Conversely, if $B \models TA'$ then there is a model $K \models I\Sigma_0'$ and an $a \in M$ such that B is $M \upharpoonright a$.

2.3 Corollary. $I\Sigma'$ and TA' prove the same Π_1'-sentences. Thus both theories extend BA' Π_1'-conservatively.

2.4 *Proof of* 2.2 – *beginning.* (1) is trivial; we prove (2). For one-element B the assertion is evident; so we assume that B has at least two elements. Work with elements of B as with generalized digits and consider finite sequences s of elements of B whose last element is not 0. (The length of s is an element of N.) This is the domain of our K. It is ordered lexicographically; $s \leq t$ iff $lh(s) < lh(t)$ or $[lh(s) = lh(t)$ and if i is minimal such that $(s)_i \neq (t)_i$ then $B \models (s)_i < (t)_i]$. This defines the meaning of S. A and M (addition and multiplication) are defined by algorithms of school mathematics, i.e. $K \models A(s,t,w)$ iff there are sequences q (of carries) and u (sum) of length $\max(lh(s), lh(t)) + 1$ satisfying "bitwise" the obvious conditions for addition (q consists only of zeros and ones and several cases are distinguished, e.g., for each $0 < i \leq \max(lh(s), lh(t))$,

$$\begin{aligned}
&(q)_{i-1} = 0 \,\&\, A((s)_i, (t)_i, (u)_i) \,\&\, (q)_i = 0 \ \mathit{or} \\
&(q)_{i-1} = 0 \,\&\, \neg(\exists z)A((s), (t), z) \\
&\qquad\qquad \&\, (\forall x)(x \text{ minimal such that } \neg(\exists z)A((s)_i, x, z) \rightarrow A(x, (u)_i, (t)_i)) \\
&\qquad\qquad \&\, (q)_i = 1 \ \mathit{or} \\
&\quad (q)_i = 1 \& \ldots .
\end{aligned}$$

and w results from u by omitting the last element if it is 0, otherwise $w = u$. Conditions for multiplication are clumsier (use $lh(t)$ auxiliary sequences of length $lh(s) + lh(t)$) but are easy to describe. The proof is completed by proving the following lemmas:

2.5 Lemma. Let B_n be the substructure of K consisting of all sequences of length at most n ($n \geq 0$). Then $B_n \subseteq_e K$ and B_n is definable in B using n-tuples of objects of B to code objects of B_n. (Thus we have a formula with $2n$ free variables defining the successor for n-tuples etc.)

Proof by formalizing the above definitions – for a fixed n. $\square$

2.6 Lemma. For each n, B_n is a model of TA'.

Proof. The verification of properties of $\leq$, S, A, M is tedious but straightforward. To verify the least number principle use the fact that by 2.5, each L'-formula with k free variables speaking about B_n can be equivalently replaced by an L'-formula with kn free variables speaking about B. Then use the lexicographic ordering of n copies of the universe inside B. □

2.7 Lemma. $K \models I\Sigma'$, $B_1 \subseteq_e K$ and B_1 is isomorphic to B.

Proof. Since $K = \bigcup B_n$ is a union of an increasing chain of initial segments, basic properties of S, A, M, $\leq$ are verified easily. The least number principle in K reduces to the least number principle in a suitable B_n. Clearly, K has no greatest element. This completes the proof of Theorem 2.2. □

(b) Cuts

2.8 Definition. (1) *Standard models* of BA' are N and non-empty initial segments of N (finite standard models). All other models of BA' are *non-standard:* by the previous, these are just non-standard models of $I\Sigma'$ and their segments given by non-standard elements.

(2) Let $M \models BA'$ and let $I \subseteq M$. I is a *cut* in M if it is a non-empty initial segment without a largest element in the ordering $\leq_M$. I is *closed under the operations* if, for each $a, b \in I$, there are $c, d \in I$ such that $M \models A(a, b, c)$ and $M \models M(a, b, d)$ (addition and multipliation are total on I).

2.9 Remark. (1) Clearly, a $M \models TA'$ is non-standard iff M has a proper cut. A cut $I \subseteq M$ is closed under operations iff $I \models I\Sigma'$.

(2) Recall the Σ_0 predicate $exp(x, y)$ defining exponentiation in $I\Sigma_0$ as a partial operation. We shall use the same notation for an equivalent Σ'-predicate.

2.10 Definition. A cut $I \models M$ is *short* if I is closed under operations and there is an $a \in M - I$ such that 2^{2^a} exists in M, i.e.

$$M \models (\exists b, c)(exp(2, a, b) \,\&\, exp(2, b, c))$$

2.11 Remark. (1) Observe that if $M \models I\Sigma_1$ (or even if $M \models I\Sigma_0 + (exp)$) then each proper cut is short.

(2) Recall that in $I\Sigma_0$ we have a Σ_0 predicate $bit(x, z, y)$ saying "the z-th bit in the binary expansion of x is z". Informally, we shall use bit as a symbol for the corresponding partial function.

2.12 Lemma. For each Σ_0-formula $\varphi(x, \mathbf{y})$ BA' proves the following: For each y and each z such that 2^z exists, there is a $w < 2^z$ such that, for each $x \leq z$

$$bit(w, x) \equiv \varphi(x, \mathbf{y})$$

Proof. Take a z_0 such that 2^{z_0} exists and prove the assertion by induction for $z \leq z_0$ (observing that it is Σ_0): assume that for z we have the respective $w \leq 2^{z_0}$ and that $z + 1 \leq z_0$ and consider $z + 1$. If $\neg\varphi(z + 1, y)$ then take $w' = w$; otherwise take $w' = w + 2^{z+1}$. □

2.13 Definition. (1) Let I be a short cut in $M \models BA'$, let $A \subseteq I$ and $a \in M - I$. *a codes* A in M if for each $x \in I$,

$$x \in A \text{ iff } bit(x, a) = 1 \, .$$

(2) $SS_I(M)$ is the set of all $A \subseteq I$ coded in M.

(3) a codes an $R \subseteq I^k$ iff it codes the set A of all $(a_1, \ldots, a_k)$ such that $R(a_1, \ldots, a_k)$. $((\ldots)$ is the definable k-tupling function; recall that I is closed under it since $I \models I\Sigma_0)$.

2.14 Theorem. Let I be a short cut $M \models BA'$. Then

(1) $SS_I(M)$ is a Boolean subalgebra of the algebra of all subsets of I.
(2) If $A \in SS_I(M)$ and B is $\Delta'(A)$ in I then $B \in SS_I(M)$.

Proof. In fact, we only need an a such that $I < a < 2^a \in M$; clearly, each $x \in SS_I(M)$ has a code $< 2^a$.

It follows immediately by 2.12 that $SS_I(M)$ is closed under Boolean operations and that if $X \in SS_I(M)$ (coded by b) and Y is $\Sigma'(X)$ (in I or in M, which is the same) then $Y \in SS_I(M)$. It remains to assume Y to be both $\Sigma'(X)$ and $\Pi'(X)$ (both in I), $x \in SS_I(M)$. Let $y \in Y \equiv I \models (\exists u)P(u, y) \equiv I \models (\forall v)Q(v, y)$ where $P, Q \in SS_I(M)$. Use the "Rosser device": put $\varphi(y) \equiv (\exists u)P(u, y) \preccurlyeq (\exists v)\neg Q(v, y)$, i.e. $\varphi(y) \equiv (\exists u)[P(u, y) \ \& \ (\forall v < u)Q(v, y)]$. Clearly, $y \in Y$ iff $I \models \varphi(y)$ iff $M \models \varphi(y)$. Thus take a c such that, for all $y < a$,

$$M \models bit(y, c) = 1 \equiv (\exists u \leq a)(P(u, y) \,\&\, (\forall v < u)Q(v, y))\, .$$

This c codes Y. □

2.15 Definition. (1) A short cut $I \in M$ is *semiregular* if I satisfies strong collection with respect to coded functions, i.e. if $F \in SS_I(M)$ is a function and $a \in I$ then $F''(\leq a)$ is not cofinal in I.

(2) I is a model of $I\Sigma_n^*$ (as a cut in M) if it satisfies induction for all formulas that are $\Sigma_n(X)$ with X ranging over elements of $SS_I(M)$. Similarly for $B\Sigma_n^*$.

2.16 Theorem. Let $M \models BA'$ and let $I \subseteq M$ be a short cut. I is semiregular iff $I \models I\Sigma_1^*$.

Proof. The implication $\Leftarrow$ is obvious since $I\Sigma_1(X)$ proves $S\Sigma_1(X)$. (Recall that $I\Sigma_1(X)$ has a new class variable X and induction for all $\Sigma_1(X)$ formulas, cf. I.2.54) For the converse observe that, by 2.14 (2), we see that I (which is clearly a model of $I\Sigma_0^*$) satisfies $S\Pi_0^*$, therefore satisfies $S\Sigma_1^*$ and hence $I\Sigma_1^*$. (See Chap. I, Sect. 2). □

(c) Extendable, Restrainable and Ramsey Cuts

2.17 Definition. (1) Make the following definition in $I\Sigma_1$: X is 1-*unbounded* if it is unbounded, i.e. $(\forall x)(\exists y > x)(y \in X)$.

A set Y of $(n+1)$-tuples is $(n+1)$-*unbounded* if the set

$$\{x_0 \mid \{\langle x_1, \ldots, x_n\rangle \mid \langle x_0, x_1, \ldots, x_n\rangle \in Y\} \text{ is } n\text{-unbounded } \}$$

is 1-unbounded.

(2) Similarly, let $M \models BA'$, let I be a short cut in M and let $X \subseteq M^{n+1}$ be coded. We call X $(n+1)$-*unbounded in* I if the set

$$\{x_0 \mid \{\langle x_1, \ldots, x_n\rangle \mid \langle x_0, \ldots, x_n\rangle \in Y\} \text{ is } n\text{-unbounded in } I\}$$

is unbounded in I.

Remark. (1) One can prove that if $k+1 = m+n$ and $m, n \geq 1$ then Y is k-unbounded iff

$$\{\langle x_0, \ldots x_{m-1}\rangle \mid \{\langle x_m, \ldots, x_k\rangle \mid \langle x_0, \ldots, x_k\rangle \in Y\} \text{ is } n\text{-unbounded}\}$$

is m-unbounded.

(2) The formula saying that the set of all $x_1, \ldots, x_n$ satisfying $\varphi(x_1, \ldots, x_n)$ is n-unbounded is equivalent to $(Cx_1)\ldots(Cx_n)\varphi(x_1, \ldots, x_n)$.

2.18 Lemma. For $m, n \geq 1$, let $Rams(n,m)$ be the formula $(\forall F \in \Delta_m)((F : V^n \to a) \to (\exists i < a)F^{-1}(i)$ is n-unbounded). (This makes sense in $I\Sigma_1$.) In each $y, w > 1$, $I\Sigma_1$ proves

$$B\Sigma_{n+m} \equiv Rams(n,m).$$

Proof. Similarly as in II.1.26, show in $I\Sigma_m$ that $Rams(n,m)$ implies $Rams(n-1, m+1)$ observing that if $F : V^{n-1} \to a$ is Δ_{m+n}, $F = \lim G$, $G : V^n \to a$, $G \in \Delta_m$, $\{(x, \mathbf{y}) \mid G(x, \mathbf{y}) = b\}$ is n-unbounded then

$$\begin{aligned}\{\mathbf{y} \mid F \mid \mathbf{y}) = b\} &= \{\mathbf{y} \mid \lim_x G(x, \mathbf{y}) = b\}\\ &= \{\mathbf{y} \mid \{x \mid G(x, \mathbf{y}) = b\} \text{ is unbounded})\end{aligned}$$

is $(n-1)$-unbounded (cf. the remark in 2.17). Then

$$(\forall m) I\Sigma_1 \vdash Rams(n,m) \rightarrow B\Sigma_{n+m}$$

by induction on n: for $n = 1$ see I.1.23 and for $n > 1$ II.1.27.

($\Rightarrow$) Assume $B\Sigma_{n+m}$ and let $F \in \Delta_m$, $F : V^n \rightarrow a$, i.e. F acts on all ordered n-tuples of numbers. Restrict F to increasing n-tuples; you get an $F' : [V]^n \rightarrow a$ and, by II.1.6, there is an unbounded A homogeneous for F'. Show that $[A]^n$ is n-unbounded; clearly F is constant on $[A]^n$. $\square$

2.19 Definition. (1) Let $M \models BA'$, $k \geq 1$ and let $I \subseteq M$ be a short cut. (1) I is *k-Ramsey* in M if for each $a \in I$ and coded $F : I^k \rightarrow a$ there is a $b < a$ such that $F^{-1}(b)$ is k-unbounded in I^k.

