4. Fine Structure of Uncountably Categorical Theories

In the preceding chapter, for T a countable uncountably categorical theory, we solved problems concerning the number of models of T in a fixed cardinality. However, this study leaves many unanswered questions about uncountably categorical theories, and raises others. Here are a few such questions.

- In [Vau61] Vaught asked if an uncountably categorical theory can be finitely axiomatizable. (It was through Zil'ber's work on this problem that geometrical stability theory, the area in which the subject matter of this chapter belongs, was born.)
- Can we isolate a broad class of uncountably categorical theories which have a strongly minimal formula (or at least a formula of Morley rank 1) over $\emptyset$? (While working on the Baldwin-Lachlan Theorem we recognized that an easier proof would be possible in such theories.)
- Are there strongly minimal sets which are radically different from the examples given in Example 3.1.1?

What is surprising is that work on each of these questions has given insight into the others. The issues underlying this connection are the following imprecisely worded problems concerning the definable relations in models of uncountably categorical theories. Recall that algebraic closure restricted to the subsets of strongly minimal set defines a pregeometry.

(1) Find a natural and meaningful dividing line between "simple" pregeometries and "complex" pregeometries among those which occur as the pregeometry on a strongly minimal set.
(2) Prove that whenever the pregeometry on a strongly minimal set is simple, the Morley rank dependence relation on tuples is also simple in a meaningful way.

In order to formulate the properties which will meet these requirements we need the notion of M^{eq} (M a model), which is developed in the next section. In the expansion M^{eq} we have not only the elements of M but elements which act as names for the definable relations in M. This expansion is used in most of model theory today.

4.1 T^{eq}

Most of the theorems we have proved so far make few distinctions between tuples from a model and elements of the model, the standard hypothesis being: "Let $\bar{a}$ be a tuple from M and ...". It is a slight deficiency in our notion of a model that we cannot use more uniform terminology for elements of M and tuples from M. Another annoyance is the nonuniqueness of the parameters involved in defining sets. As we look more deeply at the relationships between the definable subsets of a model a natural question (in a t.t. theory) might be: Given a definable set $D = \varphi(\mathfrak{C}, \bar{a})$, what is the Morley rank of the type of the parameters used to define D? This is an ambiguous question since there may be $\bar{a}$ and $\bar{b}$ with $D = \varphi(M, \bar{a}) = \varphi(M, \bar{b})$ and $MR(\bar{a}) \neq MR(\bar{b})$. Both of these deficiencies are removed by expanding the model M to M^{eq}. Shelah calls the additional elements "imaginary elements", in analogy to numbers which we add to the reals to form the complex numbers. As with the complex numbers, the most efficient proof of a theorem about the real elements of a theory may involve imaginaries. This expansion is formulated using many-sorted logic.

It is common in mathematics for the universe of a structure to consist of several disjoint classes of elements. A simple example is a projective plane $\mathcal{P}$ which consists of a set P of "points", a set L of "lines" and a binary incidence relation ε between points and lines. In expressing properties of these planes variables are restricted to ranging over either points or lines. Adopt the convention that p, p', ... denote arbitrary points and l, l', ... denote lines. Then, one of the axioms for a projective plane can be stated as: $\forall p \forall p' \exists l (p \varepsilon l \wedge p' \varepsilon l)$. To formulate this plane as a model of a first-order language we would add unary predicates P_0 and L_0 and let the universe be the disjoint union of the interpretations of these two. In any useful formula involving the variable v we would have an occurrence of $P_0(v)$ or $L_0(v)$. The following approach offers a more natural formalization.

Let I be a nonempty set whose elements are called *sorts.* The logical symbols of *$I-$sorted logic* are the same as first-order logic, except that for each sort i there are variables v_1^i, v_2^i, ... of sort i (and each variable is tagged with a sort). An *$I-$sorted language* L consists of predicate, constant and function symbols. For each $n-$ary predicate symbol P there is an $n-$tuple of sorts $(i_1, \ldots, i_n)$ and P is said to be a predicate of sort $(i_1, \ldots, i_n)$. Similarly, a constant symbol is of a particular sort and the arguments of a function symbol have specified sorts. We leave it to the reader to define the terms and formulas of L. (For example, if P is a predicate symbol of sort $(i_1, \ldots, i_n)$ and $x_1, \ldots, x_n$ are variables of sorts $i_1, \ldots, i_n$, respectively, then $Px_1 \ldots x_n$ is an atomic formula.)

An $I-$sorted structure $\mathcal{M}$ consists of the following.

1. For each $i \in I$ there is a nonempty set M_i called the universe of sort I.
2. For each predicate symbol P of sort $(i_1, \ldots, i_n)$ there is a relation $P^{\mathcal{M}} \subset M_{i_1} \times \ldots \times M_{i_n}$.

3. For a constant symbol c of sort i there is an element $c^{\mathcal{M}}$ of M_i.
4. For each function symbol f of sort ... (the obvious clause).

The definitions of truth and satisfaction are the predictable ones, given that $\forall v^i$ means "for all elements of M_i." The submodel and elementary submodel relations are defined much like the 1−sorted versions, as are elementary maps and isomorphisms. (Ordinary first-order logic as described in Chapter 1 is called 1−sorted logic.)

Let T be a complete I−sorted theory. Given $\sigma = (i_1, \ldots, i_n)$ a sequence of sorts, $S_\sigma(\emptyset)$ denotes the set of complete types in a sequence of variables of sorts $i_1, \ldots, i_n$. In situations where we used $S_n(\emptyset)$ in a 1-sorted theory we will use $S_\sigma(\emptyset)$ in an I−sorted theory. $S(\emptyset)$ denotes $\bigcup_\sigma S_\sigma(\emptyset)$.

So far, we have only stated definitions and theorems for 1-sorted logic. However, everything we have done extends trivially to many-sorted logics. For example, the term categorical in λ is defined by exactly the same statement. We chose to work in 1-sorted logic only to simplify the notation. We will, however, freely apply past results to many-sorted theories and models.

It is possible to transform a many-sorted structure into an ordinary one-sorted structure much as we did above for projective planes. The reader is referred to [End72] for the details.

For L a language and T a theory in L, L^{eq} and T^{eq} are defined as follows. As before, we assume for notational simplicity alone that L and T are 1-sorted. Let $\mathcal{E}$ be the set of all formulas $E(\bar{x}, \bar{y})$ such that for some n and every model M of T, E defines an equivalence relation on M^n. Let $I = \{ i_E : E \in \mathcal{E} \}$ be a collection of (distinct) sorts. For each $E \in \mathcal{E}$ let f_E be a function symbol taking n−tuples from the sort $i_=$ into the sort i_E. Finally, let L^{eq} be the I−sorted language which contains $\{ f_E : E \in \mathcal{E} \}$ and for each element of L a corresponding element whose arguments are required to range over the sort $i_=$. (For example, if P is an n−ary relation symbol of L then L^{eq} contains a relation symbol P of sort $(i_=, \ldots, i_=)$, where there are n copies of $i_=$.) The axioms for T^{eq} are the axioms for T restricted to the sort $i_=$, together with all statements expressing: f_E is a surjective map of n−tuples from $i_=$ onto i_E such that $\forall \bar{x}\bar{y}(E(\bar{x}, \bar{y}) \longleftrightarrow f_E(\bar{x}) = f_E(\bar{y}))$. From hereon we will identify T with its copy on $i_=$ in T^{eq}.

Statements made in T^{eq} can always be reduced to statements in T. This is made precise in the following lemma, which is proved by induction on formulas (left to the reader).

Lemma 4.1.1. *For any formula $\varphi(v_0, \ldots, v_n)$ of L^{eq}, with v_j a variable of sort i_{E_j}, there is a formula $\varphi^*(\bar{w}_0, \ldots, \bar{w}_n)$ of L such that*

$$T^{eq} \models \forall \bar{w}_0 \ldots \bar{w}_n (\varphi(f_{E_0}(\bar{w}_0), \ldots, f_{E_n}(\bar{w}_n)) \longleftrightarrow \varphi^*(\bar{w}_0, \ldots, \bar{w}_n)).$$

Let T be a complete theory in L with universal domain $\mathfrak{C}$. Let $\mathfrak{C}^{eq}$ be an expansion of $\mathfrak{C}$ to a model of T^{eq}. (For E a formula defining an equivalence relation on $n-$tuples let $(\mathfrak{C}^{eq})_{i_E} = \mathfrak{C}^n/E(\mathfrak{C})$ = the $E(\mathfrak{C})-$equivalence classes on $\mathfrak{C}^n$, and let f_E be the quotient map.) Notice that $\mathfrak{C}^{eq}$ is obtained from $\mathfrak{C}$ simply by closing under the functions of the language L^{eq}. This observation makes it clear that $\mathfrak{C}^{eq}$ is the unique model N of T^{eq} with $\mathfrak{C} = N_{i_=}$. Furthermore, an automorphism f of $\mathfrak{C}$ can be extended uniquely to an automorphism of $\mathfrak{C}^{eq}$. Given $A \subset \mathfrak{C}$ let A^{eq} denote the closure of A under the maps f_E, $E \in \mathcal{E}$.

Corollary 4.1.1. *Let T be a complete theory in L with universal domain $\mathfrak{C}$.*

(i) T^{eq} is complete.

(ii) Any relation on $\mathfrak{C}$ definable in $\mathfrak{C}^{eq}$ is definable in $\mathfrak{C}$.

(iii) $\mathfrak{C}^{eq}$ is a saturated model of T^{eq}.

(iv) T^{eq} is $\lambda-$stable if and only if T is $\lambda-$stable (for all $\lambda \geq |T|$).

(v) T^{eq} is t.t. if and only if T is t.t. Also, for φ a formula of T, $MR(\varphi)$, computed in T, is the same as $MR(\varphi)$, computed in T^{eq}.

Proof. (i) and (ii) follow immediately from Lemma 4.1.1.

(iii) To see that $\mathfrak{C}^{eq}$ is a saturated model let $A \subset \mathfrak{C}^{eq}$ have cardinality $< |\mathfrak{C}^{eq}|$ and $p \in S_1(A)$. Let B be a subset of $\mathfrak{C}$ of cardinality $\leq |A|+|T| < |\mathfrak{C}^{eq}|$ such that $A \subset B^{eq}$. Supposing that the variable in p ranges over the sort i_E there is a type q over B (in L) such that if $\bar{b}$ realizes q then $f_E(\bar{b})$ realizes p. Since $\mathfrak{C}$ is saturated, q (hence p) is realized in $\mathfrak{C}^{eq}$.

(iv) follows from much that same argument used to prove (iii). (v) is left to the reader.

Thus, we can use $\mathfrak{C}^{eq}$ as the universal domain of T^{eq}.

Not only does this expansion to $\mathfrak{C}^{eq}$ not add any new structure to $\mathfrak{C}$, but there is a one-to-one correspondence between the elementary submodels of the two models. The reader can verify that if M is an elementary submodel of $\mathfrak{C}$ then M^{eq} is an elementary submodel of $\mathfrak{C}^{eq}$. Conversely, if $N \prec \mathfrak{C}^{eq}$ then $N = M^{eq}$ for some elementary submodel M of $\mathfrak{C}$.

Definition 4.1.1. *Let T be a complete theory, possibly many-sorted, with universal domain $\mathfrak{C}$.*

(i) If D is a definable set in $\mathfrak{C}^n$ (for some n), d is called a name *for D if $f(D) = D \iff f(d) = d$, for all $f \in \mathrm{Aut}(\mathfrak{C})$.*

(ii) If every definable set has a name in $\mathfrak{C}$, we say that T has built-in imaginary elements.

Proposition 4.1.1. *Given a complete theory T, T^{eq} has built-in imaginaries.*

Proof. Let $\mathfrak{C}$ be the universal domain of T and $D = \varphi(\mathfrak{C}, \bar{a})$, where $\varphi(\bar{x}, \bar{y})$ is a formula of L. Let $E(\bar{y}, \bar{y}')$ be the equivalence relation: $E(\bar{y}, \bar{y}') \iff \forall \bar{x}(\varphi(\bar{x}, \bar{y}) \leftrightarrow \varphi(\bar{x}, \bar{y}'))$. Then, for all $\bar{b}$ and $\bar{c}$, $\models E(\bar{a}, \bar{b}) \iff \varphi(\mathfrak{C}, \bar{b}) = \varphi(\mathfrak{C}, \bar{c})$. An automorphism of $\mathfrak{C}^{eq}$ permutes the set D if and only if it fixes $\bar{a}/E$. Thus, T^{eq} has a name for every definable set in $\mathfrak{C}$. We leave it to the reader to show that if D is a definable subset of $(\mathfrak{C}^{eq})^n$, for some n, then there is also a name for D in $\mathfrak{C}^{eq}$. This proves the proposition.

This is the fundamental property of T^{eq} arising in most applications. Instead of "T has built-in imaginaries" we may say T *has imaginaries* or T *has elimination of imaginaries.* By the proposition, T^{eq} has imaginaries, for any complete theory T. However, we use this term even when the theory is not T^{eq} for some other theory. For example, when k is an algebraically closed field and we restrict k^{eq} to the structure whose sorts are the sets k^n, $n < \omega$, we obtain a theory with elimination of imaginaries. (This was proved by Poizat in [Poi83b]; see also [Hod93, 4.4.6].) Informally, the passage from $\mathfrak{C}$ to $\mathfrak{C}^{eq}$ is described as "adding names for definable sets".

Definition 4.1.2. *Let T be a complete theory with universal domain $\mathfrak{C}$. For A a set the* definable closure of A, *denoted $dcl(A)$ is* $\{ a :$ for all b, $tp(b/A) = tp(a/A) \implies a = b \}$. *Sets B and C are* interdefinable over A *if $dcl(B \cup A) = dcl(C \cup A)$.*

Of course, $A \subset dcl(A) \subset acl(A)$. Note: $a \in dcl(A)$ if and only if there is a formula $\varphi(v)$ over A such that $\models \exists! v \varphi(v)$ and $\models \varphi(a)$.

Recall that a formula φ is almost over A if it has finitely many conjugates over A, up to equivalence. Thus, if φ is almost over A in $\mathfrak{C}^{eq}$ there are finitely many elements which are the names for the conjugates over A of $\varphi(\mathfrak{C})$. Continuing with this observation yields

Lemma 4.1.2. *Suppose that T has built-in imaginary elements.*

(i) d is a name for the definable set D if and only if D is definable over d and $d \in dcl(A)$ for any set A such that D is definable over A.

(ii) A formula φ is almost over a set A if and only if $\varphi(\mathfrak{C})$ is definable over $acl(A)$.

Proof. (i) Let d be a name for D. By Lemma 3.3.8(i), D is definable over d. Suppose that D is definable over A, and f is an automorphism of $\mathfrak{C}$ fixing A. Then $f(D) = D$, so $f(d) = d$, from which it follows that $d \in dcl(A)$. To prove the converse, let e be a name for D. Since D is definable over e, $d \in dcl(e)$. By the first part of the proof, $e \in dcl(d)$; i.e., $dcl(d) = dcl(e)$. Thus, d is a name for D.

(ii) Suppose that $\varphi(\mathfrak{C})$ is definable over $\bar{a} \subset acl(A)$. Since there are only finitely many possible images of $\bar{a}$ under automorphisms that fix A, there are only finitely many conjugates of φ over A.

Conversely, suppose that φ is almost over A and a is a name for $\varphi(\mathfrak{C})$. If f is an automorphism of $\mathfrak{C}$, $f(a)$ is a name for $f(\varphi(\mathfrak{C}))$. Thus, $\{ f(a) : f \in \mathrm{Aut}(\mathfrak{C})$ fixes $A \}$ is finite, implying that $a \in acl(A)$.

This lemma is one indication of how working in $\mathfrak{C}^{eq}$ smooths out certain arguments. Intuitively, the parameters defining a formula which is almost over A are closely tied to A. However, to make this precise in the original theory we needed to introduce an equivalence relation over A having finitely many classes, using this to show, e.g., that when φ is almost over A and M is a model $\supset A$ there is a formula ψ over M equivalent to φ. If we work in $\mathfrak{C}^{eq}$ we simply observe that every model containing A also contains $acl(A)$, from which it is clear that a formula almost over A is equivalent to a formula over any model containing A.

When working in $\mathfrak{C}^{eq}$ we can also replace finite tuples by elements in most settings without changing the validity of an argument. For $\bar{a}$ a finite sequence let b be a name for $\bar{a}$ as a definable set over $\bar{a}$. Then, $dcl(\bar{a}) = dcl(b)$. Proving a property about a definable relation satisfied by $\bar{a}$ quickly reduces to proving a similar property about a formula satisfied by b. Along the same lines, proving a property of the definable subsets of $\mathfrak{C}^{eq}$ implicitly proves the same property for the definable relations on $\mathfrak{C}^{eq}$. In settings where we would have said "Given a tuple $\bar{a}$ from $\mathfrak{C}^{eq}$ ..." we will say "Given a in $\mathfrak{C}^{eq}$ ..." Other advantages of working in $\mathfrak{C}^{eq}$ will be uncovered in later applications.

From hereon, unless stated otherwise, we restrict our attention to theories with built-in imaginaries.

The term "T is a theory" will mean "T is a theory with built-in imaginary elements". Since T^{eq} of any theory has built-in imaginaries any theory appears as a sort in a theory with built-in imaginaries. If we want to know what a certain theorem says about an ordinary 1−sorted theory, when it is proved for theories with built-in imaginaries, we need only read off what the result says about a particular sort of the theory. In jargon this assumption is known as "working in T^{eq}" or "working in $\mathfrak{C}^{eq}$". The only time we may abandon this convention is when we are analyzing a natural example, such as a module or one of the theories built on equivalence relations. Then we may become sloppy and say, e.g., "Let T be the the theory of $(\mathbb{Z}, +)$." Even in this setting, where "element" means an element of $\mathfrak{C}$, we assume the elements of $\mathfrak{C}^{eq}$ are available in proofs.

This passage from ordinary theories into theories with imaginaries has the following effect on our standard examples. Suppose that $T = T_0^{eq}$, where T_0 is a theory of equivalence relations. Then for E one of the equivalence relations and a an element (of the right sort), $f_E(a)$ is an element of the universe.

Now suppose $T = T_0^{eq}$, where T_0 is a theory of (infinite) vector spaces, and V denotes the universal domain of T_0. Let W be a linear (hence definable) subspace of V^n. There is a sort of $\mathfrak{C}$ consisting of V^n/W. Since dimension and Morley rank are the same in a vector space we can write the expected identity, $MR(W) + MR(V^n/W) = MR(V^n)$, for these definable sets. In general, for G any group, G^{eq} contains the quotient of G^n by any definable normal subgroup. For example, when $G = \mathrm{GL}_n(K)$ (where K is some field) $\mathrm{PGL}_n(K)$, which is the quotient of G by its center, is definable in G^{eq}.

Historical Notes. All of this is by Shelah [She90], although T^{eq} was first treated as a many-sorted theory (in writing) by Makkai [Mak84].

4.1.1 Totally Transcendental Theories Revisited

In this subsection totally transcendental theories are studied further under the built-in imaginaries hypothesis. Previous results are restated to set the current viewpoint and to emphasize items particularly relevant to his chapter. Also, the proof of Theorem 3.3.1(i) is completed and a new tool (the canonical parameter) is introduced.

The first lemma is little more than a combination of previous results stated under the built-in imaginaries requirement.

Lemma 4.1.3. *Let $\mathfrak{C}$ be the universal domain of a t.t. theory, a an element and A a set. Then,*

(i) $tp(a/acl(A))$ is stationary.

(ii) Moreover, there is an $e \in dcl(A \cup \{a\}) \cap acl(A)$ such that $\deg(a/A \cup \{e\}) = 1$.

Proof. (i) Let $p^* \in S(\mathfrak{C})$ be a free extension of $tp(a/acl(A))$. By Theorem 3.3.1(ii), there is a defining scheme for p^* consisting of formulas almost over A. Any formula almost over A is equivalent to a formula over $acl(A)$, by Lemma 4.1.2(ii). Thus, p^* is definable over $acl(A)$. We conclude from Theorem 3.3.1(i) that $tp(a/acl(A))$ is stationary.

(ii) By Exercise 4.1.5, $tp(a/acl(A))$ is implied by $tp(a/dcl(A \cup \{a\}) \cap acl(A))$. Since the theory is t.t. there is a finite $B \subset dcl(A \cup \{a\}) \cap acl(A)$ such that $1 = \deg(a/dcl(A \cup \{a\}) \cap acl(A)) = \deg(a/B)$. In other words there is an $e \in dcl(A \cup \{a\}) \cap acl(A)$ such that $\deg(a/A \cup \{e\}) = 1$.

Lemma 4.1.4. *Let T be t.t., $p \in S(\mathfrak{C})$ and A a set. If p does not split over A then p is a free extension of $p \restriction A$ and $p \restriction A$ is stationary.*

Proof. Let $B = acl(A)$. By Lemmas 3.3.2(iii), $p \restriction B$ is a free extension of $p \restriction A$. Hence, to show that p is a free extension of $p \restriction A$ it suffices to show that p is a free extension of $p \restriction B$. Suppose, to the contrary, that for some b, $p \restriction (B \cup \{b\})$ is not a free extension of $p \restriction B$. Let $r = tp(b/B)$, which is stationary by the previous lemma, and let I be an infinite Morley sequence in r over B. Let a realize $p \restriction (B \cup I)$. Let J be a finite subset of I such that a is independent from I over $B \cup J$ and let $c \in I \setminus J$. Then c is independent from a over $B \cup J$, in fact, c is independent from a over B (by the transitivity of independence). Since p does not split over B, $tp(a/B \cup \{c\}) = p \restriction (B \cup \{c\})$ is conjugate to $p \restriction (B \cup \{b\})$ over B. Thus, a depends on c over B, a contradiction which proves that p is a free extension of $p \restriction A$.

Turning to the stationarity of $p \restriction A$, observe that $p \restriction A$ has a unique extension over B (since p does not split over A). Hence, if $q \in S(\mathfrak{C})$ is a free extension of $p \restriction A$, $q \supset p \restriction B$. Since $p \restriction B$ is stationary, q must be p. In other words, $p \restriction A$ is stationary, proving the lemma.

This completes the proof of Theorem 3.3.1(i).

In this chapter it is more natural to work with sets of realizations of types than types; i.e., $\bigwedge$ −definable sets (see Definition 3.5.10).

It is worth restating some previously defined notions in an equivalent form involving definable sets. For any sets A and B, $A \vartriangle B$ denotes the symmetric difference of A and B.

Let $\mathfrak{C}$ be the universal domain of a totally transcendental theory. Let D be an $\bigwedge$ −definable set, specifically, $D = p(\mathfrak{C})$.

- $MR(D)$ and $\deg(D)$, the Morley rank and degree of D, are defined to be $MR(p)$ and $\deg(p)$, respectively.

Now suppose D to be the definable set $\varphi(\mathfrak{C})$.

- D is called a *strongly minimal set* if φ is strongly minimal.
- D is a strongly minimal set if and only if every definable subset of D is finite or cofinite.
- D has Morley rank $\geq \alpha$ if for all $\beta < \alpha$ there are definable subsets X_i of D, for $i < \omega$, such that (a) $MR(X_i) \geq \beta$ and (b) $MR(X_i \cap X_j) < \beta$, for $i < j < \omega$.
- If D has Morley rank α, then the degree of D is the maximal k such that there are definable subsets $X_1, \ldots, X_k$ of D satisfying (a) $MR(X_i) = \alpha$, for $i = 1, \ldots, k$, and (b) $MR(X_i \cap X_j) < \beta$, for $1 \leq i < j \leq k$.

Let D be $\bigwedge$ −definable over A and have Morley rank α. There may be elements of D which belong to A−definable sets of Morley rank $< \alpha$. For example, some elements of the universal domain of algebraically closed fields of characteristic 0 are in $acl(\emptyset)$, namely the algebraic elements. Motivated by the terminology used in algebraic geometry we attach the label "generic" to the elements of D having maximal Morley rank over A.

Definition 4.1.3. *Let $\mathfrak{C}$ be the universal domain of a totally transcendental theory, D a subset which is $\bigwedge$ −definable over A, $B \supset A$ and $a \in D$. We call a* generic over B *if $MR(a/B) = MR(D)$; otherwise a is* nongeneric over B.

Remark 4.1.1. If G is an ω−stable group, A a set and $a \in G$, then a is generic over A in the sense of Definition 3.5.6 if and only if a is generic over A in the sense of Definition 4.1.3.

For example, if D is an $\emptyset$−definable strongly minimal set, $a \in D$ is generic over B if and only if $a \notin acl(B)$. For any $\bigwedge$ −definable set D and set B, D contains an element generic over B (because every type in a t.t. theory has a free extension). Intuitively, "most" of the elements of D are generic over any set B. In fact, if X and Y are definable over B and $X \vartriangle Y$ contains only elements nongeneric over B, then X and Y are "almost equal". The "almost equal" relation between sets is explicitly defined as follows.

Definition 4.1.4. *Let $\mathfrak{C}$ be the universal domain of a totally transcendental theory, X an $\bigwedge$-definable set over A, Y an $\bigwedge$-definable set over B and $\alpha = \max\{MR(X), MR(Y)\}$. We write $X \sim^* Y$ if for all $a \in X \vartriangle Y$, $MR(a/A \cup B) < \alpha$. The restriction of $\sim^*$ to sets of degree 1 is denoted $\sim$. That is, if X, Y, A and B are as above and, additionally,* $\deg(X) = \deg(Y) = 1$, *$X \sim Y$ if for all $a \in X \vartriangle Y$, $MR(a/A \cup B) < \alpha$.*

Remark 4.1.2. The detailed verifications of the following are left to the reader. Let X, Y, A and B be as in the definition of $\sim^*$.

(i) If $X \sim^* Y$, then $MR(X) = MR(Y)$. (Suppose, to the contrary, that $MR(X) < MR(Y) = \alpha$. Then any element of Y generic over $A \cup B$ is in $X \vartriangle Y$; i.e., $MR(Y \setminus X) = \alpha$; contradiction.)

(ii) Suppose that $MR(X) = MR(Y) = \alpha$. The domains A and B play no active role in the definition. That is to say, for sets $A' \supset A$ and $B' \supset B$, $X \sim^* Y$ (over A and B) if and only if $X \sim^* Y$ (over A' and B'). (There is an $a \in X \vartriangle Y$ such that $MR(a/A \cup B) = \alpha$ if and only if there is an $a \in X \vartriangle Y$ such that $MR(a/A' \cup B') = \alpha$.)

(iii) If $X = p(\mathfrak{C})$ and $Y = q(\mathfrak{C})$ both have degree 1, then $X \sim Y$ if and only if p and q have the same free extension in $S(\mathfrak{C})$. (This follows quickly from (ii).)

(iv) $\sim$ is an equivalence relation on the $\bigwedge$-definable sets of degree 1.

Since $\mathfrak{C}$ has built-in imaginaries, the quotient set of any definable equivalence relation is a definable subset of $\mathfrak{C}$. This property was used to show that every definable set X in $\mathfrak{C}$ has a name in $\mathfrak{C}$; i.e., an element x such that for all $f \in \mathrm{Aut}(\mathfrak{C})$, $f(X) = X$ if and only if $f(x) = x$. While $\sim$ is not a definable equivalence relation we will show that for each $\sim$-class, $\mathfrak{C}$ contains an element that acts like a "name" for the class (formalized as follows).

Definition 4.1.5. *Let $\mathfrak{C}$ be the universal domain of a t.t. theory and let X be an $\bigwedge$-definable set of degree 1. An element $c \in \mathfrak{C}$ is a* canonical parameter *of X if*

$$\forall f \in \mathrm{Aut}(\mathfrak{C}),\ f(X) \sim X \text{ if and only if } f(c) = c.$$

If $X = p(\mathfrak{C})$ a canonical parameter of X is also called a canonical parameter of p.

Remark 4.1.3. (i) By Remark 4.1.2(iii), degree 1 sets $X = p(\mathfrak{C})$ and $Y = q(\mathfrak{C})$ are $\sim$-equivalent if and only if p and q have the same free extension in $S(\mathfrak{C})$. Thus, for $f \in \mathrm{Aut}(\mathfrak{C})$ and p^* the free extension of p in $S(\mathfrak{C})$, $f(X) \sim X$ if and only if $f(p^*) = p^*$. So, a canonical parameter of X is an element c such that

$$\forall f \in \mathrm{Aut}(\mathfrak{C}),\ f(p^*) = p^* \text{ if and only if } f(c) = c. \tag{4.1}$$

This equivalence is the key to the proof of the next theorem.

