5. Stability

In this chapter we state and prove the basic definitions and theorems relevant to all stable theories. The first section contains the most fundamental material. Here a freeness relation (see Definition 3.3.1) called forking independence is developed which agrees with Morley rank independence on a t.t. theory. Many of the theorems proved earlier for t.t. theories can be generalized to stable theories, the class of theories on which forking independence exists.

Sections 5.1 to 5.3 contain material which anyone working in stable theories must know. The first-time reader should feel free to skip the proofs in Section 5.5, although it is important to know the statements of the results found there. The forking independence relation is analyzed more deeply in Section 5.6. A class of types (namely those having weight 1) is isolated on which a well-behaved dimension theory exists.

5.1 Stability

Here we define a broad class of theories (called the stable theories) on which there is a freeness relation satisfying the conditions specified in Definition 3.3.1. As with t.t. theories, the freeness relation is defined via a rank (more accurately, a family of ranks). Intuitively, each of these ranks could be described as "Morley rank relative to a finite set of formulas". The overall goal of the section is to develop the relevant ranks and notion of freeness, prove the definability of types in stable theories and relate its existence to the number of types over sets.

Remember: Every complete theory discussed is assumed to have built-in imaginaries.

5.1.1 Ranks and Definability

Writing the formula φ in the form $\varphi(\bar{x}, \bar{y})$ indicates that the free variables in φ are in $\bar{x}\bar{y}$, $\bar{x}$ should be regarded as a sequence of free variables in the usual sense, but $\bar{y}$ is a placeholder for a sequence of parameters. For example, a general quadratic polynomial in x can be written as $\varphi(x, abc) = ax^2 + bx + c$, where a, b and c range over the possible coefficients. We call $\bar{x}$ the *object variables* and $\bar{y}$ the *parameter variables* in φ. When Δ is a set of formulas we write $\Delta = \Delta(\bar{x})$ when the object variables of any $\varphi \in \Delta$ are in $\bar{x}$. Following the conventions previously adopted for theories with built-in imaginaries, we will usually drop the bars from the variables and just write, e.g., $\Delta = \Delta(x)$.

When $\Delta = \Delta(x)$ is a set of formulas and A is a set we call the type p over A a *Δ–type* if each formula in p is of the form $\varphi(x, a)$ or $\neg\varphi(x, a)$ for some $a \in A$ and $\varphi(x, y) \in \Delta$. A Δ–type p is called *complete* if for all $a \in A$ and $\varphi(x, y) \in \Delta$, $\varphi(x, a)$ or $\neg\varphi(x, a)$ is in p. When $\Delta = \{\varphi\}$, for some formula φ, a Δ–type is called a *φ–type.* Let $S_\Delta(A)$ denote the set of complete Δ–types over A, $S_\varphi(A) = S_{\{\varphi\}}(A)$. When p is a type over A in the variable x, $p \restriction \Delta$ denotes the Δ–type

$$\{\varphi(x, a) \in p : \varphi(x, y) \in \Delta\} \cup \{\neg\varphi(x, a) \in p : \varphi(x, y) \in \Delta\}.$$

Our notion of freeness will be defined with the following class of ranks.

Definition 5.1.1. *Let T be a complete theory, $\Delta = \Delta(x)$ a set of formulas over $\emptyset$ and S the elements of $S(\mathfrak{C})$ in the variable x. For φ a formula in x and α an ordinal (or -1) the relation $R_\Delta(\varphi) = \alpha$, is defined as follows by recursion.*

(1) $R_\Delta(\varphi) = -1$ *if φ is inconsistent;*
(2) $R_\Delta(\varphi) = \alpha$ *if*

$$\{p \restriction \Delta : p \in S,\ \varphi \in p \text{ and } \neg\psi \in p \text{ for all formulas } \psi \text{ with } R_\Delta(\psi) < \alpha\}$$

is finite and nonempty.

For p any type in x, $R_\Delta(p)$ is defined to be

$$\inf\{R_\Delta(\varphi) : \varphi \text{ is implied by } p\}.$$

(Thus, for $p \in S$, $R_\Delta(p)$ is $\inf\{R_\Delta(\varphi) : \varphi \in p\}$*.) The relation $R_\Delta(p) = \alpha$ is read* the Δ–rank of p is α. *If there is no α with $R_\Delta(p) = \alpha$ we write $R_\Delta(p) = \infty$ and say that the Δ–rank of p does not exist.*

By convention, we only write $R_\Delta(p)$ when there is an x such that $\Delta = \Delta(x)$, Δ is a set of formulas over $\emptyset$ and p is a type in x.

As with Morley rank, Δ–rank is preserved under conjugacy. The rank $R_\Delta(-)$ is what Shelah calls $R(-, \Delta, \aleph_0)$ (see [She90, p.21]).

The following is little more than a restatement of Lemma 3.3.1.

Lemma 5.1.1. *Let T be a complete theory, $\Delta = \Delta(x)$ a set of formulas over $\emptyset$, p a type in x, S the set of elements of $S(\mathfrak{C})$ in x and α an ordinal.*

(i) If φ is a formula in x and Δ contains $x = y$, $R_\Delta(\varphi) = 0$ if and only if φ is algebraic.

(ii) $R_\Delta(p) = \alpha$ *if and only if there is a formula* φ *implied by* p *such that* $\{\, q \restriction \Delta : \ q \in S, \ \varphi \in q \text{ and } \ R_\Delta(q) = \alpha \,\}$ *is finite, nonempty and equal to* $\{\, q \restriction \Delta : \ q \in S, \ q \supset p \text{ and } \ R_\Delta(q) = \alpha \,\}$.

(iii) If $R_\Delta(p) = \alpha$ *there is a* $q \in S$ *such that* $q \supset p$ *and* $R_\Delta(q) = \alpha$.

(iv) If $p \in S$ *and* $R_\Delta(p) = \alpha$ *there is a* $\varphi \in p$ *such that* $p \restriction \Delta$ *is the only element of* $\{\, q \restriction \Delta : \ q \in S, \ \varphi \in q \text{ and } \ R_\Delta(q) \geq \alpha \,\}$.

(v) $R_\Delta(p) \geq \alpha$ *if and only if for all* $\beta < \alpha$ *and all* φ *implied by* p, $\{\, q \restriction \Delta : \ q \in S, \ \varphi \in q \text{ and } \ R_\Delta(q) \geq \beta \,\}$ *is infinite.*

(vi)

$$\begin{aligned} &R_\Delta(\varphi) \text{ is the least ordinal } \ \alpha \text{ such that} \\ &\qquad \{\, q \restriction \Delta : \ q \in S, \ \varphi \in q \text{ and } \ R_\Delta(p) \geq \alpha \,\} \text{ is finite.} \end{aligned} \tag{5.1}$$

(vii) If $\Gamma(x) \supset \Delta(x)$, $R_\Gamma(p) \geq R_\Delta(p)$. *When* Γ *is the set of all formulas with object variable* x, $R_\Gamma(p) = MR(p)$. *Thus* $MR(p) \geq R_\Delta(p)$.

Proof. (i) If φ is algebraic, then the set of complete $\Delta-$types over $\mathfrak{C}$ consistent with φ is finite, hence $R_\Delta(\varphi) = 0$. Suppose φ is nonalgebraic, satisfied by the distinct elements a_i, $i < \omega$. For each $i < \omega$, $\{x = a_i\}$ extends to a complete $\Delta-$type r_i over $\mathfrak{C}$ consistent with φ. Then $\{\, r_i : \ i < \omega \,\}$ consists of infinitely many contradictory $\Delta-$types consistent with φ, proving that $R_\Delta(\varphi) > 0$.

Each of (i)–(vi) is proved like the corresponding part in Lemma 3.3.1. The proof of part (vii) is assigned as Exercise 5.1.2.

We will tacitly assume that any finite set $\Delta(x)$ of formulas under consideration contains $x = y$. This ensures that a formula has $\Delta-$rank 0 exactly when it is algebraic.

Definition 5.1.2. *Let* Δ *be a set of formulas in* x *and* p *a type in* x *with* $R_\Delta(p) = \alpha < \infty$. *Then the* $\Delta-$multiplicity *of* p, *denoted* $\text{Mult}_\Delta(p)$, *is the maximum* m *such that there are* $q_1, \ldots, q_m \in S(\mathfrak{C})$ *with* $q_i \supset p$, $R_\Delta(q_i) = \alpha$, *for* $1 \leq i \leq m$, *and* $i \neq j \implies q_i \restriction \Delta \neq q_j \restriction \Delta$. *(Equivalently, the* $\Delta-$*multiplicity of* p *is the maximal* m *such that there are* m *complete* $\Delta-$*types* $r_1, \ldots, r_m$ *over* $\mathfrak{C}$ *such that* $R_\Delta(p \cup r_i) = R_\Delta(p)$, *for* $1 \leq i \leq m$.*) Let* $(R, \text{Mult})_\Delta(p)$ *denote the pair* $(R_\Delta(p), \text{Mult}_\Delta(p))$.

Similar to the behavior of Morley rank and degree, for any set of formulas Δ and any type p there is a formula φ implied by p such that $(R, \text{Mult})_\Delta(\varphi) = (R, \text{Mult})_\Delta(p)$. Repeating the argument in Remark 3.3.1, when p is closed under finite conjunctions there is a $\varphi \in p$ such that $(R, \text{Mult})_\Delta(\varphi) = (R, \text{Mult})_\Delta(p)$.

Notation. If Δ is a finite set of formulas, p is a type and $X = p(\mathfrak{C})$, $(R, \text{Mult})_\Delta(X) = (R, \text{Mult})_\Delta(p)$.

Definition 5.1.3. *A complete theory* T *is called* stable *if for all* $p \in S(\mathfrak{C})$ *and all finite* Δ, $R_\Delta(p) < \infty$.

For any finite set of formulas Δ and any formula φ, $R_\Delta(\varphi) \leq MR(\varphi)$, hence a totally transcendental theory is stable. Here are some more examples.

Example 5.1.1. Let $L = \{E_i : i < \omega\}$ and T be T_0^{eq}, where T_0 is the theory in L saying that each E_i is an equivalence relation with only infinite classes, $E_0(x, y)$ is $x = x$ and for every $i \geq 0$, E_{i+1} refines each E_i–class into two E_{i+1}–classes. Below, x denotes a variable in the sort of the equivalence relations. Let $\Delta = \{x = y, E_{i_1}, \ldots, E_{i_j}\}$ where $i_1 < \ldots < i_j$. Given a formula φ the number of nonalgebraic complete Δ–types over $\mathfrak{C}$ consistent with φ is equal to the number E_{i_j}–classes consistent with φ. Thus $R_\Delta(p) \leq 1$ for any $p \in S(\mathfrak{C})$ in the variable x. If $\Gamma(x)$ is any finite set of formulas there is some set $\Delta = \{x = y, E_{i_1}, \ldots, E_{i_j}\}$ such that every Γ–type is a Δ–type (by elimination of quantifiers). Thus, for any $p \in S(\mathfrak{C})$, $R_\Gamma(p) \leq 1$. Again using that T has elimination of quantifiers, for z a variable of any sort and $\Gamma(z)$ a finite set of formulas, $R_\Gamma(z = z) < \omega$. This proves the stability of T. The reader should notice that T is not totally transcendental.

Example 5.1.2. Alter the last example by requiring each E_i–class to be refined into *infinitely* many E_{i+1}–classes. Suppose $\Delta = \{x = y, E_0, \ldots, E_i\}$.

Claim. (i) If $j \geq i$ and a is an element, $(R, \text{Mult})_\Delta(E_j(x, a)) = (1, 1)$.
(ii) If $j < i$ and a is an element, $R_\Delta(E_j(x, a)) = i - j$.

All nonalgebraic completions of $E_j(x, a)$ over $\mathfrak{C}$ have the same Δ–type, so (i) holds.

(ii) First let $j = i - 1$. Then $Q = \{q \restriction \Delta : q \in S(\mathfrak{C}),\ E_{i-1}(x, a) \in q$ and $\neg\varphi \in q$ for all φ with $R_\Delta(\varphi) < 2\}$ is contained in $\{q \restriction \Delta : q \in S(\mathfrak{C}),\ E_{i-1}(x, a) \in q$ and $\neg E_i(x, b) \in q$ for all $b \in \mathfrak{C}\} = P$. Since $|P| = 1$, Q has cardinality $< |\mathfrak{C}|$, so $R_\Delta(E_{i-1}(x, a)) \leq 2$. Since $E_{i-1}(x, a)$ is contained in infinitely many elements of $S_\Delta(\mathfrak{C})$ of Δ–rank 1, $R_\Delta(E_{i-1}(x, a)) = 2$.

The previous paragraph can be generalized to a downward induction which proves (ii) for all $j < i$, proving the claim.

Since T is quantifier-eliminable the claim can be used to show that for any formula $\varphi(x)$ and finite set Γ there is a set $\Delta = \{x = y, E_0, \ldots, E_i\}$ such that $R_\Gamma(\varphi) \leq R_\Delta(\varphi)$ and $R_\Delta(\varphi) \leq i$. Thus, T is stable. Moreover, for every $i < \omega$, there is a finite $\Delta(x)$ such that $R_\Delta(x = x) \geq i$.

This example also shows that the Δ–rank of a formula depends quite heavily on Δ. As Δ becomes larger the Δ–rank of $x = x$ increases without a finite bound.

Example 5.1.3. Let M be an infinite module over a ring R formulated in the natural language for R–modules. We will show in Corollary 5.3.4 that $T = Th(M)$ is stable.

Definition 5.1.4. *Let T be a stable theory.*

(i) We say that $p \in S(A)$ does not fork *over $B \subset A$ if for all finite Δ, $R_\Delta(p) = R_\Delta(p \restriction B)$. When p does not fork over B, p is called a* nonforking *extension of $p \restriction B$. The negation of nonforking is* forking.

(ii) A type q over A (perhaps incomplete) is said to fork over $B \subset A$ *if every $p \in S(A)$ containing q forks over B.*

(iii) For A, B and C sets we say that A is forking independent from B over C *and write $A \underset{C}{\perp} B$ if for all finite tuples a from A, $tp(a/B \cup C)$ does not fork over C. The negation of forking independent is* forking dependent *and is denoted $A \underset{C}{\not\perp} B$. We usually shorten these terms to "independent" or "dependent" since it is clear from context that we mean "forking independent" "forking dependent".*

Remark 5.1.1. In a t.t. theory we have already adopted "independent" to mean Morley rank independent. We will show in Corollary 5.1.4 however that Morley rank independence and forking independence are equivalent relations in a t.t. theory, eliminating the apparent conflict. Below we also redefine other terms (like "stationary") later showing the equivalence (in a t.t. theory) of this property with the one defined earlier.

Remark 5.1.2. If T is stable and $p \in S(A)$ forks over $B \subset A$ there is a formula $\varphi \in p$ such that $p \restriction B \cup \{\varphi\}$ forks over B. (Find a finite Δ such that $R_\Delta(p) < R_\Delta(p \restriction B)$ and a $\varphi \in p$ such that $R_\Delta(p) = R_\Delta(\varphi)$.)

Definition 5.1.5. *A collection of sets $\mathcal{A}$ is called* independent over B *or* B–independent *if each $A \in \mathcal{A}$ is independent from $\bigcup(\mathcal{A} \setminus \{A\})$ over B.*

Remark 5.1.3. Conditions (1), (2), (3), (5) and (7) in the definition of a freeness relation (Definition 3.3.1) hold for forking independence in a stable theory.

Verifications. Finite character and monotonicity (1) are clear, as is transitivity of independence (3). Since Δ–rank is invariant under automorphisms of $\mathfrak{C}$ so is independence (i.e., (5) holds). For any complete p and finite Δ there is a $\varphi \in p$ with $R_\Delta(p) = R_\Delta(\varphi)$. Thus, there is a set $B \subset dom(p)$ of cardinality $\leq |T|$ such that p does not fork over B, proving (2). If Δ contains the formula $x = y$ and $p \in S(\mathfrak{C})$, then $R_\Delta(p) = 0$ if and only if p is algebraic. Thus, $b \notin acl(A)$ implies that b depends on b over A; i.e., reflexivity (7) holds.

One part of (6) also holds: If $p \in S(A)$ and Δ is finite, $\{q \restriction \Delta : q \in S(\mathfrak{C})$ is a nonforking extension of $p\}$ is finite. Thus, the number of nonforking extensions of p in $S(\mathfrak{C})$ is $\leq 2^{|T|}$. What is not clear is that every complete type has at least one nonforking extension in $S(\mathfrak{C})$. This existence result as well as symmetry (4) will require real effort to verify.

Lemma 5.1.2. *Given a stable T and set A, any $p \in S(acl(A))$ does not fork over A.*

Proof. See Exercise 5.1.3.

Definition 5.1.6. *In a stable theory a complete type is called* stationary *if it has a unique nonforking extension. When p is complete, does not fork over $A \subset dom(p)$ and $p \restriction A$ is stationary we say that p is* stationary over A.

The following slightly extends Definition 3.3.7.

Definition 5.1.7. *Let T be a complete theory, $\Delta = \Delta(x)$ a set of formulas and $p \in S_\Delta(B)$. Then p is* definable over A *if for all formulas $\varphi(x, y) \in \Delta$, there is a formula $\psi(y)$ over A such that for all $b \in B$,*

$$\varphi(x, b) \in p \iff \models \psi(b).$$

Definability and nonforking are linked with

Theorem 5.1.1 (Definability Theorem). *Let T be stable and $p \in S(\mathfrak{C})$. Then p does not fork over A if and only if p is definable over $acl(A)$.*

The definability of nonforking extensions will be proved first. The analogue of this result for t.t. theories was proved using Morley sequences. There is a similar proof in this setting, however the alternative argument given here gives more insight into the properties of $\Delta-$rank. Order lexicographically the collection $I = \{ (\beta, k) : \beta$ is an ordinal and $1 \leq k < \omega \}$; i.e., $(\beta, k) < (\gamma, l)$ if $\beta < \gamma$ or $\beta = \gamma$ and $k < l$.

Lemma 5.1.3. *Let T be a complete theory, Δ a finite set of formulas, α an ordinal and $m < \omega$.*

(i) For any formula $\varphi(x, y)$ there is a set of formulas $\Gamma(y)$ such that for all a, $(R, \mathrm{Mult})_\Delta(\varphi(x, a)) \geq (\alpha, m)$ if and only if a realizes Γ.

(ii) For $\varphi(x)$ a formula over A with $(R, \mathrm{Mult})_\Delta(\varphi) = (\alpha, m)$ and $\delta(x, y) \in \Delta$ there is a formula $\psi(y)$ over A such that for all b, $\models \psi(b)$ if and only if $(R, \mathrm{Mult})_\Delta(\varphi(x) \wedge \delta(x, b)) = (\alpha, m)$.

Proof. The proof of the following preliminary fact is left to the reader.

Claim. Given $m > 1$ and α an ordinal,

$$(R, \mathrm{Mult})_\Delta(\varphi(x)) \geq (\alpha, m)$$

if and only if

there are m_1, $m_2 \geq 1$ with $m_1 + m_2 = m$, $\delta \in \Delta$ and b such that
- $(R, \mathrm{Mult})_\Delta(\varphi(x) \wedge \delta(x, b)) \geq (\alpha, m_1)$ and
- $(R, \mathrm{Mult})_\Delta(\varphi(x) \wedge \neg\delta(x, b)) \geq (\alpha, m_2)$.

Part (i) is proved by induction on the pairs (β, k), where β is an ordinal and $1 \leq k < \omega$. The minimal element of the order I is $(0, 1)$ and $(R, \mathrm{Mult})_\Delta(\varphi(x, a)) \geq (0, 1)$ exactly when $\models \exists x \varphi(x, a)$. Assume (i) holds for all elements of I less than (α, m). For $(\beta, n) < (\alpha, m)$ let $\Gamma_{(\beta,n)}(y)$ be a set of formulas such that for all a, $(R, \mathrm{Mult})_\Delta(\varphi(x, a)) \geq (\beta, n)$ if and only if a realizes $\Gamma_{(\beta,n)}$. First suppose that $m = 1$. Note:

$$(R,\mathrm{Mult})_\Delta(\varphi(x,a)) \geq (\alpha,1) \text{ if and only if}$$
$$\forall\beta<\alpha,\ \forall n(\ (R,\mathrm{Mult})_\Delta(\varphi(x,a)) \geq (\beta,n)\ ).$$

Thus, $(R,\mathrm{Mult})_\Delta(\varphi(x,a)) \geq (\alpha,1)$ if and only if a realizes $\bigcup_{\beta<\alpha,n<\omega}\Gamma_{(\beta,n)} = \Gamma_{(\alpha,1)}$. Supposing that $m>1$ the claim yields $m_1,\ m_2 \geq 1$ with $m_1+m_2 = m$, $\delta\in\Delta$ and b such that $(R,\mathrm{Mult})_\Delta(\varphi(x,a)\wedge\delta(x,b)) \geq (\alpha,m_1)$ and $(R,\mathrm{Mult})_\Delta(\varphi(x,a)\wedge\neg\delta(x,b)) \geq (\alpha,m_2)$. Let Γ_1 and Γ_2 be sets of formulas such that for all a and b,

- $(R,\mathrm{Mult})_\Delta(\varphi(x,a)\wedge\delta(x,b)) \geq (\alpha,m_1)$ if and only if ab realizes $\Gamma_1(y,z)$ and
- $(R,\mathrm{Mult})_\Delta(\varphi(x,a)\wedge\neg\delta(x,b)) \geq (\alpha,m_2)$ if and only if ab realizes $\Gamma_2(y,z)$.

Let

$$\Theta_{(m_1,m_2)}(y) = \{\,\exists z(\bigwedge\Gamma_1'(y,z)\wedge\bigwedge\Gamma_2'(y,z)) :\ \Gamma_1'\subset\Gamma_1,\ \Gamma_2'\subset\Gamma_2 \text{ finite }\},$$

a set of formulas which holds if and only if $\Gamma_1(y,z)\cup\Gamma_2(y,z)$ is consistent. Let $\Gamma(y)$ be a set of formulas such that a realizes $\Gamma(y)$ if and only if for some $m_1,\ m_2>1$ with $m_1+m_2=m$, a realizes $\Theta_{(m_1,m_2)}(y)$. This type Γ satisfies the requirements.

(ii) For all b and $\delta\in\Delta$,

$$\begin{array}{rcl} (R,\mathrm{Mult})_\Delta(\varphi(x)\wedge\delta(x,b)) & = & (\alpha,m) \\ \iff (R,\mathrm{Mult})_\Delta(\varphi(x)\wedge\neg\delta(x,b)) & < & (\alpha,1). \end{array}$$

(If $(R,\mathrm{Mult})_\Delta(\varphi(x)\wedge\neg\delta(x,b)) < (\alpha,1)$, then any r in $P=\{p\restriction\Delta :\ p\in S(\mathfrak{C}),\ \varphi\in p \text{ and } R_\Delta(p)\geq\alpha\}$ contains $\delta(x,b)$. Since $|P| = m$, $(R,\mathrm{Mult})_\Delta(\varphi(x)\wedge\delta(x,b)) = (\alpha,m)$. The converse follows immediately from the claim.)

By compactness and (i) there is a formula $\psi(y)$ over A such that $\models\psi(b)$ if and only if $R_\Delta(\varphi(x)\wedge\neg\delta(x,b))<\alpha$. Thus, $\models\psi(b)$ if and only if $(R,\mathrm{Mult})_\Delta(\varphi(x)\wedge\delta(x,b)) = (\alpha,m)$.

Lemma 5.1.4. *Let T be stable, Δ is a finite set of formulas and $p\in S_\Delta(\mathfrak{C})$ such that $R_\Delta(p\cup\{\gamma\}) = R_\Delta(\gamma)$ for some formula γ over A. Then p is definable over $acl(A)$.*

Proof. Let $q=p\cup\{\gamma\}$, $\alpha=R_\Delta(q)$ and notice that $\mathrm{Mult}_\Delta(q)=1$. Let $\xi(x)$ be a formula implied by q such that $(R,\mathrm{Mult})_\Delta(q) = (R,\mathrm{Mult})_\Delta(\xi) = (\alpha,1)$. In other words, p is the only $p_0\in S_\Delta(\mathfrak{C})$ such that $(R,\mathrm{Mult})_\Delta(p_0\cup\{\xi\}) = (\alpha,1)$. By Lemma 5.1.3(ii), for any $\delta(x,y)\in\Delta$ there is a formula $\psi(y)$ such that for all b, $\models\psi(b)$ if and only if $(R,\mathrm{Mult})_\Delta(\xi(x)\wedge\delta(x,b)) = (\alpha,1)$. Thus, $\delta(x,b)\in p$ if and only if $\models\psi(b)$.

It remains to show that ψ is equivalent to a formula over $acl(A)$, By Lemma 4.1.2, it suffices to show that ψ is almost over A. Since $R_\Delta(p\cup\{\gamma\}) = R_\Delta(\gamma)$ and γ is over A, $P=\{\,r\restriction\Delta :\ r \text{ is conjugate over } A \text{ to } q\,\}$ is finite. Furthermore, any formula conjugate over A to ψ defines some element of P. Since formulas defining the same Δ–type are equivalent, ψ is almost over A.

One direction of the Definability Theorem follows immediately:

Lemma 5.1.5. *If T is stable and $p \in S(\mathfrak{C})$ does not fork over A, then p is definable over $acl(A)$.*

Proving the other direction of the Definability Theorem, the Symmetry Lemma and the existence of nonforking extensions all involve the order property.

Definition 5.1.8. *Let T be complete and $\varphi(x, y)$ a formula. Then φ has the* order property *if there are sets of elements $\{ a_i : i < \omega \}$ and $\{ b_i : i < \omega \}$ such that $\models \varphi(a_i, b_j)$ if and only if $i \leq j < \omega$. We say that T has the* order property *if there is a formula of T with the order property.*

It can be shown that T is stable if and only if T does not have the order property. In fact, some authors take "T does not have the order property" to be the definition of stable. One direction of the equivalence is

Lemma 5.1.6. *A stable theory T does not have the order property.*

Proof. Suppose that T is stable. For Δ finite, A a set and p in $S_\Delta(A)$ there is a formula φ implied by p such that $R_\Delta(p) = R_\Delta(\varphi)$. Furthermore, for any such φ there are finitely many $r \in S_\Delta(A)$ with $R_\Delta(r \cup \{\varphi\}) = R_\Delta(\varphi)$. A formula implied by a $\Delta-$type over A is implied by a finite $\Delta-$type over A. There are $\leq |A| + \aleph_0$ finite $\Delta-$types over A, so

$$|S_\Delta(A)| \leq |A| + \aleph_0. \tag{5.2}$$

Now assume, towards a contradiction, that there are $\varphi(x, y)$, $\{ a_i : i < \omega \}$ and $\{ b_i : i < \omega \}$ such that $\models \varphi(a_i, b_j)$ if and only if $i \leq j < \omega$. We will contradict (5.2) for $\Delta = \{\varphi\}$. Let $(Y, \leq)$ be a dense linear order without endpoints which has a dense subset X of cardinality $\kappa < |Y|$. (Note: κ must be infinite.) Let $C = \{ c_i : i \in Y \}$ and $D = \{ d_i : i \in X \}$ be sets of constant symbols and Φ the set of sentences $\{ \varphi(c_i, d_j) : i \in Y, j \in X$ and $i \leq j \} \cup \{ \neg\varphi(c_i, d_j) : i \in Y, j \in X$ and $i > j \}$. Compactness proves the consistency of Φ since for any finite $\Psi \subset \Phi$ the constants appearing in Ψ can be interpreted by some of the a_i's and b_j's to obtain a model of it. Thus, without loss of generality, C and D are subsets of the universe. However, the density of X forces each c_i to have a different $\varphi-$type over D, contradicting (5.2) since $|Y| > |X| \geq \aleph_0$. This proves the lemma.

Lemma 5.1.7. *Let T be stable, $\varphi(x, y)$ a formula over $\emptyset$, and $\varphi'(y, x)$ the formula φ with y as the object variable and x as the parameter variable. Suppose that $p(x) \in S_\varphi(\mathfrak{C})$ and $q(y) \in S_{\varphi'}(\mathfrak{C})$ are definable over A and consistent with p_0, $q_0 \in S(A)$, respectively. Then for all a realizing p_0 and b realizing q_0, $\varphi(x, b) \in p$ if and only if $\varphi'(y, a) \in q$; i.e.,*

$$\varphi(x, b) \in p \text{ if and only if } \varphi(a, y) \in q.$$

Proof. Assume to the contrary there are a' realizing p_0 and b' realizing q_0 such that $\neg\varphi(x, b') \in p$ and $\varphi(a', y) \in q$. By the definability over A of p and q, for all a realizing p_0, $\varphi(a, y) \in q$ and similarly for realizations of q_0. That is,

$$\forall a \text{ realizing } p_0, \forall b \text{ realizing } q_0(\neg\varphi(x, b) \in p \text{ and } \varphi(a, y) \in q).$$

Define sets of elements $\{ a_i : i < \omega \}$ and $\{ b_i : i < \omega \}$ as follows. Assuming that a_i and b_i have been defined for $i < k$ let a_k realize the restriction of p to $A \cup \{ a_i : i < k \} \cup \{ b_i : i < k \}$ and b_k realize the restriction of q to $A \cup \{ a_i : i \leq k \} \cup \{ b_i : i < k \}$. Then, $\models \varphi(a_i, b_j)$ if and only if $i \leq j$. This contradicts Lemma 5.1.6, to prove the lemma.

Lemma 5.1.8. *Given $A = acl(A)$, Δ a finite set of formulas and $p \in S(A)$, there is at most one $q \in S_\Delta(\mathfrak{C})$ which is definable over A and consistent with p.*

Proof. Assume to the contrary that p is consistent with q, $q' \in S_\Delta(\mathfrak{C})$, both definable over A, and for some formula $\varphi \in \Delta$, $\varphi(x, b) \in q$ and $\neg\varphi(x, b) \in q'$. Let $r_0(y) = tp(b/A)$. Let φ' be the formula φ with object variable y and parameter variable x. Let $r(y) \in S_{\varphi'}(\mathfrak{C})$ be such that $R_{\varphi'}(r(y) \cup r_0(y)) = R_{\varphi'}(r_0(y))$. By Lemma 5.1.4, r is definable over A. Applying Lemma 5.1.7 to both q and q',

- for all a realizing p, $\varphi(a, y) \in r$ and,
- for all a realizing p, $\neg\varphi(a, y) \in r$.

This contradiction proves the lemma

From here we can quickly prove the existence of nonforking extensions, the other direction of the Definability Theorem and the Symmetry Lemma.

Corollary 5.1.1. *If T is stable and $p \in S(A)$ there is a $q \in S(\mathfrak{C})$, $q \supset p$, such that q does not fork over A.*

Proof. Let $p \in S(A)$. Every extension of p in $S(acl(A))$ is a nonforking extension (by Lemma 5.1.2) so may as well assume that $A = acl(A)$. For any finite set of formulas Δ there is a $q_\Delta \in S_\Delta(\mathfrak{C})$ with $R_\Delta(q_\Delta \cup p) = R_\Delta(p)$. By Lemma 5.1.4, q_Δ is definable over A, hence (by Lemma 5.1.8) q_Δ is the unique complete Δ–type over $\mathfrak{C}$ such that $R_\Delta(q_\Delta \cup p) = R_\Delta(p)$. If $\Gamma \supset \Delta$ are finite sets of formulas, q_Γ contains a Δ–type which is definable over A, hence $q_\Gamma \supset q_\Delta$. Thus, $q = \bigcup_\Delta q_\Delta$ is a nonforking extension of p in $S(\mathfrak{C})$.