(2) Let $I \models I\Sigma'$. A sequence $I \subseteq K_0 \subseteq K_1 \ldots \subseteq K_k$ is a *k-extending chain* over I (notation: $K_0 \prec_I K_1 \prec_I \ldots \prec_I K_k$) if

(i) $K_i \models BA'$ for each $i \leq k$,
(ii) K_i is a Σ'-elementary substructure of K_{i+1} for each $i < k$ (notation $K_i \prec_0 K_{i+1}$),
(iii) I is a short cut in K_i for each $i \leq k$,
(iv) for each $i < k$, there is a $b_{i+1} \in K_{i+1}$ such that $I < b_{i+1} < (K_i - I)$. (Thus, when going from K_i to K_{i+1} we get no new elements below elements of I but we do get new elements between elements of I and elements of $K_i - I$.)

(3) Let I be a short cut in a model $M \models BA'$. I is *k-extendable* over M if there is a k-extending chain over I whose 0-th element is M; I is *k-restrainable* in M if there is a k-extending chain over I whose last element is M.

2.19 Remark. Note that 1-Ramsey cuts are also called *regular*. The notion of a k-extendable and k-restrainable cut is the key notion in Paris's model-theoretic investigation of the strength of instances of combinatorial problems. Observe that if $K_0, \ldots, K_n$ is a k-extending chain over I, $a \in K_0 - I$ and 2^{2^a} exists in K_0 then for $d = 2^{2^a}$ $(K_0 \upharpoonright d), \ldots, (K_n \upharpoonright d)$ is a k-extending chain over I which is formed by models of TA' with the same top.

2.20 Theorem. Let $I \subseteq M$ be a short 1-Ramsey cut; then I is semiregular.

Proof. Assume I not semiregular and let H be a coded mapping from $(\leq a)$ into I cofinal in I. H may be partial but we may assume it to be one-one (if not, replace it by $H'(b) = \langle b, H(b) \rangle$, clearly H' is coded and cofinal). Now let, for each $x \in I$, $F(x)$ be the pre-image of the maximal $y \leq x$, $y \in range(H)$. This is $\Delta_1(H)$, thus F is in $SS_I(M)$; $F : I \rightarrow a$ and clearly, each $F^{-1}(b)$ is bounded in I. $\square$

2.21 Theorem. A short cut $I \subseteq M \models BA'$ is k-Ramsey in M iff $I \models B\Sigma^*_{k+1}$ in M (for each $k \geq 1$).

Proof. Since both conditions imply $I \models I\Sigma^*_1$ the theorem follows from 2.18. □

2.22 Theorem. If a short cut $I \subseteq M \models BA'$ is k-extendable over M then it is k-Ramsey.

Proof. By induction on k.

(1) $k = 1$. Let M, K be a 1-extending sequence over I. We prove $I \models B\Sigma^*_2$. Assume $I \models (\forall x < a)(\exists y)(\forall z)\varphi(x, y, z)$ where φ is $\Sigma_0(X)$ for some coded X. Thus

$$(\forall x < a)(\exists y \in I)(\forall z \in I)I \models \varphi(x, y, z).$$

Here, $I \models$ may be replaced by $M \models$ and therefore, by overspill,

$$(\forall x < a)(\exists y \in I)(\exists z_0 \in M - I)M \models (\forall z < z_0)\varphi(x, y, z).$$

Since $M \prec_0 K$, $M \models$ can be replaced by $K \models$. Thus, for $d \in K$ such that $I < d < M - I$, we have

$$(\forall x < a)(\exists y \in I)K \models (\forall z < d)\varphi(x, y, z).$$

Now let e be minimal in K such that

$$(\forall x < a)(\exists y < e)K \models (\forall z < d)\varphi(x, y, z).$$

Then $e \in I$ (since otherwise $e - 1$ could replace e) and thus

$$I \models (\forall x < a)(\exists y < e)(\forall z)\varphi(x, y, z).$$

This shows $I \models B\Sigma^*_2$ and thus I is 1-Ramsey in M.

(2) Now assume $K_0 \prec_I K_1 \prec_I \cdots \prec_I K_{k+1}$ and let $F : I^{k+1} \to q \in I$ be coded by $f < 2^a$. Thus,

$$(\forall z \in I)K_0 \models (\forall x_0 \ldots x_k \leq z)(\exists! u < q)(\langle x_0 \ldots x_k, u\rangle \in f).$$

By overspill, there is a $d \in K_0$, $I < d_0$, such that

$$K_0 \models (\forall x_0, \ldots, x_k \leq d)(\exists! u < q)(\langle x_0, \ldots, x_k, u\rangle \in f).$$

Define, in K_1, h as the function such that, for $\mathbf{x} = x_0, \ldots, x_{k-1}$,

$$K_1 \models (\forall x_0, \ldots, x_{k-1} \leq d)(h(\mathbf{x}) = f(\mathbf{x}, d)).$$

This h codes a $H : I^k \to q$ and by the induction assumption applied to $K_1, \ldots, K_{k+1}$, there is a $b < q$ such that $H^{-1}(b)$ is k-unbounded in I^k. We show that $F^{-1}(b)$ is $(k+1)$-unbounded in I^{k+1}. For this it suffices to show

$$\{\mathbf{x} \mid H(\mathbf{x}) = b\} \subseteq \{\mathbf{x} \mid \{y \mid F(\mathbf{x}, y) = b\} \text{ is unbounded}\}.$$

Thus assume $\mathbf{x} \in I$, $f(\mathbf{x}, d) = b$ and let e be an arbitrarily large element of I. Let $p = \min(f(\mathbf{x}, y) = b$ and $y \geq e)$ (computed in K_1). By $K_0 \prec_I K_1$, this p must be in K_0 and since $K_1 \models p \leq d$ we have $p \in I$. Thus $\{y \mid F(\mathbf{x}, y) = k\}$ is unbounded in I. This completes the proof. □

2.23 Definition. Let $M \models BA'$, let $I \subseteq M$ be a short cut, let $c \in I$. I is *regular (1-Ramsey) up to* c in M if for each $a \leq c$ and each coded $F : I \to a$ there is an $i < a$ such that $F^{-1}(i)$ is unbounded in I.

I is *semiregular up to* c in M if for each function $F \in SS_I(M)$ and each $a \leq c$, the F-image of $(\leq a)$ is not cofinal in I.

Remark. Compare these definitions with 2.19 and 2.13. It is easy to check that Theorem 2.20 generalizes: if I is regular up to c then it is semiregular up to c.

2.24 Theorem. Let $K_0 \prec_I \cdots \prec_I K_k$, $a, b \in K_0$, $a < I < b$, $c \in I$, I semiregular up to c in K_k. Then $K_0 \models [a, b] \underset{*}{\to} (k+2)^{k+1}_c$.

This theorem will be used in subsection (e). First we prove a lemma.

2.25 Lemma. Let $K_0 \prec_I K_1$, $K_i \models TA'$, $a, b \in K_0$, $a < I < b$, $f \in K_0$, $K_0 \models f : [a, b]^{k+1} \to c$, $c \in I$. Let $C \subseteq I$ be coded in K_0 and o.t.u. in I, i.e. for each $q \in I$ there is a sequence $s \in I$ coding the first q elements of C. Then there is a $B \subseteq C$, B coded in K_1, o.t.u. in I and prehomogenous for f, i.e. for arguments from $[B]^{k+1}$, the value of f does not depend on the last coordinate.

Proof. Note that $K_1 \models f : [a, b]^{k+1} \to c$ and $I \models I\Sigma_1$. (See 2.22, 2.21: $K_0 \prec_I K_1$ implies that I is short, by 2.22 I is 1-Ramsey in K_0 and by 2.21 $I \models B\Sigma_2$.) Let $\pi_0 \in K_1$, $I < \pi_0 < K_0 - I$. Let e code C in K_0. We show that there is a $\pi \in K_1$ such that $I < \pi < K_0 - I$ and $K_1 \models \pi \in e$. Indeed, for each $x \in I$, $K_1 \models (\exists y \in e)(y < \pi_0 \ \& \ y > x)$, thus, by overspill, there is an $x \in K_1 - I$ such that $K_1 \models (\exists y \in e)(y < \pi_0 \ \& \ y > x)$. Let π be such an y.

If $\beta = \langle \beta_0, \ldots, \beta_q \rangle \in K_1$ is an increasing sequence of length $> I$ we call β o.t.u. in I if, for each $r \in I$, $\beta_r \in I$. (Then clearly the set of all elements of the sequence belonging to I is o.t.u. in I.)

(1) In K_1, define an increasing sequence $\beta = \langle \beta_0, \ldots, \beta_q \rangle$ to be a *designated sequence* (d.s.) if $q \geq k$, $\beta_0, \ldots, \beta_k$ are the first k elements of e bigger than a

and, for $k < p < q$,

$$K_1 \models \beta_{p+1} = \min_{z \in e}(\forall \mathbf{s} \in [\beta_0, \ldots, \beta_p]^k)(f(\mathbf{s}, z) = f(\mathbf{s}, \pi)).$$

(2) We want to show that in K_1, there is a d.s. β of the length $q > I$ which is o.t.u. in I. This will prove the lemma: B will be the set of elements of β belonging to I.

(3) It follows easily that, in K_1, for each q there is at most one d.s. of the length q.

(4) Since being a d.s. is Σ', take in K_1 the *maximal* d.s. $\beta < \pi$; each other d.s. is an initial segment of β. We show that it has the desired properties.

(5) We show that each d.s. β such that $\beta \in I$ has a proper prolongation which is a member of I. Let $\beta = \langle \beta_0, \ldots, \beta_p \rangle \in I$; let $\beta_{p+1} = \min_{z \in e}(K_1 \models (\forall \mathbf{s} \in [\beta_0, \ldots, \beta_p]^k)(f(\mathbf{s}, z) = f(\mathbf{s}, \pi))$. We show that β_{p+1} is in I.

For $\mathbf{s} \in [\beta_0, \ldots, \beta_p]^k$, let $g(\mathbf{s}) = f(\mathbf{s}, \pi)$; observe that $g \subseteq [\beta_0, \ldots, \beta_p]^k \times (< c)$, thus $g \in I$ and hence $g \in K_0$. For each $d \in K_0$, $d > I$, we get

$$K_1 \models (\exists y < d)(\forall \mathbf{s} \in [\beta_0, \ldots, \beta_p]^k)(f(\mathbf{s}, y) = g(\mathbf{s})),$$

and hence $K_0 \models$ the same. Let u be the smallest witness of the last formula in K_0; then u is the smallest witness of it in K_1, therefore $K_1 \models u < \pi$. But then $u \in I$ and thus $\beta \frown \langle u \rangle$ is the desired prolongation of β in I.

(6) Now we show that if $\beta \in K_1$ is a d.s. and $lh(\beta) \in I$ then $\beta \in I$. Indeed, let $\langle \beta_0, \ldots, \beta_q \rangle$ be a d.s. in K_1 and let $q \in I$. For $x_1 < \cdots < x_k \leq q$ let $h(x_1, \ldots, x_k) = f(\beta_{x_1}, \ldots, \beta_{x_k}, \pi)$. Then $h : [q]^k \to c$ in K_1, thus $h \in I$ (recall again $I \models I\Sigma_1$).

Since I satisfies even Σ_1-induction with classes coded in K_0 (due to $K_0 \prec_I K_1$) we may define designated sequences inside I using C and $F = f \restriction [I]^k$. Let $DS'(\beta', a, h)$ be a formula such that $I \models DS'(\beta', s, h)$ iff, in I_1, $\beta' = \langle \beta'_1, \ldots, \beta'_k \rangle$ is a sequence, $\beta'_1, \ldots, \beta'_k$ are the first k elements of C bigger than a and, for $k < r < p$, β'_{r+1} is the least z such that

$$(\forall s_1 < \cdots < s_k < r)(F(\beta'_1, \ldots, \beta'_k) = h(s_1, \ldots, s_k)).$$

Clearly, if $p \leq q$ and $\beta' = \langle \beta_0, \ldots, \beta' \rangle \in I$ then $I \models DS'(\beta', a, h)$ iff $K_1 \models (\beta'$ is a d.s.).