(ii) If X is a degree 1 set and c and d are both canonical parameters of X, then $dcl(c) = dcl(d)$. (If $f \in \mathrm{Aut}(\mathfrak{C})$ and $f(c) = c$, then $f(X) \sim X$ and $f(d) = d$. Thus, $d \in dcl(c)$. Similarly, $c \in dcl(d)$.)

(iii) While a degree 1 set will not have a unique canonical parameter, by virtue of (ii), any two such are interdefinable over $\emptyset$. Thus, it is common to say *the* canonical parameter instead of *a* canonical parameter.

Theorem 4.1.1. *Let $\mathfrak{C}$ be the universal domain of a t.t. theory and X an $\bigwedge$-definable set of degree 1. Then X has a canonical parameter.*

Proof. Suppose X is $p(\mathfrak{C})$ and let p^* be the free extension of p in $S(\mathfrak{C})$.

Claim. A canonical parameter for X exists if there is a formula ψ such that

$(*)$ $\forall f \in \mathrm{Aut}(\mathfrak{C})$, $f(p^*) = p^*$ if and only if $f(\psi) = \psi$.

By the previous remark an element c is a canonical parameter for X if it satisfies (4.1). If ψ satisfies $(*)$, a name c for $\psi(\mathfrak{C})$ satisfies (4.1), proving the claim.

The definability of types is the key to finding such a ψ. Let $\varphi(x, a)$ be a formula in p with $MR(\varphi(x, a)) = MR(p) = \alpha$ and $\deg(\varphi(x, a)) = 1$, where $\varphi(x, y)$ is over $\emptyset$. By the definability of types in t.t. theories (Theorem 3.3.1) there is a formula $\psi(y)$ such that for all $b \in \mathfrak{C}$, $\varphi(x, b) \in p$ if and only if $\models \psi(b)$.

Claim. For all $f \in \mathrm{Aut}(\mathfrak{C})$, $f(p^*) = p^*$ if and only if $f(\psi) = \psi$.

Let $f \in \mathrm{Aut}$ $(\mathfrak{C})$. First suppose that $f(p^*) = p^*$. Then

$$\begin{aligned} f(\psi(\mathfrak{C})) &= f(\{\, b : \varphi(x, b) \in p^* \,\}) \\ &= \{\, b : \varphi(x, b) \in f(p^*) \,\} \\ &= \{\, b : \varphi(x, b) \in p^* \,\} = \psi(\mathfrak{C}). \end{aligned}$$

On the other hand, if $f(\psi(\mathfrak{C})) = \psi(\mathfrak{C})$, then $\varphi(x, f(a))$ is in p^* as well as in $f(p^*)$. Since $MR(p^*) = MR(\varphi(x, f(a))) = MR(f(p^*))$ and $\deg(\varphi(x, f(a))) = 1$, $f(p^*)$ must be p^*. This proves the claim and the theorem.

Corollary 4.1.2. *Let $\mathfrak{C}$ be the universal domain of a t.t. theory, $X = p(\mathfrak{C})$ a set of degree 1, p^* the free extension of p in $S(\mathfrak{C})$ and c the canonical parameter of X.*

(i) If p^* is definable over A, then $c \in dcl(A)$.

(ii) If p is over A, then $c \in dcl(A)$.

Proof. (i) Since p^* is definable over A, $f(p^*) = p^*$ for any $f \in \mathrm{Aut}(\mathfrak{C})$ which fixes A pointwise (by Theorem 3.3.1 and Lemma 3.1.8). Hence, $f(c) = c$ for any $f \in \mathrm{Aut}(\mathfrak{C})$ which fixes A pointwise. We conclude that $c \in dcl(A)$.

(ii) Since p has degree 1, $p^* \restriction A$ has degree 1. Thus, p^* is definable over A and we conclude from (i) that $c \in dcl(A)$.

Corollary 4.1.3. *Let $\mathfrak{C}$ be the universal domain of a t.t. theory, $X = r(\mathfrak{C})$ an $\bigwedge$-definable set of degree 1, $p^* \in S(\mathfrak{C})$ the free extension of r and c a canonical parameter of X.*

(i) If Y is an $\bigwedge$-definable set of degree 1 and $Y \sim X$, then c is a canonical parameter of Y.

(ii) p^* is definable over c.

(iii) There is a degree 1 formula $\varphi(v,c)$ over c such that p^* is the unique free extension of $\varphi(x,c)$. Moreover, if q is any type over c of degree 1 having p^* as a free extension, then for all $f \in \mathrm{Aut}(\mathfrak{C})$, $f(q(\mathfrak{C})) \sim q(\mathfrak{C})$ if and only if $f(q(\mathfrak{C})) = q(\mathfrak{C})$.

Proof. (i) Let $Y = r'(\mathfrak{C})$ and $f \in \mathrm{Aut}(\mathfrak{C})$. Since $Y \sim X$, p^* is the free extension of r' in $S(\mathfrak{C})$. Thus, $Y \sim f(Y)$ if and only if $p^* = f(p^*)$. Since $p^* = f(p^*)$ if and only if $c = f(c)$, c is a canonical parameter of Y.

(ii) Let $\psi(x,y)$ be a formula over $\emptyset$ and let $\theta(y)$ be a formula such that $\psi(x,a) \in p^* \iff \models \theta(a)$. If $f \in \mathrm{Aut}(\mathfrak{C})$ fixes c, $f(p^*) = p^*$, hence $f(\theta(\mathfrak{C})) = \theta(\mathfrak{C})$. In other words, θ is invariant under the automorphisms of $\mathfrak{C}$ which fix c. By Lemma 3.3.8(i), θ is equivalent to a formula over c.

(iii) Since p^* is definable over c, p^* is the unique free extension of $p^* \restriction c$. Hence, there is a formula $\varphi(v,c) \in p^* \restriction c$ with $MR(\varphi(v,c)) = MR(p^*)$ and $\deg(\varphi(v,c)) = 1$. Now let q be any type over c of degree 1 such that p^* is a free extension of q. For any $f \in \mathrm{Aut}(\mathfrak{C})$,

$$\begin{aligned} f(q(\mathfrak{C})) \sim q(\mathfrak{C}) &\iff f(p^*) = p^* \\ &\iff f(c) = c \\ &\iff f(q(\mathfrak{C})) = q(\mathfrak{C}), \end{aligned}$$

completing the proof.

Remark 4.1.4. Let X be an $\bigwedge$-definable set of degree 1 and c the canonical parameter of X. There is (over c) a definable Y of degree 1 such that $Y \sim X$ and for all $f \in \mathrm{Aut}(\mathfrak{C})$, $f(Y) \sim Y$ if and only if $f(Y) = Y$. In this way the set Y acts as a canonical representative for its $\sim$-class.

The next result only ties together numerous previous results to give easily referenced tools for later use.

Lemma 4.1.5. *Let $\mathfrak{C}$ be the universal domain of a t.t. theory, a an element and A a set.*

(i) Let B be an algebraically closed set containing A. Then, a is independent from B over A if and only if the canonical parameter c of $tp(a/B)$ is in $acl(A)$.

(ii) Let $p = tp(a/acl(A))$ and c the canonical parameter of p. Then, there is a Morley sequence I in p such that $c \in dcl(I)$.

Proof. (i) First notice that $tp(a/B)$ is stationary, by Lemma 4.1.3, hence it does have a canonical parameter. Let p^* be the unique free extension of $tp(a/B)$ in $S(\mathfrak{C})$. If a is independent from B over A, p^* is a free extension of $p^* \restriction A$, hence $c \in acl(A)$ by Corollary 4.1.2(i).

Conversely, if $c \in acl(A)$ then p^* is definable over $acl(A)$ (since p^* is definable over c). Thus, a is independent from B over A by Theorem 3.3.1.

(ii) Let p^* be the free extension of $tp(a/B)$ in $S(\mathfrak{C})$. By Corollary 3.3.3, there is a Morley sequence I in p such that p^* is definable over I. Thus $c \in dcl(I)$ by Corollary 4.1.2.

Corollary 4.1.4. *Let $\mathfrak{C}$ be the universal domain of a t.t. theory and a, b elements of $\mathfrak{C}$. There is a c such that*

(1) *$c \in acl(a)$,*
(2) *b is independent from a over c,*
(3) *$tp(b/c)$ is stationary, and*
(4) *there is a finite c−independent set $\{b_0, \ldots, b_n\}$ of realizations of $tp(b/c)$ such that $c \in dcl(b_0, \ldots, b_n)$.*

A final word about notation:

Notation. In this chapter we may state a result about $\emptyset$−definable sets in a t.t. theory, instead of A−definable sets for an arbitrary A. However, if $\mathfrak{C}$ is the universal domain of a t.t. theory and A is a finite set then $\mathfrak{C}_A$, the model with constants added to the language for the elements of A, is also t.t. Thus, a statement proved for the $\emptyset$−definable sets in an arbitrary t.t. theory is true of all definable sets in an arbitrary t.t. theory. (Except, of course, statements explicitly mentioning the parameters over which the set is defined.)

4.1.2 D^{eq} for a Strongly Minimal D

In subsequent sections much attention will be given to definable relations on a fixed definable set D and the canonical parameters of degree 1 relations on D, especially when D is strongly minimal. The elements of $\mathfrak{C}^{eq}$ most relevant to D are isolated in

Definition 4.1.6. *Let $\mathfrak{C}$ be the universal domain of a t.t. theory and let D be a set which is $\bigwedge$−definable over A. Then $D^{eq} = \{ x \in \mathfrak{C}^{eq} : x \in dcl(D \cup A) \}$.*

Lemma 4.1.6. *Let D be an A−definable set in the universal domain of a t.t. theory and X a degree 1 definable relation on D. Then the canonical parameter of X is in D^{eq}.*

Proof. By Proposition 3.3.3 there is a $B \subset D$ such that X is definable over $A \cup B$. By Corollary 4.1.2(ii), $c \in dcl(A \cup B)$.

Let D be a strongly minimal set. Recall from Remark 3.1.4 that dimension on D satisfies

(Additivity) For $\bar{a}$ and $\bar{b}$ finite sequences from D,
$$\dim(\bar{a}\bar{b}) = \dim(\bar{a}/\bar{b}) + \dim(\bar{b}).$$

Since $\dim(\bar{a}) = MR(\bar{a})$ when $\bar{a}$ is a finite sequence from D (by Lemma 3.3.4), Morley rank on D satisfies the corresponding additivity condition. In fact, the elements of D^{eq} are tied closely enough to D to prove

Proposition 4.1.2. *Let $\mathfrak{C}$ be the universal domain of a t.t. theory and let D be a strongly minimal set, definable over A. Then for all a, $b \in D^{eq}$*

$$MR(ab/A) = MR(a/\{b\} \cup A) + MR(b/A).$$

Proof. Without loss of generality, $A = \emptyset$. Let $\bar{c}$ and $\bar{d}$ be finite sequences from D such that $a \in dcl(\bar{c})$ and $b \in dcl(\bar{d})$. Let $\bar{c}_0$ be a maximal subsequence of $\bar{c}$ which is independent from a and $\bar{c}_1 = \bar{c} \setminus \bar{c}_0$. By the maximality of $\bar{c}_0$, any $e \in \bar{c}_1$ is in $acl(a, \bar{c}_0)$. Hence, a and $\bar{c}_1$ are interalgebraic over $\bar{c}_0$. By Lemma 3.3.2(ii), $MR(a/\bar{c}_0) = MR(\bar{c}_1/\bar{c}_0)$. The sequence $\bar{c}$ can be chosen so that $\bar{c}_0$ is independent from $\{b, \bar{d}\}$. (Given $\bar{c}_0\bar{c}_1 = \bar{c}$ let $\bar{e}_0$ be a realization of $r = tp(\bar{c}_0/acl(\emptyset))$ independent from $\{a, b, \bar{d}\}$. Since r is stationary, $tp(\bar{e}_0/a) = tp(\bar{c}_0/a)$. Thus there is an $\bar{e}_1$ from D such that $a \in dcl(\bar{e}_0\bar{e}_1)$ and $\bar{e}_1 \subset acl(\bar{e}_0, a)$.) Similarly, for $\bar{d}_0$ a maximal subsequence of $\bar{d}$ which is independent from b and $\bar{d}_1 = \bar{d} \setminus \bar{d}_0$, b is interalgebraic with $\bar{d}_1$ over $\bar{d}_0$ and $MR(b/\bar{d}_0) = MR(\bar{d}_1/\bar{d}_0)$. Without loss of generality, $\bar{d}_0$ is independent from $\{a, \bar{c}, b\}$.

The following sequence of equations shows that $MR(ab) = MR(a/b) + MR(b)$. (The details are left to the reader.)

1. $MR(ab) = MR(ab/\bar{c}_0\bar{d}_0) = MR(\bar{c}_1\bar{d}_1/\bar{c}_0\bar{d}_0)$;
2. $MR(a/b) = MR(a/b\bar{d}_0) = MR(a/b\bar{d}) = MR(a/b\bar{d}\bar{c}_0)$ (since $\bar{c}_0$ is independent from $\{a, b, \bar{d}\}$) and $MR(a/b\bar{d}\bar{c}_0) = MR(\bar{c}_1/b\bar{d}\bar{c}_0) = MR(\bar{c}_1/\bar{d}\bar{c}_0)$;
3. $MR(b) = MR(b/\bar{d}_0\bar{c}_0) = MR(\bar{d}_1/\bar{d}_0\bar{c}_0)$;
4. $MR(\bar{c}_1\bar{d}_1/\bar{c}_0\bar{d}_0) = MR(\bar{c}_1/\bar{d}\bar{c}_0) + MR(\bar{d}_1/\bar{c}_0\bar{d}_0)$.

This proves the proposition.

When working with sequences from a strongly minimal set we prefer $\dim(-)$ over $MR(-)$ to emphasize the additivity property. Because of the previous proposition we can use $\dim(-)$ to denote Morley rank on D^{eq} and keep the property that $\dim(-)$ is additive where it is defined:

Definition 4.1.7. *Let $\mathfrak{C}$ be the universal domain of a t.t. theory and let D be a strongly minimal set. For $\bar{a}$ a finite sequence from D^{eq} we define* $\dim(\bar{a})$ *to be $MR(\bar{a})$.*

Historical Notes. All of this is by Shelah [She90], although T^{eq} was first treated as a many-sorted theory (in writing) by Makkai [Mak84].

Exercise 4.1.1. Prove Lemma 4.1.1.

Exercise 4.1.2. Show that $p \in S(A)$ (in $\mathfrak{C}$) has a unique extension over A^{eq} (in $\mathfrak{C}^{eq}$). Use this observation to show that when T is t.t. and p is a type, $MR(p)$ is the same, whether computed in T or T^{eq}.

Exercise 4.1.3. Prove that T^{eq} is quantifier eliminable whenever T is quantifier eliminable.

Exercise 4.1.4. Suppose that T has built-in imaginaries and A is a finite set. Show that there is an a and a formula $\varphi(x, a)$ such that $b \in A$ if and only if $\models \varphi(b, a)$.

Exercise 4.1.5. Let $\mathfrak{C}$ be the universal domain of a complete theory, A a set, a an element and $A' = acl(A) \cap dcl(A \cup \{a\})$. Show that $tp(a/A')$ implies $tp(a/acl(A))$. (We are working in mt^{eq} here.)

Exercise 4.1.6. Let T be the 1−sorted theory in a language with a single binary relation E saying that E is an equivalence relation with infinitely many infinite classes and no finite classes. Let $\mathfrak{C}/E$ denote the sort in $\mathfrak{C}^{eq}$ consisting of the E−classes of the elements of $\mathfrak{C}$. Prove that $\mathfrak{C}/E$ is strongly minimal. (HINT: Use automorphisms of $\mathfrak{C}^{eq}$.)

4.2 The Pregeometries on Strongly Minimal Sets

In this section we introduce the property, namely local modularity, which divides the "geometrically simple" and "geometrically complex" strongly minimal sets. This property will be defined in the context of arbitrary pregeometries.

Definition 4.2.1. *Let (S, cl) be a pregeometry. The* localization of S at $A \subset S$ *is defined to be the pregeometry (S, cl'), where $cl'(X) = cl(X \cup A)$ for all $X \subset S$. (The reader can verify that S is indeed a pregeometry under cl'.) An* isomorphism *between (S, cl) and another pregeometry (S_0, cl_0) is simply a bijection f from S onto S_0 which respects the closure operators; i.e., $X \subset S$ is closed if and only if $f(X)$ is a closed subset of S_0. As usual, an isomorphism of a pregeometry onto itself is called an automorphism. S is said to be* homogeneous *if for any closed $A \subset S$ and $a, b \in S \setminus A$, there is an automorphism of S which is the identity on A and maps a to b.*

In the exercises the reader is asked to verify that the pregeometry on a strongly minimal set is homogeneous. Now to the more substantive definitions.

Definition 4.2.2. *Let (S, cl) be a pregeometry.*
(i) S is trivial *if for all nonempty $X \subset S$, $cl(X) = \bigcup\{ cl(a) : a \in X \}$.*

(ii) S is modular *if for all closed X, $Y \subset S$,*

$$\dim(X) + \dim(Y) = \dim(X \cup Y) + \dim(X \cap Y) \quad \textit{(Modularity Law).} \quad (4.2)$$

(iii) S is projective *if S is nontrivial and for all a, $b \in S$ and $X \subset S$ such that $a \in c\ell(X \cup \{b\})$, there is a $c \in c\ell(X)$ such that $a \in c\ell(\{b, c\})$.*

(iv) S is locally modular (locally projective) *if for some $a \in S$ the localization of S at $\{a\}$ is modular (projective).*

(v) For any of the properties defined in (i)-(iv), a strongly minimal set $D = \varphi(\mathfrak{C})$ is said to have the property if the pregeometry associated to D has the property. Similarly with strongly minimal formulas and types containing strongly minimal formulas.

Remark 4.2.1. Let $(S, c\ell)$ be a pregeometry.

(i) It is easy to show that S possesses one of the properties defined above if and only if the geometry associated to S also has that property. (See Exercise 4.2.2.) Also, each of the properties is invariant under isomorphism (in the class of pregeometries).

(ii) If S is trivial then S is modular.

(iii) The Modularity Law is equivalent to

Any two closed subsets X and Y of S are independent over $X \cap Y$.

Proof. Without loss of generality, X and Y have finite dimension. By the additivity of dimension (see Exercise 3.1.8), $\dim(X \cup Y) = \dim(X/Y) + \dim(Y)$. X and Y are independent over $X \cap Y$ if and only if $\dim(X/Y) = \dim(X/X \cap Y)$. Thus, X and Y are independent over $X \cap Y$ if and only if

$$\begin{aligned} \dim(X \cup Y) &= \dim(X/X \cap Y) + \dim Y \\ &= \dim X - \dim(X \cap Y) + \dim Y. \end{aligned}$$

Example 4.2.1. (i) Let D be the universal domain of the theory in the empty language with only infinite models. Then D is a trivial strongly minimal set.

(ii) Let F be a division ring, V a vector space over F and $\langle - \rangle$ the linear span operator on V. Then, $S = (V, \langle - \rangle)$ is a modular pregeometry. If V has dimension ≥ 2, S is nontrivial and projective. The geometry associated to V, P, is called a *projective geometry over F*.

A remark about the dimension of P is in order. As a geometry, the dimension of P equals the dimension of V. Strictly in the context of projective geometries over a division ring, however, it is customary to define the dimension of P to be $\dim(V) - 1$ (or ∞, if $\dim(V) = \infty$). For example, a projective plane over $\mathbb{R}$ has dimension 2 as a real projective space, but dimension 3 as a geometry. In this book, $\dim(P)$ will always denote the dimension of P as a geometry (hence $\dim(P) = 3$ when P is a projective plane).

Turning to model-theoretic considerations, formulate V as a structure in the natural language of vector spaces and suppose that it is infinite. Let

$c\ell(-)$ be algebraic closure on V. It was proved earlier that $Th(V)$ is quantifier-eliminable, hence $c\ell(-) = \langle - \rangle$ and $Th(V)$ is strongly minimal. Thus, V (when it's the universal domain of its theory) is a modular and projective strongly minimal set.

(iii) Affine spaces provide examples of locally modular strongly minimal sets which are not modular, however it will take some time to formulate these structures as strongly minimal sets. Remember (from Definition 3.5.2) that a group action (G, X) is called *regular* if for each pair x, $y \in X$ there is a unique $g \in G$ such that $gx = y$. Notice that if X is a coset of the group G in a supergroup of G, then the group operation defines a regular group action of G on X.

Let V be a vector space of dimension ≥ 1 over a division ring F. Following [BM67], an *affine space derived from* V is defined to be a regular group action of V on a set P. For a fixed group G, if G acts regularly on both X and Y, then (G, X) and (G, Y) are isomorphic as group actions. Thus, all affine spaces derived from V are isomorphic. An *affine space over* F is an affine space derived from some vector space over F.

Let W be a vector space over F properly containing V, $a \in W$ and $A = a + V$. As stated above, (V, A) is a regular group action under $+$, hence an affine space derived from V. There are various ways to formulate an affine space as a structure in a first-order language. The most natural formulation is in a two-sorted language L^* with the symbols needed to define a vector space on the first sort and a binary operator $\star$ such that given v_1 in the first sort and v_2 in the second sort, $v_1 \star v_2$ is an element of the second sort. Then, interpreting the first sort by V, the second sort by A and $\star$ by the group action turns (V, A) into a structure for L^*. It is easy to show that the theory of $M = (V, A)$ is quantifier-eliminable in this language. From hereon suppose (V, A) is the universal domain of its theory in L^*.

The relations on V definable in M are simply those definable in the vector space V. For any $x \in A$ there is a bijection between V and A definable over x (see the definition of a regular group action). Thus, A is a strongly minimal set and the localization of A at any element x is isomorphic (as a pregeometry) to the pregeometry on V. Since V is modular we conclude that A is locally modular.

Claim. When $a \notin V$, $A = a + V$ is not modular.

Let $c\ell$ denote algebraic closure restricted to A and (in the proof of the claim) let $\dim(-)$ be dimension in the pregeometry $(A, c\ell)$. Let b be an element of A, x a nonzero element of V and c an element of A which is independent from $\{b, x + b\}$. Let $X = c\ell(b, x + b)$, $Y = c\ell(c, x + c)$ and notice that $\dim(X) = \dim(Y) = 2$ and $\dim(X \cup Y) = 3$. If $\dim(X \cap Y)$ were $2 = \dim(X)$ we would have $X = Y$, contradicting that $\dim(X \cup Y) = 3$. Thus $\dim(X \cap Y) \leq 1$. Since $x \in acl(X) \cap acl(Y)$ any element of $X \cap Y \setminus c\ell(\emptyset)$ is interalgebraic with x. By the elimination of quantifiers in the model (V, A), no element of

V is algebraic in an element of A. Thus, $\dim(X \cap Y) = 0$, proving that the modularity law (4.2) fails for X and Y. This proves the claim.

The (1−sorted) structure A' whose universe is A and whose definable relations are those definable in M is also known as an affine space over F. A natural 1−sorted language in which $Th(A')$ is quantifier-eliminable is specified as follows. Let V, W and A be the objects defined above. We need to find relations on A from which the vector space V and the action of V on A can be recovered. Replace the action of V on A by the ternary operation f:

$$f(x, y, z) = x + y - z, \quad \text{for all } x, y, z \in A.$$

The action of F on V induces a family of binary operators g_α, $\alpha \in F$, on A given by the rule:

$$g_\alpha(x, y) = \alpha x + (1 - \alpha)y, \quad \text{for all } x, y \in A.$$

It is left to the reader to see that A in the language $\{f, g_\alpha\}_{\alpha \in F}$ is quantifier-eliminable and has the same definable relations as A'. Note: The vector space V and its action on A are definable in $(A')^{eq}$. (See Exercise 4.2.3.)

(iv) (A strongly minimal set which is not locally projective.) Let K be the universal domain for the theory of algebraically closed fields of some fixed characteristic. It was noted previously that K is a strongly minimal set on which field-theoretic closure is the same as algebraic closure. To see that K is not even close to being locally projective, let $\{a, c_0, \ldots, c_n\}$ be an algebraically independent set of elements of K and let $b = c_0 a^n + c_{n-1} a^{n-1} + \ldots + c_1 a + c_0$. Not only is there no $d \in acl(c_0, \ldots, c_n) \cap K$ such that $b \in acl(a, d)$, but there is no set $\{d_0, \ldots, d_{n-1}\} \subset acl(c_0, \ldots, c_n) \cap K$ such that $b \in acl(a, d_0, \ldots, d_{n-1})$. In particular, this shows that no localization of K at a finite set is projective.

In the example the only nontrivial modular strongly minimal set (a vector space) is also projective. The next lemma shows that this is no accident.

Lemma 4.2.1. *A pregeometry $(S, c\ell)$ is modular if and only if*

(P) *for all $a, b \in S$ and $X \subset S$ such that $a \in c\ell(X \cup \{b\})$, there is a $c \in c\ell(X)$ such that $a \in c\ell(\{b, c\})$.*

Thus, a pregeometry is projective if and only if it is nontrivial and modular.

Proof. The proof of this lemma is elementary but will serve to familiarize the reader with the definitions. First, suppose that S is modular, $X \subset S$ is closed and $a \in c\ell(X \cup \{b\})$. We need to find a $c \in X$ such that $a \in c\ell(\{b, c\})$. Let $Y = c\ell(\{a, b\})$ and assume, without loss of generality, that both a and b are not in X, hence $\dim(Y/X) = 1$. If $a \in c\ell(\{b\})$ we are done, so we can also assume that $\dim(Y) = 2$. By the Modularity Law on S, $\dim(X \cap Y) = 1$. Let c be an element of $(X \cap Y) \setminus c\ell(\emptyset)$. Since $b \notin X$, $c \notin c\ell(\{b\})$, hence $a \in c\ell(\{b, c\})$ by the exchange property.

Now suppose that S satisfies **(P)**.

Claim. For all closed $X, Y \subset S$, if $c \in cl(X \cup Y)$ then there are $a \in X$ and $b \in Y$ such that $c \in cl(\{a, b\})$.

This is proved by induction on $\dim(X \cup Y)$, which we can assume to be finite. Let $c \in cl(X \cup Y)$. Without loss of generality, there are $a \in Y$ and a closed $Z \subset Y$ such that $a \notin cl(X \cup Z)$, $Y = cl(Z \cup \{a\})$ and $c \notin cl(X \cup Z)$. Since S satisfies **(P)** and $c \in cl(X \cup Z \cup \{a\})$ there is a $b \in cl(X \cup Z)$ such that $c \in cl(\{a, b\})$. The conditions on Z force $\dim(X \cup Z)$ to be less than $\dim(X \cup Y)$, hence the inductive hypothesis yields $d \in X$ and $e \in Z$ such that $b \in cl(\{d, e\})$. Thus, $c \in cl(\{d, a, e\})$. Since a and e are both in the closed set Y the projectivity of S produces an element $f \in Y$ such that $c \in cl(\{d, f\})$. This proves the claim.

Assume, towards a contradiction, that S is not modular and let X and Y be closed subsets of S which are dependent over $X \cap Y = Z$. From this dependence we get a closed Y', $Z \subset Y' \subset Y$ and an $a \in Y$ such that $a \in cl(X \cup Y') \setminus Y'$. By the claim there are $b \in Y'$ and $c \in X$ such that $a \in cl(\{b, c\})$. Since $a \notin Y'$ the exchange property implies that $c \in cl(\{a, b\}) \subset Y$. Thus, $c \in Z \subset Y'$, contradicting that $a \notin Y'$. This proves the lemma.

A natural problem is: Characterize the infinite projective geometries which are (a) isomorphic to a strongly minimal set, or at least (b) isomorphic to the geometry associated to a strongly minimal set. In this introductory section only a fraction of what is known will be stated. The restrictions on the geometries are less stringent in part (b) of the problem, so it is discussed first. In the main example above we showed that any infinite projective geometry over a division ring F is the geometry associated to some model of a strongly minimal theory, namely a vector space over F. The following classical result (see, e.g., [Hal59]) shows the converse to be true (when the dimension of the strongly minimal set is sufficiently large).

Lemma 4.2.2. *Let P be a projective geometry of dimension ≥ 4 in which each closed set of dimension 2 contains at least 3 elements. Then P is isomorphic to projective geometry over some division ring F.*

Let D be a strongly minimal set such that the geometry associated to D is isomorphic to projective geometry P over a division ring F. The geometry P is derived from a vector space V as outlined in Example 4.2.1. It is natural to ask if V is $\emptyset$–definable in D^{eq}, or at least definable in D^{eq} over some set of parameters. This, and similar questions on representing strongly minimal sets using groups, will be investigated throughout this chapter.