Proof of the Definability Theorem (Theorem 5.1.1). One direction of the theorem was already proved in Lemma 5.1.5. Conversely, given $p \in S(A)$ suppose $q \in S(\mathfrak{C})$ extends p and is definable over $acl(A)$. Let p' be the restriction of q to $acl(A)$. In the proof of Corollary 5.1.1 we show that p' has a unique extension in $S(\mathfrak{C})$ which is definable over $acl(A)$ and this extension does not fork over $acl(A)$. Since p' is a nonforking extension of p (by Lemma 5.1.2) we have proved that q is a nonforking extension of p.

The symmetry of forking dependence follows easily from Lemma 5.1.7:

Corollary 5.1.2 (Symmetry Lemma). *If T is stable then for all sets A, B and C,*

$$A \underset{C}{\perp} B \implies B \underset{C}{\perp} A.$$

Proof. Assuming the lemma to be false there are elements a and b and a set C such that a is independent from b over C, but b depends on a over C. These same relations hold when we replace C by $acl(C)$, so we may as take C to be algebraically closed. Let $q_0(y) = tp(b/C)$ and $\varphi(a, y) \in tp(b/C \cup \{a\})$ a formula such that $q_0 \cup \{\varphi(a, y)\}$ forks over C. Let $p \in S(\mathfrak{C})$ be a nonforking extension of $tp(a/C \cup \{b\})$ which by transitivity of independence also does not fork over C. Let $q \in S(\mathfrak{C})$ be a nonforking extension of q_0. By Lemma 5.1.5 and the fact that C is algebraically closed, both p and q are definable over C. Since $\varphi(x, b) \in p$, Lemma 5.1.7 implies that $\varphi(a, y) \in q$, contradicting that q does not fork over C.

This completes the proof of

Corollary 5.1.3. *In a stable theory forking independence is a freeness relation.*

The following corollaries all follow easily from the Definability Theorem and a couple other key results above. Stating them rounds out our picture of forking independence.

Corollary 5.1.4. *Suppose that T is a t.t. theory. Then $p \in S(\mathfrak{C})$ does not fork over A if and only if $MR(p) = MR(p \restriction A)$. A type is stationary if and only if it has Morley degree 1.*

Proof. Left to the reader in the exercises.

Corollary 5.1.5. *Let T be stable.*

(i) Every $p \in S(acl(A))$ is stationary.

(ii) If $p \in S(\mathfrak{C})$ is definable over A, then p does not fork over A.

(iii) If $p \in S(\mathfrak{C})$ is stationary over A, then p is definable over A.

Proof. (i) This follows immediately from the Definability Theorem and Lemma 5.1.8.

(ii) Assuming that $p \in S(\mathfrak{C})$ is definable over A, p does not fork over $acl(A)$ by the Definability Theorem. Since every $q \in S(acl(A))$ does not fork over A, p does not fork over A by the transitivity of independence.

(iii) Simply unraveling the definitions shows that if $q \in S(\mathfrak{C})$, $f \in \mathrm{Aut}(\mathfrak{C})$, φ is a formula over $\emptyset$ and ψ defines $q \restriction \varphi$, then $f(\psi)$ defines $f(q) \restriction \varphi$. If

$f \in \mathrm{Aut}(\mathfrak{C})$ fixes A then p and $f(p)$ are both nonforking extensions of $p \upharpoonright A$, hence $p = f(p)$. Thus, given a formula φ and ψ which defines $p \upharpoonright \varphi$, ψ is equivalent to $f(\psi)$ for any $f \in \mathrm{Aut}(\mathfrak{C})$ fixing A. By Lemma 3.3.8, ψ is equivalent to a formula over A.

Remark 5.1.4. In a stable theory it is possible to find sets $B \supset A$ and $p \in S(B)$ which is definable over A but forks over A. (Compare this with (ii) of the previous corollary.) However, when $p \in S(M)$, M a model and p is definable over $A \subset M$, p does not fork over A (see Exercise 5.1.7).

Remark 5.1.5. Let T be stable and $p \in S(A)$ stationary. Then for any finite set of formulas Δ, $\mathrm{Mult}_\Delta(p) = 1$. (The proof is assigned as Exercise 5.1.8.)

Corollary 5.1.6. *If T is stable, then any $p \in S(A)$ is definable over A.*

Proof. Let $q \in S(\mathfrak{C})$ be a nonforking extension of p. By the Corollary 5.1.1 there is a defining scheme for q, hence for p, consisting of formulas over $acl(A)$. By Lemma 3.3.11, p is definable over A.

As with t.t. theories, this definability of types yields

Corollary 5.1.7. *Let T be stable and D a subset of $\mathfrak{C}$ definable over A. Then, for any k and $H \subset D^k$ definable over $\mathfrak{C}$ there is a $B \subset D$ such that H is definable over $A \cup B$.*

(The proof of the corollary is the same as that giving Proposition 3.3.3.)
A related property (stated for t.t. theories as Corollary 3.3.7) is:

Lemma 5.1.9. *Let T be stable, M a model, φ a formula over $A \subset M$ and a a tuple form $\varphi(\mathfrak{C})$. Then $tp(a/\varphi(M) \cup A)$ implies $tp(a/M)$.*

Proof. Let b be a tuple from M and $p = tp(b/\varphi(M) \cup A)$. For $\psi(x, y)$ a formula over $\emptyset$, p_ψ is defined by some formula $\psi'(y)$ over $\varphi(M) \cup A$ (by Corollary 5.1.6). We claim that ψ' also defines $tp(b/\varphi(\mathfrak{C}) \cup A)$. (Otherwise, there is tuple c from $\varphi(\mathfrak{C})$ such that $\models \neg(\psi'(c) \leftrightarrow \psi(b, c))$. Since M is a model there is such a c in M, a contradiction.)

Now let a' be a realization of $tp(a/\varphi(M) \cup A)$, b, $\psi(x, y)$ and $\psi'(y)$ as above. Then $\models \psi'(a')$ (since $\psi' \in tp(a/\varphi(M) \cup A)$), so $\models \psi(b, a')$, as required to prove the lemma.

The following set of results refines our knowledge of the defining scheme of a type.

Definition 5.1.9. *Let T be stable.*

(i) The stationary complete types p and q are called parallel, *denoted $p \parallel q$, if they have the same nonforking extension in $S(\mathfrak{C})$. A stationary type p is* based *on a set A if there is a $q \in S(A)$ parallel to p, in which case we let $p|A$ denote q.*

(ii) A type p is called a strong type over A *if $p \in S(acl(A))$. For any a, $tp(a/acl(A))$ is called the* strong type of a over A *and is denoted $stp(a/A)$, simply writing $stp(a)$ when $A = \emptyset$.*

Remark 5.1.6. (i) Unraveling the definitions, when p is a stationary complete type and $r \in S(\mathfrak{C})$, the unique nonforking extension of p, is definable over A, $p|A = r \restriction A$. Do not confuse $|$ with $\restriction$.

(ii) At least in regard to terminology we identify types which have the same set of realizations. If $p \in S(A)$ is stationary it has a unique extension q in $S(acl(A))$, hence is equivalent to q. Thus any stationary type may be called a strong type. Many of the properties we prove of strong types hold for all types in a fixed parallelism class. Indeed, Hrushovski (in [Hru86]) defines a strong type to be an equivalence class of types under parallelism.

In a t.t. theory when X is a degree 1 $\emptyset$−definable set, $a \in X$ and $MR(a/A) = MR(X)$ we say "a is generic over A" (see Definition 4.1.3). When T is stable, $p \in S(A)$ is stationary and $B \supset A$, we say "a is generic over A" if a realizes p and a is independent from B over A (in other words a realizes $p|B$).

The following elegant notation for the defining scheme of a strong type (due to Harrington and promoted by Hrushovski) greatly improves the readability of some proofs.

Notation. Let T be stable, $p \in S(A)$ a strong type in the variable x, $q = p|\mathfrak{C}$, $\varphi(x, y)$ a formula over $\emptyset$ and $\psi(y)$ the formula over A defining $p \restriction \varphi$. If b is any element then $\models \psi(b)$ if and only if $\varphi(x, b) \in q$ if and only if $\models \varphi(a, b)$ whenever a is a realization of p generic over b. We will denote $\psi(y)$ by $(d_p x)\varphi(x, y)$, which is read "for generic x realizing p, $\varphi(x, y)$ holds."

Suppose T is stable, p is stationary and based on A and a realizes $p|A$. Equivalent ways of describing the relationships between φ, p and $(d_p x)\varphi(x, y)$ are:

- If $a \underset{A}{\perp} b$, $\models \varphi(a, b) \iff \models (d_p x)\varphi(x, b)$.
- If $\models \varphi(a, b)$, then $a \underset{A}{\perp} b \iff \models (d_p x)\varphi(x, b)$.
- If $\models (d_p x)\varphi(x, b)$, then $a \underset{A}{\perp} b \iff \models \varphi(a, b)$.

Since all formulas defining $q \restriction \varphi$ are equivalent, $(d_p x)\varphi(x, y)$ is uniquely determined and $p' \parallel p \implies (d_{p'} x)\varphi(x, \mathfrak{C}) = (d_p x)\varphi(x, \mathfrak{C})$. The variable of the type is not always the first one appearing in the formula. We will write $(d_r y)\varphi(x, y)$ for the formula in x defining $r \restriction \varphi'$, where $\varphi'(y, x) = \varphi(x, y)$ and $r = r(y)$.

Let $p \in S(A)$ be a type in a stable theory and $X = p(\mathfrak{C})$. Define an equivalence relation $\sim$ on X by: $a \sim b$ if for all $B \supset A$ such that a and

b are both generic over B, $tp(a/B) = tp(b/B)$. Notice that $a \sim b$ if and only if $tp(a/acl(A)) = tp(b/acl(A))$; i.e., the $\sim$ −class of a is the locus of a over $acl(A)$. This equivalence relation is expressible entirely in terms of A−definable equivalence relations as follows. Let $FE(A)$ denote the set of equivalence relations which are definable over A and have finitely many classes. The elements of $FE(A)$ are called *finite equivalence relations over A*. Then,

Lemma 5.1.10. *Let T be a complete theory. For all sets A and elements a, b,*

$$tp(a/acl(A)) = tp(b/acl(A))$$

if and only if $\models E(a,b)$, for all $E \in FE(A)$ (of the appropriate sort).

Proof. If $E \in FE(A)$ then the name e for the E−class of b is in $acl(A)$ and there is a formula $\eta(x,e)$ defining this class. Thus, if $tp(a/acl(A)) = tp(b/acl(A))$, $\models E(a,b)$, for all $E \in FE(A)$. Conversely, let $\varphi(x,y)$ be a formula over A such that $\exists x\varphi(x,y)$ is algebraic. The equivalence relation $E(x,x')$ defined by: $\forall y(\varphi(x,y) \leftrightarrow \varphi(x',y))$, is in $FE(A)$. Thus, assuming that $\models E(a,b)$ for all E in $FE(A)$, $tp(a/acl(A)) = tp(b/acl(A))$. This proves the lemma.

Corollary 5.1.8. *Let T be stable.*

(i) Suppose $p \in S(\mathfrak{C})$ does not fork over A and a realizes $p \restriction acl(A)$. Then $p \restriction A \cup \{a\}$ is stationary, hence p is definable over $A \cup \{a\}$.

(ii) If p, $p' \in S(\mathfrak{C})$ are both nonforking extensions of some $q \in S(A)$, then p and p' are conjugate over A.

(iii) When $a \underset{A}{\not\perp} b$, there is a formula $\psi(x,b) \in tp(a/A \cup \{b\})$ such that any c satisfying $\psi(x,b)$ depends on b over A.

Proof. (i) Let $q = p \restriction A \cup \{a\}$ and b be a realization of q. Then $\models E(b,a)$, for all $E \in FE(A)$, hence (by Lemma 5.1.10) $tp(b/acl(A)) = tp(a/acl(A))$. Thus, q is stationary and p is definable over $A \cup \{a\}$ (by Corollary 5.1.5(iii)).

(ii) Let a and a' be realizations of $p \restriction acl(A)$ and $p' \restriction acl(A)$, respectively. Then p is the unique nonforking extension of q containing $\{ E(x,a) : E \in FE(A) \}$ and p' is the unique nonforking extension of q containing $\{ E(x,a') : E \in FE(A) \}$. Since a and a' both realize q there is an automorphism f of $\mathfrak{C}$ which pointwise fixes A and takes a to a'. Then $f(\{ E(x,a) : E \in FE(A) \}) = \{ E(x,a') : E \in FE(A) \}$, so $f(p) = p'$.

(iii) Let $p = tp(b/A)$ and $q = stp(b/A)$. By the Symmetry Lemma, b depends on a over A, so there is a formula $\varphi(a,y) \in tp(b/A \cup \{a\})$ such that $p(y) \cup \{\varphi(a,y)\}$ forks over A. Since $\varphi(a,y)$ is not in the unique nonforking extension of q in $S(\mathfrak{C})$, a cannot satisfy the formula $(d_q y)\varphi(x,y)$. By (i) q is based on $A \cup \{b\}$, hence $(d_q y)\varphi(x,y)$ is equivalent to a formula over $A \cup \{b\}$. The formula $\psi(x) = \varphi(x,b) \wedge \neg(d_q y)\varphi(x,y)$ is a formula over $A \cup \{b\}$ such that $\models \psi(c) \implies c \underset{A}{\not\perp} b$, proving (iii).

Recalling Definition 5.1.4(ii), a (possibly incomplete) type p over B forks over $A \subset B$ if whenever c realizes p, c depends on B over A. Part (iii) of the corollary can be reworded as: If $p \in S(B)$ forks over $A \subset B$, there is a $\varphi \in p$ which forks over A.

The following direct consequence of definability is called the Open Mapping Theorem because it asserts that a certain map between topologies is open. The reader is referred to [Las86] or [Bal88] for an explanation.

Lemma 5.1.11 (Open Mapping Theorem). *Suppose that T is stable, $B \supset A$ and $\varphi(x)$ is a formula over B. Then there is a formula $\psi(x)$ over A such that $p \in S(A)$ has a nonforking extension containing $\varphi(x)$ if and only if $\psi \in p$.*

Proof. Without loss of generality, $B \setminus A$ contains a single element b. Let $q = stp(b/A)$ as a type in y and $\psi_0(x) = (d_q y)\varphi(x,y)$. Then, given a independent from b over A, $\models \psi_0(a) \iff \models \varphi(a,b)$. Let $\psi_0 = \psi_0(x,e)$, where $\psi_0(x,z)$ is over A, $e \in acl(A)$ and $\exists x \psi_0(x,z)$ isolates $tp(e/A)$. Let $\psi(x) = \exists z \psi_0(x,z)$. If a is independent from b and $\models \varphi(a,b)$ then $\models \psi_0(a,e)$, so $\models \psi(a)$. In other words, $\psi(x) \in tp(a/A)$. Now suppose $p \in S(A)$ and $\psi \in p$. Then $p \cup \{\psi_0(x,e')\}$ is consistent for some e'. Since $\exists x \psi_0(x,z)$ isolates a complete type over A, $p \cup \{\psi_0(x,e)\}$ is consistent. Since $e \in acl(A)$ there is a nonforking extension p' of p over $acl(A) \cup \{b\}$ containing $\psi_0(x,e)$. That is, for some a realizing p which is independent from b over A, $\psi_0(x,e)$. Thus, $\models \varphi(a,b)$ as needed to complete the proof.

Using this lemma we can generalize Lemma 3.3.10.

Corollary 5.1.9. *Let T be stable, $p \in S(A)$ nonisolated and $q \supset p$ is an isolated complete type. Then q forks over A.*

Proof. Suppose, to the contrary, that $q \in S(B)$ does not fork over A and $\varphi(x)$ isolates q. Let $\psi \in p$ be such that $r \in S(A)$ has a nonforking extension containing $\varphi(x)$ if and only if $\psi \in r$. Any $p' \in S(A)$ containing ψ has an extension $q' \in S(B)$ containing φ. Since q is the only element of $S(B)$ containing φ, $q \restriction A = p$ is the only element of $S(A)$ containing ψ; i.e., ψ isolates p. This contradiction proves the corollary.

Canonical parameters were introduced for t.t. theories in Section 4.1.1. The canonical parameter of a degree 1 type has much in common with the name for a formula. Recall that $\ulcorner\varphi\urcorner$ denotes the name for the formula φ. (Thus, for every automorphism f of $\mathfrak{C}$, $f(\varphi(\mathfrak{C})) = \varphi(\mathfrak{C})$ if and only if $f(\ulcorner\varphi\urcorner) = \ulcorner\varphi\urcorner$.)

Definition 5.1.10. *Let T be stable and p be a stationary type in x. Let*

$$D(p) = \{\, \ulcorner(d_p x)\varphi(x,y)\urcorner : \varphi(x,y) \textit{ is a formula over } \emptyset \,\}.$$

The canonical base *of p, denoted $Cb(p)$, is $dcl(D(p))$.*

Clearly, if p is a stationary type definable over A then $Cb(p) \subset dcl(A)$.

Lemma 5.1.12 (Canonical Bases). *Let T be stable and p a stationary type.*

(i) p is based on $Cb(p)$.

(ii) If the stationary type q is parallel to p then $Cb(p) = Cb(q)$.

(iii) If $p \in S(\mathfrak{C})$, $Cb(p)$ is the largest set C such that for all $f \in \mathrm{Aut}(\mathfrak{C})$, $f(p) = p$ if and only if f fixes C pointwise.

Proof. (i) Clearly p is definable over $D(p)$, hence p is based on $Cb(p)$.

(ii) For $r = p|\mathfrak{C} = q|\mathfrak{C}$, $D(p) = D(r) = D(q)$, so $Cb(p) = Cb(q)$.

(iii) Let C be a maximal set such that for all $f \in \mathrm{Aut}(\mathfrak{C})$, $f(p) = p$ if and only if f fixes C pointwise. If $f \in \mathrm{Aut}(\mathfrak{C})$ and $f(p) = p$ then $f((d_p x)\varphi(x,y)) = (d_p x)\varphi(x,y)$, for all φ. Hence f fixes $Cb(p) = dcl(D(p))$ pointwise and (since $C = dcl(C)$) $Cb(p) \subset C$. To prove that $C \subset Cb(p)$, suppose $f \in \mathrm{Aut}(\mathfrak{C})$ fixes $Cb(p)$ pointwise. Since p is definable over $Cb(p)$, $f(p) = p$, hence f fixes C. Thus $C \subset dcl(Cb(p)) = Cb(p)$, completing the proof.

Remark 5.1.7. (i) Some authors take the property proved in (iii) of the previous lemma as the definition of a canonical base. Indeed, most of the properties we prove about canonical bases follow directly from this condition. Our definition makes it clear that each stationary type has a canonical base.

(ii) In Exercise 5.1.18 the reader is asked to show that for $p \in S(\mathfrak{C})$, $Cb(p) = dcl(C)$ if and only if for all $f \in Aut(\mathfrak{C})$, $f(p) = p$ if and only if f fixes C pointwise.

Corollary 5.1.10. *Let T be t.t., p a stationary type and c a canonical parameter of p. Then $Cb(p) = dcl(c)$.*

Remark 5.1.8. Requiring a canonical base to be definably closed guarantees the maximality condition in Lemma 5.1.12(iii). When defining the canonical parameter of a stationary type p in a t.t. theory we sacrificed this uniqueness in favor of p having a canonical parameter which is an element of the universe. Given an arbitrary stable theory and stationary p there may not be an element $c \in Cb(p)$ such that $dcl(c) = Cb(p)$. (Examples are given below.) See also Exercise 5.1.11.

Example 5.1.4. To illustrate this notion we examine canonical bases in an arbitrary theory of equivalence relations. Let $L = \{ E_i : i \in I \}$ be a collection of binary relations, T_0 a complete 1-sorted theory in L saying that each E_i is an equivalence relation and let $T = T_0^{eq}$. Then T has elimination of quantifiers regardless of the relationships between the E_i's axiomatized in T_0. Let $p \in S(\mathfrak{C})$ be a type in the same sort as the equivalence relations. By the elimination of quantifiers p is implied by

$$\{ E_i(x,a) : i \in I,\ a \in \mathfrak{C} \text{ and } E_i(x,a) \in p \} \cup$$
$$\{ \neg E_i(x,a) : i \in I,\ a \in \mathfrak{C} \text{ and } \neg E_i(x,a) \in p \}.$$

The data needed to determine p is:

(1) $\mathcal{E}(p) = \{ E_i : E_i(x,a) \in p \text{ for some } a \}$, and
(2) for $E_i \in \mathcal{E}(p)$, the set $e_i(p)$ which is the E_i–class of the elements b such that $E_i(x,b) \in p$.

(Given $q \in S(\mathfrak{C})$, if $\mathcal{E}(q) = \mathcal{E}(p)$ and $e_i(q) = e_i(p)$, for all $E_i \in \mathcal{E}(q)$, then $q = p$.) How can this data be used to find the canonical base of p or sufficiently, a set C such that $f(p) = p$ if and only if $f(C) = C$, for all $f \in \mathrm{Aut}(\mathfrak{C})$?

Claim. For all $f \in \mathrm{Aut}(\mathfrak{C})$, $f(p) = p$ if and only if f pointwise fixes $C(p) = \{ c_i : c_i \text{ is a name for } e_i(p), \text{ for } E_i \in \mathcal{E}(p) \}$.

Clearly, if $f \in \mathrm{Aut}(\mathfrak{C})$ and $f(p) = p$ then f pointwise fixes $C(p)$. Suppose, conversely, that $f \in \mathrm{Aut}(\mathfrak{C})$ pointwise fixes $C(p)$. By (1) and (2), $f(p) = p$ if and only if $\mathcal{E}(p) = \mathcal{E}(f(p))$ and $e_i(p) = e_i(f(p))$. For any $g \in \mathrm{Aut}(\mathfrak{C})$, $\mathcal{E}(g(p)) = \mathcal{E}(p)$. Let c_i be the element of $\mathfrak{C}$ which is a name for the equivalence class $e_i(p)$. We see, then, that

$$f(p) = p \iff e_i(f(p)) = e_i(p) \iff f(c_i) = c_i, \quad \text{for all } E_i \in \mathcal{E}(p).$$

This proves the claim.

Thus, $Cb(p) = dcl(C(p))$.

Pick T_0 to be the theory expressing that each E_i, $i < \omega$, has infinitely many class, E_{i+1} refines E_i and each E_i class contains infinitely many E_{i+1}–classes. The reader should find a $p \in S(\mathfrak{C})$ such that there is no $c \in Cb(p)$ with $Cb(p) = dcl(c)$.

Besides filling out our picture of the nonforking extensions of a complete type the following illustrates how canonical bases are used in proofs.

Lemma 5.1.13. *If T is stable and $p \in S(\mathfrak{C})$, then p does not fork over A if and only if p has $< |\mathfrak{C}|$ conjugates over A.*

Proof. Let $\mathrm{Aut}_A(\mathfrak{C})$ denote the set of automorphisms of $\mathfrak{C}$ which fix A pointwise. Given $f \in \mathrm{Aut}(\mathfrak{C})$, $f(p)$ is the unique nonforking extension of $f(p|Cb(p))$. Moreover, if $f, g \in \mathrm{Aut}(\mathfrak{C})$ and $f(Cb(p)) = g(Cb(p))$ then $f(p|Cb(p)) = g(p|Cb(p))$. Thus, there is a one-to-one correspondence between $\{ f(p) : f \in \mathrm{Aut}_A(\mathfrak{C}) \}$ and $\{ f(Cb(p)) : f \in \mathrm{Aut}_A(\mathfrak{C}) \}$.

Suppose p does not fork over A. Then $Cb(p) \subset acl(A)$, $\{ f(Cb(p)) : f \in \mathrm{Aut}_A(\mathfrak{C}) \}$ has cardinality $\le 2^{|T|} < |\mathfrak{C}|$, so $\{ f(p) : f \in \mathrm{Aut}_A(\mathfrak{C}) \}$ has cardinality $< |\mathfrak{C}|$. If, on the other hand, $\{ f(p) : f \in \mathrm{Aut}_A(\mathfrak{C}) \}$ has unbounded cardinality, then so does $\{ f(Cb(p)) : f \in \mathrm{Aut}_A(\mathfrak{C}) \}$. Thus $Cb(p) \not\subset acl(A)$ and p must fork over A.

5.1.2 Stability and the Number of Types

We saw in the case of totally transcendental theories that there is a tight connection between the number of complete types over sets and the existence of ranks (a countable complete theory is t.t. if and only if it is $\aleph_0$–stable).

In this subsection we establish a similar connection for stable theories. The main result is

Proposition 5.1.1. *For T a complete theory and Δ a finite set, the following are equivalent.*

(i) *There is a $p \in S(\mathfrak{C})$ with $R_\Delta(p) = \infty$.*
(ii) *There is a $p \in S(\mathfrak{C})$ with $R_\Delta(p) \geq \omega$.*
(iii) *For every infinite cardinal λ there is a set A of cardinality λ such that $|S_\Delta(A)| = 2^\lambda$.*
(iv) *For some infinite set A, $|S_\Delta(A)| > |A|$.*

Proof. Fix T and Δ throughout the proof. Recall that R_φ denotes $R_{\{\varphi\}}$ when φ is a single formula. Most of the work is contained in

Claim. Suppose there is a $p \in S(\mathfrak{C})$ with $R_\Delta(p) \geq \omega$. Then for every infinite cardinal λ there is a set A of cardinality λ such that $|S_\Delta(A)| = 2^\lambda$.

First note that $R_\Delta(x = x) \geq \omega$. The proof of the claim involves the following sets of formulas. (Remember: ${}^\alpha 2$ is the set of functions from α into $2 = \{0, 1\}$ and ${}^{\alpha>}2 = \bigcup_{\beta<\alpha} {}^\beta 2$. Also, for ψ a formula, $\psi^0 = \psi$ and $\psi^1 = \neg\psi$.)

> Let α be an ordinal and, for each $s \in {}^{\alpha>}2$, let c_s be a new constant symbol. Given a formula φ, let $\Gamma(\varphi, \alpha)$ is $\{\, \varphi(x_\tau, c_{\tau\restriction i})^{\tau(i)} \;:\; \tau \in {}^\alpha 2,\ i < \alpha \,\}$.

An easy induction on α shows that $\Gamma(\varphi, \alpha)$ is consistent whenever $R_\varphi(x = x) \geq \alpha$.

Subclaim. There is a formula $\varphi \in \Delta$ such that $R_\varphi(x = x) \geq \omega$.

For $m \leq \omega$ let $\mathcal{W}_m$ be the collection of all sets of formulas W of the form

$$W = \{\, \varphi_{\tau\restriction i}(x_\tau, a_{\tau\restriction i})^{\tau(i)} : \tau \in {}^m 2,\ i < m \,\}$$
$$\text{for new constants } a_{\tau\restriction i} \text{ and } \varphi_{\tau\restriction i} \in \Delta.$$

We also require that for all $m \leq \omega$ and for all $\sigma, \tau \in {}^m 2$, if $\sigma \restriction i = \tau \restriction i$ and $\varphi_{\sigma\restriction i} = \varphi_{\tau\restriction i}$ then $a_{\sigma\restriction i} = a_{\tau\restriction i}$. As with the $\Gamma(\varphi, \alpha)$'s, $R_\Delta(x = x) \geq m$ implies the consistency of some element of $\mathcal{W}_m$. When $m < \omega$ every element of $\mathcal{W}_{m+1}$ contains an element of $\mathcal{W}_m$ (after renaming the variables). There are consistent elements of $\mathcal{W}_m$ for arbitrarily large finite m (since $R_\Delta(x = x) \geq \omega$). Since each $\mathcal{W}_m$ (for $m < \omega$) is finite there is a consistent element of $\mathcal{W}_\omega$. This yields $2^{\aleph_0}$ many Δ−types over some countable set A. A simple counting argument (using the finiteness of Δ) produces a $\varphi \in \Delta$ for which there are $2^{\aleph_0}$ many φ−types over A. Since there are only countably many φ−types over A with finite φ−rank, $x = x$ must have infinite φ−rank, proving the subclaim.

Fix a $\varphi \in \Delta$ with $R_\varphi(x = x) \geq \omega$. By compactness, $\Gamma(\varphi, \kappa)$ is consistent for all κ. Thus, for any infinite λ there is a set A of cardinality λ (namely the elements which interpret the constants in $\Gamma(\varphi, \lambda)$) such that $|S_\Delta(A)| = 2^\lambda$. This proves the claim.

Turning to the main body of the proof, notice that (i) $\Longrightarrow$ (ii) and (iii) $\Longrightarrow$ (iv) are trivial, while (ii) $\Longrightarrow$ (iii) is a restatement of the claim. For (iv) $\Longrightarrow$ (i), suppose (i) does not hold and A is an infinite set. Any Δ–type over A is contained in only finitely many elements of $S_\Delta(A)$ of the same Δ–rank. Furthermore, any element of $S_\Delta(A)$ contains a finite Δ–type of the same rank. Since there are $|A|$ many finite Δ–types over A, $|S_\Delta(A)| = |A|$; i.e., (iv) fails. This proves the proposition.

Corollary 5.1.11. *For T a complete theory, the following are equivalent.*

(i) *T is stable.*
(ii) *For all finite Δ and $p \in S(\mathfrak{C})$, $R_\Delta(p) < \omega$.*
(iii) *For all finite Δ and all sets A, $|S_\Delta(A)| = |A| + \aleph_0$.*
(iv) *For all sets A, $|S(A)| \leq |A|^{|T|}$.*
(v) *T is λ–stable for any λ such that $\lambda = \lambda^{|T|}$.*
(vi) *For some infinite λ, T is λ–stable.*

Proof. The equivalence of (i), (ii) and (iii) follows immediately from the proposition. Since any $p \in S(A)$ is simply the union of $p \restriction \varphi$, for φ a formula, (iii) $\Longrightarrow$ (iv). Trivially, (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (vi). Finally, (ii) $\Longrightarrow$ (iii) of Proposition 5.1.1 shows that (vi) $\Longrightarrow$ (ii).

As stated earlier there are many definitions of the term "stable" in the literature. The earliest definition was: T is stable if T is λ–stable for some $\lambda \geq |T|$. Corollary 5.1.11 proves the equivalence of this definition with the one used here. Other standard definitions are:

(a) T is stable if T does not have the order property, and
(b) T is stable if for all sets A, every element of $S(A)$ is definable over A.

The reader will prove the equivalence of (b) with our definition in the exercises. We proved in Lemma 5.1.6 that when T is stable in our sense it does not have the order property. The converse, which is significantly more difficult to prove, is in [She90, §I.2]. Another equivalent involving the so-called fundamental order will be mentioned in Section 5.1.4. (See Theorem 5.1.2, specifically.)

5.1.3 Morley Sequences and Indiscernibles

As in t.t. theories indiscernible sets can be constructed by taking successive nonforking extensions of stationary types. The precise definition is

Definition 5.1.11. *Let T be stable, p a stationary type and B a set on which p is based. We call I a* Morley sequence over B in p *if I is a B–independent set of realizations of $p|B$.*

Remark 5.1.9. (i) Notice that being a Morley sequence is invariant under parallelism: if p and q are stationary and parallel then a set is a Morley sequence in p if it is a Morley sequence in q.

(ii) Let $p \in S(A)$ be a strongly minimal type. Then for B a set on which p is based, I is a Morley sequence in p if it is a set of realizations of $p|B$ which is algebraically independent over B.

(iii) More generally, any Morley sequence in a t.t. theory as defined in Definition 3.3.6 is a Morley sequence (by Corollary 5.1.4).