Now it follows by induction in I that for each $p \leq q$ there is a β' such that $DS'(\beta', s, h)$ ((5) giving the induction step); thus the original d.s. $\beta = \langle \beta_0, \ldots, \beta_q \rangle$ must be in I.

(7) Consequently, the maximal d.s. β^* cannot be in I, $lh(\beta^*) \notin I$ and each initial segment of β^* of length $q \in I$ is in I; this means that β is o.t.u. in I. The set B of all β_q such that $q \in I$ is the desired prehomogenous set.

This completes the proof of the lemma. Now we prove Theorem 2.24. Let $K_0 \prec_I \cdots \prec_I K_k$, $a < I < b \in K_0$, $c \in I$. By iterating the construction of the lemma k times we get an $A \subseteq C$, o.t.u. in I and such that, for $s \in [A]^{k+1}$, the value $f(s)$ depends only on the first coordinate of s. Let g be the corresponding unary function and let h be the function associating with each $i < c$ the first $x \geq a$ such that $g(x) = i$. Since I is semiregular up to c, $h''(\leq c)$ is bounded in I; let $a_0 \in I$ be an upper bound and put $p = ca_0 + 1$ (recall that $I \models I\Sigma_1$). Let s be the sequence of the p first elements of A; clearly, $s \in I$. Thus there is a $j < c$ such that $g^{-1}(i)$ is a finite set having at least a_0 elements and, by the choice of a_0, the minimal element of $g^{-1}(i)$ is $\leq a_0$. Thus $g^{-1}(i)$ is relatively large and homogenous for f. □

(d) Satisfaction in Finite Structures with an Application to Models of $I\Sigma_1$

We shall now formalize our notion of standard models of arithmetic with a top. We gain a notion used in various places in the later text. In the present section we use this notion to prove that each non-standard model of $I\Sigma_1$ is isomorphic to a proper cut of itself.

Recall that in Chap. I, Sect. 1–2 we formalized various sorts of partial satisfaction in the universe: for each k we defined in $I\Sigma_1$ formulas $Sat_{\Sigma,k}$ and $Sat_{\Pi,k}$ (satisfaction for Σ_k and Π_k formulas) and showed that in $B\Sigma_m$ ($m \geq 2$) we have a well-behaving satisfaction for $\Sigma_k(\Delta_m)$-formulas. Moreover, we showed that in $I\Sigma_1$ we have a well-behaving satisfaction for $\Sigma_k(\Sigma_1)$ formulas. All this was satisfaction in the universe. (Using self-reference one can show that there is no full satisfaction in the universe – this is routine, see III.2.3.)

Now we shall focus our attention to finite structures of the form $[O, x]$ endowed with the standard structure given by the language L' (successor, addition, multiplication as predicates). In $I\Sigma_1$ we show that full satisfaction for such structures exists and is Δ_1:

2.26 Theorem. ($I\Sigma_1$) There is a Δ_1 relation $x \models z[e]$ (ternary) such that for each x, each L'-formula z and each evaluation e of free variables of z by elements of $[0, x]$, Tarski's conditions for satisfaction in $[0, x]$ hold true.

The *proof* is more or less evident: take x as a parameter and define a notion of partial satisfaction in $[0, x]$ for formulas $\leq z$ as a finite set mapping pairs (z', e'), where $z' \leq z$ is a L'-formula and e' an evaluation as above into $[0, 1]$ such that Tarski's conditions are true; by induction on z show that for each z there is a unique partial satisfaction on $[0, x]$ for formulas $\leq z$ and if q_i is such a satisfaction for z_i ($i = 1, 2$) and $z_1 \leq z_2$ then q_1 is a restriction of q_2. Then $x \models z[e]$ means that the unique partial satisfaction for $[0, x]$ and formulas $\leq z$ gives the value 1 to the pair (z, e). □

2.27. We now define an important class of formulas generalizing Σ'-formulas. $\Sigma^G_{1,0}$ formulas are open L' formulas in conjunctive normal form; $\Sigma^G_{1,2k+1}$ formulas have the form $(\exists x)\varphi$ where φ is a conjunction of disjunctions of $\Sigma^G_{1,2k}$ formulas; $\Sigma^G_{1,2k+2}$ formulas have the form $(\forall x \leq y)\varphi$ where φ is a conjunction of disjunctions of $\Sigma^G_{1,2k+1}$ formulas.

A formula is Σ^G_1 (generalized Σ') if for some k it is $\Sigma^G_{1,k}$. Thus Σ^G_1 formulas result from open L' formulas by finite conjunctions and disjunctions, existential quantification and bounded universal quantification.

The whole definition formalizes easily in $I\Sigma_1$.

2.28 Theorem ($I\Sigma_1$). The Σ^G_1 formulas persist upwards on structures of the form $[0, x]$, i.e. for each $z \in \Sigma^G_1$, if $x \leq y$ and $x \models z[e]$ then $y \models z[e]$.

The *proof* by induction is easy. □

2.29 Definition. ($I\Sigma_1$) We define satisfaction for Σ^G_1 formulas in the universe ($Sat_{\Sigma,1,G}(z, e)$ or $V \models z[e]$) as follows: for each Σ^G_1 formula z and each evaluation e of its free variables by some numbers, $V \models z[e]$ if, for some x (containing all elements of the range of e), $x \models z[e]$.

2.30 Lemma ($I\Sigma_1$). The satisfaction just defined fulfils Tarski's conditions.

The *proof* by induction is easy, observing that the formula for which induction is applied is $\Sigma_0(\Sigma_1)$. Some care is necessary for handling finite conjunctions: if z is $\bigwedge\{\varphi_i, i < a\}$ and for each $i < a$ there is an x such that $x \models \varphi_i[e]$ then, by $B\Sigma_1$, there is an x such that for all $i < a$, $x \models \varphi_i[e]$. Similarly for bounded quantifiers. □

Clearly, each Σ' formula is (equivalent to) a Σ^G_1 formula. Conversely, we have the following

2.31 Lemma ($I\Sigma_1$). Each Σ^G_1 formula is equivalent to a Σ' formula, i.e. Σ^G_1 sets coincide with Σ' sets (and therefore with Σ_1 sets).

Proof evident. □

2.32 Lemma ($I\Sigma_1$). For each x there is a b such that for each $\Sigma^G_{1,x}$ sentence z,

$$V \models z \text{ iff } b \models z\,,$$

i.e. z is true in the universe iff z is true in $[0, b]$.

Proof. By possibly renaming variables we may restrict ourselves to $\Sigma^G_{1,x}$ sentences z such that all variables occurring in z are among the first x

variables. Furthermore, we may assume some normal form with respect to conjunctions and disjunctions: if $\bigwedge_{i<a} \bigvee_{j<b} \varphi_{ij}$; is a subformula of z then for each i, all φ_{ij} are pairwise distinct and also all the disjunctions $\bigvee_j \varphi_{ij}$ are pairwise distinct. It follows by induction that all $z \in \Sigma^G_{1,x}$ in this normal form form a finite set c. By $S\Sigma_1$, there is a q_0 such that

$$(\forall z \in c)[(\exists q)(q \models z) \rightarrow (\exists q \leq q_0)(q \models z)];$$

thus, for each $z \in c$, $V \models z$ iff $q_0 \models z$. $\square$

2.33 Remark. We can slightly generalize the preceding by expanding L' by finitely many constants for elements $0, 1, \ldots, t$ where t is a fixed number. The resulting language will be denoted $L(t)$.

2.34 Definition ($I\Sigma_1$). Let $\mathbf{a}, \mathbf{b}$ be tuples of numbers of the same length and let q be a number. We write

$$(V, \mathbf{a}) \rightarrow_x (q, \mathbf{b})$$

if for each $z \in \Sigma^G_{1,x}$ with the appropriate number of variables,

$$V \models z[\mathbf{a}] \text{ implies } q \models z[\mathbf{b}].$$

2.35 Corollary of 2.32 ($I\Sigma_1$). $(\forall x)(\exists q)(V \rightarrow_x q)$. (Clearly, $V \rightarrow_x q$ means the same as $(V, \emptyset) \rightarrow_x (q, \emptyset)$, i.e. sentences are concerned.)

2.36 Lemma ($I\Sigma_1$). Assume $(V, \mathbf{a}) \rightarrow (q, \mathbf{b})$ and $x > 0$.

(1) If x is even then

$$(\forall c)(\exists d < q)((V, \mathbf{a}, c) \rightarrow_{x-1} (q, \mathbf{b}, d);$$

(2) if x is odd then

$$(\forall d \leq \max(\mathbf{b}))(\exists c \leq \max(\mathbf{a}))((V, \mathbf{a}, c) \rightarrow_{x-1} (q, \mathbf{b}, d)).$$

(This is a prolongation lemma; think of $\mathbf{a}, \mathbf{b}$ as partial isomorphism. The lemma enables us to prolong it.)

Proof. (1) Let c be given and let Φ be the finite set of all $\varphi \in \Sigma^G_{1,x-1}$ formulas in the above-described normal form such that $V \models \varphi(\mathbf{a}, c)$. Then $V \models (\exists v) \bigwedge \Phi(\mathbf{a}, v)$, hence $q \models (\exists v) \bigwedge \Phi(\mathbf{b}, v)$ and thus there is a d such that, for all $\varphi \in \Phi$, $q \models \varphi(\mathbf{b}, d)$.

(2) Let $d \leq b_i$ and assume $(V, \mathbf{a}, c) \rightarrow_{x-1} (q, \mathbf{b}, d)$ to be false for all $c \leq a_i$. Then for each $c \leq a_i$ there is a formula $u \in \Sigma^G_{1,x-1}$ such that

$$V \models u(\mathbf{a}, c) \text{ but } q \models \neg u(\mathbf{b}, d).$$

The last condition is Σ_1; by $S\Sigma_1$ there is a set Φ such that for each $c \leq a_i$, a formula u satisfying the above can be found in Φ. We can further assume that Φ contains only such formulas u. Then $V \models (\forall x \leq a_i)(\bigvee \Phi(\mathbf{a}, x))$ but $q \models \neg \bigvee \Phi(\mathbf{b}, d)$, i.e. $q \models (\exists x \leq b_i) \neg \bigvee \Phi(\mathbf{b}, x)$, which contradicts $(V, \mathbf{a}) \rightarrow_x (q, \mathbf{b})$. □

2.37 Remark. Again, we may generalize to $\Sigma_1^G(t)$, i.e. admit constants for numbers up to t.

2.38 Lemma. If $M \models I\Sigma_1$, $e \in M$ is non-standard and $M \models V \rightarrow_e q$ then there is a cut $I \subseteq M$ not containing q and isomorphic to M.

Proof. By the zig-zag method: let $M = \{a_o, a_1, \ldots\}$ be an enumeration of M. Let $c_0 \in M$ be non-standard and let d_0 be such that $d_0 \leq_M q$ and $M \models (V, c_0) \rightarrow_{e-1} (q, d_0)$. Having constructed $c_0, \ldots, c_k$ and $d_0, \ldots, d_k$ such that

$$M \models (V, c_0, \ldots, c_k) \rightarrow_{e-k-1} (q, d_0, \ldots),$$

continue as follows: if k is even then take for d_{k+1} the first $a_i \in M$ less than q and not among $d_0, \ldots, d_k$; by 2.36 take for c_{k+1} the first a_j such that

$$M \models (V, a_0, \ldots, c_{k+1}) \rightarrow_{e-k-2} (V, d_0, \ldots, d_{k+1}).$$

(Observe that d_{k+1} is distinct from $c_0, \ldots, c_k$ because the corresponding open formula that says it must be true due to the arrow above.)

If k is odd, take for c_{k+1} the first $a_i \in M$ not among $c_0, \ldots, c_k$ and find similarly d_{k+1} less than q using 2.36.

If k varies over standard numbers we construct two sequences $(c_o, \ldots)$, $(d_o, \ldots)$; clearly the mapping $f(c_i) = d_i$ is one-one and maps M onto a cut $I \subseteq M$ not containing q. And f is an isomorphism since for each k, $M \models (V, c_o, \ldots c_k) \rightarrow_0 (q, d_o, \ldots, d_k)$ since $e - k - 1$ is non-standard); thus atomic formulas are preserved by f. □

2.39 Remark. Again, we may generalize keeping elements less than a given t fixed. Thus by 2.35 and 2.38 we have proved the following

2.40 Theorem. If $M \models I\Sigma_1$ is non-standard (countable) and $t \in M$ then there is a proper cut $I \subseteq M$ containing t and an isomorphism $f : M \rightarrow I$ identical on $(\leq t)_M$.