A pregeometry (S, cl) is *locally finite* if for all closed $X \subset S$ of finite dimension there is a finite $A \subset X$ such that $X = \bigcup\{ cl(a) : a \in A\}$. (Thus, (S, cl) is locally finite if in the associated geometry the closure of a finite set is finite.)

The strongest classical result about locally projective, locally finite geometries is

Lemma 4.2.3 (Doyen-Hubaut). *Let P be a locally projective, locally finite, geometry of dimension ≥ 4 in which all closed sets of dimension 2 have the same cardinality. Then P is affine or projective geometry over a finite field.*

Let D be a locally projective, locally finite, strongly minimal set and let P be the geometry associated to D. Then all closed sets of dimension 2 in P have the same cardinality because D (hence P) is homogeneous. Thus, P is affine or projective geometry over some finite field. Again, the problem of defining the relevant affine space or vector space in D^{eq} is difficult and will be discussed later.

4.2.1 Plane Curves

From a model-theoretic standpoint a deficiency of the definition of modularity is that it is stated in terms of closed sets, which are potentially undefinable objects. Our next goal is to find an equivalent of modularity which is a property of definable relations and rank instead of closed sets and dimension. This will make it easier to study modularity and local modularity with model-theoretic techniques.

Definition 4.2.3. *Let D be a strongly minimal set, definable over $\emptyset$ in the universal domain $\mathfrak{C}$ of a t.t. theory. A strongly minimal subset of D^2 is called a* plane curve *in D. If C and C' are plane curves in D we write $C \approx C'$, and say that C and C' are* equivalent curves, *if the symmetric difference of C and C' is finite. Slightly abusing the terminology, we identify a $\approx$ −class of plane curves and say that C and C' are the* same *plane curve if $C \approx C'$. A strongly minimal formula φ such that $\varphi(D)$ is a plane curve in D is also called a plane curve in D.*

Remark 4.2.2. Let D be a strongly minimal set as in the definition and C, C' plane curves in D. Then, $C \approx C'$ if and only if, in the notation of Definition 4.1.4, $C \sim C'$. The new notation is introduced only to emphasize that the relation will only be applied to plane curves. By Corollary 4.1.3, C and C' are considered to be the same plane curve if and only if they have the same canonical parameter. Furthermore, there is a plane curve $C_0 \approx C$ which acts as a canonical representative for the $\approx$ −class of C in the sense that, for all $f \in \mathrm{Aut}(\mathfrak{C})$, $f(C_0) \approx C_0$ if and only if $f(C_0) = C_0$. Often we will express the equivalence of C and C' by saying "C equals C' (up to a finite set)."

Example 4.2.2. (i) Let D be a trivial strongly minimal set defined over the empty set in the universal domain of a t.t. theory. Let C be a plane curve in D, defined over $A \subset D$, and let $(a,b) \in C \setminus acl(A)$. Since $tp(ab/A)$ is strongly minimal, $\{a, b, A\}$ cannot be independent, and this set cannot be pairwise independent since D is trivial. First suppose that $a \in acl(A)$. Then, the reader can verify that C equals $\{(c,d) \in D^2 : c = a\}$ (up to a finite set). Similarly, when $b \in acl(A)$. Now suppose that a depends on b; i.e., a and b are interalgebraic and (a,b) is independent from A over $\emptyset$. Thus, there is a strongly minimal set C' which is equal to C up to a finite set and has finitely many conjugates over $\emptyset$, equivalently, the canonical parameter of C is in $acl(\emptyset)$. The reader can verify that these are the only plane curves in D.

(ii) Let V be the universal domain of infinite vector spaces over some division ring F. Let C be a plane curve in V, defined over $A \subset V$. Then, up to a finite set, C is defined by a linear equation $f(x,y) = 0$ of the form $\alpha x + \beta y + \gamma_0 a_0 + \ldots + \gamma_n a_n = 0$, where $a_0, \ldots, a_n \in A$, $\alpha, \beta, \gamma_0, \ldots, \gamma_n \in F$ and the nullity of $f(x,y) = 0$ is 1. The element $b = \gamma_0 a_0 + \ldots + \gamma_n a_n$ is a canonical parameter of C.

Thus, any plane curve C in V is defined (up to a finite set) by an equation of the form $\alpha x + \beta y = b$, where α, $\beta \in F$ and $b \in V$.

(iii) Let K be the universal domain of algebraically closed fields of a fixed characteristic, and let C be a plane curve in K. Then, C is defined (up to a finite set) by an equation of the form $f(x,y) = 0$, where f is an irreducible polynomial over K. (This follows from the elimination of quantifiers and some basic facts about varieties found in, for example, [Har80, I.1.13].) Let $\{c_0, \ldots, c_n\}$ be algebraically independent. The equation $y = c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0$ defines a plane curve C' whose canonical parameter is interdefinable with the set $\{c_0, \ldots, c_n\}$. In particular, for each $n < \omega$ there is a plane curve whose canonical parameter has dimension $= n$.

In the example each plane curve in a modular strongly minimal set is relatively simple in that its canonical parameter has dimension ≤ 1. Strongly minimal sets with this property deserve a special name.

Definition 4.2.4. *Let D be an $A-$definable strongly minimal set in a t.t. theory with universal domain $\mathfrak{C}$. D is called* linear *if for every plane curve C in D the canonical parameter of C has dimension (over A) ≤ 1. If $D = \varphi(\mathfrak{C})$ is linear, φ is also called linear.*

An algebraically closed field K (which is not locally modular) fails to be linear, in fact, for any $k < \omega$ there is a plane curve in K whose canonical parameter has dimension $\geq k$. (See (iii) in the previous example.)

The next lemma not only connects local modularity and linearity but shows that for D a strongly minimal set, if *any* localization of D is modular then D is locally modular.

Lemma 4.2.4. *Let D be a strongly minimal set over A^* in a t.t. theory with universal domain $\mathfrak{C}$. The following are equivalent.*

(1) *D is linear.*
(2) *D is locally modular.*
(3) *For some set $A \supset A^*$, the localization of D at A is modular.*

Proof. Without loss of generality, $A^* = \emptyset$.

(1) $\Longrightarrow$ (2). Assume D to be linear and let e be any element of $D \setminus acl(\emptyset)$. We need to show that the localization of D at e (denoted D_e) is modular. It suffices to show that D_e satisfies **(P)** of Lemma 4.2.1. To this end, let B be a subset of D and a, $b \in D$ such that $a \in acl(B \cup \{b, e\})$. To satisfy **(P)** we need a

($\diamond$) $d \in acl(B \cup \{e\}) \cap D$ such that $a \in acl(b, d, e)$.

If $b \in acl(B \cup \{e\})$ or $a \in acl(b, e)$ we are done. Thus we can assume that $b \notin acl(B \cup \{e\})$ and $\{a, b, e\}$ is independent. Letting $B' = acl(B \cup \{e\})$, $p = tp(ab/B')$ is strongly minimal. Let c be a canonical parameter of p. By the linearity of D, $\dim(c) \leq 1$. The element d satisfying ($\diamond$) is found via

Claim. (i) $a \in acl(b, c)$.

(ii) There is a $d \in D$ such that $acl(c, e) = acl(d, e)$.

From the data: c is the canonical parameter of p, $MR(b/B \cup \{e\}) = 1$ and $MR(ab/B \cup \{e\}) = 1$, we derive $MR(b/c) = MR(ab/c) = 1$, establishing (i). Since $a \in acl(b, c) \setminus acl(b)$, a depends on c over b. Combining this dependence with $\dim(c) \leq 1$ yields: $c \in acl(a, b)$, c is independent from b and c is independent from e. Since D is strongly minimal there is an automorphism f of the universe such that $f(c) = c$ and $f(b) = e$. Setting $d = f(a)$ yields an element meeting the requirements in (ii) and completes the proof of the claim.

Simply because c is the canonical parameter of a free extension of a type over $B \cup \{e\}$, $c \in acl(B \cup \{e\})$. Thus, $d \in acl(B \cup \{e\})$ and $a \in acl(b, d, e)$ (by the claim); i.e., ($\diamond$) holds for this d. This completes the proof that D_e satisfies **(P)**, hence D is locally modular.

(2) $\Longrightarrow$ (3). This case is trivially true.

(3) $\Longrightarrow$ (1). This case is proved in the two steps delineated in

Claim. (i) If the localization of D at some set B is linear, then D is linear.

(ii) A modular strongly minimal set is linear.

Suppose that D is not linear. Then, there are $(a, b) \in D^2$ and c such that $p = tp(ab/c)$ is strongly minimal, c is the canonical parameter of p and $\dim(c) > 1$. By applying an automorphism to (a, b, c) if necessary, we can require B to be independent from (a, b, c). The type $q = tp(ab/B \cup \{c\})$ is simply a free extension of p, hence q is strongly minimal with canonical parameter c. Since $\dim(c/B) > 1$, the localization of D at B is not linear, proving (i).

Turning to (ii) let D_0 be a modular strongly minimal set (in the universal domain of some t.t. theory). Let a, $b \in D_0$ and $C \subset D_0$ such that $p = tp(ab/C)$ is strongly minimal. First suppose $a \in acl(C)$. Then $1 = \dim(ab/C) = \dim(ab/C \cup \{a\}) = \dim(ab/a)$; i.e., the free extension of p over $C \cup \{a\}$ is definable over $acl(a)$. The canonical parameter of p is algebraic in a (by Lemma 4.1.5), hence has dimension ≤ 1. Similarly, if $b \in acl(C)$.

We are left with the case when a and b are not in $acl(C)$. Then $a \in acl(C \cup \{b\})$, so the modularity of D_0 yields a $c \in acl(C) \cap D_0$ such that $a \in acl(b, c)$. From $\dim(ab/C) = \dim(ab/C \cup \{c\}) = \dim(ab/c)$ we conclude as above that a canonical parameter of p has dimension ≤ 1 since it is algebraic in c. This completes the proof of (i), the claim and this final case of the lemma.

As a first application of the lemma we show that it is impossible for an uncountably categorical theory to contain both a locally modular strongly minimal set and a strongly minimal set which is not locally modular.

Lemma 4.2.5. *Let D_1 and D_2 be strongly minimal sets in the universal domain $\mathfrak{C}$ of an uncountably categorical theory. Then D_1 is locally modular if and only if D_2 is locally modular.*

Proof. Let M be an $\aleph_0$−saturated model over which both D_1 and D_2 are definable. Assume that D_1 is locally modular. By Lemma 4.2.4, for D any strongly minimal set over M, D is locally modular if and only if the localization of D over M is locally modular. Let D_i' be the localization of D_i over M (for $i = 1, 2$). Then, D_1' is locally modular and it suffices to show that D_2' is locally modular. Let a_1 be any element of $D_1' \setminus M$. By Exercise 3.3.18, there is an element $a_2 \in D_2$ which is interalgebraic with a_1 over M. It follows that the geometry associated to D_1' is isomorphic to the geometry associated to D_2'. Hence D_2' is locally modular (see Remark 4.2.1(i)). This proves the lemma.

A plane curve can be thought of as an element of the universe by identifying it with its canonical parameter. This identification supports the following concept.

Definition 4.2.5. *For D a strongly minimal set in a t.t. theory, a* definable ($\bigwedge$−definable) family of plane curves *in D is a definable ($\bigwedge$−definable) set X such that each element of X is the canonical parameter of a plane curve.*

Such families are common in the study of both vector spaces and algebraically closed fields. For instance, in Example 4.2.2(ii), the collection of equations $\{ \alpha x + \beta y = b : b \in V \}$ (for fixed $\alpha, \beta \in F$) is a definable family of plane curves (since b is the canonical parameter of $\alpha x + \beta y = b$). In Example 4.2.2(iii), where K denotes the universal domain of algebraically closed fields of a given characteristic, let C be the plane curve defined by $y = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where $a_0, \ldots, a_n$ are arbitrary elements. Since $(a_0, \ldots, a_n)$ is the canonical parameter of C a definable family of plane curves is obtained by letting the coefficients vary over all $n+1$−tuples in K.

An elementary but fundamental fact about plane curves is

Lemma 4.2.6. *Let D be an $\emptyset$−definable strongly minimal set in a t.t. theory, C a plane curve in D, c the canonical parameter of C and a a generic of C.*

(i) The following are equivalent:

(1) $\dim(a) = 2$.
(2) $\dim(c) > 0$.
(3) *a depends on c.*
(4) *C is not contained in an $\emptyset$−definable set of Morley rank* 1.

(ii) If $\dim(a) = 2$, $\dim(c/a) = \dim(c) - 1$.

(iii) If $\dim(a) = 2$, *C may be chosen (from among the collection of equivalent plane curves) so that* $b \in C \implies \dim(b) = 2$.

(iv) Suppose that $k = \dim(c) > 0$ *and let* $\{a_1, \ldots, a_k\}$ *be a set of generics of C independent over c. Then,* $\{a_1, \ldots, a_k\}$ *is an independent set of generics of* D^2.

Proof. (i) Since a is a generic of a strongly minimal subset of D^2, $\dim(a/c) = 1$ and $\dim(a) \leq 2$. Thus, $\dim(a) = 2$ if and only if a depends on c. From here, part (i) follows from simple facts about canonical parameters.

(ii) Assuming that $\dim(a) = 2$, $\dim(ac) = \dim(a/c)+\dim(c) = 1+\dim(c)$. Also, $\dim(ac) = \dim(c/a)+\dim(a) = \dim(c/a)+2$, so $\dim(c/a) = \dim(c)-1$.

(iii) By (i) a depends on c. Thus, there is a formula $\varphi(x, c) \in p = tp(a/c)$ such that any $b \in \varphi(\mathfrak{C}, c)$ depends on c. We may chose $\varphi(x, c)$ to be strongly minimal, hence defining a plane curve C' equivalent to C. Then, $b \in C' \implies$ b depends on $c \implies \dim(b) = 2$ (by (i)).

(iv) Let $p \in S(c)$ be the (strongly minimal) type realized by generic elements of C. Since $\{a_1, \ldots, a_k\}$ is a Morley sequence in p and all Morley sequences in p are conjugate it suffices to find one Morley sequence in p of length k which is an independent set of generics of D^2. Let $I \subset C$ be an infinite Morley sequence in p. By Corollary 3.3.3, p is definable over I, hence $c \in acl(I)$. Let $B = \{b_1, \ldots, b_n\}$ be a minimal subset of I in which c is algebraic. By (i) each b_i is a generic of D^2. To complete the proof of (iv) we need only show

Claim. B is independent and $n = k$.

Assume, to the contrary, that B is dependent. Then b_n depends on $B' = \{b_1, \ldots, b_{n-1}\}$ (since B is indiscernible). Since $\dim(b_n) = 2$, the dependence of b_n on B' forces $\dim(b_n/B')$ to be 1. Since B is a Morley sequence in p, $\dim(b_n/B' \cup \{c\}) = 1$, hence b_n and c are independent over B'. Since $c \in acl(B)$, this independence forces c to be algebraic in B', contradicting the minimality assumption on B. Thus B is independent.

The following straightforward dimension calculation shows that $n = k$. We know that $\dim(c) = k$ and $\dim(B \cup \{c\}) = \dim(B) = 2n$ (since B is an independent set of generics of D^2 and $c \in acl(B)$). Furthermore, $\dim(B \cup \{c\}) = \dim(B/c)+\dim(c) = n+k$ (since B is a Morley sequence in a strongly minimal type over c). Thus $n = k$, completing the proof of (iv).

A linear strongly minimal set is relatively simple in that a localization at some element is modular. How does linearity effect the behavior of the collection of all plane curves simply as a family of subsets of D^2?

Lemma 4.2.7. *Let D be a linear strongly minimal set and C a plane curve in D with canonical parameter c.*

(i) If $a \in C$ and $\dim(a/c) = 1$, then $c \in acl(a)$.

(ii) If $C' \neq C$ is another plane curve in D, $\dim(C \cap C') \leq 2$.

Proof. (i) Certainly, this is true when $c \in acl(\emptyset)$. In the remaining case $\dim(c) = 1$ and $\dim(a) = 2$, hence $c \in acl(a)$ by Lemma 4.2.6(ii).

(ii) Let c' be a canonical parameter for C'. Since C and C' are distinct plane curves, $C \cap C' \subset acl(c, c')$. If $\dim(c) = \dim(c') = 0$ then $C \cap C' \subset acl(\emptyset)$, so (ii) holds in this case. Now suppose c or c' has dimension > 0, say $\dim(c) = 1$. Certainly, $\dim(C \cap C') \leq 1$ if $C \cap C' \subset acl(c)$, hence we can assume there is an $a \in C \cap C'$ with $\dim(a/c) = 1$. By Lemma 4.2.6(i), $\dim(a) = 2$ and c depends on a, hence $c \in acl(a)$. Applying the same lemma to the curve C' (which also contains a), $c' \in acl(a)$. Thus, $C \cap C' \subset acl(c, c') \subset acl(a)$, proving that $\dim(C \cap C') \leq 2$ in this final case.

Remark 4.2.3. When D is a linear strongly minimal set and C, C' are arbitrary plane curves in D, it is quite possible for $C \cap C'$ to be empty.

A collection of plane curves in a strongly minimal set is said to be *independent* if the corresponding collection of canonical parameters is independent.

The next lemma shows that the plane curves in a nonlinear strongly minimal set can have rather complicated intersections. First an example to illustrate this situation.

Example 4.2.3. Let D be the universal domain of algebraically closed fields of characteristic 0. Let C^* be the plane curve defined by the equation $y = ax + b$, where a and b are algebraically independent. Let X be the family of conjugates of C^* over $\emptyset$. If C is a plane curve in X defined by $y = a'x + b'$, then the pair (a', b') is a canonical parameter for C. Thus, each element of X has a canonical parameter of dimension 2. As a collection of subsets of D^2 the family X has the following two properties.

- If C and C' are independent elements of X then $\dim(C \cap C') \geq 2$. (This may fail in a linear strongly minimal set.)
- If a and b are independent generic elements of D^2, there is an element of X containing both a and b. (In the linear case no plane curve can contain an independent pair of generics of D^2.)
- If a is a generic of D^2 there are infinitely many elements of X containing a.

Lemma 4.2.8. *Suppose that D is an $\emptyset$–definable strongly minimal set in a t.t. theory containing the canonical parameter c^* of a plane curve C^* in D*

such that $\dim(c^*) = k > 1$. *Let* X *be the collection of conjugates of* C^* *over* $acl(\emptyset)$.

(i) If C *and* C' *are independent elements of* X, *then* $\dim(C \cap C') \geq 2$.

(ii) If a, b *is an independent pair of generics of* D^2 *there is a* $C \in X$ *containing both* a *and* b.

(iii) For $a \in D^2$ *generic,* $Y = \{ C \in X : a$ *is a generic of* $C \}$ *is infinite.*

Proof. (i) Since any two independent pairs of elements from X are conjugate over $acl(\emptyset)$, it suffices to find one independent pair of elements of X whose intersection has dimension ≥ 2. Let C be a generic of X with canonical parameter c, and let $a \in C$ be generic over c. Since $\dim(c) > 0$, $\dim(a) = 2$. By Lemma 4.2.6, $\dim(c/a) = k-1$. Let c' be a realization of $tp(c/\{a\} \cup acl(\emptyset))$ which is independent from c over a. Let C' be the element of X with canonical parameter c'. Since $a \in C' \cap C$ and $\dim(a) = 2$, to complete the proof of (i) it suffices to show that c' is independent from c over $\emptyset$. Since $\dim(c'/c) \geq k-1 > 0$, C' is distinct from C. Hence, $C \cap C'$ is finite, in particular $a \in acl(c, c')$. By the additivity of dimension, $\dim(c'ca) = \dim(c'/ca) + \dim(c/a) + \dim(a) = (k-1) + (k-1) + 2 = 2k$. Also, $\dim(c'ca) = \dim(a/c'c) + \dim(c'c) = \dim(c'c)$. Hence, $\dim(c'c) = 2k$, proving that c' and c are independent, as required.

(ii) Since all independent pairs of generics of D^2 are conjugate over $acl(\emptyset)$ and X is $\bigwedge$ −definable over $acl(\emptyset)$ it suffices to find one $C \in X$ which contains an independent pair $\{a, b\}$ of generics of D^2. Since the canonical parameter of any $C \in X$ has dimension > 1 this follows directly from Lemma 4.2.6.

(iii) The proof that Y is infinite is left as an exercise to the reader. This proves the lemma.

Lemma 4.2.4 is such a basic result in the geometry of strongly minimal sets that from hereon it will be quoted tacitly. The term "linear" will be dropped in favor of the exclusive use of "locally modular".

Returning to the introductory discussion at the beginning of the chapter, it is local modularity that we will use as the dividing line between geometrically simple and geometrically complex strongly minimal sets. This choice for the dividing line is supported by the previous two lemmas and will be further justified in later sections. In these later sections we see that an uncountably categorical universal domain containing a strongly minimal set which is not locally modular is recognizably more complicated than one which does not.

How common are locally modular strongly minimal sets? Many of the examples of strongly minimal sets we've given so far are trivial, vector spaces or affine spaces. The next theorem suggests that this is not simply due to a lack of imagination; locally modular strongly minimal sets are the rule under some model-theoretic hypotheses.

Theorem 4.2.1 (Cherlin-Mills-Zil'ber). *A strongly minimal set in an* $\aleph_0$−*categorical theory is locally modular.*

For $\mathfrak{C}$ a universal domain a definable algebraically closed field K is called *pure* if every relation on K definable in $\mathfrak{C}$ is definable in the field language on K. Motivated by the known examples Zil'ber asked in [Zil84c]:

> Is there a strongly minimal set D which is not locally modular and does not have a definable pure algebraically closed field in D^{eq}?

This was answered affirmatively by Hrushovski in [Hru90a]:

Theorem 4.2.2. *There is a strongly minimal set D which is not locally modular such that D^{eq} does not contain an infinite definable group.*

Later (in Section 4.3.2) we will see that any nontrivial locally modular strongly minimal set D has a definable group in D^{eq} which is close to being a vector space.

In this chapter we only scratch the surface of what is known about strongly minimal sets. The reader is referred to Pillay's book [Pil] for further results.

Historical Notes. Local modularity, as a property of a strongly minimal set, was isolated by Zil'ber in [Zil80]. Lemma 4.2.4 is an alternate version of a theorem in Zil'ber's [Zil84a]. Theorem 4.2.1 was proved independently by Cherlin, Mills and Zil'ber; a good history can be found in [CHL85]. This result is an essential ingredient in the proof that a totally categorical theory is not finitely axiomatizable [CHL85].

Exercise 4.2.1. Prove Lemma 4.2.8(iii).

Exercise 4.2.2. Let S be a pregeometry. Show that S possesses one of the properties in Definition 4.2.2 if and only if the geometry associated to S possesses the property.

Exercise 4.2.3. Following the notation of the end of Example 4.2.1(iii), show that he vector space V and its action on A are definable in $(A')^{eq}$.

4.3 Global Geometrical Considerations

In this section we turn our attention from strongly minimal sets to the entire universe of an uncountably categorical theory. This study, which will occupy the remainder of the chapter, will be organized around the following admittedly vague questions. We begin with the premise that strongly minimal theories are the simplest uncountably categorical theories.

1. To what degree is every uncountably categorical universe built from strongly minimal sets?

2. If D_1 and D_2 are strongly minimal sets in an uncountably categorical universe, can we characterize the possible relations between elements from D_1 and elements from D_2? In other words, how much freedom do we have in specifying an uncountably categorical universe containing both D_1 and D_2?
3. If some strongly minimal set in the universe is locally modular, do we obtain sharper answers to the first two questions?

We first address Question 1 motivated by the behavior illustrated in the following examples.

Example 4.3.1. (i) This is a rather trivial example, but it defines what we consider to be the ideal situation. Let D be the universe of a strongly minimal theory and $X = D^n$ for some n. In this theory the set X (which has Morley rank n) can easily be decomposed in terms of strongly minimal sets. Explicitly, there are definable functions (the coordinate maps) $\pi_i : X \longrightarrow D$ $(1 \leq i \leq n)$ such that for any $a \in D$, a is in the definable closure of $(\pi_1(a), \ldots, \pi_n(a))$.

(ii) In this second example, the coordinatizing strongly minimal sets are a little harder to find. To begin with, let F be a field with more than two elements, V an infinite vector space over F and $X = V^2$. Let α and β be distinct nonzero elements of F. Define a binary relation $R(v, w)$ on X by the formula: for $a = (x_1, y_1)$, $b = (x_2, y_2) \in X$, $R(a, b) \iff y_1 - \alpha x_1 = y_2 - \alpha x_2$. Similarly, $S(v, w)$ is defined on X by: for $a = (x_1, y_1)$, $b = (x_2, y_2) \in X$, $S(a, b) \iff y_1 - \beta x_1 = y_2 - \beta x_2$. Let M be the model with universe X in a language consisting of two binary predicate symbols interpreted by R and S, respectively. The reader can show that $Th(M)$ is quantifier eliminable, uncountably categorical, the universal domain $\mathfrak{C}$ has Morley rank 2, and for any element a, $R(\mathfrak{C}, a)$ and $S(\mathfrak{C}, a)$ are strongly minimal. The sets of the form $R(\mathfrak{C}, a)$ and $S(\mathfrak{C}, a)$, as a ranges over elements of $\mathfrak{C}$, will be called curves of type R and type S, respectively. The canonical parameters of curves of type R form a strongly minimal set D_1 over $\emptyset$, and similarly the canonical parameters of curves of type S make up a strongly minimal set D_2. The sets D_1 and D_2 provide us with a coordinatization of the universe as follows. Let a be an element of $\mathfrak{C}$. There is a unique curve C_1 of type R containing a and a unique curve C_2 of type S containing a. Let $c_i \in D_i$ be the canonical parameter of C_i, for $i = 1, 2$. The axioms for T imply that a is the unique element of $C_1 \cap C_2$, hence $a \in dcl(c_1, c_2)$ and $(c_1, c_2) \in dcl(a)$. In this way the universe is decomposed into strongly minimal sets.

(iii) In this example (as in the previous two) the universe is the algebraic closure of a strongly minimal set (and some finite set of parameters). Here, however the coordinatizing strongly minimal sets are necessarily not $\emptyset$−definable. Let P be the projective plane over an infinite division ring F formulated in a language with a ternary relation symbol I and the interpretation: $I(a, b, c)$ if and only if a, b and c are collinear, for all a, b, $c \in P$. To keep the notation simple, assume P to be the universal domain of $Th(P)$.

Then, P has Morley rank 2 and for each $a \neq b \in P$, $I(P, a, b)$ (which is called a *line of* P) is strongly minimal. Let X be any line in P, ℓ the canonical parameter of X, and $a_1 \neq a_2$ two elements of $P \setminus X$. The following claim shows that P is almost strongly minimal.

Claim. For all $a \in P$ there are $x_1, x_2 \in X$ such that a and (x_1, x_2) are interdefinable over $A = \{\ell, a_1, a_2\}$.

For a given $a \in P$ let ℓ_i be the line containing a and a_i, for $i = 1, 2$. Let x_i be the element in the intersection of ℓ and ℓ_i. Since ℓ_i is the unique line containing a_i and x_i, and a is the unique element in the intersection of ℓ_1 and ℓ_2, a is in the definable closure of $A \cup \{x_1, x_2\}$. By a similar argument x_i is in the definable closure of $A \cup \{a\}$, proving the claim.

The next claim shows that there is no coordinatizing strongly minimal set over $acl(\emptyset)$.

Claim. There is no strongly minimal subset D of P^{eq} such that D is definable over $acl(\emptyset)$ (in P^{eq}) and for some $a \in P$, $acl(a) \cap D \neq acl(\emptyset)$.

A basic fact about any projective plane over a division ring is that its automorphism group is 2−transitive. In other words, for any $a_1 \neq a_2$ and $b_1 \neq b_2$ in P, there is an automorphism α of P such that $\alpha(a_i) = b_i$, for $i = 1, 2$. Suppose, to the contrary, that D is a strongly minimal subset of P^{eq} which is definable over $acl(\emptyset)$, and $a \in P$ is such that $acl(a) \cap D \neq acl(\emptyset)$. The 2−transitivity of $\mathrm{Aut}(P)$ implies that $MR(a) = 2$. Hence, a cannot be algebraic in any $x \in (acl(a) \cap D) \setminus acl(\emptyset)$. This yields a $b \in P$, $b \neq a$, such that $x \in acl(b)$. If c in P is independent from a over $\emptyset$, then $acl(a) \cap acl(c) = acl(\emptyset)$. This contradicts the existence of an $\alpha \in \mathrm{Aut}(P)$ such that $\alpha(a) = a$ and $\alpha(b) = c$, to prove the claim.