Lemma 5.1.14. *Let T be stable and p a stationary type based on B.*

(i) Given $n < \omega$ and $a = (a_0, \ldots, a_n)$, $b = (b_0, \ldots, b_n)$ independent sequences of realizations of $p|B$, $tp(a/B) = tp(b/B)$. Moreover, $tp(a/B)$ is stationary.

(ii) A Morley sequence over B in p is an indiscernible set over B.

Proof. Since (ii) follows immediately from (i) we only need to prove the first part, which is done by induction on n. Let $a' = (a_0, \ldots, a_{n-1})$ and $b' = (b_0, \ldots, b_{n-1})$. By induction there is an automorphism f fixing B and taking a' to b'. Since $f(tp(a_n/B \cup a')) = p|(B \cup b') = tp(b_n/B \cup b')$, $tp(a/B) = tp(b/B)$. The stationarity of $tp(a/B)$ follows from:

– the first sentence in (i) is true when B is replaced by $acl(B)$, and
– a complete type over an algebraically closed set is stationary.

Most of the properties established for indiscernibles in t.t. theories generalize directly to stable theories. In fact, combining Lemma 3.3.5 and Corollary 5.1.11 proves

Lemma 5.1.15. *Let T be stable and $(I, <)$ an infinite indiscernible sequence over A. Then*

(i) I is an indiscernible set over A.

(ii) For any formula $\varphi(x, y)$ over A there is an $n < \omega$ such that for all a,

$$|\{\, b \in I : \models \varphi(a, b) \,\}| \leq n \text{ or } |\{\, b \in I : \models \neg\varphi(a, b) \,\}| \leq n.$$

We will see momentarily that for any indiscernible set I in a stable theory all but a "small" subset J of I is a Morley sequence over J in some type. The relevant stationary type is defined here:

Definition 5.1.12. *Let T be stable, I an infinite set of indiscernibles and A a set. The* average type *of I over A, denoted $Av(I/A)$, consists of*

$$\{\, \varphi(x) : \varphi \text{ is a formula over } A \text{ and } \models \varphi(a) \text{ for all but finitely many } a \in I \,\}.$$

Lemma 5.1.16. *Let T be stable, I and infinite set of indiscernibles and A a set. Then $Av(I/A)$ exits and is complete.*

Proof. Left to the reader in Exercise 5.1.19.

Average types are quite easy to understand when dealing with Morley sequences in a t.t. theory. Let T be t.t., p stationary, I a Morley sequence in p over A and $B \supset A$. For any $b \in B$ there is a finite $J \subset I$ such that $I \setminus J$ is a Morley sequence in p over $A \cup \{b\}$ (by Corollary 3.3.1). Thus, the average of I over any set $B \supset A$ is $p|B$.

Remark 5.1.10. Recall that $FE(A)$ denotes the set of finite equivalence relations over A (in a fixed sort determined by context). Let I be an infinite set of indiscernibles over A in a stable theory, $a \in I$ and $E \in FE(A)$. Then $E(x,a) \in Av(I/A \cup \{a\})$ since E has only finitely many classes and I is infinite and indiscernible.

Arbitrary indiscernible sets are reduced to Morley sequences with the following result.

Lemma 5.1.17. *If T is stable and I is an infinite set of indiscernibles, then $p = Av(I/\mathfrak{C})$ is based on any infinite $J \subset I$. Moreover, for any infinite $J \subset I$ $I \setminus J$ is a Morley sequence in p over J.*

Proof. The following claim (whose proof is left to the reader) indicates how to enlarge a set of indiscernibles without changing the average type.

Claim. If a realizes $p \restriction I$, then $I_0 = I \cup \{a\}$ is indiscernible and $Av(I_0/\mathfrak{C}) = p$.

By repeated applications of the claim we can assume, without loss of generality, that $|I| \geq |T|^+$. Let A be a set of cardinality $\leq |T|$ over which p does not fork. There is a set $I' \subset I$ of cardinality $\leq |T|$ such that

- A is independent from I over I' and
- for any formula φ over $A \cup I'$, if $|\{\, a \in I : \models \varphi(a) \,\}|$ is finite then $\{\, a \in I : \models \varphi(a) \,\} \subset I'$.

Then any $a \in I \setminus I'$ is independent from A over I' and $tp(a/A \cup I') = Av(I/A \cup I')$; i.e., $p \restriction (A \cup I') = tp(a/A \cup I')$ and this type does not fork over I'. That is to say, p does not fork over I'.

Now, let J be any infinite subset of I. For Δ a finite set of formulas there is a $\varphi(x, a_0, \ldots, a_n) \in p$, where $a_0, \ldots, a_n \in I'$ are distinct, such that $R_\Delta(p) = R_\Delta(\varphi(x, a_0, \ldots, a_n))$. By the indiscernibility of I, for any distinct $b_0, \ldots, b_n \in J$, $\varphi(x, b_0, \ldots, b_n) \in p$ and $R_\Delta(p) = R_\Delta(\varphi(x, b_0, \ldots, b_n))$. Thus, p does not fork over J. It remains to show that $p \restriction J$ is stationary. Fix $b \in J$ and $J' = J \setminus \{b\}$. Since J' is also infinite, p does not fork over J'. If $E \in FE(J')$, then $E(x,b) \in p$ by Remark 5.1.10. Thus, not only is $tp(b/J') = p \restriction J'$, but $tp(b/acl(J')) = p \restriction acl(J')$. By Corollary 5.1.8(i), $p \restriction J$ is stationary, proving that p is based on J. That $I \setminus J$ is a Morley sequence in p over J follows immediately.

Corollary 5.1.12. *Let T be stable, $p \in S(A)$ a stationary type and I a Morley sequence in p. Then $Av(I/\mathfrak{C})$ is parallel to p.*

Proof. See Exercise 5.1.16.

When T is t.t. this lemma can be improved to:

Corollary 5.1.13. *If T is t.t. and I is an infinite set of indiscernibles, then $p = Av(I/\mathfrak{C})$ is based on some finite $J \subset I$. Moreover, for any such J, $I \setminus J$ is a Morley sequence in p over J.*

We stated in Remark 5.1.3 that for any $p \in S(\mathfrak{C})$ (where T is stable) there is a set A of cardinality $\leq |T|$ over which p does not fork. With the above lemma this can be improved to

Corollary 5.1.14. *For T a stable theory and $p \in S(\mathfrak{C})$ there is a countable set A such that p is based on A.*

Proof. Let B be any set over which p is based and I a countably infinite Morley sequence over B in p. By Lemma 5.1.17, p is based on I.

Corollary 5.1.15. *Let T be stable, $p \in S(\mathfrak{C})$, I an infinite set of indiscernibles with $Av(I/\mathfrak{C}) = p$ and A any set. Then for all $\varphi(x,y)$, $d_p x\varphi(x,y)$ is equivalent to a formula over I. In particular, p is definable over $p \upharpoonright A(\mathfrak{C})$.*

The final result connects average types to type diagrams.

Lemma 5.1.18. *If T is stable and I and J are infinite sets of indiscernibles with the same average type over $\mathfrak{C}$ then $D(I) = D(J)$.*

5.1.4 The Fundamental Order

Our intuition is that Morley rank in a t.t. theory and the $\Delta-$ranks in stable theories provide a measure of the complexity of types. The fundamental order is an alternative such measure.

Definition 5.1.13. *Let $\mathfrak{C}$ be the universal domain of a complete theory T.*

(i) Given $B \subset A \subset \mathfrak{C}$, $p \in S(A)$ a type in the variable v and $\varphi(v,w)$ a formula over B, we say that $\varphi(v,w)$ is represented *in p if there is an $a \in A$ such that $\varphi(v,a) \in p$. The* representation class of p over B, *$\chi_B(p)$, is* $\{\varphi(v,w) :\ \varphi$ is over B and φ is represented in $p\}$. *When $B = \emptyset$ we write $\chi(p)$ for $\chi_B(p)$.*

(ii) The fundamental order *of $\mathfrak{C}$ is*

$$\mathcal{O} = \{\chi(p) : p \in S(M) \text{ for } M \text{ a model of } T\}$$

under the partial order $\leq$ of reverse inclusion. That is, $\chi(p) \leq \chi(q)$ if and only if $\chi(q) \subset \chi(p)$.

(iii) For M, N models of T and $p \in S(M)$, $q \in S(N)$, we write $p \leq q$ if $\chi(p) \leq \chi(q)$ in the fundamental order. The relation $\leq$ on the collection of types over models is also called the fundamental order of T.

(iv) For $M \subset N$ models of T and $p \in S(M)$, $q \in S(N)$ with $p \subset q$, we call q an heir *of p if $\chi_M(q) = \chi_M(p)$.*

Remark 5.1.11. Let $\mathcal{P} = \{p : p \text{ is a complete type over a model }\}$.

(i) As usual when working with types over models we apply the definitions and results to the elements of $S(\mathfrak{C})$ as well.

(ii) The fundamental order over B is defined in the obvious way.

(iii) For $p \in S(M)$, M a model, there is a model $N \subset M$ of cardinality $|T|$ such that $\chi(p) = \chi(p \restriction N)$. Clearly, $\mathcal{O}$ has cardinality $\leq 2^{|T|}$. It does not follow (immediately) that p is an heir of $p \restriction N$ since that relation requires $\chi_N(p) = \chi_N(p \restriction N)$. However, there is a chain of models $N_0 \subset N_1 \subset \ldots$ such that for each $i < \omega$, $|N_i| = |T|$ and every formula over N_i represented in p is represented in $p \restriction N_{i+1}$. Then p is an heir of $p \restriction (\bigcup_{i<\omega} N_i)$.

(iv) $p \in \mathcal{P}$ is minimal in the fundamental order if and only if $v = w$ is represented in p; i.e., p is realized in its domain.

(v) Given $M \subset N$ models and $p \in S(M)$, $q \in S(N)$ with $p \subset q$, q is an heir of p if and only if whenever $\varphi(v, w)$ over M is represented in q there is an $a \in M$ such that $\varphi(v, a) \in q$.

(vi) For any given complete theory there is a fundamental order corresponding to each sort. For simplicity we usually assume the order is on the sort of equality.

The representation class of $p = tp(a/M)$ is one measure of the amount of information p determines about a. When $\chi(p) < \chi(q)$, then to some degree p gives more information about a realization than does q. If $q \in S(N)$ is an heir of p then all of the information contained in q (given by representation) is already contained in p; i.e., all of the information in q is inherited from p.

Example 5.1.5. (i) Let T be the theory of one equivalence relation E with infinitely many infinite classes and no finite classes. In the sort of equality the fundamental order $\mathcal{O}$ contains 3 elements. The unique minimal class is the one containing $v = w$. There is a unique maximal class which can be described as the one not containing $E(v, w)$. Strictly between these two classes in $\mathcal{O}$ is the class containing $E(v, w)$ and not containing $v = w$.

(ii) Let $(I, <)$ be a linear order and E_i a binary relation symbol for each $i \in I$. Let T be the theory expressing for i, $j \in I$:

(a) Each E_i is an equivalence relation with infinitely many infinite classes and no finite classes.

(b) If $i < j$ E_i refines E_j and each E_j−class contains infinitely many E_i−classes.

Then T is quantifier-eliminable and stable. For each cut J of I there is a $p_J \in S(\mathfrak{C})$ such that p_J represents E_j if and only if $j \notin J$. There is a type

$p_0 \in S(\mathfrak{C})$ representing $v = w$, hence every E_i. There is also a $p_1 \in S(\mathfrak{C})$ not representing $v = w$ or any E_j. Let $\mathcal{J}$ be the set of cuts of I ordered by inclusion, and let $\mathcal{J}_1$ be $\mathcal{J}$ with the addition of a minimal element. Then, the fundamental order is isomorphic to $\mathcal{J}_1$.

(iii) Let K be the universal domain of algebraically closed fields of characteristic 0. The fundamental order $\mathcal{O}$ on 2−types is described as follows. The reader can verify that every representation class in $\mathcal{O}$ contains an element of $S_2(K)$. The algebraic types in $S_2(K)$ are the minimal elements of $\mathcal{O}$ and there are $\aleph_0$ many different representation classes of algebraic types. There is a unique element of $S_2(K)$ having Morley rank 2. For strongly minimal $p, q \in S_2(K)$, $\chi(p) \subset \chi(q)$ if and only if $\chi(p) = \chi(q)$ if and only if p is conjugate to q. There are $\aleph_0$ many strongly minimal elements of $S_2(K)$ up to conjugacy. Thus, $\mathcal{O}$ has a unique maximal element, $\aleph_0$ many minimal elements and the remainder is a set of $\aleph_0$ many pairwise incomparable elements.

In each of the above (stable) examples, when M is a model and $p \in S(\mathfrak{C})$, p is an heir of $p \restriction M$ if and only if p does not fork over M.

(iv) Let $\mathfrak{C}$ be the universal domain of dense linear orders without endpoints. The fundamental order $\mathcal{O}$ on 1−types has one minimal element. A nonminimal $o \in \mathcal{O}$ is determined by whether it contains $v < w$ and not $v > w$, $v > w$ and not $v < w$, or both $v < w$ and $v > w$.

Let M be a model, $p_{-\infty} = \{ v < a : a \in M \}$ and $p_{+\infty} = \{ v > a : a \in M \}$. The type $p_{-\infty}$ has a unique heir in $S(\mathfrak{C})$, namely $\{ v < a : a \in \mathfrak{C} \}$. Similarly, $p_{+\infty}$ has a unique heir in $S(\mathfrak{C})$. Let J be a cut of M such that $M \setminus J \neq \emptyset$ and $\sup J$ does not exist in M. Let $p_0 = \{ v > a : a \in J \} \cup \{ v < a : a \in M \setminus J \}$. Any nonalgebraic extension of p_0 in $S(\mathfrak{C})$ is an heir of p_0. Thus, p_0 has $2^{|\mathfrak{C}|}$ many heirs in $S(\mathfrak{C})$.

The first lemma connects the fundamental order with simple inclusion.

Lemma 5.1.19. *Let $\mathcal{O}$ be the fundamental order of a complete theory and $o_1 \leq o_2$ elements of $\mathcal{O}$. Then, for $i = 1, 2$ there is p_i a complete type over a model such that $\chi(p_i) = o_i$ and $p_1 \supset p_2$.*

Proof. The proof is omitted for brevity. The reader can find it in [LP79, 2.3].

Lemma 5.1.20. *Let $\mathfrak{C}$ be the universal domain of a complete theory, M an $\aleph_0$−saturated model and $p \in S(M)$. Then there is an heir of p in $S(\mathfrak{C})$.*

Proof. It suffices to show that p has an heir in $S(N)$ for an arbitrary model $N \supset M$. Let $q(v)$ be the set of formulas over N which contains $p(v)$ and $\neg\varphi(v, a)$ for any $\varphi(v, w) \notin \chi_M(p)$, and any $a \in N$. In outline the consistency of $q(v)$ is proved as follows. Let $b \in M$, $a \in N$, $\varphi_0(v, w_0), \ldots, \varphi_k(v, w_k) \notin \chi_M(p)$ and $q_0(v) = p \restriction b \cup \{\neg\varphi_0(v, a), \ldots, \neg\varphi_k(v, a)\}$. Since M is $\aleph_0$−saturated there is an elementary map f fixing b and $\varphi_0(v, w_0), \ldots, \varphi_k(v, w_k)$, and taking a to $a' \in M$. Then $f(q_0) \subset p$, so q is consistent.

Any completion of q over N is an heir of p over N, proving the lemma.

In a stable theory we get a stronger existence theorem for heirs.

Lemma 5.1.21. *Let $\mathfrak{C}$ be the universal domain of a stable theory, M a model, $p \in S(M)$ and $q \in S(\mathfrak{C})$ an extension of p. Then q is an heir of p if and only if q is a nonforking extension of p.*

Proof. First suppose q is a nonforking extension of p. Let $\varphi(v, w)$ be a formula over M represented in q. Let $\psi(w)$ be a formula over M defining $q \restriction \varphi$. Since φ is represented in q, ψ is consistent, hence ψ is satisfied by some element of M. Thus q is an heir of p.

Conversely suppose q forks over M and $\varphi(v, a) \in q$ witnesses this forking, where $\varphi(v, w)$ is over M. Let $\theta(w)$ be a formula over M which defines $p \restriction \varphi$. Then θ has the property:

$$\forall b(\ p \cup \{\varphi(v, b)\} \text{ does not fork over } M \iff \models \theta(b)\ ).$$

Thus, $\varphi(v, w) \wedge \neg\theta(w)$ is represented in q but not represented in p, proving that q is not an heir of p.

The fundamental order is tied to stability with

Theorem 5.1.2. *The complete theory T is stable if and only if*

(∗) *for all models M and $p \in S(M)$, p has at most one heir in $S(\mathfrak{C})$.*

Proof. If T is stable, M is a model and $p \in S(M)$, then $q \in S(M)$ is an heir of p if and only if q is a nonforking extension of p by Lemma 5.1.21.

Now suppose that (∗) holds. Let M be an $\aleph_0$−saturated model and $\lambda = |M|$. Since any $p \in S(M)$ has an heir in $S(\mathfrak{C})$ (by Lemma 5.1.20), when N is a submodel of M, each $p \in S(N)$ has at most one heir in $S(M)$. For each $p \in S(M)$ there is (by Remark 5.1.11(iii)) a model $N \subset M$ of cardinality $|T|$ such that p is an heir of $p \restriction N$. By (∗) each element of $S(M)$ is in one-to-one correspondence (by the heir relation) to a type over a set of cardinality $|T|$. Thus, $|S(M)| \leq \lambda^{|T|}$. For some choice of λ, $\lambda^{|T|} = \lambda$, hence T is stable (by Corollary 5.1.11).

The fundamental order can be used in conjunction with the forking relation to deepen our understanding of stable theories. See, for example, [Bue85b] and [HLP+92].

Historical Notes. Globally speaking all of these results are due to Shelah [She90]. In detail our development of stable theories follows the first section of [Hru86], which is based on notes from a course by Harrington. Lemma 5.1.11 (the Open Mapping Theorem) is due to Lascar and Poizat [LP79], however Corollary 5.1.9 is stated for superstable theories in [Las76]. The results in the second subsection are explicitly due to Shelah and can be found in [She71].

The fundamental order was developed by Lascar and Poizat in [LP79].

Exercise 5.1.1. Let T be the theory of a single equivalence relation E with infinitely many infinite classes and no finite classes. Let $\Delta = \{E(x,y), x = y\}$. Prove that for all $p \in S_1(\mathfrak{C})$, $MR(p) = R_\Delta(p)$.

Exercise 5.1.2. Show that when $\Gamma(x) \supset \Delta(x)$, $R_\Gamma(\varphi) \geq R_\Delta(\varphi)$, for all formulas φ. Also, when Γ is the set of all formulas with object variable x, $R_\Gamma(p) = MR(p)$. Thus $MR(p) \geq R_\Delta(p)$.

Exercise 5.1.3. Given a stable theory show that any $p \in S(acl(A))$ does not fork over A. (Prove this without using the existence of nonforking extensions, whose proof depends on this property.)

Exercise 5.1.4. Prove Corollary 5.1.4.

Exercise 5.1.5. Suppose that T is a complete theory with the property that for all A, every element of $S(A)$ is definable over A. Prove that T is stable. (HINT: Use Corollary 5.1.11.)

Exercise 5.1.6. Let T be stable, M a model and $p \in S(M)$. Let $\varphi(x,y)$ and $\psi(y)$ be formulas over M such that ψ defines $p \restriction \varphi$. Show that ψ defines $q \restriction \varphi$, where $q \in S(\mathfrak{C})$ is the nonforking extension of p.

Exercise 5.1.7. Prove: Given a stable theory, a model M and $p \in S(M)$, if p is definable over $A \subset M$, then p does not fork over A.

Exercise 5.1.8. Prove Remark 5.1.5.

Exercise 5.1.9. Let $p \in S(A)$ be a stationary type in a stable theory, $B \supset A$, and $q \in S(B)$ a forking extension of p. Show that q is also a forking extension of $p|C$ for $C \subset B$ any set on which p is based.

Exercise 5.1.10. Prove that a countable stable theory has a saturated model of cardinality κ^+, when $\kappa^+ \geq \kappa^{\aleph_0}$.

Exercise 5.1.11. Suppose that T is t.t., p is a stationary type and $C = Cb(p)$. Prove that there is a $c \in C$ such that $C = dcl(c)$.

Exercise 5.1.12. Let T be the theory in the Example 5.1.1 and $p \in S(\mathfrak{C})$ a type in x, where x has the same sort as the equivalence relations. Describe $Cb(p)$.

Exercise 5.1.13. Prove: If T is stable and I is an infinite set of indiscernibles over A, then $a, b \in I \implies stp(a/A) = stp(b/A)$. Prove, in fact, that I is indiscernible over $acl(A)$.

Exercise 5.1.14. Give a quick proof of the Open Mapping Theorem when $A = acl(A)$.

Exercise 5.1.15. Prove the claim in the proof of Lemma 5.1.17.

Exercise 5.1.16. Prove Corollary 5.1.12.

Exercise 5.1.17. Prove Corollary 5.1.15.

Exercise 5.1.18. Let T be stable and $p \in S(\mathfrak{C})$. Prove that $Cb(p) = dcl(C)$ if and only if for all $f \in \mathrm{Aut}(\mathfrak{C})$, $f(p) = p$ if and only if f fixes C pointwise.

Exercise 5.1.19. Prove Lemma 5.1.16.

Exercise 5.1.20. Prove Lemma 5.1.18.

5.2 The Stability Spectrum and $\kappa(T)$

In Corollary 5.1.11 we proved that a complete theory T is stable if and only if it is λ−stable for some infinite λ.

Definition 5.2.1. *Let T be stable.*
(i) The stability spectrum of T *is* $\{\lambda : T$ *is* λ*−stable*$\}$.
(ii) The first stability cardinal of T, $\lambda(T)$, *is the minimum infinite cardinal* λ *such that* T *is* λ*−stable.*

For a stable theory T what are the possibilities for the stability spectrum? We will see that the possibilities are controlled by $\lambda(T)$ and another important invariant of the theory, $\kappa(T)$. Many subsequent results have hypotheses involving these numbers.

The next lemma follows from Corollary 5.1.11.

Lemma 5.2.1. *If T is stable then $|T| \leq \lambda(T) \leq 2^{|T|}$.*

The following invariant helps to measure the complexity of the forking relation on a stable theory.

Definition 5.2.2. *Let T be a stable theory. The invariant $\kappa(T)$ is the least infinite cardinal κ such that whenever $\{A_i : i < \kappa\}$ is a sequence of sets with $i < j < \kappa \implies A_i \subset A_j$ and $p \in S(\bigcup_{i<\kappa} A_i)$, there is an i such that $p \restriction A_{i+1}$ does not fork over A_i.*

We let $\kappa_r(T)$ denote the least regular cardinal $\geq \kappa(T)$ (thus, $\kappa_r(T)$ is $\kappa(T)$ or $\kappa(T)^+$).

Remark 5.2.1. We leave it to the reader to see that $\kappa(T) \leq |T|^+$. Thus, for countable theories $\kappa(T)$ can only be $\aleph_0$ or $\aleph_1$. When T is t.t. $\kappa(T)$ is $\aleph_0$.

It is possible for $\kappa(T)$ to be singular when T is uncountable, creating technical difficulties which require using $\kappa_r(T)$ instead of $\kappa(T)$ in some settings.

Independence is further related to $\kappa(T)$ in the following proposition (whose proof is left to the exercises).

Proposition 5.2.1. *Let T be stable.*

(i) For all elements b and sets C there is $A \subset C$ of cardinality $< \kappa(T)$ such that $b \underset{A}{\perp} C$.

(ii) For all sets B and C there is $A \subset C$ such that $B \underset{A}{\perp} C$ and

– $|A| < \kappa(T) + |B|^+$ *if $\kappa(T)$ is regular, and*
– $|A| \leq \kappa(T) + |B|$ *otherwise.*

For λ and κ cardinals let $\lambda^{<\kappa} = \sup\{\lambda^\mu : \mu < \kappa\}$. Let ${}^{\kappa\geq}\lambda$ be the set of all functions $f : \alpha \to \lambda$, where $\alpha \leq \kappa$ (which is denoted $lh(f)$). Our eventual goal is

Theorem 5.2.1 (Stability Spectrum). *A stable theory T is λ–stable if and only if $\lambda = \lambda(T) + \lambda^{<\kappa(T)}$.*

The bulk of the proof is contained in

Lemma 5.2.2. *If T is stable and $\lambda < \lambda^{<\kappa(T)}$ then T is not λ–stable.*

Proof. Typical of such problems, we will construct many types over a set of cardinality λ by recursion. Let κ be the least cardinal such that $\lambda^\kappa > \lambda$. Since $\kappa < \kappa(T)$ there is a sequence of sets $\{A_i : i < \kappa\}$ and a $p \in S(\bigcup_{i<\kappa} A_i)$ such that

– $i < j < \kappa \implies A_i \subset A_j$, and
– $p \restriction A_{i+1}$ forks over A_i, for all $i < \kappa$.

Without loss of generality, we can require that $A_{i+1} \setminus A_i$ is a finite set a_i and $A_\delta = \bigcup_{i<\delta} A_i$, when δ is a limit ordinal. Let $A_\kappa = \bigcup_{i<\kappa} A_i$ and notice that $|A_\kappa| = \kappa \leq \lambda$. There is a tree of sets such that each branch is conjugate to $\{A_i : i \leq \kappa\}$, each node has λ many successors and these λ many successors are independent over their predecessor. It is left to the reader to see that this can be accomplished with the construction of a family of elementary maps f_ξ, $\xi \in {}^{\kappa\geq}\lambda$, such that for all $\xi, \zeta \in {}^{\kappa\geq}\lambda$,

(1) $dom(f_\xi) = A_{lh(\xi)}$,
(2) if $\zeta \subset \xi$, $f_\zeta \subset f_\xi$,
(3) if $\delta = lh(\xi)$ is a limit ordinal, $f_\xi = \bigcup_{\beta<\delta} f_{\xi\restriction\beta}$, and
(4) if $\alpha = lh(\xi)$, $B_\xi = \{f_{\xi\frown i}(A_{\alpha+1}) : i < \lambda\}$ is independent over $f_\xi(A_\alpha)$ and B_ξ is independent from $\bigcup\{f_\eta(A_\alpha) : lh(\eta) = \alpha\}$ over $f_\xi(A_\alpha)$.

Let $\Phi = {}^\kappa\lambda$, $F = \{f_\xi : \xi \in \Phi\}$ and $A = \bigcup_{\xi\in\Phi} f_\xi(A_\kappa)$ (a set of cardinality $\lambda^{<\kappa} = \lambda$). For $\xi \in \Phi$ let $p_\xi \in S(A)$ be any nonforking extension of $f_\xi(p)$ and let $P = \{p_\xi : \xi \in \Phi\}$.

Claim. $\xi \neq \zeta \in \Phi \implies p_\xi \neq p_\zeta$.

Let b_ξ and b_ζ realize p_ξ and p_ζ, respectively. Let α be the maximal ordinal for which $\xi \restriction \alpha = \zeta \restriction \alpha = \eta$ and let $C = f_\eta(A_\alpha)$. By (4) and the transitivity of independence, $f_\xi(A_\alpha)$ is independent from $f_\zeta(A_\kappa)$ over C. Since b_ζ is independent from A over $f_\zeta(A_\kappa)$, the transitivity of independence again implies that b_ζ is independent from $f_\xi(A_\alpha)$ over C. However, b_ξ depends on $f_\xi(A_\alpha)$ over C (since $p_\xi = f_\xi(p)$). Thus, $p_\xi \neq p_\zeta$, as claimed.

We conclude that $|P| = \lambda^\kappa$. Since $|A| = \lambda < \lambda^\kappa$, T is not λ−stable.

Remark 5.2.2. It follows immediately from this lemma that $\lambda(T) \geq \kappa(T)$.

Definition 5.2.3. *If $p \in S(A)$ is a complete type in a stable theory we define the* multiplicity of p, $\mathrm{Mult}(p)$, *to be*

$$|\{\, q \in S(\mathfrak{C}) : \ q \supset p \text{ and } \ q \text{ does not fork over } \ A \,\}|.$$

Let $\mu(T)$ be the supremum of $\{\,\mathrm{Mult}(p) : \ p \text{ a complete type}\,\}$.

As stated in Remark 5.1.3, $\mathrm{Mult}(p) \leq 2^{|T|}$. A complete type is stationary if and only if it has multiplicity 1.

Lemma 5.2.3. *If T is stable, $\mu(T) + \kappa(T) \leq \lambda(T)$.*

Proof. By Remark 5.2.2, $\lambda(T) \geq \kappa(T)$. Let $p \in S(A)$ be any complete type and $B \subset A$ a set of cardinality $< \kappa(T)$ over which p does not fork. Then, $\mathrm{Mult}(p) \leq$ the multiplicity of $q = p \restriction B$. Since $\kappa(T)$ and $|T|$ are both $\leq \lambda(T)$ there is a model $M \supset B$ of cardinality $\lambda(T)$. Every nonforking extension of q in $S(\mathfrak{C})$ is parallel to an element of $S(M)$, so $|S(M)| \geq \mathrm{Mult}(q)$. Since T is $\lambda(T)$−stable, $\mathrm{Mult}(p) \leq \mathrm{Mult}(q) \leq \lambda(T)$, as required.

To complete the proof of the Theorem 5.2.1 we need only prove

Lemma 5.2.4. *If T is stable and $\lambda \geq \lambda(T)$ is a cardinal such that $\lambda = \lambda^{<\kappa(T)}$, then T is λ−stable.*

Proof. Let A be a set of cardinality λ. Any $p \in S(A)$ is a nonforking extension of $p \restriction B$ for some $B \subset A$ of cardinality $< \kappa(T)$. Furthermore, there are $\leq \mu(T)$ elements of $S(A)$ which are nonforking extensions of this type $p \restriction B$. Thus,

$$\begin{aligned}
|S(A)| \ &\leq \ \text{(the number of subsets of } A \text{ of cardinality } < \kappa(T)) \\
&\quad \times\text{(the number of types over a given set of cardinality } < \kappa(T)) \\
&\quad \times \mu(T) \\
&\leq \ \lambda^{<\kappa(T)} \cdot \lambda(T) \cdot \mu(T) \\
&= \ \lambda.
\end{aligned}$$

This proves that T is λ−stable.

This completes the proof of Theorem 5.2.1.

Corollary 5.2.1. *If T is $\kappa(T)$–stable then $\kappa(T)$ is regular.*

Proof. Left as an exercise.

For countable theories the Stability Spectrum Theorem leads to a particularly simple partitioning of the stable theories.

Proposition 5.2.2. *For a countable complete theory T one of the following mutually exclusive conditions holds.*

(1) *T is λ–stable for all infinite λ.*
(2) *T is λ–stable if and only if $\lambda \geq 2^{\aleph_0}$.*
(3) *T is λ–stable if and only if $\lambda = \lambda^{\aleph_0}$.*
(4) *T is unstable.*

Proof. Suppose T is stable. If $\lambda(T) = \aleph_0$; i.e., T is $\aleph_0$–stable, then T is λ–stable for all infinite λ (by Proposition 3.3.1). Otherwise, there is a countable set A with $S(A)$ uncountable. Since the only uncountable possibility for $|S(A)|$ is $2^{\aleph_0}$ (see Lemma 2.2.4) $\lambda(T)$ is its maximum possible value $= 2^{\aleph_0}$. Since $\kappa(T) \leq |T|^+$, $\aleph_0$ and $\aleph_1$ are the only two possibilities for $\kappa(T)$ (when T is countable and stable). If $\kappa(T) = \aleph_0$ and $\lambda(T) = 2^{\aleph_0}$, then T is λ–stable if and only if $\lambda \geq 2^{\aleph_0}$. If $\kappa(T) = \aleph_1$, then T is λ–stable if and only if $\lambda = \lambda^{\aleph_0}$ ($\lambda(T)$ is necessarily $2^{\aleph_0}$ in this case).