(Remark: in fact, we have used $L\Sigma'$ and $I\Sigma'$; but this is immaterial.)

3. Provably Recursive Functions and the Method of Indicators

In this section we are going to investigate means necessary to prove that a function is provably recursive. We shall see that the amount of induction necessary to prove that a function is total is related to its growth. We shall give two characterizations of $I\Sigma_k$-provably recursive functions and two characterizations of PA-provably recursive functions: one using largeness ($\underset{*}{\rightarrow}$) and one using the Schwichtenberg-Wainer hierarchy. As a particular case, we show that $I\Sigma_1$-provably recursive functions are just primitive recursive functions (Takeuti-Minc). To achieve this, we shall use the method of indicators developed by Paris and Kirby; this method has various other uses but ours is typical. We obtain various independence results (and some few other results) as corollaries. The section is structured as follows: (a) We formulate basic definitions concerning provably recursive functions and present the main result on them. (b) We define indicators and prove the main theorem on them. We also elaborate a technical notion of a Paris sequence (more general than that of Chap. II, Sect. 2) as a means of constructiong restrainable cuts. In (c) and (d) we investigate two kinds of indicators, one based on $*$-largeness and the other on α-largeness. This will give the major results of (a). (e) contains two other corollaries: provability of $(W)_k \rightarrow (PH)_k$ in $I\Sigma_1$ and the fact that each non-standard model of $I\Sigma_1$ has a non-standard cut satisfying PA.

(a) Provably Recursive Functions, Envelopes

3.1 Definition. Let $T \supseteq I\Sigma_1$ and let $\varphi(x, y)$ be a formula; φ is said to *define a* (partial) *function* in T if

$$T \vDash (\forall x, y_1, y_2)(\varphi(x, y_1) \,\&\, \varphi(x, y_2) \rightarrow y_1 = y_2);$$

φ *defines a total function* in T if, in addition,

$$T \vDash (\forall x)(\exists y)\varphi(x, y).$$

We also say that the function defined by φ is *T-provably total* (or just that φ is *T-provably total*). (Cf. I.1.51).

3.2 Remark. (1) If φ defines a partial/total function in T then it defines such a function in each model of T; in particular, if $N \vDash T$, φ defines a function in N. We shall focus our attention on Σ_1 formulas defining functions in N (recall that we call a function partial recursive iff it has a Σ_1-definiton in N). Clearly, a function may have different definitions and their equivalence may be unprovable in a theory T. Thus, even if we shall freely speak on T-provably total functions, it should be always clear that we investigate a particular Σ_1-definition of the function in question. Recall that a Σ_1-function provably total in T is Δ_1 in T and is called *T-provably recursive.*

(2) Definition 3.1 evidently generalizes to functions with several arguments.

3.3 Definition. Let $T \supseteq I\Sigma_1$. A two-argument function $F(-,-)$ Σ_1-defined in T is an *envelope* for T-provably recursive functions if

(1) For each n, the function F_n defined by $F_n(x) = F(n,x)$ is T-provably total and

(2) For each T-provably recursive function H there is an n such that $T \vdash (\forall x)(H(x) < F_n(x))$; thus H is T-provably majorized by F_n.

Remark. To avoid any misunderstanding, let us stress again that in fact the definition concerns a particular Σ_1-definition $\varphi(u,x,y)$ of F and the corresponding definitions $\varphi(\overline{n},x,y)$ of F_n.

On the other hand, if $N \models T$ then φ defines a system $\{F_n \mid n\}$ of total recursive functions such that whenever H is a T-provably recursive function then H is majorized by one of the F_n (and more than that: the last fact is T-provable).

3.4 Lemma. If $F = \{F_n \mid n\}$ is an envelope for T-provably recursive functions then a Σ_1-definable function H is T-provably total iff, for some n, H is T-provably primitive recursive in F_n (i.e a definition T-provably equivalent to the definition of H may be obtained from the definition of H and definitions of basic primitive recursive functions by composition and primitive recursion).

Proof. $\Leftarrow$ is clear since we assume $T \supseteq I\Sigma_1$ and therefore T-provably recursive functions are closed under primitive recursion. To prove $\Rightarrow$ let $(\exists z)\alpha(x,y,z)$ be a Σ_1 definition of H (α being Σ_0); define $H^*(x) = u$ iff u is a pair $\langle y,z\rangle$ such that $\alpha(x,y,z)\,\&\,(\forall z' < z)\neg\alpha(x,y,z')$. Evidently, H^* is a function Σ_0-defined in T and is T-provably total, hence, for some n, T-provably majorized by F_n. Thus

$$T \models y = H(x) \equiv (\exists z < F_n(x))\alpha(x,y,z)\,.$$

This shows that H is T-provably primitive recursive in F_n since Σ_0 relations are and bounded minimization (T-provably) preserves (relative) primitive recursiveness. □

Now we formulate our main theorems.

3.5 Theorem. For each $k,n \geq 1$ define (in $I\Sigma_1$)

(a) $F_{k,n}(x) = \min_y([x,y] \underset{*}{\rightarrow} (k+2)^{k+1}_n)$;

$F_n(x) = \min_y([x,y] \underset{*}{\rightarrow} (n+2)^{n+1}_{n+1})$;

(b) $G_{k,n}(x) = \min_y([(x,y)]$ is ω_k^n-large)
(thus $G_{k,n}(x) = f_{\omega_{k-1}^n}(x+1)$ where f_α is the α-th function of the Schwichtenberg-Wainer hierarchy)
$G_n(x) = \min_y([(x,y)]$ is ω_n-large)
(thus $G_{n+1}(x) = f_{\omega_n}(x+1)$.

Then

(1) for each k, $\{F_{k,n} \mid n\}$ is an envelope for $I\Sigma_k$-provably recursive functions.
(2) for each k, $\{G_{k,n} \mid n\}$ is another such envelope;
(3) both $\{F_n \mid n\}$ and $\{G_n \mid n\}$ are envelopes for (PA)-provably recursive functions.

3.7 Corollary. $I\Sigma_1$-provably recursive functions are exactly all primitive recursive functions. (In more detail, φ defines an $I\Sigma_1$-provably recursive function iff there is a ψ resulting from the definitions of basic primitive recursive functions by finitely many applications of the rule of composition and primitive recursion and such that $I\Sigma_1 \vdash \varphi \equiv \psi$.)

Proof. Each primitive recursive (definition of a) function is $I\Sigma_1$-provably recursive. On the other hand, it is easy to show for each n in $I\Sigma_1$ that $f_n(x)$ is primitive recursive. Thus the result follows by 3.5 (b) and 3.4. ☐

3.8 Theorem. (1) For each $k \geq 1$, the following formulas are unprovable in $I\Sigma_k$:

(a) $(\forall x)(\forall z)(\exists y)([x,y] \underset{*}{\rightarrow} (k+2)_z^{k+1})$,
(b) $(\forall \alpha \preccurlyeq \omega_k)(f_\alpha$ is total).

(2) The following formulas are unprovable in PA:

(a) $(\forall x)(\forall z)(\exists y)([x,y] \underset{*}{\rightarrow} (z+2)_{z+1}^{z+1})$,
(b) $(\forall \alpha < \varepsilon_0)(f_\alpha$ is total).

We postpone the proofs to subsections (c) and (d).

(b) Indicators and Paris Sequences

3.9 Definition. Let $M \models I\Sigma_1$ be countable and non-standard; let χ be a set of cuts in M. A Σ_1-formula $\varphi(x,y,z)$ is an *indicator* for χ in M if

(i) φ defines a total two-argument function Y in M,
(ii) for each $a, b \in M$ such that $a < b$, $Y(a,b)$ is non-standard iff

$$(\exists I \in \chi)(a \in I \,\&\, b \notin I),$$

(iii) for $a, b, c, d \in M$ such that $c \leq a \leq b \leq d$ we have

$$Y(a,b) \leq Y(c,d).$$

We shall be mostly interested in indicators for cuts $I \models T_2$ in a model $M \models T_1$. If φ is an indicator for such cuts in each (countable) model of T_1 we simply say that φ is an indicator *for models of* T_2 *in* T_1.

The main theorem on indicators follows.

3.10 Theorem. Let $T \supseteq I\Sigma_1$ be a theory (in the language of arithmetic). Let φ be an indicator for T in T and write $Y(x,y) = z$ instead of $\varphi(x,y,z)$. Then

(i) $T \nvdash (\forall x, z)(\exists y)(Y(x,y) \geq z)$;
(ii) $(\forall n) T \vdash (\forall x)(\exists y)(Y(x,y) \geq n)$;
(iii) If $N \models T$ then the sentence $(\forall x, z)(\exists y)(Y(x,y) \geq z)$ is neither provable nor refutable in T;
(iv) The functions $g_n(x) = \min_y(Y(x,y) \geq n)$ form an envelope for T-provably recursive functions.

Proof. (i) Let $M \models T$ be non-standard, let $N < c < a$. If there is no b in M such that $M \models Y(a,b) > c$ we are done; otherwise let b be minimal with this property. Then there is a cut $a < I < b$ in M such that $I \models T$ and in T there is no b' such that $I \models Y(a,b') > c$ (since otherwise we would have $M \models Y(a,b') > c$, a contradiction).

(ii) Since in each non-standard countable model M of $I\Sigma_1$ proper cuts isomorphic to M are cofinal, given $a \in M$ we find an $a \in I \subseteq_e M$ with $I \models T$; for each $I \leq b \in M$, $Y(a,b)$ must be non-standard, thus $M \models Y(a,b) > n$. We showed $M \models (\forall x)(\exists y)(Y(a,b) > \overline{n}$ and M was arbitrary; thus $T \vdash (\forall x)(\exists y)(Y(a,b) > \overline{n})$.

(iii) is evident.

(iv) Let f be T-provably recursive and assume $T + \{(\exists x)(f(x) \geq g_n(x)) \mid n\}$ to be consistent. Then $T + \{(\exists x)(Y(x, f(x)) \geq \overline{n} \mid n\}$ is consistent; let M be a model of the latter theory. By overspill, M contains an element a such that the value $Y(a, f(a))$ is non-standard. Consequently, there is a cut $a < I < f(a)$ satisfying T; in I, f is not total since $f(a)$ is undefined. This contradicts to f being T-provably total. □

3.11 Discussion. We shall be mainly interested in indicators for $I\Sigma_k$ (or for $B\Sigma_{k+1}$). This is why we are interested in a method allowing us to conclude, under some conditions, that between two elements $a < b$ of a model M there is a cut which is k-restrainable in $[0,b]$ (such a cut necessarily satisfies $B\Sigma_{k+1}$, see 2.21–22). If I is such a cut then there are $I \subseteq_e B_k \prec_I \cdots \prec_I B_0 = [0,b]$; for each $i < k$, there is an $h_i \in B_i$ such that $I < h_i < B_{i+1} - I$. Paris exhibited two similar constructions of such a chain and we shall try to isolate their common structure. Both constructions then become particular cases. The idea is to find in $[a,b]$ a coded increasing sequence $\{g_i \mid i < \nu\}$ of non-standard length and another increasing sequence $\langle h_0, \ldots, h_{k-1}\rangle$ of elements bigger than the g_i's and less than b such that if we put $I = \bigcup_{n\in N}[0, g_n]$ and let B_j be the set of all elements of $[0, b]$ "definable" from $I \cup \{h_j, \dots, h_{k-1}\}$ (w.r.t. an appropriate notion of definability) then $B_{j+1} \prec B_j$, $I \subseteq_e B_j$ and $h_j \in B_j$ for each j and we have to assure that $h_j \notin B_{j+1}$. In more details, we shall have a Δ_1 operation $df(b, i, g, par) \subseteq [0, b]$ and for each j, $B_j = \bigcup_{n\in N} df(b, n, g_n, par_j)$ where $par_j = \langle h_j, \dots, h_{k-1}\rangle$; we have to guarantee $\{u \in B_{j+1} \mid u \notin I \,\&\, u < h_j\}$ to be empty. To this end it suffices to have, for each $i < \nu$, $[g_{i+1}, h_j] \cap df(b, i, g_i, par_{j+1}) = \emptyset$, as one easily sees. It will turn out that it is enough to have

$$[(g_{i+1}, h_j)] \cap df(b, i, g_i par_{j+1}) = \emptyset\,.$$

This leads us to the following definition.