(iv) In this example, the universe can still be viewed as being constructed from strongly minimal sets, however, no finite collection of strongly minimal sets suffices. Let $M = \bigoplus_{i<\omega}(\mathbb{Z}_4)_i$, the direct sum of $\aleph_0$ copies of the additive group $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$. Let M^* be the universal domain of $Th(M)$. The theory of M^* is quantifier-eliminable and categorical in every infinite cardinality. By this quantifier-eliminability, $2M^*$ is a vector space over $\mathbb{Z}_2$ with no additional definable relations. In particular, $2M^*$ is a strongly minimal set. Furthermore, for each $a \in M^*$, $\{\, b \in M^* : 2b = 2a \,\} = a + 2M^*$, is strongly minimal. In this way, M^* is constructed from strongly minimal sets: Given $a \in M^*$, $2a$ is in the strongly minimal set $2M^*$ and a is in a strongly minimal set definable over $2a$. More globally, this could be written as $M^* = \bigcup_{x \in 2M^*}\{\, b \in M^* : 2b = x \,\}$, the union of a strongly minimal family of strongly minimal sets.

It is left as an exercise to the reader to see that there is *not* a collection of strongly minimal sets $D_1, \ldots, D_n$ over a finite $A \subset M^*$ such that every $a \in M^*$ is interalgebraic with a subset of $D_1 \cup \ldots \cup D_n$ over A. In this sense infinitely many strongly minimal sets are needed to construct M^*.

These examples help us formulate in a more specific way the question raised under 1 at the beginning of the section, and place limits on the possible answers. Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory. In the ideal situation there are (in $\mathfrak{C}^{eq}$) strongly minimal sets $D_1, \ldots, D_n$, with each D_i definable over $acl(\emptyset)$, such that each element a is interdefinable with a subset $C(a)$ of $D_1 \cup \ldots \cup D_n$. We think of $C(a)$ as a "set of coordinates for a with respect to $D_1, \ldots, D_n$". This ideal is attained in the first two examples, but fails in (iii) and (iv). In the projective plane we must settle for strongly minimal sets which are not definable over $acl(\emptyset)$. In the model M^* of (iv) no finitely many strongly minimal sets suffice to "coordinatize" the entire model. However, each $a \in M^*$ is interdefinable with the set $\{a, 2a\}$, and both $tp(2a)$ and $tp(a/2a)$ are strongly minimal. In other words, M^* is the union of a strongly minimal family of strongly minimal sets. These facts leave us with the hope that some useful reduction to strongly minimal sets will be possible. The final results (Proposition 4.3.2 and Corollary 4.3.4) will require a few preliminary definitions and lemmas.

In (i)–(iii) of the previous example, while the universe is not strongly minimal, it is the algebraic closure of a strongly minimal set over some finite set.

Definition 4.3.1. *The complete theory T is called* almost strongly minimal *if there is a strongly minimal set D, definable over a set A, such that $\mathfrak{C} = acl(D \cup A)$.*

Lemma 4.3.1. *The countable complete theory T is almost strongly minimal if and only if*

(∗) *there is a formula $\varphi(x, \bar{y})$ over $\emptyset$ and an isolated type $q \in S(\emptyset)$ such that for any model M there is an $\bar{a} \in q(M)$ such that $\varphi(x, \bar{a})$ is strongly minimal and $M = acl(\varphi(M, \bar{a}) \cup \{\bar{a}\})$.*

Proof. That (∗) implies T is almost strongly minimal is clear. Conversely, suppose T is almost strongly minimal, A is a set and $\psi(x, \bar{a})$ is a strongly minimal formula over A such that $\mathfrak{C} = acl(\psi(\mathfrak{C}, \bar{a}) \cup A)$. Let $D = \psi(\mathfrak{C}, \bar{a})$. By compactness we may take A to be finite. The proof is carried out in the following steps.

(a) For any strongly minimal formula $\theta(x)$ there is a set $B \supset A$ such that θ is over B and for any $b \in \theta(\mathfrak{C}) \setminus acl(B)$ there is a $c \in D$ interalgebraic with b over B.
(b) T is ω−stable.
(c) T is uncountably categorical.
(d) There is a sequence $\bar{b}$ realizing an isolated type, a strongly minimal formula $\varphi(x, \bar{b})$ and a set $B \supset \bar{b}$ such that $\mathfrak{C} = acl(\varphi(\mathfrak{C}, \bar{b}) \cup B)$.
(e) There is a sequence $\bar{c} \supset \bar{b}$ realizing an isolated type such that $\mathfrak{C} = acl(\varphi(\mathfrak{C}, \bar{b}) \cup \bar{c})$.

(a) First let B_0 be any set containing A and the parameters in θ. Let b be any element of $\theta(\mathfrak{C}) \setminus acl(B)$ and $\bar{d} \subset D$ such that $b \in acl(\bar{d} \cup A)$. Without loss of generality, $\bar{d}$ is of the form $d_0\bar{d}'$, where $d_0 \in D \setminus acl(B_0 \cup \bar{d}')$ and $b \notin acl(B_0 \cup \bar{d}')$. Let $B = B_0 \cup \bar{d}'$. Since d_0 is in a strongly minimal set over B and b depends on d_0 over B, d_0 and b are interalgebraic over B. All elements of $\theta(\mathfrak{C}) \setminus acl(B)$ realize the same type over B (since θ is strongly minimal), hence every element of $\theta(\mathfrak{C}) \setminus acl(B)$ is interalgebraic over B with an element of D, proving (a).

(b) Let M be a countable model of T containing A. For any a there is a $\bar{d} \subset D$ such that $a \in acl(M \cup \bar{d})$. Since D is strongly minimal $\{ tp(\bar{d}/M) : \bar{d} \subset D$ is finite$\}$ is countable. Thus, $S_1(M)$ is countable. This proves that T is $\omega-$stable.

(c) By Theorem 3.1.2 and (b) it suffices to show that T has no Vaughtian pair. Assume to the contrary that T has a Vaughtian pair. By Lemma 3.1.7 there is a Vaughtian pair (M, N) where $M \supset N$ are $\aleph_0-$saturated and $N \supset A$. For an arbitrary $a \in M \setminus N$ there is a $\bar{d} \subset \psi(M, \bar{a})$ such that $a \in acl(\bar{d} \cup A)$. Since $a \notin N$, $\bar{d} \not\subset N$, hence $\psi(M) \not\subset N$. By Corollary 3.1.2 there is a strongly minimal formula θ over N such that (M, N, θ) is a Vaughtian triple. Let c be an element of $\psi(M) \setminus N$. By (a) and the $\aleph_0-$saturation of N there is a set $B \subset N$ and a b satisfying θ such that b and c are interalgebraic over B. Then $b \in \theta(M) \subset N$, contradicting the fact that $c \notin N$ and proving (c).

(d) Since T is uncountably categorical there is a strongly minimal formula $\varphi(x, \bar{b})$, where $tp(\bar{b})$ is isolated. By (a) there is a set $B \supset \bar{b} \cup A$ such that $D \subset acl(\varphi(\mathfrak{C}, \bar{b}) \cup B)$. Thus, $\mathfrak{C} = acl(\varphi(\mathfrak{C}, \bar{b}) \cup B)$.

(e) By compactness there are formulas $\psi_0(x, \bar{y}_0, \bar{c}_0), \ldots, \psi_n(x, \bar{y}_n, \bar{c}_n)$ such that

(1) for all $\bar{d}_i \subset \varphi(\mathfrak{C}, \bar{b})$, $\psi_i(x, \bar{d}_i, \bar{c}_i)$ is algebraic and
(2) for any a there is an $i \leq n$ and a $\bar{d}_i \subset \varphi(\mathfrak{C}, \bar{b})$ such that $\models \psi_i(a, \bar{d}_i, \bar{c}_i)$.

Taking the disjunction of the ψ_i's gives one formula $\sigma(x, \bar{y}, \bar{c})$ such that

(*) $\sigma(x, \bar{d}, \bar{c})$ is algebraic (for all $\bar{d} \subset \varphi(\mathfrak{C}, \bar{b})$) and for any a, $\models \sigma(a, \bar{d}, \bar{c})$ (for some $\bar{d} \subset \varphi(\mathfrak{C}, \bar{b})$).

There is a $\bar{b}-$definable set Z containing $\bar{c}$ such that (*) holds for any $\bar{c}' \in Z$. Thus, (*) holds for some $\bar{c}$ realizing an isolated type over $\bar{b}$, hence an isolated type over $\emptyset$.

This proves the lemma.

Corollary 4.3.1. *A countable almost strongly minimal theory is uncountably categorical.*

Note: There is an uncountably categorical theory which is not almost strongly minimal.See Example 4.3.1(iv).

Almost strongly minimal theories arise naturally in the study of $\omega-$stable groups:

Proposition 4.3.1. *If G is a simple group of finite Morley rank, then $Th(G)$ is almost strongly minimal.*

Proof. Let D be a strongly minimal set in G. Let $\mathcal{C} = \{ D \cap aQ : a \in G,\ Q$ is a definable subgroup of G and $D \cap aQ$ is infinite$\}$. Since D is strongly minimal, each element of $\mathcal{C}$ is cofinite in D. Thus, $\mathcal{C}$ is closed under finite intersections. By Proposition 3.5.1, $\bigcap \mathcal{C}$ is equal to some $D \cap aQ \in \mathcal{C}$. Clearly, $D \cap aQ$ is indecomposable. Let $g_0 \in D \cap aQ$ and $B = g_0^{-1}(D \cap aQ)$. Then B is indecomposable, strongly minimal and contains the identity 1.

Now let $\mathcal{B} = \{ g^{-1}Bg : g \in G \}$ and $N =$ the group generated by $\bigcup \mathcal{B}$. Since each element of $\mathcal{B}$ is indecomposable and contains 1, Zil'ber's Indecomposability Theorem says that $N = B_1 \cdot \ldots \cdot B_k$ for some $B_1, \ldots, B_k \in \mathcal{B}$. Since N is normal (and not $\{1\}$) and G is simple, $G = B_1 \cdot \ldots \cdot B_k$. Thus, for some $g_1, \ldots, g_k \in G$, each element of G is of the form $g_1^{-1}b_1g_1 \cdot \ldots \cdot g_k^{-1}b_kg_k$, for some $b_1, \ldots, b_k \in B$. A fortiori, for $A = \{g_1, \ldots, g_k, g, a\}$, $G = acl(A \cup D)$. Thus, G is almost strongly minimal, proving the proposition.

Definition 4.3.2. *Let $\mathfrak{C}$ be the universal domain of a t.t. theory. A subset X of $\mathfrak{C}$, definable over A, is said to be* almost strongly minimal over A *if the restriction of $\mathfrak{C}$ to X is an almost strongly minimal theory. Equivalently, X is almost strongly minimal over A if there is a $B \subset X$ and a $D \subset X$ which is strongly minimal over $B \cup A$, such that $X \subset acl(D \cup B \cup A)$. A formula over A is almost strongly minimal if the set it defines is almost strongly minimal over A. A type over A is almost strongly minimal if it contains an almost strongly minimal formula over A.*

The next few results are used to show that elements in various relationships to almost strongly minimal sets are themselves elements of almost strongly minimal sets. The first lemma shows that we needn't be careful to choose a strongly minimal subset of X in verifying that X is almost strongly minimal.

Lemma 4.3.2. *Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory. Let X be a subset of $\mathfrak{C}$ definable over A such that for some $B \supset A$ there is an almost strongly minimal set D over B with $X \subset acl(D \cup B)$. Then, X is almost strongly minimal.*

Proof. To keep the notation simple suppose that $A = \emptyset$. We will prove the lemma in the case when D is strongly minimal, leaving the proof of the full result to the reader. By Corollary 3.1.2, there is a set B' and a strongly minimal set $D' \subset X$ definable over B'. In fact, by Proposition 3.3.3, we can require B' to also be a subset of X. Exercise 3.3.18 yields a set $C \supset B \cup B'$ such that $acl(D \cup C) = acl(D' \cup C)$. Thus, $X \subset acl(D' \cup C)$. Since $D' \subset X$, Proposition 3.3.3 can again be applied to find a $C' \subset X$ such that $X \subset acl(D' \cup C')$. This proves the lemma.

Note that any definable subset of an almost strongly minimal set is finite or almost strongly minimal (see Exercise 4.3.1).

In the next two results we see that almost strong minimality is preserved under finite unions and algebraic closure.

Lemma 4.3.3. *Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory and let $X_1, \ldots, X_n$ be almost strongly minimal subsets of $\mathfrak{C}$. Then, $X_1 \cup \ldots \cup X_n$ is almost strongly minimal.*

Proof. Simply from the definition, there are strongly minimal sets $D_i \subset X_i$, for $1 \leq i \leq n$, and a set A over which each D_i is definable such that $X_i \subset acl(D_i \cup A)$. Again quoting Exercise 3.3.18, we can take A to be large enough so that $D_i \subset acl(D_1 \cup A)$, for each i. By Lemma 4.3.2, $X_1 \cup \ldots \cup X_n$ is almost strongly minimal.

Lemma 4.3.4. *Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory. Let $A \subset B$ be sets, X an almost strongly minimal set over B, and a an element which is independent from B over A and algebraic in $X \cup B$. Then, a is an element of an almost strongly minimal set over A.*

Proof. Observe that we can, without loss of generality, take both A and B to be finite. Let $A' \supset A$ be a finite subset of $acl(A)$ such that $tp(a/A') = p$ is stationary.

Claim. There is a finite set $B' \supset A'$ and an almost strongly minimal set X' over B' such that $p(\mathfrak{C}) \subset acl(B' \cup X')$.

By Proposition 3.4.1 there is a k such that whenever $\{b_0, \ldots, b_k\}$ is independent over A', a is independent from some b_i over A'. Let $\{B_0, \ldots, B_k\}$ be a set of realizations of $tp(B/A')$ which is independent over A'. For $i \leq k$ there is an $f_i \in \operatorname{Aut}(\mathfrak{C})$ which maps B to B_i and is the identity on A'. Let $X_i = f_i(X)$, an almost strongly minimal set over B_i. Let a' be any realization of p. In the next paragraph we prove

$$a' \in acl(B_i \cup X_i). \tag{4.3}$$

For some i, a' is independent from B_i over A'. Since p is stationary, the unique free extension of p over $\mathfrak{C}$ does not split over A'. Thus, $a'B_i$ is conjugate over A' to aB. (Let f_1 be an automorphism of $\mathfrak{C}$ which fixes A' pointwise and maps a' to a. Then, $f_1(B_i)$ and B realize the same type over $A' \cup \{a\}$, hence there is an automorphism f_2 of $\mathfrak{C}$ which is the identity on $A' \cup \{a\}$ and maps $f_1(B_i)$ to B. The automorphism $f_2 f_1$ is the identity on A' and maps $a'B_i$ to aB.) By this conjugacy, $a' \in acl(B_i \cup X_i)$, as required.

To complete the proof of the claim let $B' = B_0 \cup \ldots \cup B_k$ and $X' = X_0 \cup \ldots \cup X_k$. By Lemma 4.3.3, X' is almost strongly minimal. For c an arbitrary realization of p, (4.3) applied with $a' = c$, shows that $c \in acl(B' \cup X')$. This proves the claim.

By a compactness argument there is a formula $\psi_0(v) \in p$ such that $\psi_0(\mathfrak{C}) \subset acl(B' \cup X')$. By Lemma 4.3.2, $\psi_0(\mathfrak{C})$ is almost strongly minimal. Let $\psi_0, \ldots, \psi_n$ be a list of the (finitely many) conjugates of ψ_0 over A. Then, $Y = \psi_0(\mathfrak{C}) \cup \ldots \cup \psi_n(\mathfrak{C})$ is definable over A, almost strongly minimal (by Lemma 4.3.3) and contains a, completing the proof of the lemma.

The reader should think of the next proposition in two parts. First, we find in $dcl(A \cup \{a\})$ an element of an almost strongly minimal set (which we think of as a "coordinate" for a over A). Secondly, there is a coordinate for a which significantly effects the relations between a and other elements of the universe. This result is central to our understanding of uncountably categorical theories.

Proposition 4.3.2. *Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory. Then for all sets A and $a \notin acl(A)$ there is a $d \in dcl(A \cup \{a\})$ such that d is an element of an almost strongly minimal set over A, and ad is dominated by d over A.*

Proof. To simplify the notation, take A to be the empty set. First we will find an element in $acl(a)$ (rather than $dcl(a)$) which meets the other requirements. Let M be an $\aleph_0$-saturated model which is independent from a. By Corollary 3.4.1, there is a strongly minimal set D over M and a sequence $\bar{c} = (c_1, \ldots, c_k)$ from D such that a is dominated by $\bar{c}$ over M and $\bar{c} \subset acl(M \cup \{a\})$. Let $B \subset M$ be a finite set such that $a\bar{c}$ is independent from M over B. (Hence, a is dominated by $\bar{c}$ over B and $\bar{c} \subset acl(B \cup \{a\})$.) By Corollary 4.1.4 there is an element b such that

(1) $b \in acl(a)$,
(2) a is independent from $B \cup \{\bar{c}\}$ over b, and
(3) $b \in acl(\bar{c}_0 B_0, \ldots, \bar{c}_k B_k)$, for some set $\{\bar{c}_0 B_0, \ldots, \bar{c}_k B_k\}$ of realizations of $tp(\bar{c}B/a)$ which is independent over a.

Claim. ab is dominated by b over $\emptyset$.

Let C be a set which is independent from b. It suffices to show that C' is independent from a for some conjugate C' of C over $\{a, b\}$, hence we can assume that C is independent from B over $\{a, b\}$. Since B is independent from $\{a, b\}$, B is independent from $C \cup \{b\}$. That is,

$$\{C, B, b\} \text{ is independent.} \tag{4.4}$$

By (2) and the fact that $\bar{c} \subset acl(B \cup \{a\})$, $\bar{c} \subset acl(B \cup \{b\})$. Thus, C is also independent from $\bar{c}$ over B. Since a is dominated by $\bar{c}$ over B, C is independent from a over B. (by (1)). By (4.4) and the transitivity of independence, C is independent from $B \cup \{a\}$. Combining this with (1) shows that C is independent from ab.

Claim. b is an element of an almost strongly minimal set over $\emptyset$.

Lemma 4.3.4 will be used to prove the claim. Let D_i be the strongly minimal set over B_i which is conjugate to D (for $i \leq k$), $B' = B_0 \cup \ldots \cup B_k$ and $X = D_0 \cup \ldots \cup D_k$. Then X is almost strongly minimal by Lemma 4.3.3. Since B_i realizes $tp(B/a)$, B_i is independent from a for each $i \leq k$. Since $\{B_0, \ldots, B_k\}$ is independent over a, $B' = B_0 \cup \ldots \cup B_k$ is independent from a (by the transitivity of independence). Thus, b is independent from B' (because $b \in acl(a)$). By (3), $b \in acl(B' \cup X)$, hence b belongs to an almost strongly minimal set over A, by Lemma 4.3.4.

Claim. There is a $d \in dcl(a)$ such that ad is dominated by d over $\emptyset$ and d is an element of an almost strongly minimal set over $\emptyset$.

Let X^* be the set of realizations of $tp(b/a)$ in $\mathfrak{C}$. Since X^* is finite there is a name d for X^* in $\mathfrak{C}$. Also, X^* is definable over a, hence $d \in dcl(a)$. Using: $b \in acl(A \cup \{d\})$ and $d \in dcl(A \cup \{a\})$, the reader can verify that ad is dominated by d over A. Finally, d is an element of an almost strongly minimal set over A since it is interalgebraic with a finite subset of an almost strongly minimal set.

This proves the proposition.

Remark 4.3.1. The most important part of the proposition is the existence of a "coordinate" for a from an almost strongly minimal set. However, that a is dominated by a coordinate d indicates the strength of the relationship between the two elements. The corollaries below make use of and reveal the ramifications of this domination relation.

Corollary 4.3.2. *Let a and b be elements of the universe of an uncountably categorical theory such that a depends on b. Then, there are $a' \in dcl(a)$ and $b' \in dcl(b)$ such that a' and b' belong to almost strongly minimal sets and a' depends on b'.*

Proof. By Proposition 4.3.2 there are $a' \in dcl(a)$ and $b' \in dcl(b)$ such that a' and b' belong to almost strongly minimal sets over $\emptyset$, a is dominated by a' over $\emptyset$ and b is dominated by b' over $\emptyset$. Since $a \not\perp b$ these domination relations force a' to be dependent on b', proving the corollary.

Corollary 4.3.3. *Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory. Then $\mathfrak{C}$ is almost strongly minimal or there are a and d such that $d \in dcl(a)$, $a \notin acl(d)$ and a is dominated by d over $\emptyset$.*

Proof. Suppose there are no elements a and d as in the statement. Then, by Proposition 4.3.2, for any $a \notin acl(\emptyset)$ there is a d interalgebraic with a such that d belongs to an almost strongly minimal set. Thus, any $a \notin acl(\emptyset)$ belongs to an almost strongly minimal set. By compactness there are $\emptyset$–definable sets $X_0, \ldots, X_k$ such that the sort of equality in $\mathfrak{C}$ is $X_0 \cup \ldots \cup X_k$ and each X_i is finite or almost strongly minimal. Thus, $\mathfrak{C}$ is almost strongly minimal.

Proposition 4.3.2 implies that the universe of an uncountably categorical theory is built from almost strongly minimal sets. This is formalized through the following concept. For α an ordinal, $C = \{ c_i : i < \alpha \}$ an indexed family, and $i \leq \alpha$, C_i denotes $\{ c_j : j < i \}$.

Definition 4.3.3. *Let $C = \{ c_i : i < \alpha \}$ be a sequence of elements in the universe of some complete theory. We call C an* almost strongly minimal construction *(*asm-construction*, for short) if for each $i < \alpha$, $tp(c_i/C_i)$ is almost strongly minimal or algebraic. A set A is* asm-constructible *if there is an enumeration of A which is an asm-construction.*

Remark 4.3.2. If $C = \{ c_i : i < \alpha \}$ and $C' = \{ c'_i : i < \alpha' \}$ are both asm-constructions, then the enumeration of $C \cup C'$ which lists C' after all the elements of C is also an asm-construction. Thus, the union of two asm-constructible sets is asm-constructible. In fact, the union of any number of asm-constructible sets is asm-constructible.

Corollary 4.3.4. *Let A be a set in the universe of an uncountably categorical theory. Then, $dcl(A)$ is asm-constructible.*

Proof. This proof is relegated to Exercise 4.3.2. It follows quickly from Proposition 4.3.2.

4.3.1 1−based Theories

We will return to asm-constructibility in arbitrary uncountably categorical theories in later sections, where the definable relations between different almost strongly minimal subsets of the universe are studied. In the remainder of this section the above results are extended assuming the theory contains a locally modular strongly minimal set.

A strongly minimal set D is modular if and only if, for all closed $X, Y \subset D$, X and Y are independent over $X \cap Y$. The following definition and Theorem 4.3.1 extend this property to uncountably categorical theories which contain a modular strongly minimal set.

Definition 4.3.4. *An uncountably categorical theory is called* 1−based *if for all subsets A and B of the universal domain $\mathfrak{C}$, A is independent from B over $acl(A) \cap acl(B)$.*

(As usual, if T is 1−based we also call $\mathfrak{C}$ 1−based.)

Lemma 4.3.5. *The following are equivalent for $\mathfrak{C}$ the universe of an uncountably categorical theory.*

(1) *$\mathfrak{C}$ is 1−based.*
(2) *For all $a \in \mathfrak{C}$ and sets A, a canonical parameter c of $tp(a/acl(A))$ is in $acl(a)$.*

Proof. First suppose $\mathfrak{C}$ to be 1−based, let $a \in \mathfrak{C}$ and $A \subset \mathfrak{C}$. Let $p \in S(\mathfrak{C})$ be the free extension of $tp(a/acl(A))$ and c a canonical parameter of p. Since $\mathfrak{C}$ is 1−based, p is a free extension of its restriction to $B = acl(a) \cap acl(A)$, hence $c \in acl(B) \subset acl(a)$, as desired. Now suppose that (2) holds. To prove that $\mathfrak{C}$ is 1−based it suffices to show that for all elements a and b, a is independent from b over $acl(a) \cap acl(b)$. For a arbitrary elements a and b let c be a canonical parameter of $tp(a/acl(b))$. Then, a is independent from b over c, $c \in acl(b)$ (because the relevant type is definable over $acl(b)$) and $c \in acl(a)$ (by (2)). Thus, a is independent from b over $acl(a) \cap acl(b)$, as required.

Remark 4.3.3. This equivalent definition explains the term "1−based". In Shelah's terminology, a type over $\mathfrak{C}$ is "based" on a set A if it is definable over A. An uncountably categorical theory is 1−based when, given a degree 1 type p and q the free extension of p in $S(\mathfrak{C})$, q is based on $acl(a)$ for any single a realizing p.

Theorem 4.3.1. *Given $\mathfrak{C}$ the universal domain of an uncountably categorical theory, $\mathfrak{C}$ is 1−based if and only if $\mathfrak{C}$ contains a locally modular strongly minimal set.*

The proof of this theorem will take several lemmas and propositions. Starting from the fact that the theorem is true on the restriction to a modular strongly minimal subset of the universe, we will prove the result for increasingly general sets.

First we take care of the easier direction of the biconditional:

Lemma 4.3.6. *Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory containing a strongly minimal set D which is not locally modular. Then, $\mathfrak{C}$ is not 1−based.*

Proof. For simplicity, suppose D is definable over $\emptyset$. Since D is not locally modular there is, by Lemma 4.2.4, a plane curve C in D such that a canonical parameter c of C has dimension $k > 1$. Let a be an element of C such that $\dim(a/c) = 1$. Suppose, towards a contradiction, that a and c are independent over $b \in acl(a) \cap acl(c)$. Since c is a canonical parameter of $tp(a/acl(c)) \supset tp(a/bc)$ and this type is a free extension of its restriction to b, Lemma 4.1.5(i) implies that $c \in acl(b)$. Hence, $c \in acl(a)$. Using the additivity of dimensions, $2 \geq \dim(a) = \dim(ac) = \dim(a/c) + \dim(c) = 1 + \dim(c)$. Since $\dim(c) > 1$ we have reached the contradiction which proves the lemma.

The next lemma shows that dependence between the elements of a modular strongly minimal set and other elements of the universe can only occur in a very simple way. The lemma implies that sets B and C are independent over $acl(B) \cap acl(C)$, when one of B or C is contained in a modular strongly minimal set (over $\emptyset$).

Lemma 4.3.7. *Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory, D an A−definable modular strongly minimal set. Then for all sets B,* *B and D are independent over $(D \cap acl(A \cup B)) \cup A$. Furthermore, for all sets $C \subset D$, B is independent from C over $acl(A \cup C) \cap acl(A \cup B)$.*

Proof. Without loss of generality, both A and B are finite, and $D \cap acl(A \cup B) = D \cap acl(A)$. By taking A to be $\emptyset$ we can assume that $D \cap acl(B) = acl(\emptyset)$. We need to show that B and D are independent over $\emptyset$. Assuming, to the contrary, that B and D are dependent, there is a sequence $\bar{a}$ from D such that $\dim(\bar{a}/B) = \dim(\bar{a}) - 1$. As a consequence of Corollary 4.1.4 there is a set $B' \subset D$ such that

$$B' \underset{B}{\perp} \bar{a} \text{ and } B \underset{B'}{\perp} \bar{a}.$$

Thus, $\dim(\bar{a}/B') = \dim(\bar{a}) - 1$. By the modularity of D and Remark 4.2.1(iv), $\bar{a}$ and B' are independent over $D \cap acl(\bar{a}) \cap acl(B')$. Thus, there is a $c \in D \cap acl(\bar{a}) \cap acl(B')$ with $\dim(c) = 1$. Since B' and $\bar{a}$ are independent over B, $c \in acl(B)$. This contradicts the fact that $acl(B) \cap D = acl(\emptyset)$, to prove the first part of the lemma.

Turning to the furthermore clause, let $C' = D \cap acl(A \cup B)$. By the first part of the lemma, C is independent from B over $C' \cup A$. By the modularity of D, C and C' are independent over $acl(A \cup C) \cap acl(A \cup C')$. The transitivity of independence now implies that C and B are independent over $acl(A \cup C) \cap acl(A \cup B)$, completing the proof.