Examples in earlier sections show that all of these possibilities do occur. There is no such clean division for uncountable theories, however, the exact possibilities for $\lambda(T)$ are given in [She90, III.5].

Definition 5.2.4. *A stable theory T is called* superstable *if $\kappa(T) = \aleph_0$.*

The superstable theories form a major subclass of the stable theories which will be studied extensively in Chapter 6. Notice that a stable theory T is superstable exactly when T is λ–stable for all sufficiently large λ. Proposition 5.2.2 partitions the countable stable theories into the categories: (a) the $\aleph_0$–stable theories, (b) the superstable theories which are not $\aleph_0$–stable (called the *properly superstable* theories) and (c) the stable theories which are not superstable (called the *properly stable* theories).

The following illustrates how to distinguish quickly between ω–stable and properly superstable countable theories.

Lemma 5.2.5. *If T is a countable properly superstable theory, then either T is not small or T has a complete type p over a finite set with infinite multiplicity (hence multiplicity $2^{\aleph_0}$).*

Proof. We are assuming that T is not ω–stable, hence there is a countable model M with $|S(M)| = 2^{\aleph_0}$. First suppose that every element of $S(M)$ is based on a finite subset of M. Then each element of $S(M)$ is the unique nonforking extension of a type over a finite set, hence there are $2^{\aleph_0}$ complete types over finite subsets of M proving that T is not small. On the other hand, suppose $q \in S(M)$ is not based on a finite set. Let $A \subset M$ be a finite set over which q does not fork and let $p = q \upharpoonright A$. Then p has infinite multiplicity since otherwise there is a finite set A', $A \subset A' \subset acl(A)$, with q the unique nonforking extension of $q \upharpoonright A'$. That a type of infinite multiplicity (in a countable stable theory) must have multiplicity $2^{\aleph_0}$ is left to the exercises.

Corollary 5.2.2. *If T is $\omega-$categorical and superstable, then T is $\omega-$stable.*

Proof. Suppose to the contrary that T is $\omega-$categorical and properly superstable. Certainly, an $\omega-$categorical theory is small, so Lemma 5.2.5 yields a complete type $p \in S(A)$, where A is finite, which has infinite multiplicity. Recall Lemma 5.1.10 linking conjugacy over $acl(A)$ with $FE(A) =$ the set of finite equivalence relations over A. There is a subset $\{ E_i(x,y) : i < \omega \}$ of $FE(A)$ such that each E_{i+1} refines E_i and $\models E_i(a,b)$, for all i, if and only if $tp(a/acl(A)) = tp(b/acl(A))$. Let a realize p. Since p has infinite multiplicity, $p \cup \{ E_i(x,a) \wedge \neg E_{i+1}(x,a) \}$ is consistent for infinitely many i. These infinitely many types over $A \cup \{a\}$ contradict the $\omega-$categoricity of T.

Historical Notes. The original source for these results is [She71]. They are also found in [She90].

Exercise 5.2.1. Show that $\kappa(T) \leq |T|^+$, when T is stable.

Exercise 5.2.2. Prove Proposition 5.2.1.

Exercise 5.2.3. Let $\mathfrak{C}$ be the universal domain of a superstable theory and A a set. Then $Th(\mathfrak{C}_A)$ is also superstable.

Exercise 5.2.4. Give examples of countable theories in each of the classes delineated in Proposition 5.2.2.

Exercise 5.2.5. Let T be a countable stable theory and $p \in S(A)$ a type with infinite multiplicity. Show that the multiplicity of p is $2^{\aleph_0}$.

Exercise 5.2.6. Prove: If T is superstable then for every infinite set of indiscernibles I, $Av(I/\mathfrak{C})$ is based on a finite $J \subset I$.

5.3 Stable Groups and Modules

In this section we generalize the treatment of generics for $\omega-$stable groups to the stable setting. Besides the $\omega-$stable groups the stable groups include all modules (see Section 5.3.2). Group actions play a central role in stability theory today. Here we develop a theory of generic types for group actions specializing to a theory of generic types for groups (since a group acts on itself by translation). Examples of ω−stable groups and group actions were given previously.

In the subsection on modules we develop the most basic model-theoretic properties of these natural mathematical objects and interpret in this setting the tools we will develop to study stable groups.

The material on 1−based groups generalizes the subject matter in Section 4.3.2.

As with ω−stable groups our study of stable groups depends on the existence of the connected component of a group and stabilizers of types.

Given a definable group in a stable theory the groups of significant model theoretic interest, such as the connected component and stabilizers of types, may not be definable—they may only be $\bigwedge$−definable. Since we need a theory of generics for these groups as well we must work with $\bigwedge$−definable groups (and group actions) from the beginning. The explicit definitions are as follows. (Parts (i) and (ii) are simply restatements of Definition 3.5.11(i) and (ii).)

Definition 5.3.1. *Let T be a complete theory.*

(i) We call $(G,\cdot)$ an $\bigwedge$−definable group over A *if*

- *$(G,\cdot)$ is a group,*
- *G is a subset of $\mathfrak{C}$, $\bigwedge$−definable over A, and*
- *there is a function f, definable over A in $\mathfrak{C}$, such that $f \restriction G \times G$ defines the binary operation $\cdot$ on G.*

(ii) Similarly, a group action $(G,\cdot,X,\star)$ is an $\bigwedge$−definable group action over A *if $(G,\cdot)$ is an $\bigwedge$−definable group over A in $\mathfrak{C}$, X is a subset of $\mathfrak{C}$, $\bigwedge$−definable over A, and $\star$ is the restriction to $G \times X$ of an A−definable function.*

(iii) A stable group *(*stable group action*) is an $\bigwedge$−definable group (group action) in a stable theory.*

The usual conventions about dropping the A when it is $\emptyset$ are adopted. When confusion seems unlikely the $\cdot$ and $\star$ are omitted from expressions and we simply write gh for $g \cdot h$ and gx for $g \star x$. When $G = X$ and $\star$ is multiplication on the left (right), $\star$ may be called *left* (*right*) *translation.*

If the type Φ defines an $\bigwedge$−definable group action in $\mathfrak{C}$ and N is another model of the theory, then Φ defines such a group action in N (when it contains the relevant parameters). We then call $\Phi(N)$ an *$\bigwedge$−definable group action in N.*

Let G be a stable group. If G is definable we can restrict the universe to G without altering the set of definable relations. Since restriction to an $\bigwedge$−definable set is not so well-behaved we must continue to mention the ambient theory when studying an $\bigwedge$−definable group. Instead of studying the models of $Th(G)$ where this theory is stable (as we did with ω−stable groups), we study the groups $\Phi(M)$, where $\Phi(\mathfrak{C})$ is a group and M ranges over the models of the theory.

When dealing with groups we abandon some of our notational abbreviations. By $a \in G$ we really mean that a is an element of G, not that a is a finite sequence from G. When X is the set of realizations of the type $\Phi(x)$ in some model M and A is a set, $S^X(A)$ denotes the elements of $S(A)$ which extend $\Phi(x)$. Given a set of formulas $\Delta(x)$, $S^X_\Delta(A)$ denotes $\{ p \restriction \Delta : p \in S^X(A) \}$.

As with ω−stable groups, a stable group action gives rise to an action of the group on a collection of types.

Definition 5.3.2. *Let $(G, X, \star)$ be a stable group action, $\Phi(x)$ the type defining X and $p(x)$ a type over M containing Φ. Given $a \in G$, $ap = \{ \varphi(a^{-1} \star x) : \varphi \in p \}$ and is called the* left translate of p by a. *The type pa is defined similarly, and (when $X = G$) p^{-1} is obtained by replacing x by x^{-1} in p.*

This definition specializes to the earlier definition of translation in an ω−stable group G by taking $X = G$ and the action to be multiplication on the left.

Let $\bar{G} = (G, X, \star)$ be a stable group action and $\Delta(x)$ a set of formulas over $\emptyset$, where x has the same sort as X. Let $\Delta^*(x) = \{ \varphi(y \star x, z) : \varphi(x, z) \in \Delta \}$, where $\varphi(y \star x, z)$ has object variable x and a new parameter variable y. We call Δ *invariant* if for any $p \in S_\Delta(\mathfrak{C})$ and $a \in G$, $ap \in S_\Delta(\mathfrak{C})$. Notice that Δ^* is invariant and every element of $S_\Delta(\mathfrak{C})$ is a Δ^*−type (since any $\varphi(x, z) \in \Delta$ is equivalent to $\varphi(1 \star x, z)$). Warning: For Δ an arbitrary finite set of formulas and $p \in S_\Delta(\mathfrak{C})$ it is possible that $ap \notin S_\Delta(\mathfrak{C})$.

Lemma 5.3.1. *Let (G, X) be a stable group action, Φ the type defining X and Δ an invariant set of formulas. Then G acts on $S^X_\Delta(\mathfrak{C})$ and for any type p containing Φ and $a \in G$, $(R, \mathrm{Mult})_\Delta(p) = (R, \mathrm{Mult})_\Delta(ap)$.*

Proof. See Exercise 5.3.2.

Thus, if H is an $\bigwedge$−definable subgroup of the stable group G, Δ is invariant with respect to the action of G on itself and $a \in G$, then H and aH have the same Δ−rank and multiplicity. (Remember, the Δ−rank of H is, by definition, the Δ−rank of the type defining it.)

Definition 5.3.3. *Let (G, X) be a stable group action, Δ an invariant set of formulas and $p \in S^X(\mathfrak{C})$. The* Δ−stabilizer of p *is $stab(p, \Delta) = \{ a \in G : a(p \restriction \Delta) = p \restriction \Delta \}$. The* stabilizer of p, *stab(p), is $\{ a \in G : ap = p \}$; i.e., the $L(x)$−stabilizer of p, where $L(x)$ is the set of formulas over $\emptyset$ with object variable x. Equivalently, $stab(p) = \bigcap \{ stab(p, \Delta) :$* Δ invariant $\}$.

Let X be an $\bigwedge$−definable set in $\mathfrak{C}$ and $p(x, y)$ a type. A subset Y of X is called *definable-by-p* if there is a b such that $Y = p(\mathfrak{C}, b) \cap X$. Occasionally, p will have no parameter variables, in which case a definable-by-p set is just a subset of $p(\mathfrak{C})$. For $\Delta(x)$ a set of formulas as usual we call Y *definable-by-Δ* if there is a Δ−type $p(x, b)$ such that $Y = p(\mathfrak{C}, b) \cap X$. Y is called *definable-by-L* if it is definable-by-ψ for some formula ψ; equivalently Y is the intersection of X and a definable set.

Lemma 5.3.2. *If G is a stable group and $\Delta(x)$ is a finite set of formulas over $\emptyset$, where x has the same sort as G, then the collection of definable-by-Δ subgroups of G has the ascending and descending chain conditions.*

Proof. Suppose, for example, that $G_0 \supset G_1 \supset G_2 \supset \ldots$ is a strictly descending chain of definable-by-Δ subgroups. Then, for each i, G_i contains two cosets of G_{i+1}, each having the same Δ^*−rank and multiplicity as G_{i+1}. Since G_{i+1} and each coset of it is defined by a Δ^*−type, $(R, \mathrm{Mult})_{\Delta^*}(G_i) > (R, \mathrm{Mult})_{\Delta^*}(G_{i+1})$. This contradicts the existence of Δ^*−rank.

The nonexistence of an infinite ascending chain of definable-by-Δ subgroups follows from roughly the same argument using that Δ^*−rank of any type is finite.

Definition 5.3.4. *If G is a stable group and Δ is finite (and contains $x = x$, so that G is definable-by-Δ), there is a unique minimal definable-by-Δ subgroup of G having finite index in G, which is called the* Δ−connected component *of G.*

Let G be a stable group. The connected component *of G, denoted G^o, is the intersection of all of the Δ−connected components, as Δ ranges over all finite sets of formulas. G is* connected *if $G^o = G$.*

If G is $\bigwedge$−definable over A, then G^o is also $\bigwedge$−definable over A. The connected component is a normal subgroup which is itself connected.

Lemma 5.3.3. *Let $(G, X, \star)$ be a stable group action, $p \in S^X(\mathfrak{C})$ and Δ a finite invariant set of formulas. Then, for any set A over which $p \restriction \Delta$ is definable there is a formula $\psi(x)$ over A such that $stab(p \restriction \Delta)$ is definable-by-ψ. Thus, the stabilizer of p is $\bigwedge$−definable over $Cb(p)$.*

Proof. Given $g \in G$, $g \in stab(p \restriction \Delta)$ if and only if for all $\delta(x, y) \in \Delta$,

$$\forall y(\ \delta(x,y) \in p \restriction \Delta \iff \delta(g \star x, y) \in p \restriction \Delta \ )$$

Using that $p \restriction \Delta$ is definable over A this equivalence yields a formula ψ over A defining $stab(p \restriction \Delta)$.

We frequently need to work with types over sets rather than elements of $S^X(\mathfrak{C})$. The following is used to translate theorems about the action of G on $S^X(\mathfrak{C})$ into facts about the action on other types. (This generalizes Corollary 3.5.1.)

Lemma 5.3.4. *Let (G, X) be a stable group action, p, $q \in S^X(\mathfrak{C})$ and $a \in G$. Suppose that p and q are definable over A. Then,*

(1) *$q = ap$ if and only if*
(2) *there is a b realizing $p \restriction A$ such that b is independent from a over A, ab realizes $q \restriction A$ and ab is independent from a over A.*

Proof. Interpolating a few more equivalences will make the proof easy.

Claim. The following are equivalent.

(i) $q = ap$
(ii) for all sets $B \supset A \cup \{a\}$, $q \restriction B = ap \restriction B$;
(iii) there is a set $B \supset A \cup \{a\}$, $q \restriction B = ap \restriction B$.

This is proved like the claim in Corollary 3.5.1, replacing Morley rank and degree by Δ–rank and Δ–multiplicity. (In the detailed proof the reader should remember that for all finite Δ, $\mathrm{Mult}_\Delta(q) = \mathrm{Mult}_\Delta(q \restriction A) = 1$, by Remark 5.1.5.)

Turning to the lemma per se, that (1) implies (2) is just a matter of unraveling the notation. Now assume that a and b meet the conditions in (2). Let $B = A \cup \{a\}$ and let Δ be an invariant set. Since p and q are both definable over A and both b and ab are independent from a over A, b realizes $p \restriction B$ and ab realizes $q \restriction B$. Thus, $q = ap$, proving the lemma.

Since independence in a stable theory is defined with a scheme of ranks instead of a single rank a generic type cannot be defined as a type of maximal rank (as in ω–stable groups). Instead, genericity is defined in terms of forking independence (asking the reader to prove in the exercises that the two notions are equivalent for ω–stable groups).

Definition 5.3.5. *Let $(G, X, \star)$ be a stable group action, $\bigwedge$–definable over A, Φ the type defining X and $p \in S^X(\mathfrak{C})$.*

- *p is called* generic *if for all $a \in G$, ap does not fork over A.*
- *An arbitrary stationary type q is* generic *if $q|\mathfrak{C}$ is generic.*
- *An element a of X is said to be* generic over *B if $stp(a/B)$ is generic, shortening the term "generic over A" to simply "generic".*
- *If $X = G$ and $\star$ is left (right) translation we call p a left (right)* generic *of G.*

When we say "p is a generic of G", p is understood to be a left generic.

Notice that the translate of a generic is itself generic.

Lemma 5.3.5. *Let $(G, X, \star)$ be a stable group action which is $\bigwedge$–definable over A, Φ the type defining X and p a stationary type extending Φ. Then p is generic if and only if for all sets $B \supset A$ over which p is based, all a realizing $p|A$, and $g \in G$,*

$$g \mathop{\smile\!\!\!\!\!|}_{A} a \Longrightarrow g \star a \mathop{\smile\!\!\!\!\!|}_{A} g.$$

Proof. See Exercise 5.3.5.

Lemma 5.3.6. *Let (G, X) be a stable group action which is $\bigwedge$–definable over A and let Φ be the type defining X.*

(i) There is a generic type in $S^X(\mathfrak{C})$.

(ii) If $p \in S^X(\mathfrak{C})$ is generic, $G^o \subset stab(p)$. For any $p \in S^G(\mathfrak{C})$, $stab(p) \subset G^o$ (so when p is generic, $stab(p) = G^o$).

(iii) If a, $b \in G$ are $A-$independent generics then $b^{-1}a$ is $A-$independent from a and $A-$independent from b. Thus $b^{-1}a$ is a generic.

(iv) $S^{G^o}(\mathfrak{C})$ contains a left generic and a right generic.

Proof. To make the notation simpler we take A to be $\emptyset$.

(i) Let x be a variable in the same sort as X. A formula $\varphi(x)$ (over $\mathfrak{C}$) is called *small* if for some $a \in G$, $a\varphi(x)$ forks over $\emptyset$. Let $\Psi = \{ \neg\varphi : \varphi \text{ small} \} \cup \Phi$ and suppose, towards a contradiction, that Ψ is inconsistent. Then, there are small formulas $\varphi_0, \ldots, \varphi_n$ such that every type in $S^X(\mathfrak{C})$ contains one of φ_i's. Let $\varphi_i = \varphi_i(x, a_i)$, where $\psi_i = \varphi_i(x, y_i)$ is over $\emptyset$ and let $\Delta = \{\psi_0, \ldots, \psi_n\}^*$. Let p_0 be an element of $S^X(acl(\emptyset))$ with $R_\Delta(p_0)$ maximal, and let $p \in S^X(\mathfrak{C})$ be the nonforking extension of p_0. Then one of $\varphi_0, \ldots, \varphi_n$, say $\varphi_i = \varphi$, is in p. Pick $a \in G$, so that $a\varphi$ forks over $\emptyset$. Since Δ is invariant, $R_\Delta(p) = R_\Delta(ap) = k$. By the maximality of $R_\Delta(p)$, $R_\Delta(q_0) = k$, where $q_0 = ap \restriction acl(\emptyset)$. There is only one complete $\Delta-$type over $\mathfrak{C}$ consistent with q_0 and having the same $\Delta-$rank as q_0, namely the restriction to Δ of $q_0|\mathfrak{C}$ (Lemma 5.1.8). Thus, $ap \restriction \Delta$ does not fork over $\emptyset$, contradicting that this type contains $a\varphi$. This proves the consistency of Φ. Using Corollary 5.1.8(iii) the reader can verify that any completion of Φ in $S^X(\mathfrak{C})$ is a generic, proving (i).

(ii) Let Δ be a finite invariant set of formulas and $H = stab(p, \Delta)$. There is a one-to-one correspondence between the cosets of H in G and $\{ ap \restriction \Delta : a \in G \}$. Since each translate of p is a generic, and hence does not fork over $\emptyset$, there are at most $2^{|T|}$ many types in $\{ ap \restriction \Delta : a \in G \}$. Since H is definable-by-ψ for some formula ψ over $acl(\emptyset)$, this bound on $[G : H]$ forces H to have finite index in G (see the exercises). Thus H contains G^o. This is true for any invariant Δ and $stap(p) = \bigcap_\Delta stab(p, \Delta^*)$, so $stab(p) \supset G^o$.

Let $p \in S^G(\mathfrak{C})$. The key observation needed to prove that $stab(p) \subset G^o$ is the following. The proof of the claim is assigned as an exercise at the end of the section.

Claim. Suppose $H \subset G$ is a definable-by-ψ subgroup of finite index, where ψ is a formula over $acl(\emptyset)$. Then for any coset B of H in G there is a formula θ over $acl(\emptyset)$ such that B is definable-by-θ. Thus,

$$\text{for any } a \text{ and } b \text{ in } G,\ stp(a) = stp(b) \implies aG^o = bG^o. \tag{5.3}$$

The proof is assigned to the reader in Exercise 5.3.6.

Let $g \in stab(p)$, B a set on which p is based, $q = p|B$ and a a realization of q independent from g over B. Since g is in the stabilizer of p, $g \cdot a$ also realizes q (Lemma 5.3.4) hence $stp(a) = stp(g \cdot a)$. By (5.3), $g = (g \cdot a) \cdot a^{-1} \in G^o$. Thus, $stab(p) \subset G^o$, completing the proof of (ii).

(iii) Since $stp(a)$ is generic and $a \perp b^{-1}$, $b^{-1}a$ is independent from b^{-1} (by Lemma 5.3.5) hence $b^{-1}a \perp b$. Similarly, the inverse of $b^{-1}a$ ($= a^{-1}b$) is independent from a^{-1}, hence $b^{-1}a \downarrow a$. By Lemma 5.3.4, $b(stp(b^{-1}a)|\mathfrak{C}) = stp(a)|\mathfrak{C}$; i.e., $stp(b^{-1}a)|\mathfrak{C}$ is a translate of a generic, hence a generic itself.

(iv) Let a and b be independent generics with respect to left translation which realize the same strong type p over $\emptyset$. By (5.3), $b^{-1}a$ is in G^o, and this is a generic element (by (iii)). The proof that there is a generic with respect to right translation in G^o is found simply by switching from left to right in the above proof.

Proposition 5.3.1. *Let (G, X) be a transitive stable group action and Φ the type defining X.*

(i) G acts transitively on the set of generics in $S^X(\mathfrak{C})$.

(ii) For any generic p and invariant set Δ, $R_\Delta(p) = R_\Delta(\Phi)$. For any finite set of formulas $\Delta(x)$ (where x is in the sort of X), $\{p \restriction \Delta : p \in S^X(\mathfrak{C})$ is generic$\}$ is finite.

Proof. For simplicity, suppose that G and X are $\bigwedge$−definable over $\emptyset$.

(i) Let p and q be generics in $S^X(\mathfrak{C})$, $p_0 = p|acl(\emptyset)$ and $q_0 = q|acl(\emptyset)$. Let a and b be independent realizations of p_0 and q_0, respectively, and $g = ba^{-1}$. Let $h \in G^o$ be a generic with respect to right translation which is independent from $\{g, a, b\}$. Since $stab(q) \supset G^o$ and $h \downarrow b$, $hga = hb$ realizes q_0. We claim that hg is independent from a. First, h is a right generic independent from g, so $hg \downarrow g$. Moreover, hg and a are independent over g (since h is independent from $\{g, a\}$ and hg is interalgebraic with h over g). By the transitivity of independence, $hg \downarrow a$, as needed. Since a is generic, $hg \downarrow hga$; i.e., $hg \downarrow hb$. By Lemma 5.3.4, $hg(p|\mathfrak{C}) = q|\mathfrak{C}$, proving (i).

(ii) Let Δ be an invaraint finite set of formulas. That there is a generic $p^* \in S^X(\mathfrak{C})$ with $R_\Delta(p^*) = R_\Delta(\Phi)$ is implicit in the proof of Lemma 5.3.6(i). All generic types $p \in S^X(\mathfrak{C})$ have the same Δ−rank (by (i)). Thus, the cardinality of $\{p \restriction \Delta : p \in S^X(\mathfrak{C})$ is generic $\}$ is bounded by (actually equal to) the Δ−multiplicity of Φ.

The following makes a good summary of what is known about generics in stable groups.

Corollary 5.3.1. *Let G be a stable group, $\bigwedge$−definable over A, and $p \in S^G(\mathfrak{C})$.*

(i) The following are equivalent.

1. *The left stabilizer of p is G^o.*
2. *p is a right generic.*
3. *The right stabilizer of p is G^o.*
4. *p is left generic.*

(ii) If a and b are generic, $G^oa = G^ob \implies stp(a/A) = stp(b/A)$.

(iii) If a, $b \in G^o$ are generic, then $tp(a) = tp(b) = q$ and q is stationary.

(iv) If p is generic then so is p^{-1}.

(v) p is generic if and only if p is a translate of the unique generic in $S^{G^o}(\mathfrak{C})$. If p is generic and a realizes $p|acl(\emptyset)$, then $a^{-1}p$ is the generic in $S^{G^o}(\mathfrak{C})$.

Proof. (i) (1) $\Longrightarrow$ (2) Assuming that the left stabilizer of p is G^o we need to show that each right translate of p does not fork over $\emptyset$. Since each right translate of p also has G^o as its left stabilizer it suffices to show that p does not fork over $\emptyset$. Suppose that p is based on A, $q = p|A$ and a realizes q. Let $I = \{ g_i : i < \omega \}$ be an independent set of right generics in G^o which is independent from $A \cup \{a\}$. Since $g_i \in G^o$ is independent from a over A, $g_i a$ also realizes q. In fact, the $A-$independence of a and I implies that $g_i a$ realizes the nonforking extension of p over $A \cup \{a\} \cup (I \setminus \{g_i\})$. Thus, $J = \{ g_i a : i < \omega \}$ is a Morley sequence in p over A, over which p is based by Lemma 5.1.17. However, since I is an independent set of right generics, J is independent over $\emptyset$ (see the exercises). Being the average type of J, p does not fork over $\emptyset$.

(2) $\Longrightarrow$ (3) and (4) $\Longrightarrow$ (1) are by Lemma 5.3.6(ii) and (3) $\Longrightarrow$ (4) is proved by switching left and right in the proof of (1) $\Longrightarrow$ (2).

(ii) Suppose that a and b are generics and $G^o a = G^o b$. Without loss of generality, $a \perp b$. For $c = ba^{-1}$, $c \perp a$ and $c \perp b$ (since b is generic). Letting $p = stp(a)|\mathfrak{C}$ and $q = stp(b)|\mathfrak{C}$, $cp = q$. Furthermore, $c \in G^o = stab(p)$ since a and b have the same coset with respect to G^o. Thus, $p = q$, proving (ii).

(iii) This follows immediately from (ii).

(iv) If p is a left generic, then p^{-1} is a right generic, hence also a left generic by (i).

(v) This is just a summary of previous results. The details are assigned as an exercise.

5.3.1 1−based Groups and Modules

Throughout Chapter 4 groups played a key role in our detailed analysis of uncountably categorical theories. Both the strongly minimal sets and the manner in which the universe is constructed from the strongly minimal sets are “simpler” when the theory is 1−based. Here, some of the results from Section 4.3.2 (principally Theorem 4.3.3) are restated in the stable context. The purpose is not to prove the more general results in detail, but to point the reader in their direction. The details can be found in [Pil]. They are not significantly different from the uncountably categorical case.

In the next subsection theories of modules are introduced as examples of 1−based theories.

Definition 5.3.6. *A stable theory is called* 1−based *if for all sets A and B, A is independent from B over $acl(A) \cap acl(B)$.*

As usual, the universal domain of a stable theory T is called 1−based if T is 1−based.

Remark 5.3.1. The following are equivalent for $\mathfrak{C}$ the universe of a stable theory.

(1) $\mathfrak{C}$ is $1-$based.
(2) For all elements a and sets A, the canonical base of $tp(a/acl(A))$ is contained in $acl(a)$.

(The proof is virtually identical to the uncountably categorical case found in Remark 4.3.3.)

This equivalent definition explains the term "1−based". An uncountably categorical theory is $1-$based when, given a stationary type p and q the nonforking extension of p in $S(\mathfrak{C})$, q is based on $acl(a)$ for any single a realizing p.

The concept of an abelian structure was specified in Definition 4.3.6 for *definable* groups. For $\bigwedge-$definable groups we need a slightly more complicated notion.

Definition 5.3.7. *Let G be a group, $\bigwedge-$definable over A in the universal domain of a complete theory. Let*

$$\mathcal{H} = \{\, H :\ H \text{ is a subgroup of } \ G^n, \text{ for some } \ n, \\ \text{which is definable-by-}L \text{ over } \ acl(A)\,\}.$$

G is called an abelian structure *if for every $n < \omega$, every definable-by-L subset of G^n is equal to a boolean combination of cosets of elements of $\mathcal{H}$.*

Most results about $1-$based groups depend on

Theorem 5.3.1. *Let G be a group, $\bigwedge-$definable in a $1-$based (stable) theory. Then G is an abelian structure.*

The statements of the lemmas giving the proof of this theorem are virtually identical to the statements of corresponding lemmas in the proof of Theorem 4.3.3. The proofs of those earlier lemmas involved the action of a $1-$based (uncountably categorical) group G on the types over G, Morley rank independence and canonical parameters. After substituting forking independence for Morley rank independence and canonical bases for canonical parameters the same proofs yield the generalized lemmas. The reader is referred to [Pil] for the details.

One preliminary lemma that deserves to be singled out is

Lemma 5.3.7. *Let G be a stable group, $\bigwedge-$definable over A. Then, G is an abelian structure if and only if*

(∗) *for any $n < \omega$ and $p \in S_n^G(G)$ there is a connected group $H \subset G^n$, definable-by-L over $acl(A)$, such that p is a left (or right) translate of the generic type of H.*

Corollary 5.3.2. *Let G be a stable group in a $1-$based theory. Then, any connected $\bigwedge-$definable subgroup of G^n is $\bigwedge-$definable over $acl(\emptyset)$.*

Corollary 5.3.3. *A connected stable group in a $1-$based theory is abelian.*

5.3.2 Modules

The purpose of this subsection is to introduce modules as natural examples of $1-$based groups. Until stated otherwise we will work in a $1-$sorted language.

Let R be a ring with identity. Here, the term $R-$module means *right-$R-$module*. The *language of $R-$modules* is $L_R = \{0, +, r\}_{r\in R}$, where 0 is a constant symbol, + a binary operator and each r is a unary operator. It is an elementary exercise to find a theory T_R in L_R whose models are exactly the right $R-$modules.

Fixing a ring R, $\varphi(\bar{v}) = \varphi(v_1, \ldots, v_n)$ is a *positive-primitive formula (*pp-formula*) over* 0 if it is equivalent in the theory T_R to one of the form

$$\exists w_1 \cdots w_l \bigwedge_{j=1}^{m} (\sum_{i=1}^{n} v_i r_{ij} + \sum_{k=1}^{l} w_k s_{kj} = 0),$$

where r_{ij}, $s_{kj} \in R$ and 0 is a tuple of 0's of the appropriate length. In matrix notation this can be written as

$$\exists w_1 \cdots w_l (v_1, \ldots, v_n, w_1, \ldots, w_l) \begin{pmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & & \vdots \\ r_{n1} & \cdots & r_{nm} \\ s_{11} & \cdots & s_{1m} \\ \vdots & & \vdots \\ s_{l1} & \cdots & s_{lm} \end{pmatrix} = 0. \tag{5.4}$$

This is compressed to $\exists \bar{w}(\bar{v}\bar{w})H = 0$, where

$$H = \begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix}$$

and $\mathbf{r}$, $\mathbf{s}$ are the matrices of r_{ij}'s and s_{kj}'s. In yet another form $\varphi(\bar{v})$ can be written as $\exists \bar{w}(\bar{v}\mathbf{r} = -\bar{w}\mathbf{s})$. As this final form suggests, a *pp*$-$formula can be thought of as a generalized divisibility condition. (An element a is said to be *divisible by s* if there is a b such that $a = bs$, equivalently a satisfies the formula $\exists w(v = ws)$.)

For A a subset of some $R-$module the term *pp$-$formula over A* is defined as above except that on the right hand side of (5.4) there is a $1 \times m$ row vector of elements of A. For $\varphi(\bar{v})$ a *pp*$-$formula over $\bar{a}$ there is $\psi(\bar{v}, \bar{w})$ a *pp*$-$formula over 0 such that $\varphi(\bar{v}) = \psi(\bar{v}, \bar{a})$. In general, the term *pp*$-$formula refers to one with nonzero parameters.