3.12 Definition ($I\Sigma_1$). Let df be a total Σ_1 function such that, for each x, $df(x, i, g, par) \subseteq [0, x]$. Let $b > 0$, $u > 0$ and $h = \langle h_0, \dots, h_{u-1}\rangle$ be a finite increasing sequence of elements $\le b$. A finite increasing sequence $g = \langle g_0, \dots, g_{\nu-1}\rangle$ of elements $\le b$ is a *Paris sequence* for h in b w.r.t. df if

(1) each g_i is less than each h_j,
(2) for each $i < \nu - 1$ and each $j < u$,

$$[(g_{i+1}, h_j)] \cap df(b, i, g_i, par_{j+1}) = \emptyset$$

where $par_j = \langle h_j, \dots h_{u-1}\rangle$.

The number u is called the ***dimension*** of g (cf. II.2.24 and (c) below).

3.13 Definition. Let $M \models I\Sigma_1$ be non-standard and let, in M, df be a total Σ_1 function such that

$$M \models (\forall x, i, g, par)(df(x, i, g, par) \subseteq [0, x])\,.$$

Let $b \in M - N$ and let $g, h \in M$ be such that g is a Paris sequence for h in b, $lh(g) = \nu$, $lh(h) = u$. Let $J \subseteq_e M$, $J < \nu$, and let I be the cut $\sup\{g_i \mid i \in J\} = \{a \in M \mid (\exists i \in J)(a < g_i)\}$. Finally, put

$$B_j = \bigcup_{i\in J} df(b, i, g_i, par_j)$$

for $j = 0, \dots, u-1$ (par_j is as above). The operation df is *suitable* for b, g, h, I if

(1) $j < j' < u$ implies $B_j \supseteq B_{j'}$ and
(2) each B_j is closed under L'-definability from $B_j \cup I$, i.e. if x is L'-definable in $[0, b]$ by a standard formula from some elements of $B_j \cup I$ then $x \in B_j$. (Consequently, $I \subseteq_e B_j$.)

3.14 Lemma. Let M, b, g, h, I be as above and let df be suitable for b, g, h, I in M. Then for each standard k such that $M \models k \leq lh(h)$, I is k-restrainable in $[0, b]$.

Proof. The proof is easy since almost everything is in the definitions. Let B_j ($j = 0, \ldots,$) be as above. Then, evidently, $B_j \supseteq B_{j+1}$, $h_j \in B_j$, and by assumption, $B_j \prec [0, b]$. Thus each $B_{j+1} \prec B_j$. The condition (2) in 3.12 gives, for each $i \in N$ and each $j < k$,

$$[(g_{i+1}, h_j)] \cap df(b, i, g_i, par_{j+1}) = \emptyset;$$

thus there is no x such that $x \in B_{j+1} \mathbin{\&} x \notin I$ and $x < h_j$ (x would be in a $DEF(h, i, g_i, par_j)$). Moreover, h_j itself is not in B_{j+1} since if it were, h_{j-1} would also be in B_{j+1}. Thus $I < h_j < B_{j+1} - I$. $\square$

3.15 Corollary. Under the above notation, if u is standard and $u = k$, then $I \models B\Sigma_{k+1}$; if $u > N$ then $I \models PA$.

(c) Paris Sequences of the First Kind

These are the Paris sequences in the sense of II.2.24.

We slightly reformulate the definition and reprove a theorem from there (Lemma 3.17 below). The main result is Theorem 3.20 exhibiting an indicator for models of $B\Sigma_{k+1}$ and Theorem 3.23 (an indicator for models of PA).

3.16 Definition ($I\Sigma_1$). Let $i, g < b$ and let par be an increasing sequence of elements of $[0, b]$. $c \in df_1(b, i, g, par)$ if $c \leq b$ and there is a formula $\varphi(x, \mathbf{y}) \in L'$, $\varphi < i$, such that for some parameters $\mathbf{e}$ from $[0, g] \cup par$ evaluating $\mathbf{y}$, c is either the minimal or the maximal element of $[0, b]$ satisfying $\varphi(x, \mathbf{e})$ in $[0, b]$. (Clearly, df_1 is Δ_1 defined.)

3.17 Lemma ($I\Sigma_1$, cf. II.2.26). Let $u > 0$ (dimension), let $\nu \geq u$ be such that $2^\nu > \nu^4$, let $c \geq 2$ and let b be such that $[0, b] \underset{**}{\rightarrow} (u+2)_c^{u+1}$ (see II.2.7; each $f : [0, b]^{u+1} \rightarrow c$ has a homogenous set Y such that $c \leq \min Y \leq 2^{\min Y} \leq card(Y)$). Then in $[0, b]$ there are g of length ν and h of length u such that g is a Paris sequence for h w.r.t. df_1.

Proof (cf. II.2.26). We are going to define a function $F : [0, b]^{u+1} \rightarrow c$ such that if Y is the homogenous set as above and $\beta = \langle \beta_0, \ldots, \beta_\omega \rangle \in [Y]^{u+1}$ then for $h = \langle \beta_1, \ldots, \beta_u \rangle$ there is a Paris sequence g of length ν. For each

$\beta \in [0, b]^{u+1}$ define a sequence $(g_1 \mid i < \nu)$ (not depending on β_0) as follows:

$$\begin{aligned} par^{\beta}_{j} &= \langle \beta_{j+1}, \ldots, \beta_u \rangle, \\ g_0 &= 0, \\ g_{i+1} &= \max[df_1(b, i, g_i, par^{\beta}_{1}) \cap [0, \beta_1]] + 1 . \end{aligned}$$

We want to have

$$[\beta_j, \beta_{j+1}] \cap df_1(b, i, g_i, par_{j+1}) = \emptyset$$

for $j = 0, \ldots, u-1$.

Claim 1. The last condition for $j = 0$ guarantees $g_{i+1} \leq \beta_0$ and for $j = 1, \ldots, u-1$ guarantees that g is a Paris sequence for $\langle \beta_1, \ldots, \beta_u \rangle$. (Cf. II.2.26).

Define $F(\beta)$ as follows: $F(\beta) = (j, i)$ if (j, i) is lexicographically the smallest element of $u \times \nu$ such that

$$[\beta_j, \beta_{j+1}] \cap df_1(b, i, g_i, par_j) = \emptyset \text{ if there is such } (j, i) \, ;$$

Otherwise $F(\beta) = (u, 0)$. Clearly, F can be understood as a mapping into c. Let $Y = \{y_i \mid i < e\}$ be homogenous as above.

Claim 2. The common value of F on $[Y]^{u+1}$ is $(u, 0)$. We prove the claim by contradiction. Assume $F(\beta) = (j, i)$ for a $\beta \in [Y]^{u+1}$ and $j < u$. Let $(g_i \mid i < \nu)$ be the sequence corresponding to β.

(1) Observe that $[\beta_j, \beta_{j+1}] \cap df_1(b, i, g_1, par_j) \neq \emptyset$.

(2) For our i, $g_i \leq \beta_0$. This is clear for $i = 0$. For $i > 0$ observe that for $\bar{i} < i$ and $\bar{j} < j$ we have $[\beta_{\bar{j}}, \beta_{\bar{j}+1}] \cap df_1(b, \bar{i}, g_{\bar{i}}, par_{\bar{j}}) = \emptyset$; in particular, take $\bar{i} = i - 1$ and $\bar{j} = 0$ and see Claim 1 above.

(3) Since the sequence $(a_i \mid i < \nu)$ does not depend on β_0, we may conclude that $g_i < y_0$ (y_0 is the smallest member of Y).

(4) Therefore (1) implies that $[\beta_j, \beta_{j+1}] \cap df_1(b, i, y_0, par_j) \neq \emptyset$. Since $\beta \in [Y]^u$ was arbitrary we have $[y_t, y_{t+1}] \cap df_1(b, i, y_0, par_j) \neq \emptyset$ for each t such that $j < t < e-u+j-1$. Among those intervals $[y_t, y_{t+1}]$ there are $(|Y|-u)/2$ disjoint intervals, each intersected by an element of $df_1(b, i, y_0, par_j)$. Thus to get a contradiction, it is sufficient to show the following.

(5) $|df_1(b, i, y_0, par_j)| < (|Y| - u)/2$. To obtain this, observe that there are at most $(i + 1)$ formulas $\leq i$ and each may take parameters from a set of cardinality $y_0 + u + 1$; each pair (formula-parameters) can give at most two elements to $df_1(\ldots)$. Thus $|df_1(\ldots)| \leq 2(i+1)(y_0 + u + 1)^{(i+1)} \leq 2\nu(y_0 + u + 1)^{\nu}$. On the other hand, $(|Y| - u/2) \geq (2^{y_0} - u)/2 \geq 2^{y_0} - 2$ (since $y_0 \geq c \geq u$). Thus we have to prove

$$2^{y_0 - 2} > 2\nu(y_0 + u + 1)^{\nu} .$$

Put $x = y_0 + u + 1$; it suffices to prove $2^{x-u-4} > x^\nu$ or (stronger) $2^{x-u-4} \geq x^{2\nu}$. It is easy to show that this holds e.g. for $x = 2^\nu$ and all bigger x (using $2^\nu \geq \nu^4$). But $y_0 + u + 1 \geq y_0 \geq c \geq 2^\nu$. This completes the proof of Claim 2.

Thus $F(\beta) = (u, 0)$ for all $\beta \in [Y]^{u+1}$. This implies, as in (2) above, that $g_i \leq \beta_0$ for *all* $i \leq \nu$; and evidently, g satisfies the disjointness conditions from the definition of a Paris sequence for $(\beta_1, \ldots, \beta_u)$. This completes the proof. □

3.18 Remark. If we assume in the previous lemma that $c \geq 2^{2\nu}$ then we may find a Y as above and growing exponentially, i.e. $y_{i+i} \geq 2^{y_i}$. (This follows from II.2.13).

3.19 Lemma. Under the notation of 3.13, if df is df_1 (as defined in 3.16) and $J = N$ then df is suitable.

Proof. The condition 3.13 (1) is satisfied, since clearly $u \subseteq v$ implies $df_1(x, i, g_i, u) \subseteq df_1(x, i, g_i, v)$. Also $I \subseteq_e B_j$ is clear. And if $p \in N$ and $a_1, \ldots, a_p \in B_j$, and a is defined from $a_1, \ldots, a_p$ by a standard formula φ then one easily writes down a standard formula ψ witnessing, for some standard number k, that $a \in df_1(b, k, g_k, par_j)$. This shows closedness of B_j under Skolem functions. □

3.20 Theorem. For each $k \geq 1$, the function

$$Y_k(a,b) = \max_c([a,b] \underset{*}{\rightarrow} (k+2)_c^{k+1})$$
$$(\text{or} = 0 \text{ if such a does not exist})$$

is an indicator both for models of $I\Sigma_k$ and for models of $B\Sigma_{k+1}$ in $I\Sigma_1$.

Proof. Let $M \models I\Sigma_1$ and let $a < b \in M$.

(a) First assume that there is a cut $a < I < b$ such that $I \models I\Sigma_k$. By II.1.9, for each n, $I \models (\exists y)([a,y] \underset{*}{\rightarrow} (k+2)_n^{k+1}$, thus for each n, $M \models (\exists y < b)$ $([a,y] \underset{*}{\rightarrow} (k+2)_n^{k+1})$ and by overspill we get a non-standard $c < b$ such that $M \models [a,y] \underset{*}{\rightarrow} (k+2)_c^{k+1}$. Thus in M, $Y(a,b)$ is non-standard.

(b) Now assume $Y(a,b) > N$ in M. By II.2.7 (or, better: by an analysis of the *proof* of II.2.7) there is a non-standard c such that $M \models 2^{2\nu} \leq c$ and $[0,b] \underset{**}{\rightarrow} (k+2)_c^{k+1}$. Apply 3.17 and 3.18: we get g, h such that $h = (h_0, \ldots, h_{k-1})$ and g is a Paris sequence of length ν for h w.r.t. df_1 in b; h is a k-tuple of elements of a set of indiscernibles of exponential growth and non-standard cardinality contained in $[a,b]$. Thus $I = \bigcup_N [0, g_i]$ is short and k-restrainable in $[0,b]$. By 2.21–22, $I \models B\Sigma_{k+1}$. □

3.21 Corollary. Let $k \geq 1$.

(1) $I\Sigma_k$ does not prove $(\forall x, z)(\exists y)([x, y] \underset{*}{\rightarrow} (k+2)_z^{k+1})$.