As a first application of this lemma we sharpen the picture of the relationship between two locally modular strongly minimal sets supplied by Lemma 4.2.5. (This corollary is not directly involved in the proof of the Theorem 4.3.1, however its central role in the theory justifies the digression.)

Corollary 4.3.5. *Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory and D_1, D_2 strongly minimal sets over $\emptyset$.*

(i) If D_1 and D_2 are both modular, then for all generic $b_1 \in D$ there is a $b_2 \in D_2$ which is interalgebraic with b_1.

(ii) Suppose that D_1 is locally modular, $a_1, b_1 \in D_1$ are independent generics and $b_2 \in D_2$ is generic. Then, there is an $a_2 \in D_2$ such that a_1 and a_2 are interalgebraic over $\{b_1, b_2\}$.

Proof. (i) Let M be a model. By Exercise 3.3.18, there are $a_i \in D_i \setminus M$, for $i = 1, 2$, such that a_1 and a_2 are interalgebraic over M. By Proposition 3.3.3, a_1 and a_2 are interalgebraic over $(D_1 \cap M) \cup (D_2 \cap M)$. Then, the modularity of D_1 and Lemma 4.3.7 yield a $b_1 \in D_1 \setminus acl(\emptyset)$ such that $b_1 \in acl((D_2 \cap M) \cup \{a_2\})$. By the same reasoning there is a $b_2 \in D_2 \setminus acl(\emptyset)$ which is algebraic in b_1. This proves the existence of *some* pair (b_1, b_2) satisfying the necessary conditions. However, all elements of $D_1 \setminus acl(\emptyset)$ realize the same type over $\emptyset$, so (i) holds.

(ii) This part follows immediately from (i) once we observe that for D any locally modular strongly minimal set (over $\emptyset$) and $a \in D \setminus acl(\emptyset)$, the localization of D at a is modular.

Lemma 4.3.8. *Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory which contains a locally modular strongly minimal set and let A be a subset of an almost strongly minimal set over C. Then for all B, A is independent from B over $acl(A \cup C) \cap acl(B \cup C)$.*

Proof. Without loss of generality, A is finite, and for simplicity, take C to be $\emptyset$. Let b be a canonical parameter of $tp(A/acl(B))$. Since $b \in acl(B)$ and A is independent from B over b, it remains only to show that $b \in acl(A)$. Let M be an $\aleph_0$-saturated model such that

(a) M is independent from $B \cup A \cup \{b\}$, and
(b) there is a modular strongly minimal set D, definable over M, and $\bar{c} \subset D$ such that A and $\bar{c}$ are interalgebraic over M.

By Lemma 4.3.7 there is a $d \in acl(\bar{c} \cup M) \cap acl(B \cup M)$ such that $\bar{c}$ and B are independent over $\{d\} \cup M$. In fact, A and B are independent over $\{d\} \cup M$ (since A and $\bar{c}$ are interalgebraic over M). Since $MR(A/B) = MR(A/B \cup M) = MR(A/B \cup M \cup \{d\}) = MR(A/M \cup \{d\})$, and b is a canonical parameter of $tp(A/acl(B))$, $b \in acl(M \cup \{d\})$. We can conclude that $b \in acl(A)$ using the facts: $d \in acl(M \cup \bar{c}) = acl(M \cup A)$, $b \in acl(M \cup \{d\})$, and $A \cup \{b\}$ is independent from M (by (a)). This proves the lemma.

Proof of Theorem 4.3.1. One direction of the "if and only if" is Lemma 4.3.6. Assume the universal domain contains a a locally modular strongly minimal set. Let A and B be sets and $C = acl(A) \cap acl(B)$. To prove the theorem we must show that A and B are independent over C. Without loss of generality, A is finite. By Proposition 4.3.2, there is a $d \in dcl(A \cup C)$ such that $A \cup \{d\}$ is dominated by d over C and d belongs to an almost strongly minimal set over C. Since $acl(A) \supset acl(C \cup \{d\})$, $acl(C \cup \{d\}) \cap acl(C \cup B)$ is also C. Thus, by Lemma 4.3.8, d is independent from B over C. Since $A \cup \{d\}$ is dominated by d over C, A is independent from B over C. This proves the theorem.

The following is due in various parts to Cherlin, Harrington, Lachlan and Zil'ber. It follows from Theorems 4.2.1 and 4.3.1. See [CHL85].

Corollary 4.3.6. *A totally categorical theory is 1−based.*

From this result Zil'ber, and later Cherlin, Harrington and Lachlan [CHL85], proved

Theorem 4.3.2. *A totally categorical theory is not finitely axiomatizable.*

(A considerable amount of work is required to prove the theorem from the preceding corollary.)

Corollary 4.3.7. *Let $\mathfrak{C}$ be the universal domain of a uncountably categorical theory and X an infinite definable subset of $\mathfrak{C}$. Then $\mathfrak{C}$ is 1−based if and only if the restriction of $\mathfrak{C}$ to X is 1−based.*

Proof. See Exercise 4.3.3.

The following definition and results help to round out our picture of 1−based theories by improving Proposition 4.3.2 and Corollary 4.3.4.

Definition 4.3.5. *Let $C = \{c_i : i < \alpha\}$ be a sequence of elements in the universal domain of a complete theory. We call C a* rank 1 construction *(*rk1-construction*) if for each $i < \alpha$, $MR(c_i/C_i) \leq 1$. A set A is* rk1-constructible *if there is an enumeration of A which is a rk1-construction.*

Lemma 4.3.9. *Let $\mathfrak{C}$ be the universal domain of an uncountably categorical 1−based theory, A a set and $a \notin acl(A)$. Then there is a $c \in acl(A \cup \{a\})$ such that $MR(c/A) = 1$.*

Proof. By Proposition 4.3.2 it suffices to prove

Claim. Let a belong to an almost strongly minimal set over B. Then, there is a set $\bar{b} = \{b_0, \ldots, b_n\}$ such that $MR(b_i/B) \leq 1$, for $i \leq n$, and a is interalgebraic with $\bar{b}$ over B.

Let $M \supset B$ be an $\aleph_0$−saturated model which is independent from a over B. Let D be a strongly minimal set over M and $\bar{c} = \{c_0, \ldots, c_n\}$ a subset of D such that a is interalgebraic with $\bar{c}$ over M. Since the theory is 1−based, for each $i \leq n$, there is $b_i \in acl(B \cup \{a\}) \cap acl(M \cup \{c_i\})$ such that a and $M \cup \{c_i\}$ are independent over $B \cup \{b_i\}$. Let $\bar{b} = \{b_0, \ldots, b_n\}$. For each i, $b_i \in acl(B \cup \{a\})$ and a is independent from M over B, hence $MR(b_i/B) = MR(b_i/M)$. Since $b_i \in acl(M \cup \{c_i\})$ and $MR(c_i/M) \leq 1$, $MR(b_i/M) \leq 1$ and b_i is interalgebraic with c_i over M. Because a is interalgebraic with $\bar{c}$ over M, a is interalgebraic with $\bar{b}$ over M, in fact, a is interalgebraic with $\bar{b}$ over B. The claim now follows from the fact that $MR(b_i/B) = MR(b_i/M) \leq 1$, for each i. This proves the lemma.

Remark 4.3.4. The stronger version of the lemma with *acl* replaced by *dcl* is false. That is, there is a 1−based uncountably categorical theory containing a set A and $a \notin acl(A)$ such that there is no $c \in dcl(A \cup \{a\})$ with $MR(c/A) = 1$.

In [CHL85] the preceding lemma (in the totally categorical context) is called the Coordinatization Lemma.

Proposition 4.3.3. *Let A be a set in the universal domain of a 1−based uncountably categorical theory. Then $acl(A)$ is rk1-constructible.*

Proof. This is immediate by the previous lemma.

Remark 4.3.5. This finally gives us a reasonable picture of the manner in which the universal domain $\mathfrak{C}$ of a 1−based theory can be built from sets of Morley rank 1. For a any element of $\mathfrak{C}$ there is a set $\{c_0, \ldots, c_n\}$ interalgebraic with a such that $MR(c_i/C_i) \leq 1$, for $i \leq n$.

4.3.2 1−based Groups

This subsection is devoted to the study of definable groups in 1−based uncountably categorical theories. This examination will both illustrate the strength of the 1−based condition, and provide us with tools for later use in 1−based theories. A definition is needed to state the key result.

Definition 4.3.6. *Let G be an A−definable group in the universe of a complete theory. Let*

$$\mathcal{H} = \{\, H : \; H \text{ is a subgroup of } \; G^n, \text{ for some } \; n, \\ \text{which is definable over } \; acl(A)\,\}.$$

G is called an abelian structure *if for every $n < \omega$, every definable subset of G^n is equal to a boolean combination of cosets of elements of $\mathcal{H}$.*

It is left to the exercises to show that a vector space is an abelian structure. It will be shown in Section 5.3.2 that a module, formulated in the natural language for modules over a particular ring, is an abelian structure. (In fact, later we will see that any locally modular strongly minimal group is not only an abelian structure, but essentially a vector space over some division ring.) An abelian structure has an abelian subgroup of finite index, supporting the use of the term "abelian". (This is proved below in Corollary 4.3.12 in the context of uncountably categorical theories.)

It is not difficult to show directly that an algebraically closed field is not an abelian structure, although it also follows from the next theorem and Theorem 4.3.1.

Theorem 4.3.3. *Let G be an infinite definable group in the universal domain $\mathfrak{C}$ of an uncountably categorical theory. Then, G is an abelian structure if and only if $\mathfrak{C}$ is 1−based.*

The following type-oriented equivalent of being an abelian structure is easier to work with in proofs. We will only prove the lemma in the context of uncountably categorical theories, although it is true in a much broader setting. Remember, given an $\bigwedge$−definable group G and a set B, $S^G_n(B)$ denotes the set of complete n−types over B which extend the type defining G.

Lemma 4.3.10. *Let G be an A−definable group in the universal domain $\mathfrak{C}$ of an uncountably categorical theory. Then, G is an abelian structure if and only if*

(∗) *for any $n < \omega$ and $p \in S^G_n(G)$ there is a connected group $H \subset G^n$, definable over $acl(A)$, such that p is a left (or right) translate of the generic type of H.*

Proof. For simplicity, suppose $A = \emptyset$. In the proof we use the left translate version of $(*)$. The proof for right translates is the same.

First assume $(*)$ to be true. Fix an $n < \omega$ and let $\mathcal{H}_n = \{ H : H$ is a subgroup of G^n which is definable over $acl(\emptyset) \}$. We will prove by induction on Morley rank and degree that

(#) every definable subset X of G^n is equal to a boolean combination of cosets of elements of $\mathcal{H}_n$.

Let $\alpha = MR(G^n) + 1$ and $\omega^* = \omega \setminus \{0\}$. For any definable $X \subset G^n$, $(MR(X), \deg(X))$ is an element of the set of pairs $\alpha \times \omega^*$. Order $\alpha \times \omega$ lexicographically; i.e., for β, $\gamma < \alpha$ and m, $n \in \omega^*$, $(\beta, m) < (\gamma, n)$ if $\beta < \gamma$ or $\beta = \gamma$ and $m < n$. The induction will proceed using

$$\text{if } X, Y \subset G^n \text{ are definable, } X \supset Y \neq \emptyset \text{ and } MR(X) = MR(Y),$$
$$\text{then } (MR(X \setminus Y), \deg(X \setminus Y)) < (MR(X), \deg(X)). \quad (4.5)$$

Let X be a B−definable subset of G^n. If $MR(X) = 0$, then X is a finite union of cosets of $\{0\}$, hence (#) is true in this case. Suppose that $MR(X) = \beta > 0$, $\deg(X) = k$, and (#) is true for any definable $Y \subset G^n$ with $(MR(Y), \deg(Y)) < (\beta, k)$. Let a be a generic element of X and $p \in S_n(G)$ a free extension of $tp(a/acl(B))$. Let $q \in S_n(G)$ and $g \in G^n$ be such that q is a generic type of an element H of $\mathcal{H}$ and $p = gq$. By Lemma 3.5.2 we can take q to be the generic of H^o, hence we may as well assume H is connected. Hence $\deg(H) = \deg(gH) = 1$ (by Corollary 3.5.3). Since a was chosen to be a generic of X, $\beta = MR(p) = MR(q) = MR(H)$. The formula defining gH is in p, hence $MR(X \cap gH) = \beta$ and (by (4.5)) $\deg(X \setminus gH) < \deg(X)$. Since $\deg(gH) = 1$, the same reasoning gives $MR(gH \setminus X) < \beta$. Thus, by induction, both $X \setminus gH$ and $gH \cap X = gH \setminus (gH \setminus X)$ are equal to a boolean combination of cosets of elements of $\mathcal{H}$. Since $X = (gH \cap X) \cup (X \setminus gH)$, we have proved that X is equal to a boolean combination of cosets of elements of $\mathcal{H}$.

Turning to the reverse implication, suppose G is an abelian structure and let $p \in S_n(G)$ have Morley rank β. Let $\mathcal{H} = \{ H : H$ is a subgroup of G^n which is definable over $acl(\emptyset) \}$.

Claim. There is a connected group $H \in \mathcal{H}$ and an $a \in G^n$ such that $MR(H) = \beta$ and the formula defining aH is in p.

Let $\varphi \in p$ be a formula of Morley rank β and degree 1; $X = \varphi(\mathfrak{C})$, which we can take to be a subset of G^n. A series of reductions will show that we can take X to be a coset of some element of $\mathcal{H}$. For H, $K \in \mathcal{H}$ and a, $b \in G^n$, if $Y = aH \cap bK \neq \emptyset$, then Y is a coset of $H \cap K$, also an element of $\mathcal{H}$. Thus X, which is equal to a boolean combination of elements of $\mathcal{H}$, can be written as a finite union of sets of the form $Y \setminus (Z_1 \cup \ldots \cup Z_n)$, where Y and $Z_1, \ldots, Z_n$ are cosets of elements of $\mathcal{H}$. If X is a finite union $Y_1 \cup \ldots \cup Y_k$ then some Y_i has Morley rank β. So, without loss of generality,

X is equal to $a_1H_1 \setminus (b_1K_1 \cup \ldots \cup b_nK_n)$, for some $H_1, K_1, \ldots, K_n \in \mathcal{H}$ and $a_1, b_1, \ldots, b_n \in G^n$. By the same reasoning we can require H_1 to be connected. Without loss of generality, $a_1H_1 \cap b_iK_i \neq \emptyset$, hence a coset of $H_1 \cap K_i$, for $1 \leq i \leq n$. Since H_1 is connected, $MR(H_1 \cap K_i) < MR(H_1)$, for $1 \leq i \leq n$, hence $MR(b_1K_1 \cup \ldots \cup b_nK_n) < MR(H_1)$. Since $MR(X) = \beta$, we conclude that $MR(H_1) = MR(a_1H_1) = \beta$, completing the proof of the claim.

With a and H as in the claim, let $q \in S_n(G)$ be the unique generic type of H. Then aq is the unique element of $S_n(G)$ having Morley rank β and containing the formula defining aH. Thus $p = aq$, completing the proof.

Corollary 4.3.8. *Let G be an $A-$definable abelian structure in the universal domain $\mathfrak{C}$ of an uncountably categorical theory. Let $p \in S_n(G)$ have Morley rank β and canonical parameter c. Then, there is a connected group $H \subset G^n$ of Morley rank β, definable over $acl(A)$, and an $a \in G^n$ such that the formula defining aH is in p and a name for aH is interdefinable with c over A.*

Proof. By the previous lemma there is a connected group $H \subset G^n$, definable over $acl(A)$, and an $a \in G^n$ such that the formula defining aH is in p. Let a^* be a name for aH. To show that a^* is interdefinable over A with c it suffices to prove

Claim. If f is an automorphism of $\mathfrak{C}$ which is the identity on A, then $f(p) = p$ if and only if $f(a^*) = a^*$.

Let $\mathrm{Aut}_A(\mathfrak{C})$ denote the set of automorphisms of $\mathfrak{C}$ which are the identity on A. The formula ψ over a^* defining aH has Morley rank β, degree 1, and is in p. Thus, if $f \in \mathrm{Aut}_A(\mathfrak{C})$ and $f(a^*) = a^*$, then $f(p) = p$ (since p is the unique extension of ψ or Morley rank β in $S_n(G)$). Now suppose $f \in \mathrm{Aut}_A(\mathfrak{C})$ and $f(p) = p$. Then, $f(\psi) \in p$, hence $aH \cap f(aH)$ has Morley rank β. The connected group H cannot have a proper definable subgroup of Morley rank β, so $H = f(H)$. Consequently, $aH = f(aH)$ and $f(a^*) = a^*$, completing the proof of the claim and the corollary.

Corollary 4.3.9. *Let G be an $A-$definable abelian structure in the universal domain $\mathfrak{C}$ of an uncountably categorical theory. Let $X \subset G^n$ be definable and*

$$\mathcal{H} = \{\, H : \ H \text{ is a subgroup of } \ G^n \text{ definable over } \ acl(A) \,\}.$$

Then X is equal to a boolean combination of cosets of elements of $\mathcal{H}$.

Proof. See Exercise 4.3.5.

We are now in a position to prove the easy direction of Theorem 4.3.3.

Lemma 4.3.11. *Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory which contains an infinite definable abelian structure G. Then $\mathfrak{C}$ is $1-$based.*

Proof. Suppose, to the contrary, that $\mathfrak{C}$ is not $1-$based. Let G be definable over A. Since G is infinite there is a strongly minimal $D \subset G$ definable over some $B \supset A$. Since $\mathfrak{C}$ is not $1-$based, D is not locally modular (by Theorem 4.3.1). By Lemma 4.2.4, D has a plane curve C such that the dimension (over B) of the canonical parameter c of C is > 1. Let a be a generic element of C over c, $p \in S(\mathfrak{C})$ the free extension of $tp(a/c)$, and observe that c is a canonical parameter of p. Since G is an abelian structure there is (by Corollary 4.3.8) a connected strongly minimal subgroup H of G^2, definable over $acl(A)$, and a $b \in G$ such that $C \cap bH$ is also strongly minimal and c is interalgebraic over A with a name b^* for bH. Since a is a generic of C, $a \in bH$, hence b^* and c are in $acl(A \cup \{a\})$. This contradicts Lemma 4.2.8(ii), completing the proof.

The proof of Theorem 4.3.3 will be complete once we have shown

Proposition 4.3.4. *Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory which contains an infinite definable group G, and assume $\mathfrak{C}$ is $1-$based. Then, G satisfies*

(*) *for any $n < \omega$ and $p \in S_n^G(\mathfrak{C})$ there is a connected group $H \subset G^n$ definable over $acl(A)$ such that p is a translate of the generic type of H.*

A reasonable amount of work, found in subsequent lemmas, is required to prove the proposition. Most of the work revolves around stabilizers of types, introduced in Section 3.5. Indeed, the group H appearing in (*) is the stabilizer of p. For G an $\omega-$stable group, $p \in S_n^G(G)$ can also be viewed as an element of $S_1^{G^n}(G)$. Thus, the facts proved in Section 3.5 about $1-$types over an $\omega-$stable group G extend transparently to $S_n^G(G)$, for any $n < \omega$. Also, facts proved about subgroups of G extend immediately to subgroups of G^n.

Remember, when H is an infinite group defined in the universal domain $\mathfrak{C}$ of an uncountably categorical theory then $\mathfrak{C}$ is $1-$based if and only if the restriction to H is also $1-$based. (See Corollary 4.3.7.)

Stabilizers enter our proof via

Lemma 4.3.12. *Let G be an $\omega-$stable group, $p \in S_n(G)$ and $S = stab(p)$. Then*

(i) $MR(S) \leq MR(p)$, and

(ii) if $MR(S) = MR(p)$, p is a translate of a generic type of S and S is connected.

Proof. (i) This was proved in Lemma 3.5.1(ii).

(ii) Let A be a finite set over which p is definable and remember that S is a definable group over A. Let G' be a saturated model containing A, a a realization of $p \restriction G' = p'$ and g an element of S generic over $G' \cup \{a\}$. Since $tp(a/G' \cup \{g\}) = p \restriction (G' \cup \{g\})$, ga also realizes $p \restriction (G' \cup \{g\})$. By assumption, $MR(S) = MR(p)$, hence $MR(g/G' \cup \{a\}) = MR(p)$. Since g

and ga are interdefinable over $G' \cup \{a\}$, $MR(ga/G' \cup \{a\}) = MR(p)$; i.e., ga and a are G'−independent. Thus, $tp(ga/G' \cup \{a\}) = p \upharpoonright (G' \cup \{a\})$.

Claim. S is connected.

Assuming that S is not connected there is an element g' of S, generic over $G' \cup \{a\}$, such that $tp(g'/G') \neq tp(g/G')$. Repeating the above argument, $g'a$ is also a realization of $p \upharpoonright (G' \cup \{a\})$. An automorphism f of G which is the identity on $G' \cup \{a\}$ and maps ga to $g'a$ must map g to g'. This contradiction proves the claim.

Claim. Given $r \in S(G)$ the generic of S, p' is a right translate of $r \upharpoonright G'$.

Since G' is saturated there is a $b \in G'$ realizing $p \upharpoonright A$. Since S is connected, $r \upharpoonright A$ has Morley degree 1 (by Corollary 3.5.3). Then, $tp(g/G' \cup \{a\}) = r \upharpoonright (G' \cup \{a\})$, which is a free extension of $r \upharpoonright A$, does not split over A (by Theorem 3.3.1(i)). Thus, $tp(a/A \cup \{g\}) = tp(b/A \cup \{g\})$ and $tp(gb/A \cup \{g\})$ is also $p \upharpoonright (A \cup \{g\})$. Repeating the first paragraph of the proof for b instead of a, gb realizes p'. Drawing these facts together, for $r' = r \upharpoonright G' = tp(g/G')$, $p' = r'b$, proving the claim.

By the second claim and Lemma 3.5.1, p is a right translate of the generic of S in $S(G)$, completing the proof.

Lemma 4.3.13. *Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory which contains an infinite definable group G, and assume $\mathfrak{C}$ is 1−based. Let $p \in S_n^G(\mathfrak{C})$ and $S = stab(p)$. Then, $MR(S) = MR(p)$ and S is definable over $acl(\emptyset)$.*

Proof. Without loss of generality, $n = 1$. Let c be the canonical parameter of p and let $\{a, a'\}$ be a Morley sequence over c in $p \upharpoonright c$. Let $x = a'a^{-1}$ and notice that $MR(x) \geq MR(x/c) \geq MR(x/\{c, a\}) = MR(a'/\{c, a\}) = MR(p)$. Now let $g \in G$ be generic over $c \cup \{a, a'\}$ and let $q = pg$. Since right translates have the same stabilizer (Lemma 3.5.1(iii)), $stab(q) = S$. Proving that $MR(S) = MR(p) = MR(q)$ has been reduced to verifying that $x \in stab(q) = S$.

Since a and a' realize $p \upharpoonright \{c, g\}$, ag and $a'g$ realize $q_0 = q \upharpoonright \{c, g\}$. Since g is generic over $\{c, a, a'\}$, ag is generic over $\{c, a, a'\}$, hence x is independent from ag. The canonical parameter d of q is in $acl(ag)$ (since $\mathfrak{C}$ is 1−based) hence x is independent from $\{ag, d\}$. Furthermore, $a'g = xag$ by Corollary 3.5.1. Since q is the unique free extension of $tp(ag/c)$ we conclude that $x \in stab(q) = S$, completing the proof that $MR(S) = MR(p)$.

It remains to show that S is definable over $acl(\emptyset)$. Presently we only know S is definable over c, by the formula $\sigma(x, c)$, let's say. By Lemma 4.3.12(ii) S is connected, so we can take σ to have degree 1. Let $q_1 = q \upharpoonright d$. Recapping what was proved above,

(a) d and c are independent.
(b) If $e \in S$ is generic over c and b realizes $q \upharpoonright d$ and is generic over $\{e, c\}$ then $e \cdot b$ realizes q_1.

(c) For any d' realizing the type of d over $acl(\emptyset)$ independent from c, the conjugate of q_1 over d' is $pg' \restriction d'$ for some g'.

We will show that $\sigma(v, c)$ is equivalent to any conjugate of itself over $acl(\emptyset)$.

Let c' be any realization of $tp(c/acl(\emptyset))$. Choose d' realizing $tp(d/acl(\emptyset))$ independent from $\{c, c'\}$. Let q_1' be the conjugate of q_1 over d' and b a realization of q_1' independent from $\{c, c'\}$ over d'. Let $q_1'' \in S_n^G(\mathfrak{C})$ be the unique free extension of q_1'. By (c) and Corollary 3.5.1, q_1' is a right translate of p, hence $S = \sigma(G)$ is $stab(q_1'')$. Since $stab(q_1'')$ is definable over d', $\sigma(v, c)$ is equivalent to a formula over d'. By the stationarity of types over $acl(\emptyset)$, c' and c have the same type over d', hence $\sigma(v, c')$ is equivalent to the same formula over d'. This proves the equivalence of $\sigma(v, c)$ and $\sigma(v, c')$, hence $\sigma(v, c)$ is equivalent to a formula over $acl(\emptyset)$.

This proves the lemma.

Combining Lemmas 4.3.10, 4.3.12 and 4.3.13 completes the proof of Theorem 4.3.3.

The following comes in handy when proving facts about the definable subsets of an abelian structure. (It is standard to use additive notation in an abelian structure.)

Corollary 4.3.10. *Let G be a 1−based uncountably categorical group, $a \in G$ and p the unique free extension of $p' = tp(a/acl(\emptyset))$. There is a connected group S definable over $acl(\emptyset)$ (namely, the stabilizer of p) such that for any realization b of p',*

(1) *$b - a \in S$, and*
(2) *if b is independent from a, $b - a$ is a generic of S.*

Proof. Let S be the stabilizer of p, which by Lemma 4.3.13 is connected, definable over $acl(\emptyset)$ and has Morley rank $\alpha = MR(p)$.

Part (2) will be proved first (in a round about way). Let b be a realization of p' independent from a. Given $c \in S$ generic over a, $c + a$ realizes p'. Let c be a generic element of S which is independent from a. Since c and $c + a$ are interalgebraic over a, $MR(c+a/a) = MR(c/a) = \alpha$; i.e., $c+a$ is independent from a. Since p' is stationary, b and $c+a$ have the same type over $acl(\emptyset) \cup \{a\}$. Thus, $tp(b-a/acl(\emptyset) \cup \{a\}) = tp(c/acl(\emptyset))$, hence $b-a$ is a generic of S, proving (2).

Now assume only that b realizes p'. Let d be a realization of p independent from both a and b. By (2), both $a - d$ and $d - b$ are generic elements of S. Thus, $a - b = a - d + d - b$ is in S, completing the proof of the corollary.

As stated earlier, the most basic example of an abelian structure is an infinite vector space. In fact, it will be shown later that any module (formulated in the natural language for modules over a fixed ring) is an abelian structure. The next group of results investigates the degree to which every (uncountably categorical) abelian structure is a module, culminating in a proof that a strongly minimal abelian structure is (nearly) a vector space over a division ring of definable endomorphisms.

Corollary 4.3.11. *Let G be a $1-$based uncountably categorical group. Then, any connected definable subgroup of G^n is definable over $acl(\emptyset)$.*

Proof. Let H be a connected definable subgroup of G^n and let $p \in S_n(G)$ be the generic type of H. H is the stabilizer of p (by Corollary 3.5.3), hence H is definable over $acl(\emptyset)$ (by Lemma 4.3.13).

Definition 4.3.7. *A group is called* abelian-by-finite *if it has a definable abelian subgroup of finite index.*

Corollary 4.3.12. *A $1-$based uncountably categorical group G is abelian-by-finite.*

Proof. If G^o is abelian, G is abelian-by-finite, so we may take G to be connected. We will show that $Z(G)$ = the center of G, has finite index in G hence is all of G.

For $a \in G$ let $H_a = \{ (g, a^{-1}ga) : g \in G \}$, a definable subgroup of G^2.

Claim. H_a is connected.

Let $n = MR(G)$. If $(g, h) \in H_a$, h is interalgebraic with g over a. Hence, $MR(H_a)$ is also n. Suppose K is a definable subgroup of H_a of finite index, and let K_0, K_1 be the projections of K onto the first and second coordinates, respectively. Then, $n = MR(K) = MR(K_0) = MR(K_1)$, so, by the connectedness of G, $K_0 = G$. For any $g \in G$ there is a unique $x \in H_a$ whose first coordinate is x. Thus, K must be all of H_a, proving the claim.