Remark 5.3.2. Let R be a ring, $\varphi(\bar{v})$ a pp–formula over 0 and M an R–module.

(i) In the exercises the reader is asked to verify that $\varphi(M)$ is a subgroup of M^n, where n is the length of $\bar{v}$.

(ii) Furthermore, if r is in the center of R (i.e., the subring of elements commuting with every element of R) then $\models \varphi(\bar{a}) \implies \models \varphi(\bar{a}r)$, where $(a_1, \ldots, a_n)r = \bar{a}r = (a_1 r, \ldots, a_n r)$. Thus, if R is commutative $\varphi(\bar{v})$ defines a submodule of M. If R is not commutative the subgroup defined by a pp–formula need not be a submodule. (Consider, for example, the ring R of 2×2 matrices over a field K with $M = R_R$. Let

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and $\varphi(v) = \exists w(v = we)$. Then

$$\varphi(M) = \begin{pmatrix} K & 0 \\ K & 0 \end{pmatrix},$$

is a left-ideal but not a right-ideal, hence not a submodule of M.)

(iii) More generally, if $\varphi(\bar{v}, \bar{w})$ is a pp–formula over 0 and M is an R–module then $\varphi(M, \bar{a})$ is empty or a coset of the subgroup $\varphi(M, 0)$ of M^l, where l is the length of $\bar{v}$ and 0 denotes a tuple of 0's of the same length as $\bar{w}$ (exercise). When $\varphi(M, \bar{a}) \neq \emptyset$, $\varphi(\bar{v}, \bar{a})$ is equivalent to $\varphi(\bar{v} - \bar{b}, 0)$, for some $\bar{b}$.

Theorem 5.3.2 (Elimination of Quantifiers). *Let R be a ring, L_R the language of R–modules and T_R the theory of right R–modules. Then for any formula $\varphi(\bar{v})$ in L_R without parameters there is $\varphi'(\bar{v})$, a boolean combination of pp–formulas, such that $T_R \models \forall \bar{v}(\varphi(\bar{v}) \longleftrightarrow \varphi'(\bar{v}))$.*

For a proof of this theorem by Baur, Garavaglia and Monk see [Zie84] or [Pre88].

In stability theory complete types are often more useful than formulas. What does the elimination of quantifiers have to say about complete types?

Definition 5.3.8. *Let M be a module over a ring R, $A \subset M$ and $p \in S_n(A)$.*

(i) The pp–part of p, denoted p^+, is $\{ \varphi \in p : \varphi$ is a pp–formula over $A \}$, *while* $p^- = \{ \neg\varphi \in p : \varphi$ is a pp – formula over $A \}$.

(ii) If $\bar{a}$ is a sequence from M the pp–type of $\bar{a}$ over A, $pp(\bar{a}/A)$ *is $tp(\bar{a}/A)^+$. Some contexts involve modules of different complete theories (i.e., we are not working in a single universal domain), in which case the notation $pp^M(\bar{a}/A)$ is used.*

(iii) A pp–type over A *is a consistent set Γ of pp–formulas over A. Γ is* complete *(in M) if it is the pp–type over A of a sequence from the universal domain of $Th(M)$.*

Let M be a module over a ring R and p a complete type in $T = Th(M)$. By the elimination of quantifiers down to pp-formulas, p is equivalent to $p^+ \cup p^-$ in T and for $\bar{a}$, $\bar{b}$ sequences from M, $tp(\bar{a}/A) = tp(\bar{b}/A) \iff pp(\bar{a}/A) = pp(\bar{b}/A)$.

We now restrict our attention to 1-types, although the same facts hold for n-types since M^n is also an R-module.

For $p \in S_1(M)$ let

$$\Phi_p = \{\varphi(x) : \varphi \text{ is a } pp\text{-formula over } 0 \text{ and for some } a \in M,\ \varphi(x-a) \in p\}.$$

Equivalently, Φ_p is the set of pp-formulas $\varphi(x)$ over 0 such that for a realizing p, the coset $a + \varphi(\mathfrak{C})$ is represented in M. Notice that the difference of two realizations of p is in $\Phi_p(\mathfrak{C})$.

Given a pp-formula $\psi(x, \bar{b}) \in p$ there is a pp-formula $\varphi(x)$ over 0 and an $a \in M$ such that $\varphi(x - a)$ is equivalent to $\psi(x, \bar{b})$. Thus,

$$p^+ \text{ is equivalent to } \{\varphi(x-a) \in p : \varphi \in \Phi_p\}. \tag{5.5}$$

Working in M^{eq} let $C(p) = \{\ulcorner\varphi(x-a)\urcorner : \varphi(x-a) \in p \text{ and } \varphi \in \Phi_p\}$. Equivalently, $C(p)$ is the set of $b \in M^{eq}$ such that b is the name of a coset of a subgroup H of $\mathfrak{C}$, pp-definable over 0, and the formula expressing "$x+H = b$" is in p. Notice that $C(p) \subset dcl(c)$ for any c realizing $p \restriction C(p)$. By (5.5),

$$\text{For } p,\, q \in S_1(M),\ p = q \iff p^+ = q^+ \iff C(p) = C(q). \tag{5.6}$$

As a first consequence of this reduction:

Corollary 5.3.4. *The complete theory of an infinite module is stable.*

Proof. Let M be an R-module of cardinality κ, where $\kappa = \kappa^{|R|}$. For any $p \in S_1(M)$, $|C(p)| \leq |R|$. In M^{eq} there are $\leq \kappa^{|R|}$ sets of the form $C(p)$, as p ranges over $S_1(M)$. Thus, by (5.6), $|S_1(M)| \leq \kappa^{|R|} = \kappa$. Similarly, $|S_n(M)| \leq \kappa$ for all $n < \omega$, proving the corollary.

Let M be an infinite R-module, $p \in S_1(M)$ and identify p with its unique extension in $S_1(M^{eq})$. Since Φ_p is a pp-type over 0, $\Phi_p(\mathfrak{C})$ is a subgroup of $\mathfrak{C}$. Suppose there is an $a \in M$ realizing $p \restriction C(p)$ (as there will be if M is $|R|^+$-saturated). Thus, a and a realization b of p have the same coset with respect to $\varphi(\mathfrak{C})$ for any $\varphi \in \Phi_p$, in other words, $a + \Phi_p(\mathfrak{C}) = b + \Phi_p(\mathfrak{C})$.

It is a short trip from (5.6) to

Lemma 5.3.8. *For $p \in S_1(\mathfrak{C})$, $Cb(p) = dcl(C(p))$ in $\mathfrak{C}^{eq}$.*

Proof. It suffices to show that for $f \in \mathrm{Aut}(\mathfrak{C})$, $f(p) = p$ if and only if f fixes $C(p)$ pointwise. Observe that for any $f \in \mathrm{Aut}(\mathfrak{C})$, $f(C(p)) = C(f(p))$. Thus, if $f \in \mathrm{Aut}(\mathfrak{C})$ and $f(p) = p$, f pointwise fixes $C(p)$. Conversely, if $f \in \mathrm{Aut}(\mathfrak{C})$ is the identity on $C(p)$, then $C(p) = C(f(p))$, so $p = f(p)$ by (5.6).

Thus, $p \in S_1(\mathfrak{C})$ is also determined by $C(p)$ in the sense that it is definable over $C(p)$.

Corollary 5.3.5. *The theory of an infinite module is* $1-$*based.*

Proof. Suppose $p = tp(a/A)$ is stationary and let q be the nonforking extension of p over $\mathfrak{C}$. Without loss of generality, p is a $1-$type. Then $Cb(q) \subset dcl(A)$, so a realizes $q \restriction Cb(q) \supset q \restriction C(q)$. Since $C(q) \subset dcl(a)$ and $Cb(q) \subset dcl(C(q))$ (by Lemma 5.3.8), $Cb(q) \subset dcl(a)$.

One step in the proof that a $1-$based group G is an abelian structure is to show that any $p \in S^G(\mathfrak{C})$ is a translate of the generic in $stab(p)$ and $stab(p)$ is connected and definable over $acl(\emptyset)$. Below (in Proposition 5.3.2) we give an independent proof of this fact when G is a module.

The group $\Phi_p(\mathfrak{C})$ is one we are already very familiar with:

Lemma 5.3.9. *Let $\mathfrak{C}$ be the universal domain of a complete theory of modules, For $p \in S_1(\mathfrak{C})$, $\Phi_p(\mathfrak{C}) = stab(p)$.*

Proof. Let M be a saturated model on which p is based and $q = p|M$. Note: $q \supset p \restriction C(p)$ and for any $\varphi \in \Phi_p$ there is a $b \in M$ such that $\varphi(x-b) \in q$. Let s be the type over $C(p)$ such that $stab(p) = s(\mathfrak{C})$. Since M is saturated, to prove that $s(\mathfrak{C}) = \Phi_p(\mathfrak{C})$ it suffices to show that $H = s(M) = \Phi_p(M)$. If $g \in H$ and a realizes q, then $g + a$ also realizes q. Thus, $g \in \Phi_p(M)$ since the difference of any two realizations of q is in $\Phi_p(\mathfrak{C})$; i.e., $H \subset \Phi_p(M)$.

Now suppose that $g \in \Phi_p(M)$ and let $r = tp(g + a/M)$. To prove that $r = q$ we will show that $C(q) = C(r)$ and quote (5.6). Given $\varphi \in \Phi_p$, if $\varphi(x-b) \in q$, then $\models \varphi(a-b)$ and $\varphi(g)$, so $\models \varphi(g+a-b)$. Thus, $\Phi_r \supset \Phi_p$. Conversely, given $\varphi \in \Phi_r$ with $\varphi(x-b) \in r$, $\models \varphi(a-(b-g))$, hence $\varphi \in \Phi_p$. We conclude that $\Phi_p = \Phi_r$. Since $g \in \Phi_p(M)$, a and $g+a$ have the same coset with respect to $\varphi(\mathfrak{C})$ for every $\varphi \in \Phi_p$, hence $C(q) = C(r)$. This proves (by (5.6) that $r = q$. Thus, $g \in H$; i.e., $\Phi_p(M) \subset H$.

Corollary 5.3.6. *Let $\mathfrak{C}$ be the universal domain of a complete theory of modules and $p \in S_1(\mathfrak{C})$. Then $stab(p)$ is $\bigwedge-$definable over $\emptyset$.*

Proposition 5.3.2. *Let $\mathfrak{C}$ be the universal domain of a complete theory of $R-$modules and $p \in S_1(\mathfrak{C})$. Then p is a translate of a generic type in $stab(p)$. Furthermore, $stab(p)$ is connected.*

Proof. The connectedness of the stabilizer will be proved later. The bulk of the proof lies in showing

Claim. Let a and b to be realizations of $p|C(p)$ which are independent over $C(p)$. Then $a-b$ is a generic of $stab(p)$ which is independent from a and from b.

By Lemma 5.3.9, $a - b \in G = stab(p)$. It remains to show that $a - b$ is a generic of G independent from a and independent from b. This is accomplished by finding a', b' with the desired properties which realize $tp(ab/C(p))$. We will use repeatedly throughout the proof the fact that $C(p)$ is contained in the definable closure of any realization of $p|C(p)$. As a first step suppose that a' is a realization of $p|C(p)$ and g is a generic of $G = stab(p)$ which is independent from a'. Then,

$$g + a' \text{ realizes } p|C(p). \tag{5.7}$$

For a and b as given let $g \in G$ be generic over $\{a, b\}$. Then $(g + a) - b = g + (a - b)$ is a generic of G over $\{a, b\}$. Letting h be a generic of G independent from $\{a, b, g\}$, $(g + a) - (h + b) = (g + a - b) - h$ is also generic over $\{a, b, g\}$. Let $a' = g + a$ and $b' = h + b$, both realizations of $p|C(p)$ with $a' - b'$ a generic of G. We claim that

$$a' \perp a' - b',\ b' \perp a' - b' \text{ and } a' \underset{C(p)}{\perp} b'. \tag{5.8}$$

(Since $(g + a) - (h + b)$ is independent from $\{a, b, g\}$ it is independent from $g + a$. By symmetry and transitivity of independence, g is independent from $\{a, b, h\}$, hence the same argument shows that $(g + a) - (h + b)$ is independent from $h + b$. Since a and b are independent over $C(p) \subset acl(a) \cap acl(b)$ the independence of $\{g, h, \{a, b\}\}$ implies that $\{g, a\}$ and $\{h, b\}$ are independent over $C(p)$. Hence, $a' = g + a$ and $b' = h + b$ are independent over $C(p)$.)

By (5.7) a' and b' are realizations of $p|C(p)$. Since they are independent over $C(p)$, $tp(a', b'/C(p)) = tp(a, b/C(p))$. Thus there is an automorphism of $\mathfrak{C}^{eq}$ fixing $C(p)$ and taking a' to a and b' to b. The conditions in (5.8) show that a and b meet the requirements of the claim.

Now let M be a $|R|^+$–saturated model containing $C(p)$, a a realization of $p|C(p)$ in M and b a realization of $p|M$. Let $g = b - a$ and $q = tp(g/M)$. By the claim, g is a generic of $stab(p)$ and is independent from a over $\emptyset$. Since p is based on a, b is independent from M over a, hence g is also independent from M over a. By transitivity, g is independent from M over $\emptyset$ implying that q is a generic type of $stab(p)$. Since $a + q = p|M$, p is a translate of a generic of $stab(p)$, completing the proof of the main part of the proposition.

To prove the connectedness of $G = stab(p)$ first remember that a type in $S_1(\mathfrak{C})$ and its translates have the same stabilizer. Thus, if $q \in S_1(\mathfrak{C})$ is a generic of G such that q is a translate of p, $stab(q) = G$. In fact, since the action of G on the generics is transitive, any generic of G has G as its stabilizer. Let $r \in S_1(\mathfrak{C})$ be an arbitrary generic of G. By Corollary 5.3.1(v), for a a realization of $r|acl(\emptyset)$, $-a + r$ is the generic in G^o. Since $-a$ is also in the stabilizer of r, $r = -a + r$ must be the generic in G^o. That is, G is connected.

This yields the particularly simple picture of a module $\mathfrak{C}$. Any $p \in S_1(\mathfrak{C})$ is a translate of a generic of a subgroup $\Phi(\mathfrak{C})$ $(= stab(p))$, where Φ is a pp–type over $\emptyset$.

The proposition translates many properties of types into properties of their stabilizers. For example, if $p \subset q$ are stationary types in a module, then q is a forking extension of p if and only if $stab(q|\mathfrak{C})$ has infinite index in $stab(p|\mathfrak{C})$ (left to the exercises).

Historical Notes. Historically speaking, the results of this section have the same source as those in the subsection on ω−stable groups. We owe much of our knowledge about generic types outside of the ω−stable setting to Poizat [Poi81], whose work was carried on by Berline and Lascar in [BL86] and [Ber86]. Lemma 5.3.2 is due to Baldwin and Saxl [BS76]. The detailed treatment here is taken from Hrushovski's dissertation [Hru86], which is also found in [Hru90b].

The logical analysis of the theory of modules begins with Szmielew's quantifier elimination theorem for abelian groups [Szm55]. The work of Eklof, Fisher, Sabbagh and Baur (see [EF72], [ES71]) took a more model-theoretic approach. The elimination of quantifiers theorem for modules is due independently to Baur [Bau76], Garavaglia and L. Monk.

Exercise 5.3.1. Suppose the types Φ and Ψ are such that $(\Phi(\mathfrak{C}), \Psi(\mathfrak{C}), \star)$ defines a group action in $\mathfrak{C}$, where Φ, Ψ and $\star$ are all over $\emptyset$. Given any model M prove that $(\Phi(M), \Psi(M), \star)$ is a group action.

Exercise 5.3.2. Prove Lemma 5.3.1.

Exercise 5.3.3. Let G be an ω−stable group. Prove that $p \in S^G(G)$ is generic (as defined in this section) if and only if $MR(p) = MR(G)$.

Exercise 5.3.4. Suppose that G is a stable group in $\mathfrak{C}$ and H is a subgroup of G which is definable-by-ψ (for some formula ψ) with $[G : H] < |\mathfrak{C}|$. Prove that $[G : H]$ is finite.

Exercise 5.3.5. Prove Lemma 5.3.5.

Exercise 5.3.6. Prove the first claim in the proof of Lemma 5.3.6(ii).

Exercise 5.3.7. Suppose that $(G, X, \star)$ is a stable group action, I is an independent set of generics in X and $a \in G$ is independent from I. Show that $\{ a \star b : b \in I \}$ is an independent set of generics.

Exercise 5.3.8. Suppose that $(G, \cdot)$ and $(G, \odot)$ are both stable groups. Show that $(G, \cdot)$ is connected if and only if $(G, \odot)$ is connected.

Exercise 5.3.9. Prove Corollary 5.3.1(v).

Exercise 5.3.10. Prove that a pp−formula over 0 defines a subgroup of a modules. More generally, the pp−formula $\varphi(x, \bar{a})$ defines a coset of $\varphi(x, 0)$ (unless it is inconsistent).

Exercise 5.3.11. Prove: If $p \subset q$ are stationary types in a module, then q is a forking extension of p if and only if $stab(q|\mathfrak{C})$ has infinite index in $stab(p|\mathfrak{C})$.

5.4 Saturated Models

It follows directly from Lemma 2.2.6 that if T is κ-stable, there is a saturated model of T of cardinality κ^+. In fact, roughly the same proof gives the existence of a saturated model of cardinality κ when κ is regular and T is κ-stable. In the next proposition the restriction to regular cardinals will be removed and the assumption of κ-stability is proved to be necessary when κ is sufficiently large.

Proposition 5.4.1. *If the theory T is κ-stable, then T has a saturated model of cardinality κ. If, on the other hand, $\kappa \geq \lambda(T)$ and T is not κ-stable then T does not have a saturated model of cardinality κ.*

Proof. Let T be κ-stable. Then $cf(\kappa) \geq \kappa(T)$ since $\kappa^{cf(\kappa)} > \kappa$ and $\kappa^{<\kappa(T)} = \kappa$ (by Theorem 5.2.1).

A saturated model of cardinality κ is found using

Claim. There is a model M of T of cardinality κ such that for all $A \subset M$ of cardinality $< \kappa(T)$ and $p \in S(acl(A))$, M contains a Morley sequence over $acl(A)$ in p of cardinality κ.

The model M is constructed via an elementary chain, M_i, $i < \kappa$. Let M_0 be any model of T of cardinality κ. Given M_i, let M_{i+1} be an elementary extension of cardinality κ such that for all $p \in S(M_i)$, M_{i+1} contains a Morley sequence in p of cardinality κ. Such a model exists since $|S(M_i)| = \kappa$ and T is κ-stable. If j is a limit ordinal, let $M_j = \bigcup_{i<j} M_i$. Let $M = \bigcup_{i<\kappa} M_i$. If A is a subset of M of cardinality $< \kappa(T)$, then there is an $i < \kappa$ such that $A \subset M_i$ (since $cf(\kappa) \geq \kappa(T)$). By construction, any strong type over A, M contains a Morley sequence of cardinality κ, proving the claim.

To prove that M is saturated let $A \subset M$ have cardinality $\lambda < \kappa$ and let $p \in S(A)$. Let $B \subset A$ be a set of cardinality $< \kappa(T)$ over which p does not fork and let $q \in S(acl(B))$ be such that $q|acl(A) \supset p$. It suffices to find an a realizing q which is independent from A over B. By the claim there is a set $I \subset M$ of cardinality κ which is a Morley sequence over $acl(B)$ in q. By Proposition 5.2.1 there is a set $J \subset I$ of cardinality $< \kappa(T) + |A|^+$ (if $\kappa(T)$ is regular) and $\leq \kappa(T) + |A|$ (otherwise) such that A is independent from I over $B \cup J$. In any case, $|J| < \kappa$ (since $\kappa \geq \kappa(T)$ and $\kappa = \kappa(T)$ can only occur when $\kappa(T)$ is regular by Corollary 5.2.1). Then $a \in I \setminus J$ is a realization of q which is independent from A over $J \cup B$, hence independent from A over B (by the transitivity of independence). Thus, a realizes p, proving the first part of the proposition.

Now suppose that $\kappa \geq \lambda(T)$ and T is not κ-stable. By The Stability Spectrum Theorem (Theorem 5.2.1), $\kappa < \kappa^{<\kappa(T)}$ and $\kappa \geq \kappa(T)$ (since $\lambda(T) \geq \kappa(T)$). The proof of the nonexistence of a saturated model of cardinality κ is split into two cases, the first reducing largely to cardinal arithmetic.

Case 1. $cf(\kappa) \geq \kappa(T)$.

Choose $\mu < \kappa(T)$ such that $\kappa^\mu > \kappa$. Express κ as $\sup_{i<cf(\kappa)}\kappa_i$, where $\kappa_i < \kappa$. Every subset of κ of cardinality $< \kappa(T)$ is contained in one of the κ_i's by the assumption of this case, hence $\kappa^\mu = \sup_{i<cf(\kappa)}\kappa_i^\mu$. Thus, there is $\kappa_i < \kappa$ with $\kappa_i^\mu > \kappa$. Since $\mu < \kappa(T)$ the proof of Lemma 5.2.2 yields a set A of cardinality κ_i such that $|S(A)| = \kappa_i^\mu$. Thus, there cannot be a saturated model of cardinality κ.

Case 2. $cf(\kappa) < \kappa(T)$.

Let $\lambda = cf(\kappa)$ and $\kappa = \sup_{i<\lambda}\kappa_i$, where $\kappa_i < \kappa$. Suppose, towards a contradiction, that M is a saturated model of cardinality κ. Write M as $\bigcup_{i<\lambda} M_i$, where M_i is a submodel of M of cardinality κ_i and when $\delta < \lambda$ is a limit ordinal, $M_\delta = \bigcup_{i<\delta} M_i$. Since $\lambda < \kappa(T)$ there is a chain of sets A_i, $i \leq \lambda$, such that

- $A_{i+1} \setminus A_i$ is finite,
- $A_\delta = \bigcup_{i<\delta} A_i$, if δ is a limit ordinal, and
- there is a $p \in S(A_\lambda)$ such that $p \restriction A_{i+1}$ forks over A_i, for all $i < \lambda$.

Since M is saturated and $|M_i| < \kappa$ we can choose the A_i's and M_i's so that $A_i \subset M_i$, for all $i < \lambda$, and A_{i+1} is independent from M_i over A_i. Since $|A_\lambda| = \lambda < \kappa(T) \leq \kappa$ and M is saturated there is an $a \in M$ realizing p. Find $i < \lambda$ such that $a \in M_i$. This contradicts that $p \restriction A_{i+1}$ forks over A_i and A_{i+1} is independent from M_i over A_i, proving the proposition.

In the exercises the reader is asked to investigate when the union of an elementary chain of saturated models of a stable theory is also saturated.

5.4.1 a–models

One useful property of t.t. theories is that every type over a model is based on a finite subset of the model. The same may not be true in a properly superstable theory, even though every type does not fork over a finite set. Consider, for example, the theory T in the language $L = \{E_i(x,y) : i < \omega\}$ which says that each E_i is an equivalence relation with only infinite classes, $E_0(x,y)$ is $x = x$ and each E_i–class is refined into two E_{i+1}–classes. Let M be any countable model. There is a one-to-one correspondence between stationary types over $A \subset M$ and elements of $S(M)$ which are definable over A. Since T is small there can only be countably many types over M definable over a finite subset of M, while there are $2^{\aleph_0}$ types over M. If, however, we chose M to be $\aleph_1$–saturated instead of countable, $tp(a/M)$ would be based on any $b \in M$ realizing $tp(a/acl(\emptyset))$. The following property of a model M is potentially weaker than κ^+–saturation, but is enough to guarantee that every complete type over M is based on a subset of M of cardinality $< \kappa^+$.

Definition 5.4.1. *A model M is* almost κ–saturated, *(a,κ)–saturated, for short, if for all sets $A \subset M$ of cardinality $< \kappa$, every strong type over A is realized in M. When T is stable the terms almost $\kappa(T)$–saturated model,* almost saturated model, $a-$saturated model, *and* $a-$model *are used interchangeably.*

Restating the definition, M is $(a, \kappa)-$saturated if for all $A \subset M$ of cardinality $< \kappa$, every element of $S(acl(A))$ is realized in M. This form of the definition makes it clear that a model which is $(\kappa + |T|^+)-$saturated is $(a, \kappa)-$saturated and an $(a, \kappa)-$saturated model is $\kappa-$saturated. In any context using almost $\kappa-$saturated models for some $\kappa > \kappa(T)$ we can usually reach the same goal using $a-$models. The term almost $\kappa-$saturated is a little misleading because an $(a, \kappa)-$saturated model is certainly $\kappa-$saturated. The "almost" comes from the fact that an $(a, \kappa)-$saturated model must realize every type "almost over a subset A"; i.e., a type consisting of formulas almost over A.

These models do have the desired property:

Lemma 5.4.1. *Let T be stable and M an $a-$model. For any $p \in S(M)$ there is a subset of M of cardinality $< \kappa(T)$ on which p is based.*

Proof. Let $A \subset M$ be a set of cardinality $< \kappa(T)$ such that p does not fork over A. Since M is an $a-$model there is $a \in M$ realizing $p \restriction acl(A)$. By Corollary 5.1.8(i) p is based on $A \cup \{a\}$, a set of cardinality $< \kappa(T)$.

Almost saturated models will be used in Section 5.6.2 to develop a good theory of domination. The "dimension theory" developed in Section 5.6 is easiest to apply in the context of $a-$models. Indeed, many of the theorems in [She90] related to Morley's Conjecture apply to the class of $a-$models of a superstable theory.

Types over $a-$models are determined by the elements realizing a subtype (compare this with Lemma 5.1.9):

Lemma 5.4.2. *Let M be an $a-$model, A a subset of M of cardinality $< \kappa(T)$, $q \in S(A)$ and $b \subset q(\mathfrak{C})$. Then, $tp(b/A \cup q(M)) \models tp(b/M)$.*

Proof. Let e be an element of M and b' an arbitrary realization of $tp(b/A \cup q(M))$. We need to show that b and b' have the same type over e. Let E be a subset of $q(M)$ of cardinality $< \kappa(T)$ such that e is independent from $q(M) \cup A$ over $E \cup A$. In fact, e is independent from $q(\mathfrak{C}) \cup A$ over $E \cup A$. (Suppose that e depends on the finite $c \subset q(\mathfrak{C})$ over $E \cup A$. Since M is $\kappa(T)-$saturated there is a $c' \subset q(\mathfrak{C})$ realizing $tp(c/E \cup A \cup \{e\})$; a contradiction.) Let $b_0 \in M$ realize $stp(b/E \cup A)$. Since this strong type is based on $\{b_0\} \cup E \cup A \subset q(M) \cup A$, $tp(b/q(M) \cup A)$ is stationary. Thus, b and b' have the same type over $q(M) \cup A \cup \{e\}$, proving the lemma.

This alternative kind of saturation is further connected to ordinary saturation in

Lemma 5.4.3. *(i) If T is t.t. then a model M is $(a,\kappa)-$saturated if and only if M is $\kappa-$saturated. Thus, the $a-$models of a t.t. theory are exactly the $\aleph_0-$saturated models.*

(ii) If T is a countable stable theory then an $\aleph_1-$saturated model is $a-$saturated. Thus, if T is properly stable a model is $\aleph_1-$saturated if and only if it is $a-$saturated.

Proof. (i) Suppose that M is $\kappa-$saturated, $A \subset M$ has cardinality $< \kappa$ and $p \in S(acl(A))$. Since T is t.t. a complete type has only finitely many nonforking extensions. Thus, there is a set B, $A \subset B \subset acl(A)$, of cardinality $< \kappa$ such that p is implied by its restriction to B. Since M is $\kappa-$saturated there is an $a \in M$ realizing $p \restriction B$, hence also p.

(ii) If T is countable, $\kappa(T)$ is $\leq |T|^+ = \aleph_1$. Thus, an $\aleph_1-$saturated model is $(\kappa(T) + |T|^+)-$saturated, hence $a-$saturated. If T is properly stable $\kappa(T)$ is $\aleph_1$, hence an $a-$model must be $\aleph_1-$saturated.

Corollary 5.4.1. *Let T be a countable stable theory and M a model of T. If T is properly stable then M is an $a-$model if and only if M is $\aleph_1-$saturated. If T is $\aleph_0-$stable then M is an $a-$model if and only if M is $\aleph_0-$saturated.*

Remark 5.4.1. If T is a countable properly superstable theory and M is $\aleph_0-$saturated, M may or may not be an $a-$model, depending on detailed properties of T.

Historical Notes. Proposition 5.4.1 was proved for $\omega-$stable theories by Harnik in [Har73] and generalized to stable theories by Shelah in [She90, III, 3.10 and 3.12]. The notion of an $a-$model is due to Shelah [She90].

Exercise 5.4.1. If T is a countable properly superstable theory what is the least cardinal in which T has an $a-$model?

Exercise 5.4.2. Let T be stable, M an $a-$model and λ an infinite cardinal. Show that M is $\lambda-$saturated if and only if for every infinite set of indiscernibles $I \subset M$ there is a set of indiscernibles J, $I \subset J \subset M$, of cardinality λ.

Exercise 5.4.3. Let T be a countable stable theory and M_i, $i < \aleph_1$, an elementary chain of $\lambda-$saturated models (for some λ). Prove that $M = \bigcup_{i<\aleph_1} M_i$ is also $\lambda-$saturated.

Exercise 5.4.4. Let T be a stable theory and $\{M_i : i < \delta\}$ a chain of $\lambda-$saturated models, where $\kappa(T) \leq cf(\delta)$. Show that $M = \bigcup_{i<\delta} M_i$ is $\lambda-$saturated.

5.5 Prime Models

In most applications of prime models an essential ingredient is their uniqueness. For prime models in countable theories this was proved in Section 2.1, where we also remarked that an uncountable theory may have prime models which are not isomorphic or a nonatomic prime model. We proved in Lemma 3.1.5 the existence of prime models over arbitrary sets in an $\aleph_0$−stable theory without addressing the uniqueness issue. In the proofs of certain results, such as Corollary 3.1.4, the nonexistence of Vaughtian pairs was used in place of the uniqueness of prime models to see that prime models over conjugate sets are isomorphic. Outside of the context of uncountably categorical theories, no such replacement is possible. The uniqueness of prime models (over sets) in any t.t. theory is proved in this section.

Prime models over sets may not exist in theories which are not totally transcendental. However, we will see that in stable theories there are prime models over sets relative to the class of a−models. (The $\aleph_0$−prime models of Section 3.1 are an example of such models.) We will also prove the uniqueness of these so-called a−prime models.

5.5.1 Prime Models in a t.t. Theory

The proof of the existence of prime models over sets in an $\aleph_0$−stable theory did use the countability of the theory, so another proof which handles all t.t. theories is needed. The models we find are not only prime, but are constructed as such in the following sense.

Notation. If $\{ a_\beta : \beta < \alpha \}$ is a set of elements indexed by an ordinal α and $\beta < \alpha$, $A_\beta = \{ a_\gamma : \gamma < \beta \}$.

Definition 5.5.1. *(i) A set* $\{ (a_\beta, \varphi_\beta) : \beta < \alpha \}$ *is called a* t−construction over A *if, for all* $\beta < \alpha$, φ_β *is a formula over* $A \cup A_\beta$ *which isolates* $tp(a_\beta / A \cup A_\beta)$. *When the particular isolating formulas are irrelevant we may simply write* $\{ a_\beta : \beta < \alpha \}$ *for the* t*−construction. A set is* t−constructible over A *if some enumeration of it is a* t*−construction over* A.