(2) The functions $F_{k,n}(x) = \min_y([x, y] \underset{*}{\rightarrow} (k+2)_n^{k+1})$ $(n = 1, 2, \ldots)$ form an envelope for $I\Sigma_k$-provably recursive functions.

Proof. (1) follows immediately from 3.10 (i).
(2) follows easily from 3.10 (iv). □

3.22 Remark ($I\Sigma_1$). (a) Recall II.2.16: if $[a, b] \underset{*}{\rightarrow} (e+1)_d^e$ and $d = c^3$ then $[a, b] \underset{**}{\rightarrow} (e+1)_c^e$.

(b) Observe that II.2.12, together with the fact that for $c \geq 7$ we have $c \geq 1 + 2\sqrt{c}$, implies the following: $c \geq 7$ and $[a, b] \underset{*}{\rightarrow} (e+2)_c^{e+1}$ implies $[a, b] \underset{*}{\rightarrow} (e+1)_c^e$.

3.23 Theorem. The function

$$Y(a, b) = \max_c \left([a, b] \underset{*}{\rightarrow} (c+1)_c^c\right.$$

is an indicator for models of PA in $I\Sigma_1$.

Proof. If $a < b \in M \models I\Sigma_1$, $a < I < b$, $I \subseteq_e M$ and $I \models PA$ then obviously, for each standard $n > 0$, $I \models (\exists y)([a, y] \underset{*}{\rightarrow} (n+1)_n^n)$; thus $Y(a, b)$ is non-standard.

Conversely, let $Y(a, b) = d$ be non-standard; let u be non-standard such that $e = 2^{6u} \leq d$. By 3.22, $[a, b] \underset{*}{\rightarrow} (u+2)_e^{u+1}$ and thus $[a, b] \underset{*}{\rightarrow} (u+2)_c^{u+1}$ for $c = 2^{2u}$. Apply 3.17: in $[a, b]$, there are g, h of length u such that g is a Paris sequence for h in b w.r.t. df_1, and $I = \bigcup_N [0, g_n]$ is short in $[a, b]$. By 3.14 (2), it is k-restrainable in $[0, b]$ for each $k \in N$ and consequently $I \models PA$. □

3.24 Corollary. (1) PA does not prove

$$(\forall x, u)(\exists y)([x, y] \underset{*}{\rightarrow} (u+2)_{u+1}^{u+1}).$$

The functions

$$F_n(x) = \min_y \left([x, y] \underset{*}{\rightarrow} (n+2)_{n+1}^{n+1}\right)$$

form an envelope for PA-provably recursive functions.

(d) Paris Sequences of the Second Kind

Here we exhibit another indicator for models of $B\Sigma_{k+1}$ and of PA, using the notion of α-large sets and the Schwichtenberg-Wainer hierarchy.

3.25 Definition ($I\Sigma_1$). Let $g < b$ and let par be an increasing sequence of elements of $[0, b]$. $c \in df_2(b, g, par)$ if there is a formula $\varphi(x, \mathbf{y}, \mathbf{z}) \in L'$ and parameters $\mathbf{e}$ evaluating $\mathbf{y}$ such that

(1) $\langle\varphi, \mathbf{e}\rangle < g - 1$, and
(2) c is the minimal element of $[0, b]$ satisfying $\varphi(x, e, par)$ in $[0, b]$.

3.26 Remark. Compare this with 3.16 (definition of df_1): there we had two independent bounds for the formula and parameters distinct from par. Here we have one point bound for the pair formula-parameters. The main property of df_2 is that, for each b, g, par, the cardinality of $df_2(b, g, par)$ is less than g. (And clearly, df_2 is Δ_1-defined.)

3.27 Lemma ($I\Sigma_1$). Assume $A \subseteq [a, b]$ to be ω_u^v-large, where $u, v \geq 1$, $a \geq 2$ and let par be an increasing sequence of elements of $[0, b]$. Then there is an $A_1 \subseteq A$ and a $h \in A$, $h > \max A_1$ such that

(1) A_1 is ω_{u-1}^{v-1}-large and
(2) if $\{g_i \mid i < \nu\}$ is the increasing enumeration of A_1

then

$$[(g_{i+1}, h)] \cap df_2(b, g_i, par) = \emptyset$$

for each $i < \nu - 1$. (Remark: $[(x, y)]$ denotes the open interval with endpoints x, y.)

Proof. Let $g_0 = \min A$, $h_0 = \max A$. We construct a sequence of intervals $[g_i, h_i]$ $(i = 0, \ldots, \nu)$ such that $g_i < g_{i+1} < h_{i+1} \leq h_i$ as follows:
let $A \cap [g_i, h_i]$ be $\omega^{\{\omega_{u-1}^v\}(g_0-1)\ldots(g_{i-1}-1)}$-large and assume the exponent $\{\omega_{u-1}^v\}(g_0 - 1)\ldots(g_{i-1} - 1) = \alpha$ to be ≥ 2. Then use the increasing enumeration of $df_2(b, g_i, par)$ to define a decomposition of $[g_i, h_i]$ onto $\leq g_i$ intervals $[x, y]$ such that $[(x, y)] \cap df_2(b, g, par) = \emptyset$. Apply II.3.21 (4) to obtain a $[g_{i+1}, h_{i+1}] \subseteq [(g_i, h_i)]$ such that $A \cap [g_{i+1}, h_{i+1}]$ is $\omega^{\{\alpha\}(g_i-1)}$-large and $[g_{i+1}, h_{i+1})] \cap df_2(b, g_i, par) = \emptyset$.

Let u be the least number such that $\{\omega_{u-1}^v\}(g_0 - 1)\ldots(g_{\nu-1} - 1) \leq 1$. Take a g_ν $(= g_{\nu-1} + 1$, say); then $\{g_i - 1 \mid i \leq \nu\}$ is ω_{u-1}^v-large, i.e. $\{g_i - 1 \mid 1 \leq i \leq \nu\}$ is $\{\omega_{u-1}^v\}(g_0 - 1)$-large, thus $\{\omega_{u-1}^v\}(1)$-large and therefore ω_{u-1}^{v-1}-large. But then also $\{g_i \mid 0 \leq i \leq \nu - 1\}$ is ω_{u-1}^{v-1}-large. Put $h = h_{\nu-1}$; then the assertion of the lemma holds. This completes the proof. □

3.28 Lemma. There is an m such that $I\Sigma_1$ proves the following: if $a \geq \overline{m}$ and assumptions of 3.27 hold then $g_{i+1} > 2^{g_i}$ for all $i \leq \nu - 2$.

Proof. If $a > \overline{m}$ for an appropriate m then for each g_i there is an $x < g_i < 2^x$ such that $2^{2^x} \in df_2(b, g_i, \emptyset) \subseteq df_2(b, g_i, par)$. (See the sublemma below.) Now assume $g_{i+1} \le 2^{g_i}$; then $g_{i+1} \le 2^{2^x}$, thus $h_{i+1} \le 2^{2^x}$ (since $[(g_{i+1}, h_{i+1})] \cap df_2(b, g_i, par) = \emptyset$).

Case 1. $g_{i+1} < 2^x$, then $h_{i+1} \le 2^x < 2^{g_i} < 2^{g_{i+1}}$.

Case 2. $2^x \le g_{i+1}$, then $h_{i+1} \le 2^{2^x} \le 2^{g_{i+1}}$.

In both cases, $h_{i+1} < f_2(g_{i+1}) - 1$ (f_2 is the second function of the Schwichtenberg-Wainer hierarchy, cf. II.3.31), thus $[g_{i+1}, h_{i+1}]$ is not ω^2-large. Thus $\alpha = \{\omega^v_{u-1}\}(g_0 - 1) \dots (g_i - 1) \le 1$ (since $[g_{i+1}, h_{i+1}]$ is ω^α-large and $\alpha \ge 2$ implies $\alpha \underset{*}{\rightarrow} 2$). We get $i + 1 = \nu - 1$ (g_{i+1} is the last element of g). It remains to prove a sublemma to complete our proof.

3.29 Sublemma. There is an m such that $I\Sigma_1$ proves the following: if $b > g > \overline{m}$ then there is an $a < g < 2^a$ such that $2^{2^a} \in df_2(b, g, \emptyset)$.

Proof. Let $\varphi(x, y)$ define $y = 2^{2^x}$; for the standard pairing function we have $\langle \varphi, a \rangle = a^2/2 + ka + h$ for some k, h; find m such that $a > m$ implies $ka + h < a^2/2$, then $\langle \varphi, a \rangle < a^2$. Further, let $m \ge 37$ which guarantees that for $g \ge m$ there is an a such that $a^2 < g < 2^a$. (This works for g even; for g odd the proof is analogous.) □

3.30 Lemma ($I\Sigma_1$). Let $a \ge 2$, $A \subseteq [a, b]$ ω^v_u-large; then there is an increasing sequence h of elements of $[a, b]$ with $lh(h) = u$ and an increasing sequence g of elements of A with $lh(g) = v - u$ such that g is a Paris sequence for h in b w.r.t. df_2.

Proof. Apply iteratively Lemma 3.27: put $A_0 = A$, $par_u = \emptyset$; you get $A_1 \subseteq A$ ω^{v-1}_{u-1}-large and a $h_{u-1} \in A$, $h_{u-1} > A_1$ such that for two consecutive elements x, x' of A_1 we have

$$[(x', h_{u-1})] \cap df_2(b, x, par_u) = \emptyset \,.$$

Put $par_{u-1} = \{h_{u-1}\}$. If we have $A_i \subseteq A_{i-1}$, A_i ω^{v-i}_{u-i}-large, $h_{u-1}, \dots, h_{u-i} > A_i$, and $par_{u-1} = (h_{u-i} \dots h_{u-1})$ such that $x, x' \in A_i$, $x < x'$ implies $[(x', h_{u-i})] \cap df_2(b, x, par_{u-i+1}) = \emptyset$, then we can use 3.27 to get the same with i replaced by $i+1$. Finally we get A_u ω^{v-u}_0-large (i.e. $(v-u)$-large, thus having at least $v - u$ elements) and $par_u = (h_0, \dots, h_{u-1}) = h$ such that if $g = \langle g_i \mid i < v - u \rangle$ enumerates A_u (or: the first $v - u$ elements of A_u) then g is a Paris sequence for h in b w.r.t. df_2. □

3.31 Remark. Note that by 3.28 if we assume $a \ge \overline{m}$ (where m is as in 3.28) then we may conclude that the sequence grows exponentially.

3.32 Lemma. Let $M \models I\Sigma_1$ be non-standard and let a, g, h, b be as in 3.30–3.31. Assume $v - u$ non-standard and let $J \subseteq v - u$ be a cut; put $I = \sup\{g_i \mid i \in J\}$ (cf. 3.13). Then df_2 is suitable for g, h, b, I.

Proof. Cf. the proof of 3.19. We verify that for $B_j = \bigcup_{i \in J} df_2(b, g_i, par_j)$, $par_j = (h_j, \ldots, h_{u-1})$, B_j is closed under Skolem functions. Let $a_1, \ldots, a_p \in B_j$ ($p \in N$), let φ be standard and let $c \in [0, b]$ be defined in $[0, b]$ by φ from $a_1, \ldots, a_p$. Assume $a_1 \ldots a_p \in df_2(b, g_i, par_j)$; let a_i be defined by $\langle \varphi_i, e_i \rangle < g_q$ ($q \in J$, φ_i not necessarily standard).

Thanks to exponential growth of g, there is a $q' = q + k$ for some standard k such that the formula φ' saying

$$\text{“}(\exists x_1, \ldots, x_p)(\bigwedge_i \varphi_i(x_i, y_i, par_j) \,\&\, \varphi(z, \mathbf{x}))\text{”}$$

together with the juxtaposition $\mathbf{e}'$ of $e_1, \ldots, e_p$ form a pair $\langle \varphi', e' \rangle < g_q$; clearly, φ' defines c from e', par_j in $[0, b]$. This shows that B_j is closed under definable Skolem functions. □

3.33 Theorem. For each $k \geq 1$, the function

$$Z_k(a, b) = \max_c([(a, b)] \text{ is } \omega_k^c\text{-large}), \text{ or}$$
$$= 0 \text{ if such } c \text{ does not exist},$$

is an indicator both for models of $B\Sigma_{k+1}$ and for models of $I\Sigma_k$ in $I\Sigma_1$.