By Lemma 4.3.11, H_a is definable over $acl(\emptyset)$, for any $a \in G$. A compactness argument shows that $\{ H_a : a \in G \}$ is some finite set of groups $\{H_{a_1}, \ldots, H_{a_k}\}$. For $a, b \in G$, $aZ(G) = bZ(G)$ if and only if $H_a = H_b$, hence $Z(G)$ has finite index in G. Since G is connected we conclude that G is abelian, as desired.

Definition 4.3.8. *Let G be a group which is $\bigwedge-$definable over A. Then, G^- denotes $acl(A) \cap G$. If B and C are subsets of G or elements of G, we write $B =^* C$ if $B + G^- = C + G^-$.*

With notation as in the definition, G^- is a subgroup of G, which is definable exactly when it is finite. The equivalence relation $=^*$ is simply the inverse image of equality under the quotient map from G into G/G^-.

Showing that a strongly minimal abelian structure is close to being a vector space requires the introduction of definable homomorphisms, accomplished as follows.

Definition 4.3.9. *Let G_0 and G_1 be $A-$definable groups (in the universal domain $\mathfrak{C}$ of a complete theory). A subgroup H of $G_0 \times G_1$ is called a $*-$homomorphism of G_0 into G_1 if*

– *H is definable,*
– *the projection of H onto the first coordinate is all of* G_0, *and*
– $\{ a \in G_1 : (0, a) \in H \} = K$ *is finite.*

H is a $*-$endomorphism of G_0 *if it is a* $*-$*homomorphism of* G_0 *into* G_0. *H is a* $*-$isomorphism of G_0 onto G_1 *if the projection of H onto the second coordinate is* G_1 *and* $\{ a \in G_0 : (a, 0) \in H \}$ *is finite.*

With notation as in the definition, the $*-$homomorphism H is the graph of a definable homomorphism $\sigma_H : G_0 \longrightarrow G_1/K$, and, if H is $B-$definable, K is also $B-$definable. This homomorphism will also be called a $*-$homomorphism of G_0 into G_1. For $a \in G_0$, $\sigma_H(a)$ denotes the appropriate coset of K (hence a finite subset of G_1). Several elementary results and definitions are collected in

Definition 4.3.10. *Let G, H and K be* $\emptyset-$*definable abelian groups in the universal domain of a complete theory. Let* $\mathcal{A} = \{ \sigma : \sigma$ *is a* $*-$*homomorphism from G into H*$\}$ *and* $\mathcal{B} = \{ \sigma : \sigma$ *is a* $*-$*homomorphism from H into K*$\}$. *Addition on* $\mathcal{A}$ *is defined by the rule:*

$$\text{for } \sigma, \tau \in \mathcal{A} \text{ and } a \in G,\ (\sigma + \tau)(a) = \sigma(a) + \tau(a),$$

with the + on the right-hand side denoting addition on sets.

Multiplication between $\mathcal{A}$ *and* $\mathcal{B}$ *is defined by:*

$$\text{for } \sigma \in \mathcal{B}, \tau \in \mathcal{A} \text{ and } a \in G,\ \sigma \cdot \tau(a) = \sigma(\tau(a)).$$

(We will largely be interested in multiplication when $G = H = K$.*)*

For $\sigma, \tau \in \mathcal{A}$ *we write* $\sigma =^* \tau$ *if the graphs of* σ *and* τ *are* $=^*$ *as subsets of* $G \times H$; *i.e.,* $\sigma =^* \tau$ *if for all* $a \in G$, $\sigma(a) =^* \tau(a)$.

Let $\mathrm{Hom}^*(G, H) = \mathcal{A}/=^*$; *i.e.,* $\mathrm{Hom}^*(G, H)$ *is the set of equivalence classes of elements of* $\mathcal{A}$ *with respect to the equivalence relation* $=^*$. *Let* $\mathrm{End}^*(G)$ *denote* $\mathrm{Hom}^*(G, G)$. *The + operation extends to* $\mathrm{Hom}^*(G, H)$ *and* $\cdot$ *extends to* $\mathrm{End}^*(G)$ *in the obvious ways (for example,* $(\sigma/=^*) + (\tau/=^*) = (\sigma + \tau)/=^*$*). An element of* $\mathrm{Hom}^*(G, H)$ *is also called a* $*-$*homomorphism from G into H and an element of* $\mathrm{End}^*(G)$ *is called a* $*-$*endomorphism of G.*

Most statements made below involving a $*-$homomorphism σ remain valid after replacing σ by any $*-$homomorphism $\tau =^* \sigma$. This excuses the abuse of calling an element of $\mathrm{Hom}^*(G, H)$ a $*-$homomorphism. If G and H are $\emptyset-$definable groups, $a \in G$ and $\alpha \in \mathrm{Hom}^*(G, H)$ we write $\alpha(a) =^* b$ if there is a $*-$homomorphism σ such that α is $\sigma/=^*$ and $\sigma(a) =^* b$.

Remark 4.3.6. Let G and H be $\emptyset-$definable abelian groups in the universal domain of a complete theory. Suppose that σ is a $*-$isomorphism from G onto H and let $S \subset G \times H$ be the graph of σ. Let S^{-1} denote the inverse of S as a binary relation. Then, S^{-1} is the graph of a $*-$isomorphism τ from H into G and $\tau \cdot \sigma$ is a $*-$endomorphism of G which is $=^*$ the identity on G.

The straightforward proof of the following lemma is left to the reader.

Lemma 4.3.14. *Let G and H be definable abelian groups in the universal domain of a complete theory. Under the operations $+$ and $\cdot$ defined above,*
(i) $\mathrm{Hom}^*(G, H)$ *is an abelian group, and*
(ii) $\mathrm{End}^*(G)$ *is a ring.*

Definition 4.3.11. *For G and H definable abelian groups,* $\mathrm{Hom}^*(G, H)$ *is called the* group of $*-$homomorphisms from G into H *and* $\mathrm{End}^*(G)$ *is the* $*-$endomorphism ring of G.

If G and H are $\emptyset-$definable abelian groups, $a \in G$ and $\alpha \in \mathrm{Hom}^*(G, H)$, then $b =^* \alpha(a) \implies b \in acl(a)$. The next proposition shows (surprisingly) that all algebraic closure in a generic element of a connected $1-$based group is witnessed by $*-$homomorphisms.

Proposition 4.3.5. *(i) Let G and H be $\emptyset-$definable groups in a $1-$based uncountably categorical theory with G connected. Let A be a set, a an element of G generic over A and $b \in acl(A \cup \{a\}) \cap H$. Then, there is a $*-$homomorphism σ from G into H such that σ is definable over $acl(\emptyset)$ and $\sigma(a) =^* b'$ for some d, for some d independent from A with $MR(d) = MR(b)$. Furthermore, if $A = \emptyset$ we may take d to be b.*

(ii) Suppose, in addition, that $G = G_0 \times \ldots \times G_n$ and $a = (a_0, \ldots, a_n)$, where G_i is a connected group definable over $acl(\emptyset)$ and $a_i \in G_i$, for $i \leq n$. Then, there are $\sigma_i \in$ $\mathrm{Hom}^*(G, H)$*, for $i \leq n$, such that $\sigma(a) =^* \sum_{i \leq n} \sigma_i(a_i)$.*

Proof. (i) Let $\bar{G}$ be the $\emptyset-$definable group $G \times H$. Let $p = tp((a, b)/acl(A))$, $X = p(\mathfrak{C})$, and S the stabilizer of the unique free extension of p in $S(\bar{G})$.

Claim. S is the graph of a $*-$homomorphism σ from G into H, definable over $acl(\emptyset)$.

Let $K = \{y : (0, y) \in S\}$. Since $b \in acl(A \cup \{a\})$, K is finite. For (a', b') an element of X $A-$independent from (a, b), $(a, b) - (a', b') \in S$ by Corollary 4.3.10. Hence, the projection of S onto the first coordinate contains a generic element, namely $a - a'$. Since G is connected, the projection of S onto the first coordinate must be all of G. Since the stabilizer of any type in a $1-$based group is definable over $acl(\emptyset)$, σ is definable over $acl(\emptyset)$, proving the claim.

Since a is independent from A any $d \in \sigma(a)$ is independent from A. Moreover, $MR(d) = MR(b)$ since p is a translate of the generic type of S.

Now suppose $A = \emptyset$.

Claim. There is an element $-c =^* b$ such that $-c \in \sigma(a)$.

There is an element c such that $(-a, c) \in S$. Since $(-a, c) = (0, b + c) - (a, b)$, $(0, b + c)$ and (a, b) have the same coset with respect to S. Since S is definable over $acl(\emptyset)$ the difference of any two realizations of $q = tp((0, b + c)/acl(\emptyset))$

is in S. Since the group K is finite, the set of realizations of q is finite; i.e., $b+c \in H^-$. This completes the proof of the claim and (i) of the proposition.

(ii) Remember from Exercise 3.5.8 that a is a generic of G if and only if a_i is a generic of G_i (for $i \leq n$) and $\{a_0, \ldots, a_n\}$ is independent. Let S and σ be defined as in the proof of (i), bearing in mind that S is now a subgroup of $G_0 \times \ldots \times G_n \times H$. For $i \leq n$, let $S_i = \{ (x,y) : (0, \ldots, 0, x, 0, \ldots, 0, y) \in K \}$, where x is in the coordinate corresponding to G_i. As in the proof of (i), for each $i \leq n$, S_i is the graph of a $*-$homomorphism σ_i from G_i into H. It is easily verified that $\sum_{i \leq n} \sigma_i(a_i) =^* \sigma(a)$, proving the proposition.

Corollary 4.3.13. *Let G be a $1-$based uncountably categorical group. Any element of* End*(G) *is definable over acl($\emptyset$).*

(This corollary follows immediately from the preceding proposition.)

Theorem 4.3.4. *Let G be a $1-$based strongly minimal group.*

(i) $R =$ End(G) is a division ring.*

(ii) Let $b, a_0, \ldots, a_n \in G$ and suppose that b depends on $\{a_0, \ldots, a_n\}$. Then, there are $\alpha_0, \ldots, \alpha_n \in R$ such that $b =^* \sum_{i \leq n} \alpha_i a_i$.

Proof. (i) Let $\sigma \in R$ be nonzero. Both $K = ker(\sigma)$ and $H =$ the range of σ are $\emptyset-$definable subgroups of G. Since G is strongly minimal it has no infinite proper definable subgroup. Since σ is nonzero, this implies that K is a finite subgroup of G^- and $H = G$. Thus, the inverse of σ (as a relation on $G \times G$) is the graph of $*-$endomorphism of G. In other words, every nonzero element of G is invertible.

(ii) Since G is strongly minimal, $b \in acl(a_0, \ldots, a_n)$. There are $\alpha_i \in R$, $i \leq n$, such that $b =^* \sum_{i \leq n} \alpha_i a_i$ (by Proposition 4.3.5(ii)) completing the proof.

The following definition is a natural consequence of the theorem.

Definition 4.3.12. *A $1-$based strongly minimal group G is called a $*-$*vector space. *If $R \subset$ End*(G) *is a division ring of $*-$endomorphisms of G. Then G is called an $R - *-$*vector space.

An $R - *-$vector space G, for $R =$ End*(G), falls short of being a (quantifier-eliminable) vector space only in two ways.

- For $\sigma \in R =$ End*(G) and $a \in G$, $\sigma(a)$ may be a finite subset of G containing more than one element.
- G^- may contain a nonzero element.

Given $\sigma \in R$ and $a \in G$, $\sigma(a) \subset a + G^-$, hence σ induces an endomorphism of the group G/G^-. Moreover, G/G^- is an $R-$vector space. When $G^- = \{0\}$ this observation and Theorem 4.3.4 yield

Corollary 4.3.14. *Let G be a $1-$based strongly minimal such that $G^- = \{0\}$. There is a division ring R of endomorphisms of G, each definable over $acl(\emptyset)$, such that G is an $R-$vector space and every definable relation on G is equivalent to a boolean combination of $R-$linear relations.*

Proof. Left to the reader in Exercise 4.3.6.

Corollary 4.3.15. *The pregeometry on a $1-$based strongly minimal group G is projective.*

Proof. (We know simply from Theorem 4.3.1 that the pregeometry on G is locally projective.) Let $a, b, c_0, \ldots, c_n \in G$ be such that $a \in acl(b, c_0, \ldots, c_n)$. By Theorem 4.3.4 there are $\beta, \gamma_0, \ldots, \gamma_n \in \mathrm{End}^*(G)$ such that $a =^* \beta b + \gamma_0 c_0 + \ldots + \gamma_n c_n$. Any $d =^* \gamma_0 c_0 + \ldots + \gamma_n c_n$ is an element of $acl(c_0, \ldots, c_n) \cap G$ such that $a \in acl(b, d)$. This proves the projectivity of G.

Finally, we see that in the context of a $1-$based uncountably categorical theory strongly minimal groups are unique, up to $*-$isomorphism.

Corollary 4.3.16. *Let G and H be $\emptyset-$definable strongly minimal groups in the universal domain of a $1-$based uncountably categorical theory. Then, there is a $*-$isomorphism σ from G onto H which is definable over $acl(\emptyset)$.*

Proof. By Corollary 4.3.15, G and H are modular strongly minimal sets. Corollary 4.3.5 yield $a \in G \setminus G^-$ and $b \in H \setminus H^-$ which are interalgebraic over $\emptyset$. There is a $*-$homomorphism σ from G into H, definable over $acl(\emptyset)$, with $\sigma(a) =^* b$, by Proposition 4.3.5(i). Since G and H are strongly minimal, σ must be a $*-$isomorphism.

Historical Notes. Proposition 4.3.2 is more or less due to Shelah [She90, III.5]. In Zil'ber's early writings he worked with the condition "$\mathfrak{C}$ does not contain a definable pseudoplane". This property developed into a statement about canonical parameters in [CHL85]. Our main result, Theorem 4.3.1, is equivalent to one by Zil'ber in [Zil84a] and [Zil84b], and in the totally categorical context, implicit in [CHL85]. A generalization of the theorem, with up to date definitions, is found in [Bue86]. A weak version of Theorem 4.3.3 can be extracted from Zil'ber's writings. In its present form the theorem was first proved (independently) by Hrushovski and Pillay [HP87]. Proposition 4.3.5 and related results are due to Hrushovski in [Hru87].

Exercise 4.3.1. Show that any definable subset of an almost strongly minimal set is finite or almost strongly minimal.

Exercise 4.3.2. Prove Corollary 4.3.4.

Exercise 4.3.3. Prove Corollary 4.3.7.

Exercise 4.3.4. Prove that a vector space is an abelian structure.

Exercise 4.3.5. Prove Corollary 4.3.9

Exercise 4.3.6. Proof Corollary 4.3.14.

Exercise 4.3.7. Let G and H be $\emptyset-$definable strongly minimal groups in the universal domain of a $1-$based uncountably categorical theory. Show that $\mathrm{End}^*(G) \cong \mathrm{End}^*(H)$ (as rings).

4.4 Automorphism Groups of Constructions

Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory. We proved that $\mathfrak{C}$ is asm-constructible, in fact, for any $a \in \mathfrak{C}$ there are $c_0, \ldots, c_n$, with $c_n = a$, such that $tp(c_i/\{c_0, \ldots, c_{i-1}\})$ is almost strongly minimal. If $\mathfrak{C}$ is also $1-$based it is rk1-constructible. In this way $\mathfrak{C}$ is decomposed in terms of strongly minimal sets. In this section the structure gleaned from this decomposition is strengthened by describing, for X_1 and X_2 two almost strongly minimal subsets of $\mathfrak{C}$, the definable relations on $X_1 \times X_2$. We will see that (among other things) it always possible to choose the almost strongly minimal sets in a construction (like the one above) to be closely bound to one another, in a sense to be made precise momentarily. First a few motivating examples.

Example 4.4.1. (i) Let D be a definable set (over $\emptyset$) in the universal domain $\mathfrak{C}$ of a complete theory. A definable $X \subset D^{eq}$ is contained in $dcl(D)$. Any definable relation on $X \cup D$ reduces to a definable relation on D (in a way the reader is left to formalize). Notice that the condition $Y \subset dcl(D)$ is equivalent to "any $f \in \mathrm{Aut}(\mathfrak{C})$ which is the identity on D is also the identity on Y." Here the definable set X is "tightly bound" to D.

(ii) Let k_0 be an algebraically closed field of characteristic 0 and k_1 a proper elementary submodel. Let L be the language of fields together with a unary predicate P and $M_0 = (k_0, k_1)$ the model in L where k_1 interprets P and k_0 is the universe. Let (k^*, ℓ^*) be the universal domain of $Th(M_0)$. The relationship between k^* and the definable subset ℓ^* is described classically with the Galois group of k^* over l^*; i.e., the group of field automorphisms of k^* which fix ℓ^* pointwise. Below we use such automorphism groups to describe the relationships between two definable sets.

(iii) Let M be the abelian group $\bigoplus_{i<\omega}(\mathbb{Z}_4)_i$ and M^* the universal domain of $Th(M)$ (see Example 4.3.1(iv)). Let $V = 2M^*$, a a generic of M^* and $H = a + V$, which is also definable over $2a \in V$. For any $b \in H$ there is an automorphism of M^* which is the identity on V and maps a to b. Since $H \subset dcl(V \cup \{b\})$ for any $b \in H$, there is no nontrivial automorphism of M^* which fixes $V \cup \{b\}$ pointwise. In other words, the group G_0 of all automorphisms of M^* which fix V pointwise acts regularly on H. Let $G = \{\sigma : \sigma = \tau \restriction H$ for some $\tau \in G_0\}$.

Claim. $G \cong (V, +)$.

For $c \in V$ let τ_c be defined by: $\tau_c(x) = x + c$, for all $x \in H$. Observe that τ_c is in G. Fixing $a \in H$, any $\sigma \in G$ is determined by $\sigma(a)$; i.e., if $b = \sigma(a) = \sigma'(a)$, where $\sigma' \in G$, then $\sigma' = \sigma$. Since any $b \in H$ is $a + c$ for some $c \in V$, every $\sigma \in G$ is τ_c for some $c \in V$. Moreover, $\tau_c \cdot \tau_d = \tau_{d+c}$. This proves the claim.

(iv) Let P be the universal domain of the theory of a projective plane over an algebraically closed field, say the complex numbers. (P is formulated in a 2−sorted language with a single binary relation ϵ. The first sort in P is the set of "points" of P, the second the set of "lines" of P and $x \epsilon \ell$ is read "x lies on ℓ".) Let ℓ_1 and ℓ_2 be names for two distinct lines, D_i the set of points on ℓ_i, for $i = 1, 2$. Let $G_0 = \{\sigma \in \mathrm{Aut}(P) : \sigma \restriction (D_1 \cup \{\ell_1, \ell_2\}) = \text{ the identity } \}$ and $G = \{\sigma \restriction D_2 : \sigma \in G_0\}$. The reader is asked to show the following in Exercise 4.4.1.

(a) For any $a_1 \neq a_2$ and $b_1 \neq b_2$ in $D_2 \setminus D_1$ there is a $\sigma \in G$ such that $\sigma(a_1) = b_1$ and $\sigma(a_2) = b_2$.
(b) Given $a_1 \neq a_2$ in $D_2 \setminus D_1$, $D_2 \subset dcl(D_1 \cup \{a_1, a_2\})$.

In group action terminology the action of G on $D_2 \setminus D_1$ is sharply 2−transitive. (It is 2−transitive because there is only one orbit in the set of distinct pairs from $D_2 \setminus D_1$. It is sharply 2−transitive because (by (b)) the σ in (a) is unique.)

The condition stated intuitively as "D_2 is closely bound to D_1" is formalized in

Definition 4.4.1. *Let $\mathfrak{C}$ be the universal domain of a complete theory. Let D_1 be an A−definable subset of $\mathfrak{C}$ and D_2 a subset of $\mathfrak{C}$ definable over $B \subset D_1 \cup A$. D_2 is said to be* finitely generated over $D_1 \cup A$ *if there are:*

(1) *a finite $\bar{b} \subset D_2$, and*
(2) *a function f, definable over $B \cup \bar{b}$, taking D_1^n onto D_2, for some n.*

When (1) and (2) hold $\bar{b}$ is called a fundamental generator of D_2 over $D_1 \cup A$ *and f is called the* generating function of D_2 over $D_1 \cup A$.

In Example 4.4.1(i) X is finitely generated over D with fundamental generator $\emptyset$. In Example 4.4.1(ii) k^* is not finitely generated over ℓ^*. The coset H of Example 4.4.1(iii) is finitely generated over V; any $b \in H$ is a fundamental generator with generating function $+$. Finally the projective line D_2 in Example 4.4.1(iv) is finitely generated over $D_1 \cup \{\ell_1, \ell_2\}$ with any pair of distinct points of $D_2 \setminus D_1$ as fundamental generator. It is left to the reader to describe the corresponding generating function.

Remark 4.4.1. Let $\mathfrak{C}$ be the universal domain of a complete theory, D_1 an $\emptyset$−definable subset of $\mathfrak{C}$ and D_2 a subset of $\mathfrak{C}$ definable over $B \subset D_1$. Let $\mathfrak{C}$, D_1 and D_2 be as in the definition, with D_1 $\emptyset$−definable (for simplicity).

(i) If D_2 is finite it is finitely generated over D_1.

(ii) If D_2 is finitely generated over D_1 there is a finite $\bar{b} \subset D_2$ such that $D_2 \subset dcl(D_1 \cup \bar{b})$. Thus, if D_1 is almost strongly minimal, D_2 is also almost strongly minimal (by Lemma 4.3.2).

(iii) If $D_2 \subset D_1^{eq}$ there is a definable function f taking D_1^n onto D_2, for some n. Hence, D_2 is finitely generated over D_1 with $\emptyset$ as a fundamental generator and generating function f.

(iv) Suppose that D_2 is finitely generated over D_1 and let f, B, n and $\bar{b}$ witness this as in the definition. Then, there is a B–definable $Y \subset D_1^{eq}$ and a $B \cup \bar{b}$–definable bijection g between D_2 and Y.

(This shows that there is little difference between a finitely generated set and an element of D_1^{eq}, although parameters outside of D_1^{eq} may be needed to define it.) To prove this fact let $E(x, y)$ be the equivalence relation on D_1^n defined over $B \cup \bar{b}$ by the rule: for all $\bar{x}$, $\bar{y} \in D_1^n$, $E(\bar{x}, \bar{y}) \iff f(\bar{x}) = f(\bar{y})$. Let Y be the set of equivalence classes of E and g the obvious $B \cup \bar{b}$–definable bijection from Y onto D_2 derived from f. Since E is a definable relation on D_1 there is a $B' \subset D_1$ such that E is B'–definable. Hence, $Y \subset D_1^{eq}$.

(v) If D_2 is finitely generated over $D_1 \cup A$ and D_3 is finitely generated over $D_2 \cup B$, then D_3 is finitely generated over $D_1 \cup A \cup B$. (The proof is left to the reader in Exercise 4.4.2.)

(vi) Let D_1 be A–definable and D_2 definable over $A \cup D_1$. Suppose there are: $\bar{b} \subset D_2$, a definable $X \subset D_1^n$ (for some n) and a $(A \cup \bar{b})$–definable function f taking X onto D_2. It is easy to find from f a function defined on all of D_1^n, hence D_2 is finitely generated over $D_1 \cup A$.

In Example 4.4.1(iii), where a is a generic of M^* and $H = a + V$, $\{2a, a\}$ defines an asm-construction of a. Here, H is not only a strongly minimal set over $2a$, but is finitely generated over V (= the strongly minimal set containing $2a$). We will show later that for $\mathfrak{C}$ the universal domain of an uncountably categorical theory and $b \in \mathfrak{C}$, there is an asm-construction $c_0, \ldots, c_n$ of b where c_i is an element of an almost strongly minimal set X_i, definable over $C_i = \{c_0, \ldots, c_{i-1}\}$, such that, for $1 \leq i \leq n$, X_i is finitely generated over $X_{i-1} \cup C_i$. Thus, we can gain more detailed information about an uncountably categorical theory through the relation "D_2 is finitely generated over D_1".

The definable relations holding between the elements of two definable sets are best studied with the following object.

Definition 4.4.2. *Let $\mathfrak{C}$ be the universal domain of a complete theory. Let D_1 be an A–definable subset of $\mathfrak{C}$ and D_2 a subset of $\mathfrak{C}$, $\bigwedge$–definable over $D_1 \cup A$. A map $\sigma : D_2 \longrightarrow D_2$ is an* automorphism of D_2 over $D_1 \cup A$ *if σ is the restriction to D_2 of an element of* $\mathrm{Aut}(\mathfrak{C})$ *which is the identity on $D_1 \cup A$. The collection of all automorphisms of D_2 over $D_1 \cup A$ is a group denoted* $\mathrm{Aut}(D_2/D_1 \cup A)$. *When D_2 is definable and finitely generated over $D_1 \cup A$,* $\mathrm{Aut}(D_2/D_1 \cup A)$ *is called the* binding group of D_2 over $D_1 \cup A$. *When $A = \emptyset$ it is omitted.*

In Example 4.4.1(i), $\mathrm{Aut}(X/D)$ is trivial and in (ii), $\mathrm{Aut}(k^*/\ell^*)$ is $\mathrm{Gal}(k^*/\ell^*)$, the Galois group of k^* over ℓ^*. In the third example, $\mathrm{Aut}(H/V) \cong (V,+)$. In (i) and (iii), with D_1 and D_2 the relevant definable sets, D_2 is finitely generated over D_1. This degree of control over the relations between the elements of D_2 and D_1 is reflected in the simplicity of $\mathrm{Aut}(D_2/D_1)$. In both (i) and (iii) $\mathrm{Aut}(D_2/D_1)$ is a definable group in the following sense.

Definition 4.4.3. *Let $\mathfrak{C}$ be the universal domain of a complete theory. Let D_1 be an $A-$definable subset of $\mathfrak{C}$ and D_2 a subset of $\mathfrak{C}$, $\bigwedge-$definable over $D_1 \cup A$. We say that $\alpha \in G = \mathrm{Aut}(D_2/D_1 \cup A)$ is definable if α agrees with a definable function g_α on D_2. In this case α is identified with a name for g_α. G is called* definable *if every element of G is definable and (G, D_2) is a definable group action.*

Remark 4.4.2. In the definition, when each $\alpha \in G$ is definable $G \subset \mathfrak{C}$ since we identify a definable function with its name. Remember: (G, D_2) is a definable group action if G and D_2 are definable sets and both the group operation and the action of G on D_2 are definable.

The goal of this section is the following set of "Ladder Theorems" by Zil'ber. The first two are improvements of Corollary 4.3.4 and Proposition 4.3.3, respectively.

Theorem 4.4.1 (Main Ladder Theorem). *Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory and a an element. Then there is a sequence $a_0, \ldots, a_{n-1}, a_n = a$ and definable sets $D_0, \ldots, D_n$ such that for $i \le n$ and $A_i = \{a_0, \ldots, a_{i-1}\}$,*

(1) *$a_i \in dcl(a)$;*
(2) *$a_i \in D_i$;*
(3) *D_0 is $\emptyset-$definable and almost strongly minimal; D_i is finite or almost strongly minimal and D_i is definable over A_i;*
(4) *D_i is finitely generated over $D_0 \cup \ldots \cup D_{i-1}$ (when $i > 0$);*
(5) *$G_i = \mathrm{Aut}(D_i/D_0 \cup \ldots \cup D_{i-1})$ is definable.*

Theorem 4.4.2 (1−based Ladder Theorem I). *Let $\mathfrak{C}$ be the universal domain of a $1-$based uncountably categorical theory and a an element. Then there is a sequence $a_0, \ldots, a_{n-1}, a_n = a$ and definable sets $D_0, \ldots, D_n$ such that for $i \le n$ and $A_i = \{a_0, \ldots, a_{i-1}\}$,*

(1) *$a_i \in acl(a)$;*
(2) *$a_i \in D_i$;*
(3) *D_i is finite or strongly minimal and D_i is definable over A_i;*
(4) *D_i is finitely generated over $D_0 \cup \ldots \cup D_{i-1}$ (when $D_0 \cup \ldots \cup D_{i-1}$ is infinite);*
(5) *$G_i = \mathrm{Aut}(D_i/D_0 \cup \ldots \cup D_{i-1})$ is definable, has Morley rank ≤ 1 and is abelian-by-finite (when $D_0 \cup \ldots \cup D_{i-1}$ is infinite).*

An infinite definable abelian group G in a universal domain is called *minimal abelian* if there is no infinite definable subgroup of G.