(ii) A model $M \supset A$ *(of the theory in question) is called* strictly prime over A *if it is* t*−constructible.*

Remark 5.5.1. In Shelah's terminology isolated types are in the class $\mathbf{F}^{\mathbf{t}}_{\aleph_0}$, meaning that the type is isolated by a subtype of cardinality $< \aleph_0$. Moreover, a t−construction is called an $\mathbf{F}^{\mathbf{t}}_{\aleph_0}$−construction in [She90].

Remark 5.5.2. If $\mathfrak{C}$ is t.t. and A is a set, the theory of $\mathfrak{C}$ with constants for the elements of A is also t.t. Thus, a theorem stated to hold for the prime models of any t.t. theory actually holds for the prime models over sets in any t.t. theory. In practice, though, it is common to state a major result about the prime models over a set A in a t.t. theory, and say "Without loss of generality, $A = \emptyset$" when beginning the proof. In lemmas we may omit the set in the statement.

The results of the section are summarized in

Theorem 5.5.1. *Suppose that T is a t.t. theory and A is a set.*
(i) There is a strictly prime model over A.
(ii) Any two strictly prime models over A are isomorphic over A.
(iii) The following are equivalent:

(1) *M is strictly prime over A.*
(2) *M is prime over A.*
(3) *M is atomic over A and does not contain an uncountable set of indiscernibles over A.*

Corollary 5.5.1. *If T is t.t. and M, N are prime models of T, then $M \cong N$.*

The first part of the theorem is handled rather easily:

Lemma 5.5.1. *If T is a t.t. theory then there is a strictly prime model M. In addition, M is prime and atomic.*

Proof. The main point is contained in

Claim. For all sets B the isolated points are dense in $S(B)$.

Let φ be a formula over B. Let $p \in S(B)$ be an element containing φ which has minimal Morley rank α. Let $\psi \in p$ be such that p is the only element of $S(B)$ containing ψ and having Morley rank α. Since we can assume that ψ implies φ the minimal rank assumption shows that ψ isolates p, to prove the claim.

The construction of a strictly prime model and the verification that it is prime and atomic is carried out exactly as in Lemma 3.1.5.

Corollary 5.5.2. *A $t-$constructible set in a t.t. theory is atomic.*

The following basic fact about $t-$constructible sets arises repeatedly in the proof.

Lemma 5.5.2. *If $B = \{ b_\beta : \beta < \alpha \}$ is a $t-$construction over A and $B' \subset B$ is finite, then $\{ b_\beta : \beta < \alpha \}$ is a $t-$construction over $A \cup B'$.*

Proof. For each $\beta < \alpha$, $\{ b_\beta : \beta < \alpha \}$ is a $t-$construction over $A \cup B_\beta$, hence B is atomic over $A \cup B_\beta$. In particular, $tp(b_\beta \bar{b}'/A \cup B_\beta)$ is isolated, where $\bar{b}'$ is an enumeration of B'. By the transitivity of isolation, $tp(b_\beta/A \cup B_\beta \cup B')$ is isolated, as desired.

Part (ii) of the theorem is proved in the following lengthy result.

Proposition 5.5.1. *If T is t.t. and A is a set, then any two strictly prime models over A are isomorphic over A.*

Proof. Adding constants to the language for the elements of A results in another t.t. theory, so we may as well assume that $A = \emptyset$. Let $\{ (a_\beta, \varphi_\beta) : \beta < \alpha \}$ be a $t-$construction of a strictly prime model M over $\emptyset$.

A set $B \subset M$ is *closed* if for all $a_\beta \in B$ the parameters of φ_β are in B. Note: any union of closed sets is closed. In the proof we need the following basic properties of closed sets.

Claim. (i) For every $\beta < \alpha$ there is a finite closed set C containing a_β.
(ii) If B is closed M is atomic over B.

(i) This is proved by induction on β. Let $\{c_0, \ldots, c_n\} \subset A_\beta$ be the set of parameters appearing in φ_β. Each c_i is contained in a finite closed set C_i, hence $C_0 \cup \ldots \cup C_n \cup \{a_\beta\}$ is a closed set containing a_β.

(ii) Since a strictly prime model is atomic it suffices to show that for all $\beta < \alpha$, $tp(a_\beta/B \cup A_\beta)$ is isolated. This is accomplished by showing that for all $\beta < \alpha$, $a_\beta \in B$ or φ_β isolates $tp(a_\beta/B \cup A_\beta)$. Fix $\beta < \alpha$ and suppose $a_\beta \notin B$. To show that φ_β isolates $tp(a_\beta/B \cup A_\beta)$ we prove (inductively), for $\delta < \alpha$ and $B^\delta = B \cap A_\delta$, φ_β isolates $tp(a_\beta/B^\delta \cup A_\beta)$. If $\delta < \beta$, $B^\delta \subset A_\beta$, so we can assume that $\delta > \beta$ and $\delta = \gamma + 1$ (since the case for limit ordinals is easy). By the inductive hypothesis, φ_β isolates $tp(a_\beta/B^\gamma \cup A_\beta)$. Since B is closed φ_γ is over B^γ, hence it isolates $tp(a_\gamma/B^\gamma \cup A_\beta \cup \{a_\beta\})$. Thus, $\varphi_\beta(x) \wedge \varphi_\gamma(y)$ isolates $tp(a_\beta, a_\gamma/B^\gamma \cup A_\beta)$. This, in turn, implies that φ_β isolates $tp(a_\beta/B^\gamma \cup \{a_\gamma\} \cup A_\beta) = tp(a_\beta/B^\delta \cup A_\beta)$, proving the claim.

With this claim in hand we can proceed with the main body of the proof. Let $\{ (b_\beta, \psi_\beta) : \beta < \alpha' \}$ be a $t-$construction over $\emptyset$ of a model M'. Without loss of generality, $\alpha \leq \alpha'$. We define by recursion on $\beta < \alpha$ closed sets $C_\beta \subset M$ and $C'_\beta \subset M'$ and elementary maps f_β from C_β onto C'_β such that:

(1) If $\gamma < \beta$ then $C_\gamma \subset C_\beta$, $C'_\gamma \subset C'_\beta$ and $f_\gamma \subset f_\beta$.
(2) If β is a limit ordinal then $C_\beta = \bigcup_{\gamma<\beta} C_\gamma$, $C'_\beta = \bigcup_{\gamma<\beta} C'_\gamma$ and $f_\beta = \bigcup_{\gamma<\beta} f_\gamma$.
(3) If η is a limit ordinal and $\beta = \eta + (2n+2)$, then $a_{\eta+n} \in C_\beta$; if $\beta = \eta + (2n+1)$ then $b_{\eta+n} \in C'_\beta$.

Consider a typical case in the recursion such as $\beta = \gamma + 1 = \eta + (2n+2)$, where η is a limit ordinal. There is a finite closed set $B_0 \subset M$ containing $a_{\eta+n}$ (by (i) of the claim). Since M is atomic over C_γ (by (ii) of the claim) there is an elementary map g_0 extending f_γ and taking B_0 onto some $B'_0 \subset M'$. Now find a finite B'_1 with $B'_0 \subset B'_1 \subset M'$ such that B'_1 is closed and note that $C'_\gamma \cup B'_1$ is closed. Since M' is atomic over C'_γ and B'_0 is finite, M' is atomic over $C'_\gamma \cup B'_0$. Hence, there is an elementary map g_1 extending g_0 and taking $C_\gamma \cup B_1$ (for some $B_1 \subset M$) onto $C'_\gamma \cup B'_1$. Continuing in this manner produces chains of finite sets $B_i \subset M$ and $B'_i \subset M'$ and elementary maps g_i from $C_\gamma \cup B_i$ onto $C'_\gamma \cup B'_i$ such that B_i is closed if i is even and B'_i is closed if i is odd. Finally, the sets $C_\beta = C_\gamma \cup \bigcup_{i<\omega} B_i$, $C'_\beta = C'_\gamma \cup \bigcup_{i<\omega} B'_i$ and $f_\beta = \bigcup_{i<\omega} g_i$ satisfy (1)–(3).

In the end the construction yields an isomorphism $f = f_\alpha$ from M onto the closed subset B'_α of M'. Since M' is an atomic model over B'_α, M' must equal B'_α. Thus, M and M' are isomorphic, proving the proposition.

The equivalents in part (iii) of the theorem are handled in the next two lemmas. In Lemma 5.5.1 (1) $\Longrightarrow$ (2) was proved. The next lemma proves (2) $\Longrightarrow$ (3).

Lemma 5.5.3. *If T is t.t. and M is prime over A, then M is atomic over A and there is no uncountable subset of M which is indiscernible over A.*

Proof. It follows from Lemma 5.5.1 that any prime model over A is atomic. (A prime model is elementarily embeddable into a strictly prime model N over A, and any subset of N is atomic over A.) The model M is also prime over $acl(A)$. An infinite set of indiscernibles over A is indiscernible over $acl(A)$ (by Exercise 5.1.13), so we may as well assume that $A = acl(A)$. By working over A we may take A to be $\emptyset$. Suppose, towards a contradiction, that M contains an uncountable indiscernible set I. Without loss of generality, M is strictly prime (since a prime model is contained in a strictly prime model). By Corollary 5.1.13 there is a finite set $J \subset I$ such that $I \setminus J$ is a Morley sequence over J. Since J is finite M is strictly prime over J. So, we may as well assume that $J = \emptyset$ and I is a Morley sequence over $\emptyset$ in a stationary type $p \in S(\emptyset)$.

Let I_0 be a countable infinite subset of I and M' a prime (hence atomic) model over I_0 which is contained in M. Since M is prime over $\emptyset$ it can be elementarily embedded into M', hence M' contains an uncountable Morley sequence J in p. We prove that $a \in J \implies a \not\perp I_0$ as follows. There is a formula φ over a finite $I' \subset I_0$ which isolates $tp(a/I_0)$. If $a \perp I'$, then a and any $b \in I_0 \setminus I'$ have the same type over I' (since p is stationary and I_0 is independent over $\emptyset$). Thus, any $b \in I_0 \setminus I'$ also satisfies φ contradicting that φ isolates a type over I_0. Thus, $a \not\perp I_0$.

Since I_0 is countable there is a countable $J_0 \subset J$ such that I_0 and J are independent over J_0. A fortiori, any $b \in J \setminus J_0$ (which exists since J is uncountable) is independent from I_0 over J_0. However, b is independent from J_0, hence independent from I_0 by transitivity. This contradicts what was proved in the previous paragraph, to establish the lemma.

The proof of (3) $\Longrightarrow$ (1) of Theorem 5.5.1(iii) is

Lemma 5.5.4. *Suppose that T is t.t., M is atomic over A and does not contain an uncountable set of indiscernibles over A. Then M is strictly prime over A.*

Proof. The structure of the proof is set with the following claim.

Claim. It suffices to show that for any set $A' \subset M$ such that M is atomic over A' and M does not contain an uncountable indiscernible set over A' there is a set B such that $(*)$ $A' \subset B \subset M$, such that M is atomic over B and $B \neq A'$ is $t-$constructible over A'. In fact, it suffices to find a set B satisfying $(*)$ with $A' = A$.

Notice that M does not contain an uncountable set of indiscernibles over any subset containing A. Thus, assuming $(*)$, we can apply it again with A replaced by B to obtain a set $B_1 \neq B$, $B \subset B_1 \subset M$, which is $t-$constructible over B and over which M is atomic. The constructions of B over A and B_1 over B can be pieced together to give a $t-$construction of B_1 over A. If B_i, $i < \delta$, is a chain where each B_i is a $t-$construction over which M is atomic and the $t-$construction of B_{i+1} extends the $t-$construction of B_i, then $\bigcup_{i<\delta} B_i$ is also a $t-$construction over which M is atomic. From these facts we obtain a chain of sets $A \subset B \subset B_1 \subset B_2 \subset \ldots \subset M$, each of which is $t-$constructible over A, and whose union is all of M. This proves that M is strictly prime over A as desired.

A set D, $C \subset D \subset M$, is said to be *full over* C if whenever a and b in M have the same type over C, $a \in D \iff b \in D$.

Claim. Let $A' \subset M$ be any set such that M is atomic over A' and M does not contain an uncountable indiscernible set over A'. If $D \subset M$ is full over A', then M is atomic over D.

Let a be an arbitrary element of M and B a finite subset of D such that $tp(a/D) = q$ does not split over B. Since B is finite, M is atomic over $A' \cup B$, yielding a formula $\varphi(x)$ which isolates $tp(a/A' \cup B)$. Assume, towards a contradiction, that φ does not isolate $tp(a/D)$ and $\psi(x, b) \in q$ is such that $\models \exists x(\varphi(x) \wedge \neg\psi(x, b))$. Thus, for θ a formula isolating $tp(b/A' \cup B)$ there is a $c \in M$ such that $\models \theta(c)$ and $\neg\psi(a, c)$. Then $c \in D$, because D is full over A. Since $\models \psi(a, b)$ and $\models \neg\psi(a, c)$ we contradict that q does not split over B, proving the claim.

Claim. Let $A' \subset M$ be any set such that M is atomic over A' and M does not contain an uncountable indiscernible set over A'. If $D \subset M$ is full over A', then D is $t-$constructible over A'.

This is proved for all D and A' by induction on the least ordinal α such that $a \in D \implies MR(a/A') < \alpha$. If $\alpha = 1$ any enumeration of D is a $t-$construction. Suppose α is a limit ordinal and for $\beta < \alpha$ let $B_\beta = \{a \in D : MR(a/A) < \beta\}$. Since each B_β is full over A', M is atomic over B_β. In fact, $B_{\beta+1}$ is full over B_β, hence is $t-$constructible over B_β, by induction. Piecing these constructions together end to end yields a $t-$construction of $D = \bigcup_{\beta<\alpha} B_\beta$ over A'.

Now suppose that $\alpha = \beta + 1$ and $\{p_i : i < \lambda\}$ is an enumeration of the complete types over A' realized in D. We will find by recursion on $i < \lambda$ a

t-construction of $B_i = p_i(D)$ over $C_i = A \cup \bigcup_{j<i} B_j$. A t-construction of D over A' is again obtained by piecing together these constructions end to end.

At stage i in the recursion let B' be a maximal A'-independent subset of B_i which is also independent from C_i over A'. Observe that B' must be countable. (There are only finitely many strong types over A' extending p_i. Assuming B' to be uncountable there is an uncountable $B'' \subset B'$ such that the elements of B'' have the same strong type over A'. Then B'' would be an uncountable Morley sequence, hence indiscernible set over A', in contradiction to the hypotheses.) Let $\{ b_m : m < \omega \}$ be an enumeration of B' (with repetitions if B' is finite). For $m < \omega$ let $D_m = \{ b \in B_i : MR(b/C_i \cup \{b_0, \ldots, b_m\}) < \alpha \}$ and set $D_{-1} = \emptyset$. We will show that each D_m is t-constructible over $C_i \cup D_{m-1}$, and to aid in the recursion, that M is atomic over $C_i \cup D_m$. Assuming that M is atomic over $C_i \cup D_{m-1}$ it is also atomic over $C_i \cup D_{m-1} \cup \{b_m\}$. Examining the definition shows that D_m is full over $C_i \cup D_{m-1} \cup \{b_m\}$, hence M is atomic over $C_i \cup D_m$. Furthermore, for any $b \in D_m$, $MR(b/C_i \cup D_{m-1} \cup \{b_m\}) < \alpha$, so D_m is t-constructible over $C_i \cup D_{m-1} \cup \{b_m\}$ (by induction). In fact, D_m is t-constructible over $C_i \cup D_{m-1}$ (since the type of b_m over $C_i \cup D_{m-1}$ is isolated). Finally, observe that these constructions can be concatenated to produce a t-construction of $B_i = \bigcup_{m<\omega} D_m$ over C_i, as desired. This proves the claim.

The reduction obtained in the first claim will be used to complete the proof. Suppose $a \in M \setminus A$. Let $D \supset A$ be a subset of M, full over A, which contains a. By the second claim M is atomic over D, and by the third claim D is t-constructible over A. Thus, D satisfies $(*)$. By the first claim the lemma is proved.

Following is an example of a superstable theory with a prime model but no atomic model. Of course, the language is necessarily uncountable. This shows, in particular, that the restriction to t.t. theories in Theorem 5.5.1 is necessary.

Example 5.5.1. Let M_0 be the direct product of $\aleph_0$ copies of the two element group $(\mathbb{Z}_2, +, 0)$, (with universe ${}^{\omega}2$) and $M_1 = M_0 \times M_0$. Let H be the subgroup $M_0 \times \{0\}$. Let π be the composition of the projection of M_1 onto $\{0\} \times M_0$ followed by the natural identification of this group with H. For $i < \omega$ let $U_i = \{ (f,g) : f, g \in {}^{\omega}2 \text{ and } f(j) = g(j) = 0 \text{ for } j \leq i \}$, a subgroup of M_1 of index 2^{i+1}. Let $M_2 = (M_1, H, U_i, +, 0, \pi)_{i<\omega}$, a structure in a language $L = \{H, U_i, +, 0, \pi\}_{i<\omega}$. Then, $T_0 = Th(M_2)$ is quantifier-eliminable, countable and superstable (of ∞-rank 2, see Section 6.1). The connected component of $\mathfrak{C}$ is $\bigcap_{i<\omega} U_i(\mathfrak{C})$, so $\mathfrak{C}$ has $2^{\aleph_0}$ many generics. Also $H(\mathfrak{C})$ has $2^{\aleph_0}$ many generic types since $H(\mathfrak{C}) \cap \mathfrak{C}^o$ has $2^{\aleph_0}$ many cosets in $H(\mathfrak{C})$. Note that any nonzero element of $H(\mathfrak{C})$ is a generic element. Similarly, for any $a \in H(\mathfrak{C})$ the set $\pi^{-1}(a) = \{ b : \pi(b) = a \}$ is partitioned into continuum many cosets of $\mathfrak{C}^o$. Moreover, any two elements of $\pi^{-1}(a)$ with the same coset of $\mathfrak{C}^o$ have the same type over $acl(a)$. Furthermore, π induces an isomorphism π^* between the groups $H(\mathfrak{C})$ and $\mathfrak{C}/H(\mathfrak{C})$ which preserves the U_i's.

Let c^* be a nonzero element of $H(\mathfrak{C})$. Let $C = \{ c_\eta : \eta < 2^{\aleph_0} \}$ be a subset of $\pi^{-1}(c^*)$ such that $C + \mathfrak{C}^o \supset \pi^{-1}(c^*)$ and $\eta \neq \eta' \implies c_\eta + \mathfrak{C}^o \neq c_{\eta'} + \mathfrak{C}^o$. Finally, let T be the theory of N with a constant symbol for each c_η. It is easy to verify that for any $M' \models T$ and $a \in H(M')$ every strong type over a extending $\pi(v) = a$ is realized in M'.

Elimination of quantifiers gives the existence of a model M with the property that each element of $H(M)$ is the difference of two c_η's (and each element of $M/H(M)$ is π^* of an element of $H(M)$). We claim that M is a prime model which is not atomic. Let M' be any model of the theory. Certainly there is an elementary mapping F of $H(M) \cup (\pi^{-1}(c^*) \cap M)$ into M', so it remains to map the sets $\pi^{-1}(b) \cap M$ into $\pi^{-1}(Fb) \cap M'$, for $b \in H(M)$. Given $b \in H(M) \setminus \{0, c^*\} = X$ let $\hat{b}$ be an element of $\pi^{-1}(b) \cap M$. Then, $\pi^{-1}(b) \cap M$ is simply $\hat{b} + H(M)$. Elimination of quantifiers guarantees that $\{ \hat{b} : b \in X \}$ is independent over $X \cup (\pi^{-1}(c^*) \cap M)$. As we argued above, $F(stp(\hat{b}/b))$ is realized in M' by some element $\hat{b}_F$. Since $\{ \hat{b}_F : b \in X \}$ must be independent over $F(X \cup (\pi^{-1}(c^*) \cap M))$, F extends to an elementary map f taking $\hat{b}$ to $\hat{b}_F$. Since $\pi^{-1}(b) \cap M = \hat{b} + H(M)$, f extends to an isomorphism of M into M'. Hence M is prime. Furthermore, for any choice of $\hat{b}$, $tp(\hat{b}/\{b\})$ is nonisolated (left as an exercise). Thus, M is not atomic.

5.5.2 a-prime Models

In the proof of Morley's Categoricity Theorem we dealt with $\aleph_0$–prime models; i.e., prime models over sets relative to the class of $\aleph_0$–saturated models. In general,

Definition 5.5.2. *If $\mathcal{K}$ is a class of models and A is a set we call $M \supset A$* a prime model over A relative to $\mathcal{K}$ *if $M \in \mathcal{K}$ and for any $N \in \mathcal{K}$ such that $N \supset A$, M is elementarily embeddable over A into N.*

We will see that by taking $\mathcal{K}$ to be the class of a–models we obtain relatively prime models in properly superstable or properly stable theories which act somewhat like prime models in totally transcendental theories. The relevant notions of isolation are defined summarily.

Definition 5.5.3. *Let T be stable and κ an infinite cardinal, $\mathcal{K}$ the class of κ–saturated models of T and $\mathcal{K}_a$ the class of (a, κ)–saturated models of T.*

*(i) A model M is κ–*prime over *A if M is a prime model over A relative to $\mathcal{K}$. We call M (a, κ)–prime over A if M is prime over A relative to $\mathcal{K}_a$. When $\kappa = \kappa(T)$ we abbreviate (a, κ)–prime to a–*prime.

*(ii) Let $B \subset A$ have cardinality $< \kappa$. A type $p \in S(A)$ is κ–*isolated *over B if $p \restriction B \models p$. A strong type p over A is (a, κ)–*isolated over *B if $p \restriction acl(B) \models p$; equivalently, for all a realizing p, $stp(a/B) \models p$. We call*

*p κ–isolated (respectively, (a,κ)–isolated) if it is κ–isolated (respectively, (a,κ)–isolated) over some subset of A. When $\kappa = \kappa(T)$ we say a–*isolated *for (a,κ)–isolated.*

Only types over algebraically closed sets are relevant in (a,κ)–isolation, so it requires a little more care in handling than κ–isolation. However, most proofs for (a,κ)–isolation and κ–isolation are very similar. The next lemma is stated for (a,κ)–isolation, but the proofs are much the same (or easier) for κ–isolation.

Lemma 5.5.5. *Let T be stable, κ an infinite cardinal, p a strong type over A and $B \subset A$.*

(i) If p is (a,κ)–isolated over B, then p does not fork over B.

(ii) p is (a,κ)–isolated over B if and only if it is $(\kappa+|T|^+)$–isolated over $acl(B)$.

(iii) (Existence) *If $\kappa \geq \kappa(T)$ then for any A, $B \subset A$ of cardinality $< \kappa$ and strong type q over B there is an (a,κ)–isolated strong type r over A extending q.*

Proof. Part (i) is clear since p is implied by a strong type over B. Part (ii) is left as an exercise to the reader.

(iii) Without loss of generality, $A = acl(A)$. By the definition of $\kappa(T)$, there is a set C, $B \subset C \subset A$, with $|C \setminus B| < \kappa(T)$ and a type $r \in S(acl(C))$ extending p such that no extension of r over A forks over C. Then $|C| < \kappa$ and the unique nonforking extension of r over $acl(A)$ is an (a,κ)–isolated extension of p.

When $\kappa < \kappa(T)$, (a,κ)–isolated types may not exist as in (iii) of the previous lemma. There are few uses for (a,κ)–isolation when $\kappa > \kappa(T)$ and as (ii) indicates the notion collapses to κ–isolation when κ is sufficiently large. To focus on the main concept, we will only deal with $(a,\kappa(T))$–isolation; i.e., a–isolation, in the remainder of the section.

Our treatment is further specialized by considering only theories in which $\kappa(T)$ is regular (as it is when T is countable). Indeed, many of the results do not hold without this restriction. This property is used to obtain the following, which is part of Proposition 5.2.1.

> (*) Suppose that $\kappa(T)$ is regular, $\kappa \geq \kappa(T)$ and $|B| < \kappa$. Then for all sets A, there is a set $C \subset A$ of cardinality $< \kappa$ such that B is independent from A over C.

Definition 5.5.4. *Let T be stable.*

*(i) A set $B = \{ b_\beta : \beta < \alpha \}$ is called an a–*construction over *A if for all $\beta < \alpha$, $stp(b_\beta/A \cup B_\beta)$ is a–isolated. A set C is a–*constructible over *A if some enumeration of C is an a–construction over A.*

(ii) A model $M \supset A$ is strictly a–prime over *A if M is an a–model and is a–constructible over A.*

*(iii) A set B is $a-$*atomic over *A if for all finite sequences b from B, $stp(b/A)$ is $a-$isolated.*

(iv) Let $B = \{ b_\beta : \beta < \alpha \}$ be an $a-$construction over some set A and, for $\beta < \alpha$, $C_\beta \subset B_\beta \cup A$ a set of cardinality $< \kappa(T)$ over which $stp(b_\beta/B_\beta \cup A)$ is $a-$isolated. Then, a set $D \subset B$ is called closed *if whenever $b_\beta \in D$, $C_\beta \subset D \cup A$.*

In analogy to Theorem 5.5.1 we will prove below

Theorem 5.5.2. *Suppose that T is stable, $\kappa(T)$ is regular and A is any set.*

(i) There is a strictly $a-$prime model over A and every such model is $a-$prime over A.

(ii) Any two strictly $a-$prime models over A are isomorphic over A.

(iii) A strictly $a-$prime model over A is $a-$atomic over A and does not contain a set of cardinality $> \kappa(T)$ which is indiscernible over A.

(iv) An $a-$prime model over A is strictly $a-$prime over A.

Unlike the ordinary prime case we will not prove the converse of part (iii): If M is $a-$atomic over A and does not contain a set over indiscernibles over A of cardinality $> \kappa(T)$, then M is strictly $a-$prime over A. The proof of this harder result can be found in [Her92], which improves a slightly weaker result proved in [She90, IV, 4.14].

Lemma 5.5.6. *Suppose that T is stable and $\kappa(T)$ is regular.*

(i) If B is $a-$constructible over A and $C \subset B$ there is a closed set $C' \subset B$ containing C of cardinality $< |C|^+ + \kappa(T)$.

(ii) If the strong type p over A is $a-$isolated and p does not fork over $B \subset A$, then $q = p \restriction acl(B)$ is $a-$isolated. In fact, p is $a-$isolated over a subset of B.

(iii) If B is $a-$constructible over A, then B is $a-$atomic over A.

(iv) If B is $a-$atomic over A and $|B| < \kappa(T)$, then any enumeration of B is an $a-$construction over A. If B is $a-$atomic over A and $|B| \leq \kappa(T)$, then B is $a-$constructible over A.

(v) (Transitivity) *If $|B| < \kappa(T)$ and B is $a-$atomic over A, then $stp(b/A \cup B)$ is $a-$isolated if and only if $B \cup \{b\}$ is $a-$atomic over A.*

(vi) If B is $a-$atomic over A and $C \subset B$ has cardinality $< \kappa(T)$, then B is $a-$atomic over $A \cup C$.

(vii) If B is $a-$constructible over A and $C \subset B$ is closed, then C is $a-$constructible over A and B is $a-$constructible over $A \cup C$.

Proof. (i) Let $B = \{ b_\beta : \beta < \alpha \}$ be an $a-$construction of B over A. For $x \in B$ and $b_\beta = x$ let D_x be a subset of B_β of cardinality $< \kappa(T)$ such that $stp(b_\beta/B_\beta \cup A)$ is $a-$isolated over $D_x \cup A$. For $X \subset B$ let $X' = \bigcup\{ D_x : x \in X \} \cup X$. Then the set $D = \bigcup_{n<\omega} C^{(n)}$, where $C^{(0)} = C$ and $C^{(n+1)} = (C^{(n)})'$ is $a-$constructible. (Enumerate C as $\{ c_\beta : \beta < \alpha' \}$ where $\gamma < \beta < \alpha'$, $c_\gamma = b_\delta$ and $c_\beta = b_\epsilon$ implies that $\delta < \epsilon$. The reader can check that this enumeration gives an a–construction of C over A.) To check the cardinality of D we consider only when $|C| < \kappa(T)$, leaving the general case to the reader. If $\kappa(T) = \aleph_0$ then each $C^{(n)}$ is finite and for some n, $C^{(n+1)} = C^{(n)}$, hence D is finite. If $\kappa(T)$ is uncountable, then the regularity of the cardinal implies that $|C^{(n)}| < \kappa(T)$ and $\sup_{n<\omega}|C^{(n)}|$ is also $< \kappa(T)$, completing the proof of (i).

(ii) Without loss of generality, $A = acl(A)$ and $B = acl(B)$. Let $r \subset p$ be a strong type over a set A_0 of cardinality $< \kappa(T)$ which is equivalent to p. Since $\kappa(T)$ is regular there is a set $C \subset B$, the algebraic closure of a set of cardinality $< \kappa(T)$, such that A_0 is independent from B over C and p is based on C. We will show that $s = q \restriction C = p \restriction C$ is equivalent to p, which is sufficient to prove this part. Suppose, to the contrary, that a realizes s, but a does not realize q. Without loss of generality, a is independent from A_0 over B, hence a is independent from A_0 over C (by transitivity). Since the nonforking extension of s over A is p, a realizes $p \restriction A_0 = r$. Hence, a realizes $p \supset q$, a contradiction which proves (ii).

(iii) Without loss of generality, $A = acl(A)$. It suffices to show that when $B = \{ b_\gamma : \gamma \leq \beta \}$ is an a–construction over A and B_β is a–atomic over A, B is also a–atomic over A. Suppose this to be false. Let $a = b_\beta$ and let $\bar{b}$ be a finite sequence from B_β such that $stp(a\bar{b}/A)$ is not a–isolated. Let $C_0 \supset \bar{b}$ be a subset of $B_\beta \cup A$ of cardinality $< \kappa(T)$ such that $stp(a/B_\beta \cup A)$ is a–isolated over C_0. Then C_0 is a–atomic over A, in fact, $C = acl(C_0)$ is also a–atomic over A (see the exercises). Let $A_0 \subset A$ be the algebraic closure of a set of cardinality $< \kappa(T)$ such that $C \cup \{a\}$ is independent from A over A_0. Since $stp(a\bar{b}/A)$ is not a–isolated there is a sequence $a'\bar{b}'$ realizing $stp(a\bar{b}/A_0) = tp(a\bar{b}/A_0)$ which does not realize $tp(a\bar{b}/A)$. Let f be an automorphism of the universe fixing A_0 and mapping $a\bar{b}$ to $a'\bar{b}'$; let $D = f(C)$. Since C is a–atomic over A and independent from A over A_0, (ii) implies that $tp(\bar{c}/A_0) \models tp(\bar{c}/A)$ for any finite sequence $\bar{c}$ from C. This implies that for all $\bar{c}$ from C, $tp(\bar{c}/A) = tp(f(\bar{c})/A)$, hence there is an automorphism g of $\mathfrak{C}$ which fixes A and maps $f(c)$ to c, for any $c \in C$. Since $f(a) = a'$, $g(tp(a'/D \cup A_0)) = tp(a/C \cup A_0)$. Again using (ii), $tp(a/C \cup A_0) \models tp(a/C \cup A)$, hence $g(a')$ realizes $tp(a/C \cup A)$. This contradicts that $g(\bar{b}') = \bar{b}$, g fixes A and $a'\bar{b}'$ does not realize $tp(a\bar{b}/A)$, completing the proof of (iii).