Remark. This is a well-defined Δ_1 function in $I\Sigma_1$ since $a + 1 > 0$ and therefore $[a + 1, b - 1]$ is not ω_k^{b-a}-large (prove this using II.3.21 (5)). There is a *maximal* c such that $[(a, b)]$ is ω_k^c-large, since the property in question is a Δ_1 property of c.

Proof. Let $M \models I\Sigma_1$, M non-standard, and let $a < b \in M$.

(a) Similarly as in 3.20 we show that if there is a cut $a < I < b$ such that $I \models I\Sigma_k$ then $Z_k(a, b)$ is non-standard.

(b) Conversely, assume $c = Z_k(a, b) > N$ in M. Then $b > N$ (by the remark above) and we may also assume $a > N$ (otherwise take N for I). Apply 3.30: $[a + 1, b - 1]$ is ω_k^c-large, therefore there are $h = (h_0, \ldots, h_{k-1})$ in $[0, b]$ and a Paris sequence g for h w.r.t. df_2, g of length $c - k > N$. By 3.31, g grows exponentially. Thus by 3.14, $I = \bigcup_n [0, g_n]$ is k-restrainable in $[0, b - 1]$ and therefore $I \models B\Sigma_{k+1}$. □

3.34 Corollary. For $k \geq 1$, the functions $f_{\omega_{k-1}^n}$ ($n = 1, 2, 3, \ldots$) form an envelope for $I\Sigma_k$-provably total Σ_1 functions.

Proof.

$$\begin{aligned} g_n(x) &= \min_y(Z_k(x, y) \geq \overline{n}) \\ &= \min_y(\max_c([(x, y)] \text{ is } \omega_{k-1}^c\text{-large}) \geq \overline{n}) \\ &= \min_y([(x, y)] \text{ is } \omega_{k-1}^n\text{-large}) = \min_y([x + 1, y)] \text{ is } \omega_{k-1}^n\text{-large}) \\ &= f_{\omega_{k-1}^n}(x + 1). \end{aligned}$$

□

3.35 Corollary. Let $k \geq 1$; then $I\Sigma_k$ does not prove f_{ω_k} to be total.

Proof. By the indicator theorem $I\Sigma_k$ does not prove $(\forall x, z)(\exists y)([x, y]$ is ω_k^z-large). By II.3.34, this means that $I\Sigma_k$ does not prove that each z, $f_{\omega_{k-1}^z}$ is total. But this means that $I\Sigma_k$ does not prove f_{ω_k} to be total. □

3.36 Theorem. The function

$$Z(a, b) = \max_c([(a, b)] \text{ is } \omega_c^{2c}\text{-large})$$
$$= 0 \text{ if such } c \text{ does not exist}$$

is an indicator for models of PA in $I\Sigma_1$.

Proof. Clearly, if $a < I < b$ and $I \models PA$ then for each u,

$$I \models (\exists y)([a, y)] \text{ is } \omega_u^{2u}\text{-large}) .$$

Therefore $Z(a, b) > N$. Conversely, if $c = Z(a, b) > N$ then assume $a > N$; by 3.30, there are g, h (h of length c) such that g is a Paris sequence of the second kind for h of non-standard length c. By 3.31, g has exponential growth. Thus by 3.14, there is an $I \subseteq_e M$, such that for each k I is k-restrainable in $[0, b - 1]$. Thus $I \models PA$. □

(e) Further Consequences

3.37 Theorem. For each k, $I\Sigma_1$ proves that $(W)_k$ implies $(PH)_k$. (This is Theorem II.3.36; $(W)_k$ is $(\forall x, z)(\exists y)([x, y]$ is ω_k^z-large) and $(PH)_k$ is the k-th instance of the Paris-Harrington principle:

$$(\forall x, z, q)(\exists y)([x, y] \underset{*}{\rightarrow} (q)_z^{k+1}) .$$

Proof. Let $M \models I\Sigma_1$ be non-standard satisfying $(W)_k$, let $k \geq 1$, $a \in M - N$, $c \in M - N$; we want to find a b such that $M \models [a, b] \underset{*}{\rightarrow} (k+2)_c^{k+1}$. This will clearly verify $M \models (PH)_k$. We may assume $M \models c < a$ (otherwise change a to c). Find a b such that $M \models [a, b]$ is ω_k^e-large where $e = c^\nu + k$ for some non-standard ν and put $d = 2^{2^b}$; in M, $[a, d]$ is also ω_k^e-large.

Let $f \in M$, $f : [a, b]^k \rightarrow c$; then clearly f is coded in $[0, d]$. Apply 3.30 and 3.31 with b replaced by d and with A replaced by $[a, b]$; find g, h in $[0, d]$ such that $lh(h) = k$, $lh(g) = c^\nu$ and g is an exponentially growing Paris sequence for h w.r.t. df_2, consisting of elements of $[a, b]$.

Claim. There is a cut $J < c^\nu$ such that $I = \sup_J\{g_i \mid i \in J\}$ is regular up to c.

Proof of the claim. Enumerate all coded partitions of the set $[0, b]$ into c parts as $\bigcup_{j<c} q_{ij}$, $i \in N$ (this system exists due to the countability of M). Define a sequence $\{A_i \mid i \in N\}$ such that $A_0 = A$, for each i, $card(A_i) \geq c^{\nu - i}$ and for some j, $A_{i+1} \subseteq A_i \cap q_{ij}$. Then put $I = \sup_N \min(A_i)$. I is a cut in $[0, d]$ and if $\bigcup_{j<c} d_j$ is a coded partition of I then it induces a partition $\bigcup_j q_{ij}$ for some i ($d_j = q_{ij} \cap I$). Take A_{i+1} and let $A_{i+1} \subseteq q_{ij_0}$. Then clearly $A_{i+1} \cap I$ is unbounded in I, thus d_{j_0} is unbounded in I. This proves the claim.

We continue the proof of the theorem. Apply 3.14. Thus we have a chain

$$a \in I \subseteq_e K_0 \prec_I \cdots \prec_I K_k \prec [0, d];$$

by inspecting the proof of 3.30 we see that we may assume $f \in K_0$. (Put $par_u = (f)$.) Observe that $b \notin I$ and apply 2.24: we have $K_0 \models [a, b] \underset{*}{\rightarrow} (k + 2)_c^{k+1}$ and $f \in K_0$, thus in K_0 and consequently in M, f has a relatively large homogeneous set. This completes the proof. □

3.38 Corollary. For each k, the following are equivalent over $I\Sigma_1$:
(i) $Con(I\Sigma_k + Tr(\Pi_1))$, (ii) $(PH)_k$, (iii) $(W)_k$.
(See II.3.36.)

3.39 Theorem. Each non-standard model of $I\Sigma_1$ has a non-standard cut that is a model of PA.

Proof. For each standard u, $N \models (\exists y)(Y(u, y) \geq u)$, thus if M is non-standard and $M \models I\Sigma_1$ then $M \models (\exists y)(Y(u, y) \geq u)$. ($Y$ is the indicator for PA in $I\Sigma_1$.) By overspill, there is a non-standard $c \in M$ such that $M \models (\exists y)(Y(c, y) \geq c)$. Let d be such a y; then in M, $Y(c, d)$ is non-standard and therefore is a cut $c < I < d$, $I \models PA$. □

4. Formalizing Model Theory

The main aim of this short section is to exhibit a proof of Paris's conservation result (for all n, $B\Sigma_{n+1}$ is a Π_{n+2}-conservative extension of $I\Sigma_n$, see Sect. 1) in $I\Sigma_1$. This implies that for each n $I\Sigma_{n+1}$ proves $Con^\bullet(B\Sigma_{n+1}^\bullet)$ (since it proves $Con^\bullet(I\Sigma_n^\bullet)$, see I.4.34). This is a valuable piece of information on the relation of fragments of PA; for example, it implies that $I\Sigma_{n+1}$ is not interpretable in $B\Sigma_{n+1}$.

Thus the section is a free continuation of Chap. I, Sect. 4 where we formalized some parts of logic in $I\Sigma_1$. Our present formalization consists in formalizing Paris's proof of his conservation result; roughly, we show in

$I\Sigma_1$ that each model of $I\Sigma_n$ has a satisfactorily elementary extension to a model of $B\Sigma_{n+1}$ and that this implies the desired conservation. Thus we shall see that important parts of the model theory of fragments can be developed inside $B\Sigma_2$. Central devices are the Low arithmetized completeness theorem (cf. I.4) and a consistency criterion (see below). The section has the following structure: (a) some results on satisfaction and consistency, (b) Paris's conservation result in $B\Sigma_2$, (c) a proof of conservativity of ACA_0 over PA.

(a) Some Results on Satisfaction and Consistency

Recall Chap. I, Sect. 4 (c); there we formalized model theoretical notions in $I\Sigma_1$ and after having proved the Low arithmetized completeness theorem (I.4.27) we investigated (inside $I\Sigma_1$) the language of arithmetic and defined its standard model (the universe). For each k, $I\Sigma_1$ proves that the standard model has a satisfaction for $(\Sigma_k \cup \Pi_k)$-formulas; but it follows easily by self-reference that $I\Sigma_1$ proves the non-existence of a full satisfaction for the standard model (for a precise statement see below).

Now that we have studied non-standard models of fragments, we are ready to continue the development of positive results in fragments by investigating, in some appropriate fragment, non-standard models of arithmetic. The Low arithmetized completeness theorem yields models with full satisfaction; but other constructions need not. We present here a theorem enabling us to derive, in $I\Sigma_1$ (or in $I\Sigma_1(\Gamma)$ for some Γ). the consistency from the existence of a model with partial satisfaction. This will be used in the next subsection during our proof of Paris-Friedman's conservation theorem in $I\Sigma_1$.

4.1 Theorem. For each k, $I\Sigma_1$ proves the following: there is no $Sat \in \Sigma_k^\bullet$ such that Sat is a full satisfaction for the standard model. (Cf. I.4.23 for the notion of a full satisfacton.)

Proof. Let $\sigma(z,e,p)$ be a Σ_k formula defining binary $\Sigma_k^\bullet$-relations in $I\Sigma_1$; let, by the diagonal lemma III.2.1, $\varphi(p)$ be such that $I\Sigma_1 \vdash \varphi(p) \equiv \neg\sigma(\overline{\varphi(p)}, [\dot{p}], p)$. Now assume in $I\Sigma_1$ that we have a p such that $Sat = \{\langle z,e\rangle \mid \sigma(z,e,p)\}$ is a full satisfaction class for the universe. Then on the one hand, $\varphi(p)$ is equivalent to $\neg\sigma(\overline{\varphi(p)}, [\dot{p}], p)$, but on the other hand, by "it's snowing"-it's snowing applied to φ, $\varphi(p) \equiv \langle\varphi(p), [p]\rangle \in Sat \equiv \sigma(\varphi(p), [p], p)$, which is a contradiction. $\square$

4.2 Convention. In the rest of the subsection we assume that T is a theory extending $I\Sigma_1$ and Γ is a class of formulas such that $T \vdash \Delta_1(\Gamma) \subseteq \Gamma$ and $T \vdash I\Sigma_1(\Gamma)$. Our favourite choice will be $\Gamma = LL_1$ (i.e. low $\Sigma_0^*(\Sigma_1)$) and

$T = I\Sigma_1$, cf. Chap. I, Sect. (c). Instead of saying that we work in T, just say that we work in $I\Sigma_1(\Gamma)$.

4.3 Theorem ($I\Sigma_1(\Gamma)$). If L is the language of arithmetic and (M, S, Sat) is a Γ-defined model for L where Sat is a satisfaction for (M, S) and $(\Sigma_k^\bullet \cup \Pi_k^\bullet)$-formulas then the Π_{k+1} theory of (M, S) is consistent.

Proof. Note that Sat uniquely extends to a satisfaction Sat' for Π_{k+2} formulas; but Sat' need not be Γ. The Π_{k+2} theory of (M, S) is the set of all Π_{k+2} sentences true in (M, S). The theorem follows by I.4.26 (relativized to $I\Sigma_1(\Gamma)$) from the following lemma: □

4.4 Lemma ($I\Sigma_1(\Gamma)$). If (M, S, Sat) is as in 4.3 then, for each finite set T_o of Π_{k+2}-formulas true in (M, S, Sat), (M, S) can be expanded to a Γ-defined model (M, S') of skolemizations of formulas from T_o (with respect to the satisfaction guaranteed by I.4.24).