Theorem 4.4.3 (Simple Ladder Theorem). *Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory and a an element. There is a sequence of definable sets $D_0, \ldots, D_n$ such that for all $i \leq n$*

(1) *$a \in D_n$.*
(2) *D_i is finite or almost strongly minimal and finitely generated over $D_0 \cup \ldots \cup D_{i-1}$ (when $D_0 \cup \ldots \cup D_{i-1}$ is infinite).*
(3) *If $D_0 \cup \ldots \cup D_{i-1}$ is infinite, $G_i = \mathrm{Aut}(D_i/D_0 \cup \ldots \cup D_{i-1})$ is definable. When G_i is infinite it is simple or minimal abelian.*

Theorem 4.4.4 (1−based Ladder Theorem II). *Let $\mathfrak{C}$ be the universal domain of a 1−based uncountably categorical theory and a an element. Then there is a sequence $a_0, \ldots, a_{n-1}, a_n = a$ and definable sets $D_0, \ldots, D_n$ such that for $i \leq n$ and $A_i = \{a_0, \ldots, a_{i-1}\}$,*

(1) *$a_i \in acl(a)$;*
(2) *$a_i \in D_i$;*
(3) *D_i is finite or strongly minimal and D_i is definable over A_i;*
(4) *D_i is finitely generated over $D_0 \cup \ldots \cup D_{i-1}$ (when $D_0 \cup \ldots \cup D_{i-1}$ is infinite);*
(5) *When $D_0 \cup \ldots \cup D_{i-1}$ is infinite both $G_i = \mathrm{Aut}(D_i/D_0 \cup \ldots \cup D_{i-1})$ and the action of G_i on D_i are definable over $D_0 \cup \ldots \cup D_{i-1}$. Moreover, when G_i is infinite it is strongly minimal (and abelian).*

The 1−based Ladder Theorem I will follow rather quickly from the Main Ladder Theorem using Proposition 4.3.3. The Simple Ladder Theorem says that in the sequence of almost strongly minimal sets we can choose $G_i = \mathrm{Aut}(D_i/D_0 \cup \ldots \cup D_{i-1})$ to be finite, minimal abelian, or simple if we are willing to sacrifice other properties; namely, that $dcl(a) \cap D_i$ is nonempty and D_i is definable over a.

With notation as in the Main Ladder Theorem, $\{a_0, \ldots, a_n\}$ is an asm-construction of a. The existence of a sequence satisfying (1)–(3) was proved in Corollary 4.3.4. The object of this section is to obtain an asm-construction with the additional properties specified in (4)–(6). Certain results leading up to Corollary 4.3.4 (Proposition 4.3.2, for one) will be redone in this section to emphasize different points and increase the scope of the methods.

The first major result of the section indicates when we can expect $\mathrm{Aut}(D_2/D_1)$ to be definable.

Theorem 4.4.5 (Binding Group Theorem). *Let $\mathfrak{C}$ be the universal domain of a t.t. theory, D_1 an $\emptyset$−definable set and D_2 a D_1−definable set which is finitely generated over D_1. Then $\mathrm{Aut}(D_2/D_1)$ is a definable group.*

The proof of this theorem involves the notion of the type of an element over a definable set. When $\mathfrak{C}$ is a universal domain, a an element and X an

$\emptyset$−definable subset of $\mathfrak{C}$ the type of a over X is, by fiat, not a type since X is not a set. However, many properties of $tp(a/X)$ reduce to properties of types over sets by the following lemma.

Lemma 4.4.1. *Let $\mathfrak{C}$ be the universal domain of a t.t. theory T, a an element and X an $\emptyset$−definable subset of $\mathfrak{C}$.*

(i) There is a type r over a such that $tp(b/X) = tp(a/X)$ if and only if b realizes r.

(ii) r is equivalent to a type over a subset of X of cardinality $\leq |T|$.

(iii) There is $X_0 \subset X$ of cardinality $\leq |T|$ such that $tp(a/X_0)$ implies $tp(a/X)$. In fact, for any set A there is a $Y \subset X$ of cardinality $\leq |T| + |A|$ such that $(*)$ if A is conjugate to B over Y there is an elementary map from A to B which is the identity on X. (The notation $tp(A/Y) \models tp(A/X)$ will be used as shorthand for $(*)$.)

(iv) If $tp(b/X) = tp(a/X)$ there is an automorphism of $\mathfrak{C}$ which maps a to b and is the identity on X.

(v) There is a formula $\rho(x)$ over a implied by r such that any b realizing $tp(a) \cup \{\rho(x)\}$ realizes r.

Proof. (i) Let $\varphi(x, \bar{y})$ be a formula over $\emptyset$ and $E_\varphi(x, x')$ the $\emptyset$−definable equivalence relation expressing:

$$\text{for all } \bar{y} \text{ from } X\,(\,\varphi(x,\bar{y}) \leftrightarrow \varphi(x',\bar{y})\,).$$

Letting $\Xi(x, x') = \{\, E_\varphi(x, x') : \varphi \text{ is a formula over } \emptyset \,\}$ and $r = \Xi(x, a)$ produces a type meeting the requirements of (i).

Turning to (ii), since $\mathfrak{C}$ is assumed to be t.t., $tp(a/X)$ is definable over X (by Lemma 3.3.11). Thus, given a formula $\varphi(x, \bar{y})$ over $\emptyset$ there is a formula $\psi_\varphi(\bar{y})$ over $\bar{b}_\varphi \subset X$ such that

$$\text{for all } \bar{y} \text{ from } X(\models \varphi(a,\bar{y}) \text{ if and only if } \models \psi_\varphi(\bar{y})\,).$$

Then, for φ any formula over $\emptyset$, $E_\varphi(x, a)$ is equivalent to the $\bar{b}_\varphi$−definable relation:

$$\text{for all } \bar{y} \text{ from } X(\,\varphi(x,\bar{y}) \longleftrightarrow \psi_\varphi(\bar{y})\,).$$

There are $|T|$ many sets of the form $\bar{b}_\varphi$, so we have proved (ii).

(iii) This is immediate by (ii).

(iv) Since X has the same cardinality as $\mathfrak{C}$ (when it is infinite) we cannot simply use the homogeneity of $\mathfrak{C}$ to find such an automorphism. Instead an automorphism of $\mathfrak{C}$ is constructed using

Claim. There is a chain of elementary maps f_α, $\alpha < \kappa = |\mathfrak{C}|$, such that for all α,

(1) $f_\alpha \upharpoonright X$ is the identity on X;
(2) $f_\alpha(a) = b$;
(3) for all $c \in \mathfrak{C}$ there are β, $\gamma < \kappa$ such that c is in the domain of f_β and c is in the range of f_γ.

To begin let f_0 be the elementary map which is the identity on X and takes a to b. The detailed construction of the chain will be left to the reader. The essential features are contained in the proof of

($\sharp$) If f is an elementary map defined on $X \cup A$ (for some set A) and $c \in \mathfrak{C}$, then there is an elementary map g extending f which is defined on $X \cup A \cup \{c\}$.

By (iii) there is a set $Y \subset X$ such that $tp(A \cup \{c\}/Y) \models tp(A \cup \{c\}/X)$. Since $tp(f(A)/Y) = tp(A/Y)$ there is a d such that $tp(f(A) \cup \{d\}/Y) = tp(A \cup \{c\}/Y)$. Since the type of $A \cup \{c\}$ over Y implies its type over X the map g which extends f and takes c to d is elementary. This proves ($\sharp$) and the claim.

To complete the proof we need only observe that $g = \bigcup_{\alpha<\kappa} f_\alpha$ is an automorphism of $\mathfrak{C}$ which is the identity on X and takes a to b.

(v) Let $\Xi' = \{ E_i(x,x') : i < |T| \}$ be a set of formulas obtained from $\Xi(x,x')$ by closing under finite conjunctions. Since any formula implied by $\Xi'(x,a)$ is implied by $E_i(x,a)$ for some i, there is an i such that $(MR(\Xi'(x,a)), \deg(\Xi'(x,a))) = (MR(E_i(x,a)), \deg(E_i(x,a)))$. Let $p = tp(a)$.

Claim. Any b realizing $p \cup \{E_i(x,a)\}$ also realizes r.

Assuming the claim to fail there is a $j \neq i$ such that $p \cup \{E_i(x,a)\}$ does not imply $E_j(x,a)$. Let b be a realization of $p \cup \{E_i(x,a)\}$ such that $\not\models E_j(b,a)$. Then $\Xi'(x,a)$ and $\Xi'(x,b)$ are extensions of $p \cup \{E_i(x,a)\}$ which are contradictory and have the same Morley rank and degree (since they are conjugate). This contradicts that $E_i(x,a)$ has the same Morley rank and degree as $\Xi'(x,a)$, proving the claim and completing the proof of the lemma.

Before getting to the proof of the Binding Group Theorem we show that when "finitely generated" is replaced by "finite" the proof needs no assumption other than the completeness of the theory. The proof of the lemma helps to motivate certain steps in the proof of the Binding Group Theorem.

Lemma 4.4.2. *Let $\mathfrak{C}$ be the universal domain of a complete theory, D_1 a definable set and D_2 a finite D_1-definable set. Then $G = \mathrm{Aut}(D_2/D_1)$ is a D_1-definable group and the action of G on D_2 is also D_1-definable.*

Proof. The proof is clear after a few moments thought but we may as well think aloud. First observe that there is a (finite) set $A \subset D_1$ such that $G = \mathrm{Aut}(D_2/A)$. Let $\bar{D}$ be the set of all enumerations of D_2. Identify $\alpha \in G$ with $d_\alpha = \{ (\bar{c}, \alpha(\bar{c})) : \bar{c} \in \bar{D} \}$ and let $\bar{G} = \{ d_\alpha : \alpha \in G \}$. Let $\mathrm{Aut}_A(\mathfrak{C})$ denote the set of automorphisms of $\mathfrak{C}$ which fix A pointwise. If $\beta \in \mathrm{Aut}_A(\mathfrak{C})$ then $\beta(d_\alpha) = d_{\beta\alpha\beta^{-1}}$, hence $\bar{G}$ is invariant under the elements of $\mathrm{Aut}_A(\mathfrak{C})$. By Lemma 3.3.8(i), $\bar{G}$ is A-definable. Define $\cdot$ on $\bar{G}$ by: $d_\alpha \cdot d_\beta = d_{\alpha \cdot \beta}$ (for $\alpha \in G$). Arguing as above, $\cdot$ is invariant under the elements of $\mathrm{Aut}_A(\mathfrak{C})$ hence $\cdot$ is also A-definable. This proves that the group $\bar{G}$ (which we identify with

G) is A−definable. The action of $\bar{G}$ on D_2 is defined by: $d_\alpha * x = \alpha(x)$. If $\beta \in \mathrm{Aut}_A(\mathfrak{C})$, then

$$(\forall x,\, y \in D_2)(\forall d_\alpha \in \bar{G})(\, d_\alpha * x = y \iff \beta(d_\alpha) * \beta(x) = \beta(y)\,).$$

Thus, $*$ is A−definable. We conclude that through map $\alpha \mapsto d_\alpha$ from G onto $\bar{G}$ we can identify the action of G on D_2 with $(\bar{G}, \cdot, *)$.

Proof of Theorem 4.4.5 (Binding Group Theorem). Let D_2 be B−definable for $B \subset D_1$ finite. Let $\bar{b}$ be a fundamental generator of D_2 over D_1 with generating function $f(y_1, \ldots, y_n, \bar{z})$; i.e., $f(D_1^n, \bar{b}) \supset D_2$. Let $\psi_0(\bar{z})$ be a formula in $tp(\bar{b}/B)$ such that for any $\bar{c}$ satisfying ψ_0, $f(y_1, \ldots, y_n, \bar{c})$ is a function mapping D_1^n onto D_2. Let $\bar{c} \in \psi_0(\mathfrak{C})$ realize an isolated type in $S(B)$. By Lemma 4.4.1(v), $tp(\bar{c}/D_1)$ is isolated by some formula ψ. Let $X = \psi(\mathfrak{C})$. To prove the theorem it suffices to show:

Claim. There are $\tau : X \longrightarrow \mathrm{Aut}(D_2/D_1)$ and $\bar{c}$−definable operations,

$$\cdot : X \times X \longrightarrow X \text{ and } * : X \times D_2 \longrightarrow D_2$$

such that $*$ defines an action of the group $(X, \cdot)$ on D_2 and τ is an isomorphism of the group action $(X, \cdot, *)$ onto $\mathrm{Aut}(D_2/D_1)$.

Let $\theta(x, x', \bar{y}, \bar{y}')$ be a formula (over B) defining the relation:

$$\bar{y}, \bar{y}' \in X,\ x, x' \in D_2 \text{ and } \exists \bar{z} \in D_1^n(\, x = f(\bar{z}, \bar{y}) \wedge x' = f(\bar{z}, \bar{y}')\,).$$

Let $\alpha \in \mathrm{Aut}(D_2/D_1)$ and suppose $\alpha(\bar{c}) = \bar{c}'$. Then for all $x, x' \in D_2$, $\theta(x, x', \bar{c}, \bar{c}') \iff x' = \alpha(x)$, so α is a definable map which we denote $\beta_{\bar{c}'}$. Notice that $\beta_{\bar{c}'}$ is the unique element of $\mathrm{Aut}(D_2/D_1)$ which takes $\bar{c}$ to $\bar{c}'$. Since ψ isolates a complete type over D_1 every $\bar{d} \in X$ realizes $tp(\bar{c}/D_1)$. Hence for any $\bar{d} \in X$ there is an $\alpha \in \mathrm{Aut}(D_2/D_1)$ such that $\alpha(\bar{c}) = \bar{d}$.

With these facts in hand we can define the necessary mappings $\cdot$ and $*$. Let τ be the bijection from X onto $\mathrm{Aut}(D_2/D_1)$ such that $\tau(\bar{d})$ is the unique $\gamma \in \mathrm{Aut}(D_2/D_1)$ such that $\beta_{\bar{d}} = \gamma$. Define the binary operation $\cdot$ on X by: $\beta_{\bar{d}\cdot\bar{e}} = \beta_{\bar{d}}\beta_{\bar{e}}$. Define $* : X \times D_2 \longrightarrow D_2$ by: $\bar{d} * a = \tau(\bar{d})(a)$. Using the formula $\theta(x, x', \bar{y}, \bar{y}')$ a routine argument shows that $\cdot$ and $*$ are both $\bar{c}$−definable. Furthermore, τ is a group action isomorphism of $(X, \cdot, *)$ onto $\mathrm{Aut}(D_2/D_1)$. This proves the claim, hence the theorem.

Remark 4.4.3. There may be many definable group actions isomorphic to $\mathrm{Aut}(D_2/D_1)$; i.e., many binding groups of D_2 over D_1. In the proof we picked $\bar{c}$ to be any element satisfying ψ_0 and realizing an isolated type over B. A different isolated completion of ψ_0 would lead to a different binding group. The set of fundamental generators X used as the universe of the binding group will be called the *special set of fundamental generators.*

The proof of the Binding Group Theorem finds, for any $\bar{c} \in X$ (a special set of fundamental generators), a copy of the binding group defined on X over $\bar{c}$. In the following corollary we show that while the action of the binding group generally needs a parameter from X there is a single B−definable group that works for all $\bar{c}$.

Corollary 4.4.1. *Let $\mathfrak{C}$ be the universal domain of a t.t. theory, D_1 an $\emptyset$−definable set and D_2 a B−definable set, where $B \subset D_1$, which is finitely generated over D_1. Let $\bar{c}$ be a fundamental generator of D_2 over D_1 such that $r = tp(\bar{c}/D_1)$ is isolated and let $X = r(\mathfrak{C})$. For each $\bar{c} \in X$ let $(G_{\bar{c}}, \cdot_{\bar{c}}, *_{\bar{c}})$ denote the copy of the binding group definable over $\bar{c}$, and let $\tau_{\bar{c}}$ denote the isomorphism of $G_{\bar{c}}$ onto* $\mathrm{Aut}(D_2/D_1)$ *(as group actions on D_2). Then there is an B−definable group $(G, \circ)$ and a formula $\epsilon(x, \bar{y})$ such that*

(1) $G \subset D_1^{eq}$.
(2) *For each $\bar{c} \in X$, $\epsilon(x, \bar{c})$ defines an isomorphism $\epsilon_{\bar{c}}$ of $(G, \circ)$ onto $(G_{\bar{c}}, \cdot_{\bar{c}})$.*
(3) *For each $\bar{c} \in X$ let $\pi_{\bar{c}} = \tau_{\bar{c}}\epsilon_{\bar{c}}$, an isomorphism of $(G, \circ)$ onto* $\mathrm{Aut}(D_2/D_1)$. *Let $\star_{\bar{c}}$ be the definable action of G on D_2 given by: $g \star_{\bar{c}} x = \epsilon_{\bar{c}}(g) *_{\bar{c}} x = \pi_{\bar{c}}(g)x$ (for $g \in G$ and $x \in D_2$). Hence $\pi_{\bar{c}}$ is an isomorphism of $(G, \circ, \star_{\bar{c}})$ onto* $\mathrm{Aut}(D_2/D_1)$ *as group actions.*
(4) *For each $\bar{c} \in X$, $\star_{\bar{c}}$ induces a regular group action of G on X.*
(5) *If $\gamma \in$* $\mathrm{Aut}(D_2/D_1)$, *$\bar{c} \in X$ and $\bar{d} = \gamma(\bar{c})$ (also an element of X) then for all $g \in G$, $\pi_{\bar{d}}(g) = \gamma\pi_{\bar{c}}(g)\gamma^{-1}$.*

Proof. Let $\bar{c} \in X$. Since $X \subset D_2^k$ (for some k) X is finitely generated over D_1. In fact, there is a $\bar{c}$−definable function $f_{\bar{c}}$ mapping D_1^m (for some m) onto X. By Remark 4.4.1(iii) there is a B−definable $G \subset D_1^{eq}$ and a $B \cup \bar{c}$−definable bijection $\epsilon_{\bar{c}}$ mapping G onto X. Since $G_{\bar{c}}$ is defined on X there is definable binary operation $\circ$ on G such that $\epsilon_{\bar{c}}$ is an isomorphism of $(G, \circ)$ onto $(G_{\bar{c}}, \cdot_{\bar{c}})$. Since all elements of X realize the same type over D_1 (hence the same type over D_1^{eq}) $\epsilon_{\bar{d}}$ is an isomorphism of $(G, \circ)$ onto $(G_{\bar{d}}, \cdot_{\bar{d}})$ for any $\bar{d} \in X$. This proves (1) and (2).

There is really nothing to prove in (3), its role being solely to set notation and viewpoint. Turning to (4) remember that X is a subset of D_2^k, hence $\star_{\bar{c}}$ defines an action of G on X. Since all elements of X have the same type over D_1 the action is transitive. For any $\bar{c} \in X$, $X \subset dcl(D_1 \cup \bar{c})$, hence only the identity of G can fix $\bar{c}$. In other words $\star_{\bar{c}}$ defines a regular action.

(5) Let $g \in G$ and $g \star_{\bar{c}} \bar{c} = \bar{e}$. Then by definition of $\star_{\bar{c}}$, $\pi_{\bar{c}}(g)$ is the unique element of $\mathrm{Aut}(D_2/D_1)$ taking $\bar{c}$ to $\bar{e}$. Since γ is in $\mathrm{Aut}(D_2/D_1)$, $g \star_{\bar{d}} \bar{d} = \gamma\bar{e}$. That is, $\pi_{\bar{d}}(g)$ is the unique element of $\mathrm{Aut}(D_2/D_1)$ taking $\bar{d}$ to $\gamma\bar{e}$. From here it is easy to see that $\pi_{\bar{d}}(g) = \gamma\pi_{\bar{c}}(g)\gamma^{-1}$.

Remark 4.4.4. This corollary gives us the picture of binding groups most useful in applications. Specifically, $G \subset D_1^{eq}$ is a definable group and there are

– a uniformly definable family of group actions $\{ \star_{\bar{c}} : \bar{c} \in X \}$ and
– a family of maps $\{ \pi_{\bar{c}} : \bar{c} \in X \}$ such that

for each $\bar{c} \in X$, $\pi_{\bar{c}}$ is an isomorphism of $(G, \circ, \star_{\bar{c}})$ onto $\mathrm{Aut}(D_2/D_1)$ (as group actions on D_2). From now on the term "binding group" refers to this copy of $\mathrm{Aut}(D_2/D_1)$ contained in D_1^{eq}.

The Binding Group Theorem allows us to apply all of our knowledge of ω–stable groups to binding groups. In particular, when $\mathfrak{C}$ is a 1–based uncountably categorical theory the binding group is abelian-by-finite. The strength of this fact will be discussed later in the context of the Ladder Theorems.

The applicability of the Binding Group Theorem depends on the existence of "many" sets which are finitely generated over a fixed set. The following result is the key in the context of uncountably categorical theories.

Theorem 4.4.6. *Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory, D an infinite A–definable set and a an element not in $acl(A)$. Then there is a $b \in dcl(A \cup \{a\}) \setminus acl(A)$ such that b is an element of an A–definable set which is finitely generated over $D \cup A$.*

The bulk of the proof of this theorem will be done in the context of a t.t. theory satisfying an additional condition (which is always true in an uncountably categorical theory).

Definition 4.4.4. *Let $\mathfrak{C}$ be the universal domain of a t.t. theory, D an A–definable set and Y $\bigwedge$–definable over A. Then Y is* foreign to D over *A if for any set $B \supset A$ and any $a \in Y$ which is generic over B, a is independent from $D \cup B$ over A. For q a type over A, q is foreign to D if $q(\mathfrak{C})$ is foreign to D over A.*

Notice the potential asymmetry in the foreign relation; Y may be foreign to D while D is not foreign to Y. This is possible because over a set B we test for independence using an arbitrary subset of X and a *generic element* of Y.

Theorem 4.4.6 will follow quickly from

Proposition 4.4.1. *Let $\mathfrak{C}$ be the universal domain of a t.t. theory, D an A–definable set, $p = tp(a/acl(\emptyset))$ and $Y = p(\mathfrak{C})$. If Y is not foreign to X there is a $b \in dcl(A \cup \{a\}) \setminus acl(A)$ such that b is an element of an A–definable set which is finitely generated over $D \cup A$.*

The proof of the proposition will be split between two results involving the following concept.

Definition 4.4.5. *Let $\mathfrak{C}$ be the universal domain of a t.t. theory, D an A–definable set and Y $\bigwedge$–definable over A. Then Y is said to be D–*internal over A *if for all* $a \in Y$ *there is a* $B \supset A$ *such that* a *is generic over* B *and* $a \in dcl(B \cup D)$. *If* q *is a type over* A *and* $q(\mathfrak{C})$ *is* $D-$*internal over* A *we also call* q $D-$*internal over* A.

Remark 4.4.5. Let D be an $\emptyset-$definable set in the universal domain of a t.t. theory, and Y $\bigwedge-$definable over $\emptyset$. The proofs of the following observations are left to the reader.

(i) When Y is finitely generated over D, Y is $D-$internal.

(ii) If Y is $D-$internal any conjugate of Y over $\emptyset$ is $D-$internal.

(iii) $p \in S(acl(\emptyset))$ is $D-$internal if for *some* a realizing p there is a B such that a is independent from B and $a \in dcl(B \cup D)$.

(iv) If $tp(a/acl(\emptyset))$ is $D-$internal and $b \in dcl(a)$, then $tp(b/acl(\emptyset))$ is $D-$internal.

(v) If $tp(a_i/acl(\emptyset))$ is $D-$internal for $i \leq n$, and b is the name for $\{a_0, \ldots, a_n\}$, then $tp(b/acl(\emptyset))$ is $D-$internal.

Notation. An $\bigwedge-$definable set X over A which is the set of realizations of a complete type over A is called a *locus over* A. Given an element a, the *locus of* a *over* A is the set of realizations of $tp(a/A)$. Note: the locus of a over A is the orbit of a under the automorphisms of $\mathfrak{C}$ which fix A.

Lemma 4.4.3. *Let* $\mathfrak{C}$ *be the universal domain of a t.t. theory,* D *an infinite* $A-$*definable set and* Y *a set* $D-$*internal over* A *such that* Y *is a locus over* $acl(A)$. *Then there is an* $A-$*definable set* $X \supset Y$ *such that* X *is finitely generated over* $D \cup A$.

Proof. Without loss of generality, $A = \emptyset$. The proof proceeds through the following steps.

(a) Let $a^* \in Y$ be generic over $\bar{b}^*$ such that $a^* = f(\bar{d}, \bar{b}^*)$ for some definable function $f(\bar{x}, \bar{b}^*)$ and $\bar{d} \subset D$. Let $q = tp(\bar{b}^*/acl(\emptyset))$. Then for all $\bar{b}'$ realizing q and $a' \in Y$ generic over $\bar{b}'$, $a' = f(\bar{d}', \bar{b}')$ for some $\bar{d}' \subset D$. Without loss of generality, $f(\bar{x}, \bar{b}^*)$ is defined on all of D^k for some k.

(b) Let $B = \{\bar{b}_i : i < \omega\}$ be a Morley sequence in q. Then for any $a \in Y$ there is a $\bar{b}_i \in B$ such that $a = f(\bar{d}, \bar{b}_i)$ for some $\bar{d} \subset D$.

(c) There is an $n < \omega$ such that

$$(\forall a \in Y)(\exists i \leq n)(\exists \bar{d} \subset D)(\ a = f(\bar{d}, \bar{b}_i)\ ).$$

(d) For $\bar{b} = \bar{b}_0 \cup \ldots \cup \bar{b}_n$ there is a single $\bar{b}-$definable function $g(\bar{z}, \bar{b})$ such that for any $a \in Y$, $a = g(\bar{d}, \bar{b})$ for some $\bar{d} \subset D$.

(e) There is a definable set $X \supset Y$ such that the condition in (d) is true with Y replaced by X.

Proofs:

(a) Every type in $S(\emptyset)$ is stationary, hence for any $\bar{b}'$ realizing q and $a' \in Y$ generic over $\bar{b}'$, $tp(a'\bar{b}') = tp(a\bar{b}^*)$, which is sufficient.

(b) Let a be any element of Y. By Corollary 3.3.1 there is a $\bar{b}_i \in B$ which is independent from a. Then, $a = f(\bar{d}, \bar{b}_i)$ for some $\bar{d} \subset D$.

(c) By (b) and a compactness argument there is such an n.

(d) This step is accomplished with a simple trick for producing one definable function from finitely many. Without loss of generality, $n > 1$. Define the function $g(\bar{z}, \bar{b})$ on D^{nk} so that for all $\bar{d}_0 \ldots, \bar{d}_n \in D^k$, $g(\bar{d}_0 \ldots \bar{d}_n, \bar{b}) = f(\bar{d}_i, \bar{b}_i)$, where i is the minimal index such that $f(\bar{d}_i, \bar{b}_i) \neq f(\bar{d}_j, \bar{b}_j)$ for all $j \neq i$, if one exits, and $i = n$, otherwise.

To verify that $g(\bar{z}, \bar{b})$ maps onto Y let $a \in Y$, $i \leq n$ and $\bar{d}_i \in D^k$ such that $a = f(\bar{d}_i, \bar{b}_i)$. To obtain $\bar{d}_0, \ldots, \bar{d}_n$ such that $a = g(\bar{d}_0, \ldots, \bar{d}_n, \bar{b})$ it suffices to find $\bar{d}_j$ (for $j \neq i$) such that $f(\bar{d}_l, \bar{b}_l) = f(\bar{d}_{l'}, \bar{b}_{l'})$ for all $l, l' \neq i$. Let $c \in Y$ be generic over $\{\bar{b}, \bar{d}_i, a\}$. Then, for each $j \neq i$ there is a $\bar{d}_j$ such that $c = f(\bar{d}_j, \bar{b}_j)$. This proves (d).

(e) Letting $X = g(D^{nk}, \bar{b})$ meets the requirement.

This proves the lemma.

The reader should compare the following lemma and its proof to Proposition 4.3.2.

Lemma 4.4.4. *Let D be an $A-$definable set in a t.t. theory and Y a locus over $acl(A)$. If Y is not foreign to D over A there is a $b \in dcl(A \cup \{a\}) \setminus acl(A)$ such that $p = tp(b/acl(A))$ is $D-$internal over A.*

Proof. Without loss of generality, $A = \emptyset$. Let $\bar{b}_0$ be a finite set independent from a and $\bar{d}_0 \subset D$ such that a depends on $\bar{d}_0$ over $\bar{b}_0$. By Corollary 4.1.4 there is an element c such that

(1) $c \in acl(a)$,
(2) a is independent from $\bar{b}_0\bar{d}_0$ over c, and
(3) $c \in dcl(\bar{d}_0\bar{b}_0, \ldots, \bar{d}_k\bar{b}_k)$, for some set $B = \{\bar{d}_0\bar{b}_0, \ldots, \bar{d}_k\bar{b}_k\}$ which is a Morley sequence over a in $tp(\bar{b}_0\bar{d}_0/acl(a))$.