(iv) The case for $|B| < \kappa(T)$ is proved by induction on $|B|$. For finite B this follows immediately from

Claim. If $stp(ab/A)$ is a–isolated over B then $stp(a/A \cup \{b\})$ is a–isolated over $B \cup \{b\}$ and $stp(b/A)$ is a–isolated over B.

Since $stp(ab/B) \models stp(ab/A)$ it is clear that $stp(b/B) \models stp(b/A)$. Suppose that a' realizes $stp(a/B \cup \{b\})$. Then, $a'b$ realizes $stp(ab/B)$, hence $a'b$ is independent from A over B. This implies that a' (like a) is independent from $A \cup \{b\}$ over $B \cup \{b\}$, thus a' realizes $stp(a/A \cup \{b\})$, proving the claim.

Now suppose that B is infinite and let $\{ b_\beta : \beta < \alpha \}$ be any enumeration of B. Let $\beta < \alpha$, $\kappa = |B_\beta|$ and write B_β as $\bigcup_{i<\kappa} C_i$, where $|C_i| < \kappa$ and $i < j \implies C_i \subset C_j$. Since each set $C_i \cup \{b_\beta\}$ is a–atomic over A the inductive hypothesis implies that any enumeration of the set is an a–construction. Thus, $stp(b_\beta/C_i \cup A)$ is a–isolated. This allows us to write $stp(b_\beta/B_\beta \cup A)$ as $\bigcup_{i<\kappa} p_i$, where the p_i's form a chain and each p_i is a–isolated. We leave it as an exercise to the reader to show that since $\kappa < \kappa(T)$ this union is a–isolated. This proves that $\{ b_\beta : \beta < \alpha \}$ is an a–construction of B.

If $|B| = \kappa(T)$, then the first part of (iv) says that an enumeration $\{ b_\alpha : \alpha < \kappa(T) \}$ of B is an a–construction over A.

(v) Both directions of the biconditional follow from (iii) and (iv) of this lemma.

(vi) The proof of this part is left for the reader in the exercises.

(vii) Let $\{ b_i : i < \alpha \}$ be an a–construction of B and let $<$ denote the induced well-ordering of B. For $i < \alpha$ let $E_i \subset B_i \cup A$ be the distinguished set of cardinality $< \kappa(T)$ such that $stp(b_i/B_i \cup A)$ is a–isolated over E_i. Let $D = B \setminus C$ and $C = \{ c_i : i < \gamma \}$, $D = \{ d_i : i < \delta \}$ enumerations of these sets which respect the enumeration of B (i.e., if $c_i = b_\beta$ and $c_j = b_{\beta'}$ then $i < j \iff \beta < \beta'$). Let $f : \delta \longrightarrow \alpha$ and $g : \gamma \longrightarrow \alpha$ be defined by: $f(i) = j$ if $d_i = b_j$ and $g(i) = j$ if $c_i = b_j$.

Since C is closed, $i < \gamma$ implies that $E_{g(i)} \subset C_i \cup A$, hence the given enumeration is an a–construction of C over A. To prove that D is a–constructible over $A \cup C$ it suffices to show that for $i < \delta$, $stp(d_i/D_i \cup A \cup C)$ is a–isolated over $E_{f(i)}$. To prove this we fix i and show by induction on $k < \gamma$ that $stp(d_i/D_i \cup A \cup C_k \cup E_{f(i)})$ is a–isolated over $E_{f(i)}$. (This is sufficient because a quick check of the definitions shows that $E_{f(i)}$ is contained in $D_i \cup A \cup C$.) Since $C_0 = \emptyset$ and $D_i \subset B_{f(i)}$ there is nothing to prove when $k = 0$. The condition is also preserved at limit ordinals, so suppose that $k = l + 1$. If c_l precedes d_i in the ordering of B then $D_i \cup C_l \subset B_{f(i)}$ and again there is nothing to prove. In the remaining case, when $f(i) < g(l)$, we see that $E_{f(i)} \subset A \cup D_i \cup C_l$, $A \cup D_{i+1} \cup C_l \subset A \cup B_{g(l)}$ and $stp(c_l/A \cup D_{i+1} \cup C_l)$ is a–isolated over a subset of $A \cup C_l$ (since $E_{g(l)} \subset A \cup C_l$). A fortiori, $stp(c_l/A \cup C_l \cup D_i) \models stp(c_l/A \cup C_l \cup D_{i+1})$. Switching the roles of c_l and d_i in this equation proves that $stp(d_i/A \cup C_l \cup D_i) \models stp(d_i/A \cup C_{l+1} \cup D_i)$. Since $stp(d_i/A \cup E_{f(i)}) \models stp(d_i/A \cup C_l \cup D_i)$ we have shown that $stp(d_i/A \cup C_{l+1} \cup D_i)$ is a–isolated over $E_{f(i)}$, as required to complete the proof.

Parts of the following proposition are proved like the corresponding results for ordinary isolation in t.t. theories. There are differences in that finite is replaced by "of cardinality $< \kappa(T)$", but the properties proved above fill the gaps. With this proposition we prove parts (i)–(iii) of Theorem 5.5.2.

Proposition 5.5.2. *Suppose that T is stable, $\kappa(T)$ is regular and A is any set.*

(i) There is a strictly a–prime model M over A. The model M is a–prime over A and its cardinality is $\leq$ the first cardinal $\geq |A|$ in which T is stable.

(ii) Any two strictly a–prime models over A are isomorphic over A.

(iii) A strictly a−prime model over A is a−atomic over A and does not contain a set of cardinality $> \kappa(T)$ which is indiscernible over A.

It remains to prove Theorem 5.5.2(iv), which is done in

Proposition 5.5.3. *Suppose that T is stable and $\kappa(T)$ is regular. Then for all sets A, if M is a−prime over A, M is strictly a−prime over A. In fact, if $M \supset B \supset A$, B is a−constructible.*

Proof. We know that there is a strictly a−prime model over A and M can be embedded into this model over A. Thus, it suffices to show that when M' is a strictly a−prime model over A, any set B, $A \subset B \subset M'$, is a−constructible over A. Let $\{ a_\alpha : \alpha < \nu \}$ be an a−construction of M' over A. We will find, by recursion on $\alpha < \mu$ (for some $\mu < |M'|^+$) sets $D_\alpha \supset A$ such that

(1) Each D_α is closed.
(2) If $\alpha < \beta$, $D_\alpha \subset D_\beta$ and $D_\alpha = \bigcup_{\beta<\alpha} D_\beta$ when α is a limit ordinal.
(3) $|D_{\alpha+1} \setminus D_\alpha| \leq \kappa(T)$.
(4) $M' = \bigcup_{\alpha<\mu} D_\alpha$.
(5) D_α and B are independent over $B_\alpha = D_\alpha \cap B$.

That B is a−constructible will follow rather quickly from this.

The set $D_{\alpha+1}$ will be the union of D_α and the sets $D^n_{\alpha+1}$, $n < \omega$, which are defined as follows.

(i) $D^0_{\alpha+1} = \{a_\alpha\}$.
(ii) If $n > 0$ is even, $D^n_{\alpha+1}$ is a closed set of cardinality $< \kappa(T)$ containing $D^{n-1}_{\alpha+1}$.
(iii) If n is odd we first let $C \subset B$ be a set of cardinality $< \kappa(T)$ such that $D^{n-1}_{\alpha+1}$ is independent from $B \cup D_\alpha$ over $C \cup D_\alpha$, and then set $D^n_{\alpha+1} = D^{n-1}_{\alpha+1} \cup C$.

Finally, let $D_{\alpha+1} = D_\alpha \cup \bigcup_{n<\omega} D^n_{\alpha+1}$. A union of closed sets is closed, so (1) holds. Since $D_{\alpha+1} \setminus D_\alpha$ is the union of countably many sets of cardinality $< \kappa(T)$ this difference has cardinality $\leq \kappa(T)$. To verify the fifth condition suppose that d is a finite subset of $D_{\alpha+1}$ and n is minimal with $d \subset D^n_{\alpha+1}$. By construction there is a set $C \subset B \cap D_{\alpha+1}$ such that d is independent from B over $D_\alpha \cup C$. Since B and D_α are independent over $B \cap D_\alpha$, d is independent from B over $B \cap D_{\alpha+1}$, so (5) holds. The other conditions hold trivially.

We claim that $B_{\alpha+1}$ is a−atomic over B_α. Since D_α is closed, Lemma 5.5.6 says that M' is a−constructible over D_α, hence a−atomic over D_α. That is, for any finite set b from $B_{\alpha+1}$, $stp(b/D_\alpha)$ is a−isolated. Since b is independent from D_α over B_α, Lemma 5.5.6(ii) implies that $stp(b/B_\alpha)$ is a−isolated, as desired. Then $B_{\alpha+1}$ is a−constructible over B_α (by Lemma 5.5.6(iv)). Piecing together all of these constructions gives an a−construction of B, completing the proof of the proposition.

Corollary 5.5.3. *Suppose that T is stable and $\kappa(T)$ is regular. If M is a−prime over A and $M \supset B \supset A$, then M is a−prime over B.*

Historical Notes. Shelah first proved the uniqueness of prime models for $\omega-$stable theories by induction on rank in [She72]. The proof given here was inspired by an unpublished proof by Ressayre. All of the other results are by Shelah and can be found in [She90, IV].

Exercise 5.5.1. Prove the property (*) preceding the definition of an $a-$construction (Definition 5.5.4).

Exercise 5.5.2. Prove that the union of a chain of $< \kappa(T)$ many $a-$isolated types is $a-$isolated.

Exercise 5.5.3. Prove that the algebraic closure of a set C is $a-$atomic over A whenever C is $a-$atomic over A.

Exercise 5.5.4. Prove (vi) of Lemma 5.5.6.

Exercise 5.5.5. Prove Corollary 5.5.3.

5.6 Orthogonality, Domination and Weight

We saw in the proofs of the Morley Categoricity Theorem and the Baldwin-Lachlan Theorem how the dimension theory on strongly minimal sets can be used to determine when models are isomorphic. This section is one facet in the development of a dimension theory for arbitrary stationary types which is based on the forking dependence relation.

Definition 5.6.1. *Let T be stable. Given a stationary type p over A and a set B, I is a* basis for p in B *if it is a maximal Morley sequence over A in p which is contained in B.*

Some key features of the dimension theory for strongly minimal sets in uncountably categorical theories are

1. If M is a model, φ is a strongly minimal formula over $A \subset M$ and p is the unique nonalgebraic completion of φ over A, then all bases for p in M have the same cardinality (which is called the *dimension* of $p(M)$ or $\varphi(M)$).
2. If ψ is another strongly minimal formula over M and M is uncountable, then $\varphi(M)$ and $\psi(M)$ have the same dimension.

Using also the fact that M is prime over any strongly minimal set in M we obtained the Morley Categoricity Theorem. To prove the Baldwin-Lachlan Theorem (about countable models) we need a finer result on dimension: If, in addition, φ' is a strongly minimal formula conjugate to φ, then $\varphi(M)$ and $\varphi'(M)$ have the same dimension.

When working in an uncountably categorical theory strongly minimal sets have the same dimension in any uncountable model. Other t.t. theories may contain strongly minimal sets whose dimensions in models can differ widely. Here is a simple example.

Example 5.6.1. Let T be the theory of one equivalence relation E with infinitely many infinite classes and no finite classes. For any a, $E(\mathfrak{C}, a)$ is a strongly minimal set. For any pair of infinite cardinals κ, λ there is a model M containing a and b such that $E(M, a)$ has dimension κ and $E(M, b)$ has dimension λ.

This freedom to find models with varying dimensions of strongly minimal sets can be used to show there are nonisomorphic models in each uncountable cardinal. In detail, let κ be uncountable and Λ the set of cardinals $\leq \kappa$ and Λ^+ the set of infinite cardinals $\leq \kappa$. Let $\Phi = \{f : f$ is a function from Λ^+ into Λ such that $f(\kappa) \neq 0$ or $\kappa \in$ range $(f)\}$. For any model M of T of cardinality κ let F_M be the element of Φ such that for any $\lambda \in \Lambda^+$, $F_M(\lambda)$ is the number of E–classes in M of cardinality λ. Then,

- any element of Φ is F_M for some model M of T, and
- for models M, N of T of cardinality κ, $M \cong N$ if and only if $F_M = F_N$.

Thus, the number of models of T of cardinality κ, up to isomorphism, is $|\Phi|$.

In generalizing the observed behavior of strongly minimal sets to a collection P of stationary types in a stable theory the most basic questions are:

1. Given stationary types p, $q \in P$ (over, say, $\emptyset$) let $(*)$ denote the condition: for all sufficiently large cardinals κ, λ there is a model M containing a basis for p of cardinality κ and a basis for q of cardinality λ. What basic properties of p and q cause $(*)$ to hold?
2. When $(*)$ fails for a pair p, q how widely can the cardinality of bases of p and q vary as we range over models of the theory.
3. Can we isolate a broad class of types Q such that dimension is well-defined for any $p \in Q$? (That is, for any $p \in Q$ and model M containing the domain of p, all bases of p in M have the same cardinality.)

These items are the subjects (in order) of the three subsections: orthogonality, domination and weight.

Before getting to the main topics we make the conceptual jump of introducing types and strong types in infinitely many variables.

Notation. If A and B are sets, $tp(A/B)$ is the set of formulas in a potentially infinite sequence of variables obtained in the expected way. (In the background we have fixed an arbitrary enumeration of A.) Such types, called *$*-$types*, are only used to conveniently speak of the class of all sets conjugate to A over B: A' realizes $tp(A/B)$ if there is an automorphism of the universe fixing B and taking A' to A. Extending also the notation for strong types, $stp(A/B) = tp(A/acl(B))$.

Most terms associated to the forking relation on types can be generalized to $*-$types using the obvious definitions. For example, $tp(A/B)$ is stationary if it has a unique nonforking extension over $\mathfrak{C}$, equivalently, $tp(A/B) \models stp(A/B)$.

If p and q are stationary types over A then all pairs ab, where a realizes p, b realizes q and a is independent from b over A, realize the same stationary type over A. This type is denoted $p \otimes q$. More generally,

Notation. If $p_i = stp(B_i/A)$, $i \in I$, is a family of stationary $*-$types over A and $\{ B_i : i \in I \}$ is $A-$independent then $stp(\bigcup_{i \in I} B_i/A)$ is denoted $\bigotimes_{i \in I} p_i$. If p is a strong type over A and λ is a cardinal, $p^{(\lambda)}$ is the strong type over A of a Morley sequence over A in p of cardinality λ (equivalently, the $\otimes-$product of λ copies of p).

Throughout the entire section we assume the underlying theory to be stable. The stability hypothesis may be restated for emphasis in key definitions and results.

5.6.1 Orthogonality

Intuitively, stationary types are orthogonal when there are models in which bases for the types have widely varying cardinalities. The actual definition (given subsequently) specifies a property which guarantees this behavior.

Definition 5.6.2. *(i) The $*-$types p and q over A are said to be* almost orthogonal, *written $p \overset{a}{\perp} q$, if for all B realizing p and C realizing q, B and C are independent over A. The negation of the relation is denoted $p \overset{a}{\not\perp} q$.*

(ii) The stationary $*-$types p and q are called orthogonal, *written $p \perp q$, if for all sets A on which both p and q are based, $p|A \overset{a}{\perp} q|A$.*

(iii) The $*-$types p and q are orthogonal *if $p' \perp q'$ whenever $p', q' \in S(\mathfrak{C})$ are nonforking extensions of p and q, respectively.*

Example 5.6.2. As trivial examples of orthogonal and nonorthogonal types consider a single unary predicate U, the theory T_1 saying that U and $\neg U$ define infinite sets, and the theory $T_2 \supset T_1$ saying that an additional function symbol F defines a bijection between the sets defined by U and $\neg U$.

Then, for p and q the unique nonalgebraic types containing U and $\neg U$, respectively, $p \perp q$ in T_1 and $p \not\perp q$ in T_2.

Let T be the theory in Example 5.6.1. Let $a \neq b$ in $\mathfrak{C}$ and p_a, p_b the strongly minimal types extending $E(x, a)$, $E(x, b)$, respectively. Then p_a is orthogonal to p_b.

The finite character of forking allows us to prove most facts about orthogonality by considering only types (instead of $*$−types). (Use the fact that $stp(B/A)$ is almost orthogonal to $stp(C/A)$ if and only if $stp(b/A) \overset{a}{\perp} stp(c/A)$, for all finite subsets b of B and c of C.) In the sequel we will normally state results only for types, leaving the extension to $*$−types to the reader.

Lemma 5.6.1. *Let T be stable.*

(i) For stationary types p and q, the following are equivalent.

(1) $p \perp q$.

(2) *For some set A on which p and q are both based and all sets $B \supset A$, $p|B \overset{a}{\perp} q|B$.*

(3) *For some a−model M on which p and q are both based, $p|M \overset{a}{\perp} q|M$.*

(ii) If p and q_i, $i \in I$, are stationary types, $p \perp \bigotimes_{i \in I} q_i$ if and only if $p \perp q_i$, for all $i \in I$.

Proof. The proof of (ii) is left to the exercises, while (1) $\implies$ (2) $\implies$ (3) of (i) are vacuously true. To prove (3) $\implies$ (1) let p be nonorthogonal to q and M an a−model on which both p and q are based. Suppose A is a set on which both p and q are based such that $p|A \overset{a}{\not\perp} q|A$. For any $B \supset A$, $p|B \overset{a}{\not\perp} q|B$, so there is a $B \supset M$ such that $p|B \overset{a}{\not\perp} q|B$. Let a be a realization of $p|B$, b a realization of $q|B$ and c a finite subset of B such that a depends on b over $M \cup \{c\}$. Let $C \subset M$ be of cardinality $< \kappa(T)$ over which $stp(abc/M)$ is based. Since M is an a−model there is $c' \in M$ realizing $stp(c/C)$. Now choose elements a' and b' such that $stp(a'b'c'/C) = stp(abc/C)$ and $a'b'$ is independent from M over $C \cup \{c'\}$. These conditions imply that a' is not only independent from M over $C \cup \{c'\}$, but also over C; i.e., a' realizes $p|M$, and similarly, b' realizes $q|M$. Since a' and b' are dependent over $C \cup \{c'\}$, $p|M \overset{a}{\not\perp} q|M$ as required.

Remark 5.6.1. As a simple application of the lemma:

> If T is uncountably categorical and φ is a strongly minimal formula, then every nonalgebraic stationary type p is nonorthogonal to φ.

(A type is nonorthogonal to φ if it is nonorthogonal to the unique nonalgebraic $q \in S(\mathfrak{C})$ containing φ. Let M be an a−model containing the parameters in φ and over which p is based. Let a realize $p|M$ and N be the prime model over $M \cup \{a\}$. Since T has no Vaughtian pair there is a $b \in N \setminus M$ satisfying φ, hence realizing $q|M$. By Corollary 3.3.4, a and b are dependent over M, witnessing the nonorthogonality of p and q (by the lemma).)

The argument used to verify this remark can be generalized. When p and q are orthogonal stationary types over an a−model M there is a larger a−model realizing p and omitting q. (This is proved using Corollary 5.6.1 below.) We can iterate this process to find a "long" Morley sequence in p in a a−model $N \supset M$ which omits q. Hence the cardinalities of bases for p can q can vary widely as we range over a−models containing M. (See Corollary 5.6.2 below.)

Given nonorthogonal stationary types p and q the first question which comes to mind is: What must a set A contain to ensure that $p|A$ and $q|A$ are not almost orthogonal? We saw in Lemma 5.1.17 that p is based on an indiscernible set whose average type is parallel to p. The next result shows that such indiscernibles also control the manner in which the stationary type interacts with other types.

Proposition 5.6.1. *For T stable and p, $q \in S(\mathfrak{C})$ the following are equivalent:*

(1) $p \perp q$.
(2) *For some infinite sets of indiscernibles I, J such that $Av(I/\mathfrak{C}) = p$ and $Av(J/\mathfrak{C}) = q$, $p|(I \cup J) \overset{a}{\perp} q|(I \cup J)$.*
(3) *For some set A on which p and q are both based and for all n, $(p|A)^{(n)}$ is almost orthogonal to $(q|A)^{(n)}$.*

Proof. With regard to (2), recall from Lemma 5.1.17 that p and q are based on I and J, respectively, hence $p|(I \cup J)$ and $q|(I \cup J)$ exist. That (1) $\Longrightarrow$ (2) is trivial. For (2) $\Longrightarrow$ (3) suppose that there are such sets of indiscernibles I and J. Let I' and J' be infinite and coinfinite subsets of I and J, respectively. By Lemma 5.1.17 both p and q are based on $A = I' \cup J'$. Assuming that for some n $(p|A)^{(n)} \overset{a}{\not\perp} (q|A)^{(n)}$ yields minimal sets $I_0 \subset I$ and $J_0 \subset J$ such that $stp(I_0/A) \overset{a}{\not\perp} stp(J_0/A)$. Let $a \in I_0$, $b \in J_0$, $I_1 = I_0 \setminus \{a\}$ and $J_1 = J_0 \setminus \{b\}$. By the minimality assumption on I_0 and J_0,

$$I_0 \underset{A}{\downarrow} J_1 \text{ and } I_1 \underset{A}{\downarrow} J_0.$$

Combining this with the initial assumption about I_0 and J_0 shows that for $B = A \cup I_1 \cup J_1$, $stp(a/B) = p|B$ is not almost orthogonal to $stp(b/B) = q|B$. Since $B \subset I \cup J$ this contradicts that $p|(I \cup J) \overset{a}{\perp} q|(I \cup J)$, proving that (2) $\Longrightarrow$ (3).

To prove (3) $\Longrightarrow$ (1) suppose that p is nonorthogonal to q and A is any set on which p and q are based. By Lemma 5.6.1(i) there is a set $B \supset A$, a realizing $p|B$ and b realizing $q|B$ such that a and b are dependent over B. Let $I = \{a_i b_i : i < \omega\}$ be an infinite Morley sequence in $stp(ab/B)$. Then $\{a_i : i < \omega\}$ and $\{b_i : i < \omega\}$ are Morley sequences over B (and A) in p. Since ab realizes the average type of I over B and this average is based on I there must be some n and m such that a_n and b_m are dependent over $A \cup \{a_i : i < n\} \cup \{b_i : i < m\}$. The sequences $(a_0, \ldots, a_n)$ and $(b_0, \ldots, b_m)$ witness that $(p|A)^{(n+1)} \overset{a}{\not\perp} (q|A)^{(m+1)}$, hence if $n \geq m$, $(p|A)^{(n+1)} \overset{a}{\not\perp} (q|A)^{(n+1)}$, completing the proof of the lemma.

The next example exhibits a simple situation in which nonorthogonal types p and q are based on a set A, but are almost orthogonal over this set.

Example 5.6.3. Let M be an infinite direct sum of copies of the group $\mathbb{Z}_2$ (in the language $\{+, 0\}$) and let N be a subgroup of M of index 2. Add to the language a unary predicate P and let M' be the expansion of M interpreting P by N. Then, $P(x)$ and $\neg P(x)$ are strongly minimal formulas. Let p and q be the unique nonalgebraic completions of $P(x)$, $\neg P(x)$, respectively, over $\emptyset$. For any realizations a and a' of q in M' there is an automorphism of the model fixing N pointwise and taking a to a'. As the reader can verify, this implies that $p \overset{a}{\perp} q$. However, any element of $M' \setminus N$ is interalgebraic with an element of N over a, hence $q|a \overset{a}{\not\perp} p|a$.

Remark 5.6.2. Let p, $q \in S(A)$ be nonorthogonal regular types in a stable theory. By the previous proposition there are m and n such that $p^{(n)}$ and $q^{(m)}$ are not almost orthogonal. We can ask: What are the minimal such m and n. For example, when p and q are modular strongly minimal types (in an uncountably categorical theory) we can take m and n to be 1 by Corollary 4.3.5. However, when p and q are both locally modular and nonmodular the minimal pair may be $m = n = 2$. In general, the answer to this question is a deep result in geometrical stability theory worked out in [Hru89].

Definition 5.6.3. *The complete type p is orthogonal to the set A, written $p \perp A$, if $p \perp q$ for all $q \in S(A)$.*

The results in [She90] and [SHM84] show how families of nonorthogonal types can lead to many nonisomorphic models in a fixed cardinality. The presence of a type orthogonal to a set leads to arbitrarily large families of pairwise orthogonal types.

Example 5.6.4. Consider first the ω−stable, ω−categorical theory T_1 of a single equivalence relation E with infinitely many infinite classes and no finite classes. Let M be a model (which is an a−model as it is $\aleph_0$−saturated). For some $a \in M$ let $p \in S_1(M)$ be the unique nonalgebraic type containing $E(x, a)$. It is left to the reader to show (using the elimination of quantifiers) that $tp(\bar{b}/M)$ is nonorthogonal to p only if some element of the sequence $\bar{b}$ is E−equivalent to a, hence forks over $\emptyset$. If $q \in S(\mathfrak{C})$ does not fork over $\emptyset$ then q is based on M. We have shown that $q|M$, hence q, is orthogonal to p. Thus, p is orthogonal to $\emptyset$.

Now consider the theory T_2 of two equivalence relations E and E' such that

- E and E' have infinitely many infinite classes,
- E and E' have no finite classes, and
- for all a and b there are infinitely many elements which are E−equivalent to a and E'−equivalent to b.

(T_2 is known as the theory of two cross-cutting equivalence relations.) The reader can check that for any a and b, $E(x, a)$ and $E'(x, b)$ both have Morley rank 2, $E(x, a) \wedge E'(x, b)$ has Morley rank 1 and each of these formulas has degree 1. Let M be a model, $a \in M$, p the unique element of $S_1(M)$ of Morley rank 2 containing $E(x, a)$ and q the unique element of $S_1(M)$ containing

$\neg E(x,b) \wedge \neg E'(x,b)$ for all $b \in M$. Let c be a realization of q and d an element satisfying $E(x,a) \wedge E'(x,c)$. The specified equivalences imply that d realizes p and depends on c over M. Thus, p is nonorthogonal to q. Since q is based on $\emptyset$ this type witnesses that p is nonorthogonal to $\emptyset$. This gives a type which forks over $\emptyset$ but is nonorthogonal to $\emptyset$.

Unraveling the definitions shows that $p \perp A$ if and only if $p \perp acl(A)$. The next proposition is the key to understanding this relation.

Proposition 5.6.2. *For T a stable theory, p a stationary type and A a set, the following are equivalent.*

(1) *$p \perp A$.*
(2) *If p is based on B and A' is independent from B over A, then p is orthogonal to A'.*
(3) *For any set B on which p is based, if f is an automorphism fixing $acl(A)$ with $f(B) \underset{A}{\downarrow} B$, then $p|B \perp f(p|B)$.*

Proof. We can assume without loss of generality that $A = acl(A)$.

(1) $\Longrightarrow$ (2). Suppose p is based on B, $p \perp A$, $A' = acl(A')$ is independent from B over A and, to the contrary, that $q \in S(A')$ is nonorthogonal to p. By Proposition 5.6.1 there are a and b which are finite Morley sequences over $A' \cup B$ in p and q, respectively, and are dependent over $A' \cup B$. Since a is independent from A' over B, a must depend on $A' \cup \{b\}$ over B. Since B is independent from $A' \cup \{b\}$ over A this set witnesses that $stp(a/B)$ is nonorthogonal to A. For some n, $stp(a/B) = (p|B)^{(n)}$, so Lemma 5.6.1(ii) implies that p is nonorthogonal to A, a contradiction which proves the implication.

(2) $\Longrightarrow$ (3). This part holds trivially.

(3) $\Longrightarrow$ (1). Suppose $p \in S(B)$, $p \not\perp q$ for some $q \in S(A)$ and, towards a contradiction, that p is orthogonal to $p' = f(p)$, where f is as in (3). Let $\{ B_i : i \in I\}$ be an A-independent family of realizations of $stp(B/A)$, where $|I| = \kappa(T)$, and let p_i be a conjugate of p over B_i. Since all independent pairs of realizations of $stp(B/A)$ have the same type over A the pair (p_i, p_j) is conjugate to (p, p'), hence $p_i \perp p_j$, for $i \neq j \in I$. Let M be an a-model containing $A \cup \bigcup_{i \in I} B_i$, $q' = q|M$ and $p'_i = p_i|M$ (for $i \in I$). Since q is a stationary type over A and each p_i is conjugate over A to p, $q \not\perp p_i$, for each $i \in I$. By Lemma 5.6.1(i) there are b realizing q' and a_i realizing p'_i such that b depends on a_i over M, for each $i \in I$. Let $I' \subset I$ be a set of cardinality $< \kappa(T)$ such that b is independent from $\{ a_i : i \in I \}$ over $M \cup \{ a_i : i \in I' \}$. Then, for a fixed $j \in I \setminus I'$, the dependence over b and a_j over M forces a_j to be dependent on $\{ a_i : i \in I' \}$ over M. However, Lemma 5.6.1(ii) implies that p_j is orthogonal to $\bigotimes_{i \in I'} p_i$. This contradiction completes the proof of the proposition.

5.6.2 Domination

The domination relation on stationary types addresses the second motivating question at the beginning of the section. Namely: When p and q are nonorthogonal stationary types, must bases of p and q (in a given model) have the same cardinality? This question is approached by defining the domination relation on types, which forces bases of p and q to have the same cardinality, then discussing (under the subsection on weight) when domination agrees with nonorthogonality. In this subsection we also point out some useful connections between a−isolation and dependence.

The domination relation on triples of sets was introduced in Definition 3.4.2 in the context of totally transcendental theories. Before discussing domination between types we study the extension of this notion to triples of sets in stable theories.

Definition 5.6.4. *For sets A, B and C we say that A* is dominated by B over C, *and write $A \lhd B$ (C), if for all sets D,*

$$D \underset{C}{\perp} B \implies D \underset{C}{\perp} A.$$

The sets A and B are said to be domination equivalent over C, *written $A \sqsubseteq\!\!\!\sqsupseteq B$ (C), if $A \lhd B$ (C) and $B \lhd A$ (C).*

The basic properties of the relation, especially as they relate to independence are found in

Lemma 5.6.2. *(i) Suppose that $A_0 \subset A$ and C is independent from A over A_0. Then*

$$B \lhd C\ (A_0) \implies B \lhd C\ (A).$$

(ii) If $A_0 \subset A$ and $B \cup C$ is independent from A over A_0, then

$$B \lhd C\ (A) \implies B \lhd C\ (A_0).$$

(iii) If $B_0 \subset B$, $B \lhd B_0 \cup C$ (A) and $C \underset{A}{\perp} B$, then $B \lhd B_0$ (A).

(iv) If $\{ A_i : i \in I \}$ is a family of sets which is independent over C and B_i is dominated by A_i over C for all $i \in I$, then $\{ B_i : i \in I \}$ is independent over C.

(v) (Transitivity) *If $A \supset B \supset C \supset D$, $A \lhd B$ (C) and $B \lhd C$ (D) then $A \lhd C$ (D).*

Proof. (i) Let D be independent from C over A. Then $D \cup A$ is independent from C over A_0 (by the transitivity of independence). Since $B \lhd C$ (A_0), $D \cup A$ is independent from B over A_0. We conclude that D is independent from B over A.

(ii) Suppose that D is independent from C over A_0. Without loss of generality, D is independent from A over $A_0 \cup C \cup B$. (If there is such a

D which is dependent on B over A_0, there is one satisfying this condition.) Since A is independent from $B \cup C$ over A_0, $D \cup B \cup C$ is independent from A over A_0. Thus, D is independent from $A \cup C$ over $A_0 \cup C$ which, combined with the original assumption about D, shows that D is independent from $A \cup C$ over A_0. Thus, D is independent from C over A and D is independent from B over A (since $B \lhd C$ (A)). From the independence of D from A over A_0 we conclude that D and B are independent over A_0, as desired.