Proof. Observe that theorem I.4.33, stating that for $n \geq 1$, $I\Sigma_n$ proves the consistency of the set of all true $\Pi_{n+1}^\bullet$ sentences, is a particular case of our present theorem 4.3: $T = I\Sigma_n$, $\Gamma = \Delta_n$, $T \vdash I\Sigma_1(\Gamma)$, (M, S) is the standard model$^\bullet$ with the Δ_n satisfaction for $(\Sigma_{n+1}^\bullet \cup \Pi_{n+1}^\bullet)$ formulas$^\bullet$. Thus take $k = n - 1$: we get $T \vdash Con^\bullet(Tr(\Pi_{k+2}^\bullet))$, and $k + 2 = n + 1$. □

The proof of I.4.33 is just a particular case of the proof of our present lemma; the reader may generalize easily. (Hint: all least number operators are understood as taken outside the model, not in the sense of the model. Values of terms are well defined since all function symbols in question are interpreted as Γ-functions.)

4.5 Corollary. $I\Sigma_1$ proves that if $M = (M, S, Sat)$ is an LL_1 model and Sat is a satisfaction for $(\Sigma_k^\bullet \cup \Pi_k^\bullet)$ formulas ($k \geq 1$) then the $\Pi_{k+2}^\bullet$ theory of M is consistent.

Proof. By 4.3 using our favourite choice (cf. 4.2). □

(b) A Conservation Result in $I\Sigma_1$

We formalize the considerations of Sect. 1 (f); in particular, we formalize the proof of 1.61 in $I\Sigma_1$.

4.6 Theorem ($I\Sigma_1$). Let $k \geq 0$ and let $(I\Sigma_k^\bullet)^*$ be the strong Skolem extension of $I\Sigma_k^\bullet$ as described in I.4.10. Let M^* be an LL_1 model of $(I\Sigma_k)^*$ with full satisfaction and let M be the reduct of M^* to the language of arithmetic.

Then there is an LL_1 model I of $B\Sigma^\bullet_{k+1}$ with satisfaction for $\Sigma^\bullet_{k+1} \cup \Pi^\bullet_{k+1}$-formulas such that I is a cofinal, $(k+1)$-elementary extension of M.

4.7 Corollary. $I\Sigma_1$ proves that, for each $k \geq 0$, $B\Sigma^\bullet_{k+1}$ extends $I\Sigma^\bullet_k$ $\Pi^\bullet_{k+2}$-conservatively. (See the remark following 1.61 and use 4.5 with $(k+1)$ instead of k).

4.8 Corollary. For each $k \geq 1$, $I\Sigma_k$ proves $Con^\bullet(B\Sigma^\bullet_k)$. (Since, by $I\Sigma_k \vdash Con(I\Sigma^\bullet_{k-1})$ and, furthermore, $I\Sigma_k \vdash Con^\bullet(I\Sigma^\bullet_{k-1}) \rightarrow Con^\bullet(B\Sigma^\bullet_k)$.)

In the rest of the subsection, we elaborate the proof of 4.6.

4.9 Construction. We proceed in $I\Sigma_1$. Our aim is to formalize the proof of 1.61. Let M^* be as in 4.6 above. We shall carefully define an ultrapower K^* of M^* using an ultrafilter U of definable subsets of M^*. The care concerns not only the verification that our construction works in $I\Sigma_1$, but also that our special choice of U will ensure that the cut $I = \sup_K(M)$ (where K, M are reducts of K^*, M^* to the language of arithmetic) will be Δ_1 in M^*, i.e. will itself be LL_1. We shall *define* low Δ_2 satisfaction for K^* using Łoś's theorem and *define* an LL_1 satisfaction on I for $\Sigma_k \cup \Pi_k$-formulas such that $I \prec_k K$. An additional proof will show that even satisfaction for $\Sigma_{k+1} \cup \Pi_{k+1}$ on I is Δ_1(M), i.e. LL_1. The reader is advised to compare our construction with 1.65.

(1) K_0 will be the set of all parametrical definitions of mappings of M^* into itself, i.e. of formulas $\varphi(y, x, a)$ of the language of $(I\Sigma^\bullet_k)^*$ such that $M^* \models (\forall x)(\exists! y)\varphi(x, y, a)$ (a is a parameter from M). Clearly, $K_0 \in \Delta_1(M^*)$ (here M^* denotes the structure together with its full satisfaction) and hence $K_0 \in LL_1$. Similarly, B will denote the set of all parametrical definitions of subsets of M, i.e. of arbitrary formulas $\varphi(x, a)$. Again, $B \in \Delta_1(M^*)$.

Since $I\Sigma_1$ proves $I\Sigma_1(LL_1)$ and both K_0 and B are unbounded, we may work with increasing enumerations of both sets; the enumerations are $\Delta_1(M^*)$. Thus let

$$K_0 = \{f_1, f_2, \ldots\}$$
$$B = \{\varphi_1, \varphi_2, \ldots\}$$

(2) We shall construct a "generic sequence" of elements of B defining an ultrafilter U. In the even steps, we shall deal with φ_i's and in the odd steps we shall decide the membership to the cut I. Using $I\Sigma_1(LL_1)$ we construct by induction a sequence of formulas $\alpha_j(x)$ (containing parameters from M^*); for each j, $\Omega_j(x)$ abbreviates $\bigwedge_{z<j} \alpha_z(x)$. Put $\alpha_0(x) = \text{TRUE}$.

Step $j = 2i + 1$. Let $\Phi \equiv (\exists y)(0x)(\Omega_j(x) \,\&\, f_i(x) \leq y)$.

Case 1. $M^* \models \Phi$. Then let y_0 be the least element of M^* (in the ordering of the universe) such that $M^* \models (Cx)(\Omega_j(x) \,\&\, f_i(x) \leq y_0)$ and put $\alpha_j(x) = f_i(x) \leq y_0$.

Case 2. Otherwise let $\alpha_j(x) = \mathrm{TRUE}$.

Step $j = 2i+2$. If $M^* \models (Cx)(\Omega_j(x) \,\&\, \varphi_i(x))$ then $\alpha_j(x)$ is $\varphi_i(x)$, otherwise $\alpha_j(x)$ is $\neg\varphi_i(x)$.

Let us stress again that the sequence of α's is $\Delta_1(M^*)$ and hence LL_1; for each j we have $M^* \models (Cx)\alpha_j(x)$. Let U be the set of all φ_i such that in step $2i+2$ we have taken $\alpha_j(x)$ to be $\varphi_i(x)$; again $U \in \Delta_1(M^*)$.

(3) For $f, g \in K_0$ put $f =_U g$ if the formula $f \equiv g$ belongs to U (in other words, the set $\{x \in M \mid M \models f(x) = g(x)\}$ is in U). We define $f + g$ and $f.g$ "coordinate-wise", i.e. $f + g$ is defined as the function associating with each x the value $f(x) + g(x)$ etc. Similarly for $0, S, \leq$. Observe that $=_U$ is a congruence for all these. Thus we have defined K_0 as a structure for the language of arithmetic with non-absolute equality. We may convert K_0 to the structure K with absolute equality by selecting from each equivalence class of $=_U$ its least element. Clearly, $K \in \Delta_1(M^*)$.

(4) We define the full satisfaction for K as follows:

$$K \models \varphi(f_1, \ldots, f_n) \text{ iff } \{x \in M \mid M \models \varphi(f_1(x), \ldots, f_n(x))\} \in U\,.$$

This definition is $\Delta_1(M^*)$: recall that $f_1, \ldots, f_n$ are formulas of $(I\Sigma_1)^*$ and so is $\varphi(f_1(x), \ldots)$. In the above enumeration, the last formula is φ_i, say. Thus the left hand side above holds iff in step $2i+2$ of the construction of U, the first case occurs (and φ_i is put into U). We are obliged to prove that this is indeed a satisfaction:

Lemma. The definition above defines full satisfaction on K, i.e. Tarski's conditions hold true. (Easy.)

(5) We embed M into K by identifing each $a \in M$ with the constant function on M with the value a (i.e. with the formula $y = a$). Then the definition of satisfaction for K immediately gives $M \prec K$. The proof of the fact that M is not cofinal in K is as in 1.66.

(6) Define

$$I = \sup_K(M) = \{f \in K \mid (\exists y \in M)(\{x \mid f(x) < y\} \in U)\}\,.$$

By this definition, I is $\Sigma_1(M^*)$; but it is even $\Delta_1(M^*)$ since $f \in I$ if for the i such that $f = f_i$ (in the enumeration of K_0 above) we have Case 1 in step $2i+1$. Thus I is low Δ_2.

(7) Now define satisfaction for $\Sigma_k^\bullet \cup \Pi_k^\bullet$-formulas on I using the satisfaction on K, i.e. put

$$I \models \varphi(\mathbf{a}) \text{ iff } K \models \varphi(\mathbf{a})$$

for $\mathbf{a} \in I$ and $\varphi \in \Sigma_k^\bullet \cup \Pi_k^\bullet$; use the proof of 1.66 (3) to show that I is indeed a satisfaction, i.e. Tarski's conditions hold true.

(8) Extend this to a satisfaction $\models'$ on I for $\Sigma_{k+1}^\bullet \cup \Pi_{k+1}^\bullet$. We show that $\models'$ is also $\Delta_1(M^*)$. It suffices to show that satisfaction for $\Sigma_{k+1}^\bullet$ is $\Delta_1(M^*)$ so let $\models'$ mean this satisfaction. Clearly, $\models'$ is $\Sigma_1(M^*)$; we show that it is also

$\Pi_1(M^*)$. We know that K is a model of $I\Sigma_k$ (since $M \prec K$ and $M \models I\Sigma_k$) and I is a cut in K. By underspill in K, for each $\Sigma^\bullet_{k+1}$-formula $(\exists x)\varphi(x, y)$,

$$I \models \neg(\exists x)\varphi(x, a) \text{ iff } (\exists b \in K - I)(K \models \neg(\exists x < b)\varphi(x, a)),$$

which is $\Sigma_1(K, I)$ and therefore $\Sigma_1(M^*)$.

(9) Now we may prove $M \prec_{k+1} I$ exactly as in 1.66 (2), and $I \models B\Sigma_{k+1}$ as in 1.67 (i.e. in 1.22). This completes the proof of 4.27.

(c) Appendix: Another Conservation Result

We shall sketch a proof of the following

4.10 Theorem. $I\Sigma_1$ proves that ACA_0 is a conservative extension of PA.

Proof. We shall imitate the model-theoretic proof presented in III.1.16. In $I\Sigma_1$ we have to deal with LL_1 sets and a reasonably good satisfaction; this needs some care.

In $I\Sigma_1$, assume $Con^\bullet(PA^\bullet)$ and investigate PA^* – the open Δ_1 theory with a language L^* extending $PA^\bullet$ conservatively and such that for each formula$^\bullet$ $\varphi(x, \mathbf{y})$ of $PA^\bullet$ there is a PA^*-provably equivalent open $PA^\bullet$-formula$^\bullet$. Let M be a full LL_1 model of PA^*. Extend M to a model K by adding (codes of) parametrically M-definable subsets of M as sets (in such a way that the set of all such sets is LL_1). Define $K \models X = Y$ iff $M \models (\forall x)(\varphi(x) \equiv \psi(x))$ where X is defined by φ and Y by ψ. Make K into a model with absolute equality by the same trick as in subsection (c). If X is given by $\varphi(x)$ then $K \models x \in X$ means $M \models \varphi(x)$. Define a predicate Set ranging over all sets of K, a function $L(X)$ selecting the least element of X if X is non-empty and $A(X, Y)$ selecting the least element of the symmetric difference of X, Y if it is non-empty. Finally, for each *open* formula $\Phi(x, \mathbf{y}, \mathbf{Z})$ of PA^* add a function symbol $F_\Phi(\mathbf{y}, \mathbf{Z})$ and the axiom

$$Set(F_\Phi(\mathbf{y}, \mathbf{Z}) \,\&\, [\Phi(x, \mathbf{y}, \mathbf{Z}) \equiv x \in F_\Phi(\mathbf{y}, \mathbf{Z}).$$

Observe that this can be done in such a way as to make the resulting model (denoted again by K) LL_1.

Show that the open theory T_0 of K implies ACA_0 and $Th(M)$ (the theory of M). T_0 is consistent by I.4.26 (relativized; thus $ACA_0 + Th(M)$ is also consistent. This completes the proof. □