(iii) The proof of this, which is similar to (ii), is assigned in the exercises.

(iv) To simplify the notation let $C = \emptyset$. To prove the independence of the family of B_i's we prove by induction on $|X|$ that whenever X is a finite subset of I, $\{ A_i : i \in I \setminus X \} \cup \{ B_i : i \in X \}$ is independent. When $X = \emptyset$ it is true by hypothesis. For the inductive step let $j \in X$ and $Y = X \setminus \{j\}$. Since $\{ A_i : i \in I \setminus X \} \cup \{A_j\} \cup \{ B_i : i \in Y \}$ is independent and B_j is dominated by A_j, $\{ A_i : i \in I \setminus X \} \cup \{B_j\} \cup \{ B_i : i \in Y \}$ is independent, as required.

(v) The straightforward proof is left to the reader.

Definition 5.6.5. *For stationary $*-$types p and q we say that p is* dominated *by q and write $p \lhd q$ if there is a set A on which both p and q are based and realizations C and D of $p|A$ and $q|A$, respectively, such that C is dominated by D over A.*

If $p \lhd q$ and $q \lhd p$ we call p and q domination equivalent *and write $p \sqsubseteq\!\!\sqsupseteq q$.*

Remark 5.6.3. Notice that $\lhd$ is transitive on stationary types and $x \sqsubseteq\!\!\sqsupseteq y$ is an equivalence relation (exercise). Also, for p, q and r stationary, $p \lhd q \implies p \otimes r \lhd q \otimes r$. Thus, if p_i, q_i are stationary and $p_i \sqsubseteq\!\!\sqsupseteq q_i$, for $1 \leq i \leq n$, then $p_1 \otimes \ldots \otimes p_n$ is domination equivalent to $q_1 \otimes \ldots \otimes q_n$.

Again, we only prove properties for types, leaving the extension to $*-$types to the reader. Domination links two types within $a-$models through the following connections to $a-$isolation. The first result is little more than a rewording of the definitions. The subsequent proposition is a direct generalization of Lemma 3.4.7. Both proofs are left to the reader in the exercises.

Lemma 5.6.3. *If $B \supset C$ are sets and a is an element, then $\{a\} \cup B \lhd B$ (C) if and only if for all sets D independent from B over C, $stp(a/B) \models stp(a/D \cup B)$.*

Proposition 5.6.3. *Let M be an $a-$model and $B \supset A$. If B is $a-$atomic over $M \cup A$ then B is dominated by A over M. If A is finite (or of cardinality $< \kappa(T)$ when $\kappa(T)$ is regular) and B is dominated by A over M, then B is $a-$atomic over $M \cup A$.*

As an immediate consequence of the proposition we obtain:

Corollary 5.6.1. *If M is an $a-$model and B is $a-$atomic over $A \cup M$, then for any $b \in B \setminus M$, $b \not\perp_M A$.*

Corollary 5.6.2. *Suppose M is an $a-$model and p, $q \in S(M)$ are orthogonal. Then, for all cardinals κ, λ there is an $a-$model $N \supset M$ and sets I, J such that I is a basis for p in N, J is a basis for q in N, $|I| = \kappa$ and $|J| = \lambda$.*

Proof. Assigned as Exercise 5.6.9.

Proposition 5.6.4. *For p and q stationary types and M an $a-$model on which p and q are based, the following are equivalent:*

(1) $p \lhd q$.
(2) *There is a b realizing $q|M$ and an a realizing $p|M$ such that $a \lhd b\,(M)$.*
(3) *For any b realizing $q|M$ and any $a-$model $N \supset M \cup \{b\}$, $p|M$ is realized in N.*

Proof. Most of the work goes into proving (1) $\Longrightarrow$ (2). Suppose that A is a set on which p and q are based and there are c, d realizing $p|A$, $q|A$, respectively, with $c \lhd d\,(A)$. Without loss of generality, cd is independent from M over A. Let B be a subset of M of cardinality $< \kappa(T)$ over which both p and q are based. Choosing $A_0 \subset A$ a set of cardinality $< \kappa(T)$ over which $tp(cd/A)$ does not fork, Lemma 5.6.2 implies that $c \lhd d\,(A_0)$. Since M is an $a-$model there is a set $A_1 \subset M$ realizing $tp(A_0/B)$, and there are ab such that $tp(abA_1/B) = tp(cdA_0/B)$ and ab is independent from M over $A_1 \cup B$. The types of c and d over $M \cup A_0$ are $p|(M \cup A_0)$ and $q|(M \cup A_0)$, respectively, and these types are based on B, so a realizes $p|M$ and b realizes $q|M$. Furthermore, by $a \lhd b\,(A_1)$, $tp(c/M \cup A_0) = p|(M \cup A_0)$, the independence of ab from M over A_1, and Lemma 5.6.2(i), $a \lhd b\,(M)$. This completes the proof of (1) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (3) Letting a, b and M be as in (2), the proof of this part follows immediately from

Claim. There is an element a' realizing $p|M$ such that $stp(a'/M \cup \{b\})$ is $a-$isolated.

Let $A_0 \subset M$ be a set of cardinality $< \kappa(T)$ over which $tp(ab/M)$ does not fork. Let N be an $a-$prime model over $M \cup \{b\}$ and a' a realization of $stp(a/A_0 \cup \{b\})$ in N. By Lemma 5.6.2(ii) a' is dominated by b over A_0. Thus, the A_0-independence of b and M yields the A_0-independence of a' and M. We conclude that a' realizes $p|M$ and $stp(a'/M \cup \{b\})$ is $a-$isolated (simply because $a' \in N$), proving the claim.

That (3) $\Longrightarrow$ (1) follows from Proposition 5.6.3.

By the previous proposition and Proposition 5.6.3, given an $a-$model M and p, $q \in S(M)$ such that $p \lhd q$, there are a realizing p and b realizing q such that $ab \lhd b\,(M)$.

Corollary 5.6.3. *Given an $a-$model M, p, $q \in S(M)$ domination equivalent and an $a-$model $N \supset M$, there is a base for p in N of cardinality λ if and only if there is a base for q in N of cardinality λ.*

Proof. Suppose I is a basis for p in N of cardinality λ. For $a \in I$ there is a $b_a \in N$ realizing q such that $\{b_a, a\}$ is dominated by a over M. Let $J = \{b_a : a \in I\}$. By Lemma 5.6.2(iv), J is a Morley sequence in q of cardinality λ.

5.6.3 Weight

The most basic problem in the development of a dimension theory is: Given a set A, do all maximal independent subsets of A have the same cardinality? If the answer is negative is there at least a measure of how widely the cardinalities of maximal independent sets can differ? Weight addresses these issues.

Definition 5.6.6. *Let T be stable and p a complete $*-$type over A.*

(i) Let $PWT(p)$ be the set of all λ such that given B realizing p there is an $A-$independent set C such that $c \not\downarrow_A B$, for all $c \in C$ and $|C| = \lambda$.

(ii) The pre-weight of *p ($pwt(p)$) is* sup $PWT(p)$.

(iii) Suppose that p is stationary. The weight of *p ($wt(p)$) is the supremum of $\{pwt(p|C) : p$ is based on $C\}$.*

For any set B the pre-weight of *B* over *A is $pwt(tp(B/A))$, which we denote by $pwt(B/A)$. The* weight of *B* over *A ($wt(B/A)$) is $wt(stp(B/A))$. If $A = \emptyset$ we omit it as usual.*

Remark 5.6.4. In Definition 3.4.1 pre-weight was defined for complete types in a t.t. theory. The reader can verify that the two notions agree in t.t. theories.

Pre-weight and weight are invariant under conjugacy. Thus, $pwt(B/A) = pwt(stp(B/A))$, and $wt(B/A) = wt(B'/A)$ for all sets A, B and B' with $tp(B'/A) = tp(B/A)$. Also, given $B' \subset B$, $pwt(B'/A) \leq pwt(B/A)$ and $wt(B'/A) \leq wt(B/A)$.

Remark 5.6.5. Given a superstable theory and a complete type p in finitely many variables, $PWT(p) \subset \omega$. (Suppose $p \in S(A)$, a realizes p and I is an $A-$independent set such that each $b \in I$ depends on a over A. There is a finite $J \subset I$ such that a is independent from $I \cup A$ over $J \cup A$. Any $b \in I \setminus J$ is independent from $A \cup \{a\}$ over $J \cup A$, hence independent from a over A. Thus, $I = J$, proving that I is finite.)

Remark 5.6.6. A complete strongly minimal type has weight 1. (See Exercise 5.6.10.)

If λ is any cardinal, λ^- is κ if $\lambda = \kappa^+$ and $\lambda^- = \lambda$ if λ is a limit cardinal. For countable stable theories, $\kappa(T)^-$ is always $\aleph_0$.

Lemma 5.6.4. *Let T be stable.*

(i) $wt(B/A) \leq \kappa(T)^- + |B|$.

(ii) If p is a stationary type and M is an a–model over which p is based, $wt(p) = pwt(p|M)$.

(iii) If p and q are stationary types and $p \lhd q$, then $wt(p) \leq wt(q)$.

(iv) Domination equivalent stationary types have the same weight.

Proof. (i) Since A is arbitrary here it suffices to prove the inequality for pre-weight instead of weight. Let $C = \{ c_i : i \in I \}$ be an A–independent set with c_i dependent on B over A, for all $i \in I$. By Proposition 5.2.1 there is a set $J \subset I$ of cardinality $< \kappa(T) + |B|^+$ (if $\kappa(T)$ is regular) and $\leq \kappa(T) + |B|$ (otherwise) such that B is independent from $C \cup A$ over $A \cup \{ c_i : i \in J \}$. The stated conditions on C force J to equal I, hence $|I|$ satisfies the restrictions placed on $|J|$. Whether $\kappa(T)$ is regular or singular, $|I| \leq \kappa(T)^- + |B|$. Since $pwt(B/A)$ is the supremum of the cardinalities of such sets I this proves (i).

(ii) It suffices to show that whenever

- A is a set on which p is based,
- a is a realization of $p|A$, and
- $C = \{ c_i : i \in I \}$ an A–independent set such that c_i depends on a over A, for all $i \in I$,

there are

- b realizing $p|M$ and
- an M–independent set $\{ c'_i : i \in I \}$ such that c'_i depends on b over M for all $i \in I$.

Let A be a set on which p is based, a a realization of $p|A$ and $C = \{ c_i : i \in I \}$ an A–independent set such that c_i depends on a over A, for all $i \in I$. We can assume that $A \supset M$ (an exercise left to the reader). The argument in (i) shows that $|I| < \kappa(T)$, so there is an $A' \subset A$ of cardinality $\leq \kappa(T)$ such that $\{a\} \cup C$ is independent from A over A' (see Proposition 5.2.1). Let $B \subset M$ be a set of cardinality $< \kappa(T)$ over which p is based. Since M is $\kappa(T)$–saturated there is a set $A'' \subset M$ realizing $tp(A'/B)$. In fact, A'' realizes $tp(A'/B \cup \{a\})$ since p is based on B and a is independent from both A' and M over B. Let $C' = \{ c'_i : i \in I \}$ be a family of sets such that $C' \cup A''$ is conjugate to $C \cup A$ over $B \cup \{a\}$ and C' is independent from M over $A'' \cup \{a\}$. These conditions imply that C' is not only A''–independent, but M–independent and a depends on c'_i over M for all $i \in I$. Thus, $|I| \leq pwt(p|M)$, completing the proof of (ii).

(iii) Let M be an a–model on which both p and q are based. By Proposition 5.6.4 there are a realizing p and b realizing q such that a is dominated by b over M. If $\{ c_i : i \in I \}$ is an M–independent set such that c_i depends on a over M for all $i \in I$, then the domination hypothesis guarantees that c_i depends on b over M for all $i \in I$. Thus, $pwt(p) \leq pwt(q)$. By (ii) and the fact that M is an a–model, $wt(p) \leq wt(q)$, completing the proof.

The proof of (i) leads us to ask if the weight of any stationary type is actually $< \kappa(T)$. However, even when T is superstable it is not clear that $stp(a/A)$ has finite weight. It is at least conceivable that there are A–independent sets of size n for arbitrarily large n witnessing that $wt(a/A) \geq n$. This will be shown to be impossible in the next subsection in a lengthy argument.

The next result links the weight of a union of two sets to the weights of the component sets.

Proposition 5.6.5 (Additivity). *Let T be stable and $B = \bigcup\{ B_i : i < \alpha \}$ a family of sets. Then*

(i) $wt(B/A) \leq \sum wt(B_i/A \cup B_{<i})$, where $B_{<i} = \bigcup\{ B_j : j < i \}$, and

(ii) If $\{ B_i : i < \alpha \}$ is A–independent then $wt(B/A) = \sum wt(B_i/A)$.

Proof. (i) Replacing A by a larger set A' independent from B over A such that $pwt(B/A') = wt(B/A)$, it suffices to show $pwt(B/A) \leq \sum wt(B_i/A \cup B_{<i})$. Without loss of generality, $A = \emptyset$. Let $\{ c_i : i \in I \}$ be an independent family witnessing that $pwt(B) \geq |I|$. We will write I as a union of a disjoint family of sets J_i, $i < \alpha$, such that $\{ c_j : j \in J_i \}$ witnesses that $wt(B_i/B_{<i}) \geq |J_i|$. Since $|I| = \sum_{i<\alpha} |J_i|$ this proves the necessary inequality.

Preliminarily, we choose by recursion, $I_i \subset I$, for $i < \alpha$, a maximal subset of $I_i^{\cap} = \bigcap_{j<i} I_j$ such that $\{ c_j : j \in I_i \} \cup \{B_{<i+1}\}$ is independent. The sets I_i, $i < \alpha$, form a descending chain of subsets of I whose intersection is empty since every c_i depends on $B = B_\alpha$. Let $J_i = I_i^{\cap} \setminus I_i$. (Intuitively, J_i is what is lost from the chain at stage i). Since $\bigcap_{i<\alpha} I_i = \emptyset$, $\bigcup_{i<\alpha} J_i = I$ and the J_i's certainly form a pairwise disjoint family. The definition of the chain of sets directly implies that $C_{I_j} = \{ c_k : k \in I_j \}$ is independent from $B_{<i+1}$, and c_j depends on B_i over $B_{<i} \cup C_{I_j}$ for every $j \in J_i$. Thus, the pre-weight of B_i over $B_{<i} \cup C_{I_j}$ is $\geq |J_i|$. By the independence of C_{I_j} from $B_{<i+1}$ we conclude that $wt(B_i/B_{<i}) \geq |J_i|$. This proves (i).

(ii) Replacing A by a larger set if necessary and then letting $A = \emptyset$ it suffices to show (by (i)) that $pwt(B) \geq \sum_{i<\alpha} pwt(B_i)$. Let $C_i = \{ c_j^i : j \in J_i \}$ be an independent family such that c_j^i depends on B_i for all j. Without loss of generality, C_i is independent from $\{ B_j \cup C_j : j \neq i \}$ over B_i. By the transitivity of independence, $\{ B_i \cup C_i : i < \alpha \}$ is independent. Thus, $C = \bigcup_{i<\alpha} C_i$ is independent and each element of C depends on B. Since $|C| = \sum_{i<\alpha} |C_i|$, $pwt(B) \geq \sum_{i<\alpha} pwt(B_i)$.

As the following corollary suggests we can expect a reasonable dimension theory on the realizations of a weight 1 type.

Corollary 5.6.4. *Let T be stable and $\mathcal{B}$ a collection of sets such that $wt(B/A) = 1$, for all $B \in \mathcal{B}$. Let $\bar{B} = \{ B_i : i \in I \}$ and $\bar{C} = \{ C_j : j \in J \}$ be maximal A–independent subsets of $\mathcal{B}$. Then, $|I| = |J|$.*

Proof. By Proposition 5.6.5(ii), $wt(\bigcup \bar{B}/A) = |I|$. Since $\bar{B}$ is a maximal independent subset of $\mathcal{B}$, C_j depends on $\bar{B}$ over A, for all $j \in J$, hence $|J| \leq wt(\bar{B}/A) = |I|$. For the same reasons, $|I| \leq |J|$.

Definition 5.6.7. *If $p \in S(A)$ is stationary and has weight* 1, *then all bases for p in C (where C is some set) have the same cardinality by the corollary. We call this cardinality the* dimension of p in C, *denoted* $\dim(p, C)$.

In the next subsection we show that in a stable theory in which $PWT(p) \subset \omega$, for all stationary p, a stationary type p is domination equivalent to a finite product of weight 1 types. This goes a long way towards reducing all problems about types vis-a-vis orthogonality and domination to the class of weight 1 types, as well as giving a good dimension theory on $\mathfrak{C}$.

Remark 5.6.7. While weight 1 types do have dimension, they have a weakness in one area. A frequently used feature of dimension on strongly minimal sets is its additivity: If φ is a strongly minimal formula defined over A and $N \supset M$ are models with $A \subset M$, $\dim(\varphi(N)/A) = \dim(\varphi(N)/M) + \dim(\varphi(M)/A)$. There are weight 1 types on which a corresponding additivity result fails (even for $a-$models). We can eliminate this pathology by working with a special class of weight 1 types called regular types. We will prove that every weight 1 stationary type in a superstable theory is domination equivalent to a regular type, giving us this more robust dimension theory in superstable theories.

We end with two corollaries which address the issue of linking the dimensions of different weight 1 types.

Corollary 5.6.5. *If T is stable and p, q are stationary types with p of weight* 1, *then $p \not\perp q \iff p \lhd q$. Thus, if q also has weight* 1, $p \not\perp q \iff p \sqsupseteq q$.

Proof. It follows directly from the definitions that $p \lhd q \implies p \not\perp q$ for any two stationary types. Supposing p and q to be nonorthogonal let M be an $a-$model on which both p and q are based and a, b realizations of p, q, respectively, which are dependent over M (see Lemma 5.6.1(i)). Suppose, towards a contradiction, that a is not dominated by b over M. Then there is a c independent from b over M which depends on a over M. This is impossible since $wt(a/M) = 1$. Thus, $p \lhd q$.

Corollary 5.6.6. *Let T be stable, M an $a-$model and p, $q \in S(M)$ weight* 1 *types.*

(i) If $p \perp q$, then for all $\kappa \geq |M|$ there is an $a-$model $N \supset M$ such that $dim(p, N) = \kappa$ and $dim(q, N) = 0$.

(ii) If $p \not\perp q$, then for all $a-$models $N \supset M$, $dim(p, N) = dim(q, N)$.

Proof. The reader is asked to combine the relevant results in the exercises.

Thus, all of the questions on page 274 can be answered quite satisfactorily for dimension on weight 1 types over $a-$models.

5.6.4 Finite Weight

For T a stable theory let

$$PWT(T) = \bigcup\{PWT(p) : p \text{ a stationary type in } T\}.$$

(In this definition we allow only types, not $*-$types.) In this subsection we study weight and domination in a stable theory in which $PWT(T) \subset \omega$. This class of theories includes the superstable theories (see Remark 5.6.5). The key result, which follows, goes a long way towards reducing any type to weight 1 types (at least as far as orthogonality, domination and dimension go).

Theorem 5.6.1. *Let T be a stable theory with $\kappa(T)$ regular and $PWT(T) \subset \omega$.*

(i) Then every stationary type in T has finite weight.

(ii) Moreover, given a stationary type p there are weight 1 types $q_1, \ldots, q_n$ (where $n = wt(p)$) such that $p \sqsubseteq q_1 \otimes \ldots \otimes q_n$.

Corollary 5.6.7. *Every stationary type in a superstable theory has finite weight.*

All of the work in the proof goes into showing (ii); (i) will follow quickly using some previously established facts about weight and domination.

Remark 5.6.8. Suppose T is stable, $wt(a/A) = 1$ and a depends on b over A. Then a is dominated by b over A. (Suppose, to the contrary, that there is a c independent from b over A such that c depends on a over A. Then the pair b, c witnesses that $wt(a/A) \geq 2$; contradiction.)

The following proposition, due to Tapani Hyttinen [Hyt95], is the key. The proof given here is largely due to Pillay.

Proposition 5.6.6. *Let T be stable and p a stationary type. If there is no weight 1 stationary type q dominated by p then for some nonforking extension p' of p, $\aleph_0 \in PWT(p')$.*

Proof. Let A_0 be a set over which p is based. Certainly p has weight > 1, so there is a set A_1 such that $pwt(p|A_1) > 1$. Choose b realizing $p|A_1$ and elements a_1, c_1 such that

($*$) $\{a_1, c_1\}$ is A_1-independent, b depends on a_1 over A_1 and b depends on c_1 over A_1.

Suppose there is a set $A_{11} \supset A_1$ such that

$$b \underset{A_0}{\perp} A_{11}, \quad c_1 \underset{A_1}{\not\perp} A_{11} \quad \text{and} \quad c_1 A_{11} \underset{A_1}{\perp} a_1. \tag{5.9}$$

Then, $\{a_1, c_1\}$ is $A_{11}-$independent, b depends on a_1 over A_{11} and b depends on c_1 over A_{11}. Iterating this process, let $A_1 \subset A_{11} \subset A_{1\alpha} \subset \ldots$, $\alpha < \lambda$ be a chain of sets such that for $A_1' = \bigcup_{\alpha<\lambda} A_{1\alpha}$ and all $\alpha < \beta < \lambda$,

$$b \mathop{\perp}_{A_0} A_1', \quad c_1 \mathop{\not\perp}_{A_\alpha} A_\beta \quad \text{and} \quad c_1 A_1' \mathop{\perp}_{A_1} a_1.$$

Thus, $\lambda < \kappa(T)$. By choosing the $A_{1\alpha}$'s to be a maximal chain and replacing A_1 by A_1' we can require (in addition to $(*)$) that

$(**)$ there is no set $B \supset A_1$ such that

$$b \mathop{\perp}_{A_0} B, \quad c_1 \mathop{\not\perp}_{A_1} B \quad \text{and} \quad c_1 B \mathop{\perp}_{A_1} a_1. \tag{5.10}$$

Without loss of generality, A_1 is an $a-$model.

Claim. $wt(c_1/A_1) \geq 2$.

Suppose $wt(c_1/A_1) = 1$. Since b depends on c_1 over A_1, c_1 is dominated by b over A_1. This contradicts that p does not dominate a weight 1 stationary type, to prove the claim.

We chose A_1 to be an $a-$model, so $pwt(c_1/A_1) \geq 2$ (Lemma 5.6.4(ii)). Let $\{a_2, c_2\}$ be an A_1-independent set such that c_1 depends on a_2 over A_1 and c_1 depends on c_2 over A_1. Choose a_2 and c_2 so that $c_2 a_2$ is independent from $bc_1 a_1$ over Ac_1. Thus, $\{a_2, c_2, a_1\}$ is A_1-independent. Let $B = A_1 \cup \{c_2\}$. Then c_1 depends on B over A_1 and a_1 is independent from $c_1 B$ over A_1. By $(**)$, b depends on B over A_1; i.e., b depends on c_2 over A_1. As above there is a set $A_2 \supset A_1$ such that

- b is independent from A_2 over A_0,
- $\{a_1, a_2, c_2\}$ is A_2-independent,
- b depends on any element of $\{a_1, a_2, c_2\}$ over A_2 and
- there is no set $B \supset A_2$ such that

$$b \mathop{\perp}_{A_0} B, \quad c_2 \mathop{\not\perp}_{A_2} B \quad \text{and} \quad c_2 B \mathop{\perp}_{A_2} a_1 a_2.$$

Continuing in this manner yields elements $a_1, a_2, a_3, \ldots$ and sets $A_1 \subset A_2 \subset A_3 \subset \ldots$ such that, letting $\bar{A} = \bigcup_{j<\omega} A_j$, for each $i < \omega$, b is independent from $\bar{A}$ over A_0, $\{a_1, \ldots, a_i\}$ is A_i-independent and independent from $\bar{A}$ over A_i, and b depends on a_i over each A_j, $j \geq i$. Thus, $\{a_i : i < \omega\}$ is $\bar{A}-$independent and b depends on each a_i over $\bar{A}$. Since $tp(b/\bar{A})$ is a nonforking extension of p the proposition is proved.

Corollary 5.6.8. *Let T be a stable theory such that $PWT(T) \subset \omega$. Then for any $a-$models $M \subset N$, $M \neq N$, there is a $a \in N \setminus M$ such that $tp(a/M)$ has weight* 1.

Proof. See Exercise 5.6.14.

Lemma 5.6.5. *Suppose that T is stable with $\kappa(T)$ regular, M is an $a-$model, $B \supset M$ and $p \in S(B)$ is a weight* 1 *stationary type nonorthogonal to M. Then there is a weight* 1 *type $q \in S(M)$ domination equivalent to p.*

Proof. Let $B_0 \subset B$ be a set of cardinality $< \kappa(T)$ such that p is based on $acl(B_0)$. Let $A \subset M$ be a set of cardinality $< \kappa(T)$ such that B_0 and M are independent over A (which exists since $\kappa(T)$ is regular) and $p \not\perp A$. Since M is an $a-$model there is a $B_1 \subset M$ realizing $stp(B_0/A)$. Since B_0 and B_1 are $A-$independent, Proposition 5.6.2 indicates that $p|acl(B_0)$ is nonorthogonal to its conjugate p' over $acl(B_1)$. Weight is preserved under conjugacy, so p' also has weight 1. Thus, $q = p'|M$ is a weight 1 type nonorthogonal to p. On weight 1 types nonorthogonality is the same as domination equivalence (Corollary 5.6.5). This proves the lemma.

Proof of Theorem 5.6.1. We prove part (ii) first. Without loss of generality, $p \in S(M)$ for an $a-$model M.

Claim. Let $N \supset M$ be an $a-$model and $C \subset N$ a maximal $M-$independent set of realizations of weight 1 types in $S(M)$. Then N is dominated by C over M.

Let M' be a maximal subset of N which is dominated by C over M. Notice that M' is an $a-$model. (Let $M'' \subset N$ be the $a-$prime model over M'. By Proposition 5.6.3, M'' is dominated by M' over M, hence by C over M. The maximality of M' forces M'' to equal M'.) Suppose, towards a contradiction, that $M' \neq N$. By Corollary 5.6.8 there is an $a \in N \setminus M'$ such that $wt(a/M') = 1$. If $tp(a/M')$ is orthogonal to M, $M' \cup \{a\}$ is dominated by M' over M, contradicting the maximality of M'. Thus, $tp(a/M')$ is nonorthogonal to M, yielding a $q \in S(M)$ of weight 1 domination equivalent to $tp(a/M')$ (by Lemma 5.6.5). Proposition 5.6.4 then gives a $b \in N$ such that $tp(b/M')$ is a nonforking extension of q. Then $C \cup \{b\}$ is an $M-$independent set of realizations of weight 1 types, contradicting the maximality of C to prove the claim.

Let a be a realization of p, N the $a-$prime model over $M \cup \{a\}$ and $C \subset N$ a maximal $M-$independent set of realizations of weight 1 types over M. Since every element of C depends on a over M (Corollary 5.6.1) and $pwt(p)$ is finite, C is finite. Let $C = \{c_1, \ldots, c_n\}$ and $q_i = tp(c_i/M)$, for $1 \leq i \leq n$. Since C is finite Proposition 5.6.3 implies that $a \sqsupseteq C\ (M)$. The type of C over M is $r = q_1 \otimes \ldots \otimes q_n$, so $p \sqsupseteq r$, proving (ii).

(i) Let p be a stationary type and $q_1, \ldots, q_n$ weight 1 types such that $p \sqsupseteq q_1 \otimes \ldots \otimes q_n$. By Proposition 5.6.5(ii), $wt(q_1 \otimes \ldots \otimes q_n) = n$. Part (i) now follows from (ii) and Lemma 5.6.4(iv) (which says that domination equivalent types have the same weight). This proves the theorem.

As stated in Remark 5.6.7 a full-featured dimension theory requires an additivity condition which may fail for weight 1 types. Simply knowing that every type in a superstable theory has finite weight does, however, have its applications. A good example is the following theorem by Lachlan, whose original proof (before weight was developed) was much harder.

Theorem 5.6.2 (Lachlan). *A countable superstable theory has* 1 *or infinitely many countable models.*

Proof. Assume, to the contrary, that T is a countable superstable theory which is not $\aleph_0$-categorical, but has finitely many countable models. By Lemma 2.3.1, T has a countable model M which realizes every complete type over $\emptyset$ and is prime over a finite set a. Let $n = wt(a)$. Since T is not $\aleph_0$-categorical there is a nonisolated type $p \in S(\emptyset)$. Let $b = \{b_0, \ldots, b_n\}$ be an independent set of realizations of p. Since M realizes p we may as well assume that $b \in M$. Since $tp(b_i)$ is nonisolated and $tp(b_i/a)$ is isolated Corollary 5.1.9 indicates that $a \not\perp b_i$, for all $i \leq n$. The independence of $\{b_0, \ldots, b_n\}$ now contradicts that $wt(a) = n$ to prove the theorem.

Remark 5.6.9. The alert reader will notice that the Baldwin-Lachlan Theorem is a special case of this theorem. Indeed, parts of the proof of the Baldwin-Lachlan Theorem given earlier are restricted versions of the proof of Lachlan's result.

Historical Notes. The concepts of orthogonality and weight are due to Shelah and found in [She90]. The domination relation on sets and types was developed by Lascar in [Las82]. Our exposition owes a great debt to Makkai [Mak84]. Theorem 5.6.1 is found for regular types (instead of weight 1 types) in [She90]. The generalization to stable theories with $PWT(T) \subset \omega$ was done by Pillay, with the key step due to by Hyttinen. Theorem 5.6.2 was proved by Lachlan in [Lac73] with an alternative proof found in [Las76].

Exercise 5.6.1. Prove Lemma 5.6.1(ii).

Exercise 5.6.2. Let p be a stationary type based on a set A, r a stationary type nonorthogonal to p and r' a conjugate of r over A. Show that r' is also nonorthogonal to A.

Exercise 5.6.3. Prove Lemma 5.6.2(iii).

Exercise 5.6.4. Prove Lemma 5.6.3.

Exercise 5.6.5. Prove Proposition 5.6.3.

Exercise 5.6.6. Prove that $x \lhd y$ is transitive on stationary types and $x \underline{\square}\, y$ defines an equivalence relation.

Exercise 5.6.7. Suppose that p and q are strongly minimal types in a stable theory. Prove that $p \not\perp q$ if and only if $p \underline{\square}\, q$ (without using Corollary 5.6.5).

Exercise 5.6.8. Suppose that $M \supset N$ are a-models, p, $q \in S(N)$ are domination equivalent and I is a basis for p in M. Show that there is a basis J for q in M such that $|J| = |I|$.

Exercise 5.6.9. Prove Corollary 5.6.2.

Exercise 5.6.10. Prove Remark 5.6.6.

Exercise 5.6.11. Let p be a weight 1 type nonorthogonal to $\emptyset$ and p' a conjugate of p over $acl(\emptyset)$. Prove that $p \not\perp p'$.

Exercise 5.6.12. Alter the vector space example (Example 5.6.3) slightly to produce two strongly minimal types p and q such that $p^{(2)} \overset{a}{\perp} q$, $p \overset{a}{\perp} q^{(2)}$, but $p^{(2)} \overset{a}{\not\perp} q^{(2)}$.

Exercise 5.6.13. Prove Corollary 5.6.6.

Exercise 5.6.14. Prove Corollary 5.6.8.