Let $\bar{b} = \bar{b}_0 \ldots \bar{b}_k$, $\bar{d} = \bar{d}_0 \ldots \bar{d}_k$ and $q = tp(c/acl(\emptyset))$. Since a is independent from $\bar{b}_0$ and B is a Morley sequence over a, a is independent from $\bar{b}$. Thus, c is independent from $\bar{b}$. Since $\bar{d} \subset D$ and $c \in dcl(\bar{b}\bar{d})$, q is $D-$internal.

To obtain a realization of a $D-$internal type which is in $dcl(a)$ instead of only $acl(a)$ let b be a name for the (finite) set of conjugates of c over a. Since any conjugate of q is $D-$internal, b is a finite set of elements each realizing a $D-$internal type over $acl(\emptyset)$. By Remark 4.4.5, $p = tp(b/acl(\emptyset))$ is $D-$internal. This proves the lemma.

Proof of Proposition 4.4.1. The proposition follows immediately from the combination of Lemma 4.4.3 and Lemma 4.4.4.

Proof of Theorem 4.4.6. Without loss of generality, $A = \emptyset$. Let Y be the locus of a over $acl(\emptyset)$. It suffices (by Proposition 4.4.1) to show that Y is not foreign to D. Let M be a countable saturated model and b an element of Y generic over M. Then there is a $c \in D$ such that $tp(c/M)$ is strongly minimal (by Corollary 3.1.2) and $c \in acl(M \cup \{b\})$ (by Exercise 3.3.18). Thus, Y is not foreign to D.

Our first reward is a proof of the Main Ladder Theorem.

Proof of Theorem 4.4.1. Without loss of generality, $a \notin acl(\emptyset)$. By Proposition 4.3.2 there is an element $a_0 \in dcl(a)$ such that a_0 is in an $\emptyset$-definable almost strongly minimal set D_0. Now suppose $a_0, \ldots, a_i$ and $D_0, \ldots, D_i$ have been defined to satisfy (1)–(5) up to i. If $a \in acl(A_i)$ let $a_i = a$ and end the construction. Otherwise there is an $a_i \in dcl(A_i \cup \{a\}) \setminus acl(A_i)$ and an A_i-definable set D_i such that $a_i \in D_i$ and D_i is finitely generated over $D_0 \cup \ldots \cup D_{i-1}$ (by Theorem 4.4.6). Since $A_i \subset dcl(a)$, $a_i \in dcl(a)$. By the Binding Group Theorem (Theorem 4.4.5) $G_i = \mathrm{Aut}(D_i/D_0 \cup \ldots \cup D_{i-1})$ is definable, proving the theorem.

Proof of Theorem 4.4.2. The most important additional tool in this proof is Lemma 4.3.9, which says

($\sharp$) for any set A and $a \notin acl(A)$ there is a $c \in acl(A \cup \{a\})$ such that $MR(c/A) = 1$.

This fact is augmented with the following to obtain sets which are strongly minimal in addition to having Morley rank 1. (This is just a restatement of Lemma 4.1.3(ii).)

($\sharp\sharp$) For any a and finite set A there is an $e \in dcl(A \cup \{a\}) \cap acl(A)$ such that $\deg(a/A \cup \{e\}) = 1$.

Let a be any element of the universal domain. The choice of elements a_i and sets D_i proceeds as follows through several cases. The construction ends at the first step in which a_i is set to a. After defining these objects we will prove the necessary properties of the binding groups.

Case 1. $a \in acl(A_i)$. Let $a_i = a$ and D_i be the set of realizations of $tp(a/A_i)$.

Case 2. $a \notin acl(\emptyset)$ *and* $i = 0$. By ($\sharp$) there is a $c \in acl(a)$ such that $MR(c/\emptyset) = 1$. If $tp(c)$ is strongly minimal let $a_0 = c$ and D_0 be an $\emptyset$-definable strongly minimal set containing c. If, on the other hand, $\deg(c) > 0$ choose $e \in dcl(c) \cap acl(\emptyset)$ such that $\deg(c/e) = 1$ (by ($\sharp\sharp$)). In this case we let $a_0 = e$, $D_0 =$ the set of realizations of $tp(e)$, $a_1 = c$ and D_1 a strongly minimal set over a_0 which contains c.

Case 3. $a \notin acl(A_i)$ *and* $D_0 \cup \ldots \cup D_{i-1}$ *is infinite.* By ($\sharp$) there is a $c \in acl(A_i \cup \{a\})$ such that $MR(c/A_i) = 1$. Since $\bar{D} = D_0 \cup \ldots \cup D_{i-1}$ is infinite, Theorem 4.4.6 yields a $c' \in dcl(A_i \cup \{c\}) \setminus acl(A_i)$ such that c' belongs to an A_i-definable set which is finitely generated over $\bar{D}$. Thus, we may as well require c to belong to an A_i-definable set of Morley rank 1 which is finitely generated over $\bar{D}$.

If $tp(c/A_i)$ is strongly minimal we let $a_i = c$ and D_i an A_i-definable strongly minimal set which contains c and is also finitely generated over $\bar{D}$. If $\deg(c/A_i) > 1$ we interpose another element of $acl(a)$ as follows. By (♯♯) there is an $e \in dcl(A_i \cup \{c\}) \cap acl(A_i)$ such that $\deg(c/A_i \cup \{e\}) = 1$; i.e., $tp(c/A_i \cup \{e\})$ is strongly minimal. Let $a_i = e$ and D_i the (finite) set of realizations of $tp(e/A_i)$. Let $a_{i+1} = c$ and D_{i+1} an $A_{i+1}-$definable strongly minimal set. Notice that D_{i+1} is finitely generated over $D_0 \cup \ldots \cup D_i$.

The reader should observe that the described cases encompass all possibilities (until $a_n = a$ and the construction terminates). It remains to show that (when $D_0 \cup \ldots \cup D_{i-1}$ is infinite)

(♭) $G_i = \mathrm{Aut}(D_i/D_0 \cup \ldots \cup D_{i-1})$ is definable over $D_0 \cup \ldots \cup D_{i-1}$ and has Morley rank ≤ 1.

When D_i is finite this is true by Lemma 4.4.2. Suppose D_i is infinite. That G_i is definable over $D_0 \cup \ldots \cup D_{i-1}$ is simply by Corollary 4.4.1. Let X be a special set of fundamental generators for D_i over $D_0 \cup \ldots \cup D_{i-1}$ and recall that $MR(G_i) = MR(X)$, which we have assumed is > 0. Since $D_0 \cup \ldots \cup D_{i-1}$ is infinite one of $D_0, \ldots, D_{i-1}$ is strongly minimal. By Lemma 4.4.5(ii), for any $a \in D_i \setminus acl(A_i)$, $D_i \subset acl(D_0 \cup \ldots \cup D_{i-1} \cup \{a\})$. Since X is a subset of D_i^k for some k, and all elements of X realize the same type over $D_0 \cup \ldots \cup D_{i-1}$, $MR(X) = MR(a/D_0 \cup \ldots \cup D_{i-1}) \leq 1$. This proves (♭) and completes the proof of the theorem.

We turn now to the Simple Ladder Theorem, which will follow rather quickly from

Proposition 4.4.2. *Let $\mathfrak{C}$ be the universal domain of a t.t. theory, D_1 an infinite $\emptyset-$definable set and D_2 a set which is finitely generated over D_1 and definable over $B \subset D_1$. Let G be a binding group of D_2 over D_1.*

(i) Suppose that $B \subset C \subset D_1$ and f is a $C-$definable function from D_2 onto a set F. Let H be $\{h \in G : h \text{ is the identity on } F\}$. Then H is a $C-$definable normal subgroup of G. Furthermore, $H = \{1\}$ if and only if $D_2 \subset dcl(D_1 \cup F)$.

(ii) Conversely, let H be a definable normal subgroup of G. Then there is a definable set F, finitely generated over D_1 such that for any $\bar{c} \in X$, $\mathrm{Aut}(D_2/D_1 \cup F) = \pi_{\bar{c}}(H)$ and $\mathrm{Aut}(F/D_1) = \mathrm{Aut}(D_2/D_1)/\pi_{\bar{c}}(H)$. If H is $B-$definable then we can take F to be the set of realizations of an isolated type over D_1.

Proof. For the statement of (i) to make sense the reader must observe that the action of G on D_2 extends in a unique way to an action of G on $D_2 \cup F$. Let X be the special set of fundamental generators of D_2 over D_1. In the proof we freely draw on the notation used in Corollary 4.4.1. In particular,

G is a definable group in D_1^{eq} and for each $\bar{c} \in X$, $\star_{\bar{c}}$ defines an action of G on D_2 and a regular action of G on X.

(i) For any $\bar{c} \in X$ let $\varphi_{\bar{c}}(x)$ be the formula $x \in G \wedge (\forall y \in F)(x \star_{\bar{c}} y = y)$. Then, $H = \varphi_{\bar{c}}(\mathfrak{C})$, hence H is definable over $C \cup \{\bar{c}\}$. Since F is $C-$definable and the elements of X all have the same type over D_1^{eq}, $\varphi_{\bar{c}}$ is equivalent to $\varphi_{\bar{d}}$ for all $\bar{c}$, $\bar{d} \in X$. It follows that H is $C-$definable. The reader should verify that H is normal.

If $D_2 \subset dcl(D_1 \cup F)$ then any $h \in H$ must be the identity on D_2; i.e., $H = \{1\}$. On the other hand, if $a \in D_2$ and $a \notin dcl(D_1 \cup F)$ there is a $b \neq a$ realizing $tp(a/D_1 \cup F)$. By Lemma 4.4.1(iv) there is an $h \in G$ which maps a to b and fixes every element of $D_1 \cup F$; that is, $H \neq \{1\}$.

(ii) The set F will be the quotient of X by some D_1-definable equivalence relation.

Claim. Let Y be a D_1-definable set such that each $\star_{\bar{c}}$ defines an action of G on Y. For each $\bar{c} \in X$ define an equivalence relation $E_{\bar{c}}$ on Y by:

$$E_{\bar{c}}(x, y) \text{ if and only if } \exists \gamma \in H(\gamma \star_{\bar{c}} x = y).$$

Then for all $\bar{c}$, $\bar{d} \in X$, $E_{\bar{c}}$ is equivalent to $E_{\bar{d}}$.

Note: For $\bar{c} \in X$ and $x \in Y$, $E_{\bar{c}}(\mathfrak{C}, x) = \pi_{\bar{c}}(H)x$. Remember that the action of $\mathrm{Aut}(D_2/D_1)$ on X is regular. Pick $\bar{c}$, $\bar{d} \in X$ and let $\gamma \in \mathrm{Aut}(D_2/D_1)$ be such that $\bar{d} = \gamma\bar{c}$. By Corollary 4.4.1, for any $g \in G$, $\pi_{\bar{d}}(g) = \gamma \cdot \pi_{\bar{c}}(g) \cdot \gamma^{-1}$. Since H is normal $\pi_{\bar{d}}(H) = \gamma \cdot \pi_{\bar{c}}(H) \cdot \gamma^{-1} = \pi_{\bar{c}}(H)$. Thus, $E_{\bar{d}}(\mathfrak{C}, x) = \pi_{\bar{d}}(H)x = \gamma \cdot \pi_{\bar{c}}(H) \cdot \gamma^{-1}x = \pi_{\bar{c}}(H)x = E_{\bar{c}}(\mathfrak{C}, x)$, proving the claim.

Now apply the claim with $X = Y$. Let E be the equivalence relation such that for all x, $y \in X$, $E(x, y)$ holds if and only if there is a $\bar{c} \in X$ such that $E_{\bar{c}}(x, y)$. Then E is D_1-definable. Let F be the set of $E-$classes of elements of X. Since $F \subset dcl(X \cup D_1)$ any element of $\mathrm{Aut}(D_2/D_1)$ extends uniquely to an element of $\mathrm{Aut}(F/D_1)$.

Claim. For any $\bar{c} \in X$, $H = \{ g \in G : g \star_{\bar{c}} x = x \text{ for all } x \in F \}$.

Fix $\bar{c} \in X$ and $\star_{\bar{c}}$ as an action of H on F. It is immediate from the definition of E that any $g \in H$ is the identity on F. Conversely, suppose that $g \star_{\bar{c}} x = x$ for all $x \in F$. Let $\bar{d}$ be any element of X and $\bar{e} = g \star_{\bar{c}} \bar{d}$. Since g fixes every element of F (under $\star_{\bar{c}}$) $\bar{d}$ and $\bar{e}$ have the same type over F. Then $\bar{d}$ and $\bar{e}$ must be $E-$equivalent (since F is the set of $E-$classes), hence there is an $h \in H$ with $\bar{e} = h \star_{\bar{c}} \bar{d}$. Since the action of G on X is regular we conclude that $g = h$, proving the claim.

Since $\mathrm{Aut}(D_2/F \cup D_1) = \{\gamma \in \mathrm{Aut}(D_2/D_1) : \gamma \text{ is the identity on } F\}$, the claim proves that $\mathrm{Aut}(D_2/F \cup D_1) = \pi_{\bar{c}}(H)$, for any $\bar{c} \in X$.

Clearly, any element of $\mathrm{Aut}(F/D_1)$ extends to an element of $\mathrm{Aut}(D_2/D_1)$. Thus, the natural embedding of $\mathrm{Aut}(D_2/D_1)$ into $\mathrm{Aut}(F/D_1)$ is surjective. The kernel of this embedding is $\pi_{\bar{c}}(H)$ hence

$$\mathrm{Aut}(F/D_1) = \mathrm{Aut}(D_2/D_1)/\pi_{\bar{c}}(H).$$

Finally notice that when H is B–definable, so is the equivalence relation E. Remember that $F = X/E$. Since X is B–definable and all elements of X realize the same complete type over D_1, the same is true of F. This proves the proposition.

Proof of Theorem 4.4.3. To begin let $D_0, \ldots, D_l$ be a sequence of almost strongly minimal sets and $a_0, \ldots, a_l = a$ a sequence of elements meeting the requirements (1)–(5) of Theorem 4.4.1. Suppose, for example, that $G_1 = \mathrm{Aut}(D_1/D_0)$ is infinite, nonsimple and not minimal abelian. Let H be a definable normal subgroup of G. By Proposition 4.4.2 there is a D_0–definable set F, finitely generated over D_0, such that $\mathrm{Aut}(F/D_0) = G_1/H$ and $\mathrm{Aut}(D_1/D_0 \cup F) = H$. Replace the original sequence $D_0, D_1, \ldots, D_l$ by $D_0, F, D_1, \ldots, D_l$. Continuing this process produces a sequence of sets (in finitely many steps) satisfying (1)–(3) in the statement of the theorem.

A much more refined picture can be obtained when the theory is 1–based. Recall that a definable group in a 1–based theory is abelian-by-finite. Thus, in a 1–based theory the connected component of any binding group is abelian. The first part of the next lemma shows the strength of this condition.

Lemma 4.4.5. *Let $\mathfrak{C}$ be the universal domain of a 1–based uncountably categorical theory, D_1 an infinite $\emptyset$–definable set and D_2 a set, finitely generated over D_1 and definable over $B \subset D_1$. Let X be a special set of fundamental generators of D_2 over D_1, $\bar{c} \in X$, $(G, \cdot, \star_{\bar{c}})$ the binding group of D_2 over D_1 (presented as in Corollary 4.4.1) and $\pi_{\bar{c}}$ the isomorphism of $(G, \cdot, \star_{\bar{c}})$ onto* $\mathrm{Aut}(D_2/D_1)$. *Suppose G is abelian.*

(i) There is a B–definable action $\star$ of G on D_2 such that for all $\bar{c} \in X$, $\star = \star_{\bar{c}}$.

(ii) Let $a \in D_2$ and Y the set of realizations of $tp(a/D_1)$. Then $Y \subset dcl(D_1 \cup \{a\})$.

Proof. (i) For each $\bar{d} \in X$, $\pi_{\bar{d}}$ is a group action isomorphism, hence $g \star_{\bar{d}} x = \pi_{\bar{d}}(g)x$, for all $g \in G$ and $x \in D_2$. Let $\bar{d}$ and $\bar{e}$ be arbitrary elements of X. There is a $\gamma \in \mathrm{Aut}(D_2/D_1)$ such that $\bar{e} = \gamma(\bar{d})$ and, more to the point, $\pi_{\bar{e}}(g) = \gamma\pi_{\bar{d}}(g)\gamma^{-1}$. Since G is abelian we conclude that for all $g \in G$ and $x \in D_2$, $g \star_{\bar{d}} x = g \star_{\bar{e}} x$. Since the elements of X realize an isolated type over D_1 there is a D_1–definable action $\star$ of G on D_2 such that $\star = \star_{\bar{d}}$ for all $\bar{d} \in X$.

(ii) Simply because G is $\mathrm{Aut}(D_2/D_1)$, Y is the orbit of a under the action of G. Since $G \subset dcl(D_1)$ and the action of G on D_2 is definable over D_1, $Y \subset dcl(D_1 \cup \{a\})$, as needed to prove the lemma.

Proof of Theorem 4.4.4. Combining Theorem 4.4.2 with Proposition 4.4.2 will prove the theorem. For $\mathfrak{C}$ as hypothesized and a an arbitrary element let $a'_0, \ldots, a'_l = a$ and $D'_0, \ldots, D'_l$ satisfy all of the requirements of Theorem 4.4.2. We will find sets $D_0, \ldots, D_n$ and elements $a_0, \ldots, a_n = a$ satisfying the additional requirements of this theorem. These a_i and D_i will be chosen so that the a'_j and D'_j are among them. Suppose $a_0, \ldots, a_{i-1}$ and $D_0, \ldots, D_{i-1}$ have been found satisfying the conditions of the theorem "up to $i-1$" and let $\bar{D} = D_0 \cup \ldots \cup D_{i-1}$. Suppose j is minimal so that D'_j is not among $D_0, \ldots, D_{i-1}$. If D'_j is finite let $a_i = a'_j$ and note that (5) holds for $G_i = \mathrm{Aut}(D_i/\bar{D})$ by Lemma 4.4.2. Now suppose D'_j to be strongly minimal, in which case $H = \mathrm{Aut}(D'_j/\bar{D})$ has Morley rank 1. As a group H is definable over A_i by Corollary 4.4.1. If H is strongly minimal, let $D_i = D'_j$, $a_i = a'_j$ and $G_i = H$. Assuming that $\deg(H) > 1$ let G be the connected component of H, a strongly minimal normal subgroup of H which is A_i-definable. By Proposition 4.4.2 there is a definable set F such that

- F is finitely generated over $\bar{D}$,
- F is A_i-definable and the elements of F realize the same complete type over $\bar{D}$,
- $\mathrm{Aut}(D'_j/\bar{D} \cup F) \cong G$ and
- $\mathrm{Aut}(F/\bar{D}) \cong (H/G)$.

Since G has finite index in H, F is finite. Let $D_i = F$, $D_{i+1} = D'_j$, a_i any element of F and $a_{i+1} = a'_j$. Then $G_i = H/G$ is finite and $G_{i+1} = G$ is strongly minimal.

That a strongly minimal group is abelian is proved in Corollary 3.5.5. We proved in Lemma 4.4.5(i) that group action of G_i on D_i is definable over $D_0 \cup \ldots \cup D_{i-1}$ (since G_i is abelian). This proves the theorem.

Recipe. I'm sure you've worked up quite an appetite by now. After a long day of mathematics there is nothing like a big plate of lasagna. This recipe was given to me by Philipp Rothmaler in exchange for a preprint of [Bue87].

First we need a sauce bolognaise. Quickly brown 3/4 lb. of ground beef with a large chopped onion. Add salt by taste and remove most of the grease. Add 2 – 3 big chopped tomatoes, 2 – 3 tablesp. of tomato paste and the spices thyme, oregano, basil, black pepper, paprika and minced garlic (by taste). Cook under low heat until the tomatoes are saucy. (This could take quite a while; have a glass of wine and start the next section.)

When the sauce bolognaise is nearly finished it is time for the sauce bechamel. In a small sauce pan melt 3 tablesp. of butter and stir in 1 – 2 tablesp. of flour to make a smooth paste. Gradually add 1 cup of cold milk under low heat, stirring until it thickens. Add salt to taste and a few pinches of nutmeg.

The sauces, uncooked *(sic)* lasagna noodles and Mozzarella cheese are layered in a backing dish as follows. In the bottom of the dish put a thin layer of sauce bechamel, a layer of noodles and more bechamel on top. Then comes the bolognaise, Mozzarella, noodles, bechamel, bolognaise, etc. End the layering with a lot of Mozzarella on top. Cook at 350 for 30 minutes.

Historical Notes. With few exceptions the results in this section are due to Zil'ber. They originally appeared in various papers, but are compiled in [Zil93]. Binding groups are called liaison groups by some authors, most notably Poizat (see [Poi87]).

Exercise 4.4.1. Prove (a) and (b) in Example 4.4.1(iv).

Exercise 4.4.2. Prove: If D_2 is finitely generated over $D_1 \cup A$ and D_3 is finitely generated over $D_2 \cup B$, then D_3 is finitely generated over $D_1 \cup A \cup B$.

4.5 Defining a Group from a Pregeometry

The canonical example of a nontrivial modular strongly minimal set is a vector space. In fact, for any nontrivial modular strongly minimal set D there is a vector space V such that the geometry associated to D is isomorphic (as a geometry) to the geometry associated to V. In this section we show (roughly) how to find V as a definable group in D^{eq} from the pregeometry D. More precisely, from a configuration of points, that can always be found in a nontrivial modular strongly minimal set, a definable strongly minimal group is constructed. By Theorem 4.3.4 this definable strongly minimal group is a $*-$vector space. We will also analyze configurations of points leading to definable fields. This will lead to a characterization of the strongly minimal sets D containing a definable field in D^{eq}.

The configuration of points alluded to above is defined as follows. Note that the elements involved are not assumed to be from a strongly minimal set.

Definition 4.5.1. *Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory. A $6-$tuple of elements $Q = (a_1, a_2, a_3, b_1, b_2, b_3)$ is called an* algebraic quadrangle *if the following hold for any $\{i, j, k\} = \{1, 2, 3\}$ and $\ell_{ijk} = \{b_i, a_j, a_k\}$.*

(1) *Q is pairwise independent and no element of Q is in $acl(\emptyset)$.*
(2) *$a_j \in acl(b_i, a_k)$.*
(3) *$b_i \in acl(b_j, b_k)$.*
(4) *b_i is interalgebraic with the canonical parameter of $tp(a_j a_k / acl(b_i))$.*
(5) *For $\{i', j', k'\} = \{1, 2, 3\}$, ℓ_{ijk} is independent from $\ell_{i'j'k'}$ over $\ell_{ijk} \cap \ell_{i'j'k'}$.*

For A a set and Q a $6-$tuple the notion Q is an algebraic quadrangle over *A is defined with the obvious adjustments in (1)–(5).*

Remark 4.5.1. There are many variations on the above definition. All are known under the general heading of "Zil'ber's configuration", after Boris Zil'ber who first isolated the notion and proved a variant of the following theorem.

Remark 4.5.2. If Q is an algebraic quadrangle and A is independent from Q, then Q is an algebraic quadrangle over A. See Exercise 4.5.1.

The roles of the a_i's and b_i's is symmetric in the definition. Given an algebraic quadrangle $(a_1, a_2, a_3, b_1, b_2, b_3)$ and π a permutation of $\{1, 2, 3\}$,

$$(a_{\pi(1)}, a_{\pi(2)}, a_{\pi(3)}, b_{\pi(1)}, b_{\pi(2)}, b_{\pi(3)})$$

is also an algebraic quadrangle.

The main theorem of the section is

Theorem 4.5.1. *Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory and $Q = (a_1, a_2, a_3, b_1, b_2, b_3)$ an algebraic quadrangle. There is a finite set A independent from Q and $A-$definable sets X and G satisfying:*

(1) *There is a generic of X interalgebraic with a_1 over A and* $\deg(X) = 1$.
(2) $MR(G) = MR(b_2)$.
(3) *G is a connected definable group and there is a definable faithful transitive group action of G on X.*

Given an arbitrary uncountably categorical theory it is not at all clear that $\mathfrak{C}$ contains an algebraic quadrangle. However, we will see that a nontrivial strongly minimal set in $1-$based theory contains an algebraic quadrangle, quickly leading to

Theorem 4.5.2. *Let $\mathfrak{C}$ be the universal domain of a $1-$based uncountably categorical theory and D a nontrivial strongly minimal set over $\emptyset$. There is a finite set A and a strongly minimal group G definable over A such that a generic of G is interalgebraic over A with an element of $D \setminus A$.*

As an application of Theorems 4.5.1 and 3.5.2 we offer Theorem 4.5.3, which does not *a priori* have anything to do with groups.

Definition 4.5.2. *Let D be a $\emptyset-$definable strongly minimal set. Then D is said to be* pseudomodular *if there is a $k < \omega$ such that whenever $X \cup \{a, b\} \subset D$ and $a \in acl(X \cup \{b\})$, there is a $Z \subset acl(X) \cap D$ of cardinality $\leq k$ such that $a \in acl(Z \cup \{b\})$.*

Remark 4.5.3. A strongly minimal set D is pseudomodular if and only if there is a k such that $MR(c/\emptyset) \leq k$, for c the canonical parameter of a plane curve of D. For this reason some authors say pseudolinear instead of pseudoprojective.

A modular strongly minimal set is pseudomodular with $k = 1$, while an algebraically closed field is not pseudomodular. In fact,

Theorem 4.5.3. *A pseudomodular strongly minimal set is locally modular.*

Throughout the section we assume the ambient theory to be uncountably categorical, although many of the results hold in much greater generality.

An algebraic quadrangle will not lead directly to a definable group action, but to an $\bigwedge$−definable collection of maps between $\bigwedge$−definable sets. Obtaining a definable group action from this collection of maps requires the following study.

4.5.1 Germs of Definable Functions

Here a definable function is identified with a name for the defining formula in $\mathfrak{C}^{eq}$. In this way a definable function is considered to be an element of the universe.

Throughout this section the theory is assumed to be uncountably categorical

(although we may restate this assumption to make results easier to reference).

Remark 4.5.4. (i) Let R be an A−definable binary relation on the universe and $X \subset dom(R)$ a locus over A such that the restriction of R to X defines a function. Then there is an A−definable function f which agrees with R on X.

(ii) Let g be an A−definable function and $a \in dom(g)$ such that $tp(a/A)$ is stationary. Then $tp(g(a)/A)$ and $tp(ag(a)/A)$ are stationary. (The proof is left as Exercise 4.5.2.)

Definition 4.5.3. *Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory, A a set and X, Y infinite loci over A such that* $\deg(X) = \deg(Y) = 1$. *An element g is a* generic map of X to Y *if g is a definable function and for all $a \in X$ generic over g, $g(a) \in Y$ and $\{a, g, g(a)\}$ is pairwise independent over A.*

When g is a generic map of X into Y we may also say g maps X to Y generically.

Remark 4.5.5. In the definition all elements of X and Y are generic over A since X and Y are loci over A. The assumption that X and Y are infinite is made only to eliminate trivial cases.

Remark 4.5.6. Let A be a set and X, Y loci over A such that $\deg(X) = \deg(Y) = 1$. Let g be a definable function.

(i) If a and b are elements of X generic over g, then $tp(g(a)/A) = tp(g(b)/A)$ and this type is stationary. Thus, g maps X into Y generically if and only if for *some* $a \in X$ generic over g, $g(a) \in Y$ and $\{a, g, g(a)\}$ is pairwise independent over A.

(ii) If g maps X to Y (generically) and $c \in Y$ is generic over g then there is some $b \in X$ generic over g such that $g(b) = c$. Thus, g maps onto the elements of Y generic over g.

(iii) Suppose $a \in X$, $b \in dcl(a,c) \cap Y$ and $\{a, b, c\}$ is pairwise independent over A. Then there is an $h \in dcl(c)$ which is a generic map from X into Y and takes a to b. (See Exercise 4.5.4.)

Let V be an algebraic variety. Morphisms g and h are called generically equal if there is an open set U on which g and h are both defined and agree. Being generically equal defines an equivalence relation on the "local morphisms" of V. The class of a morphism g under generic equality is the "germ of g".

Definition 4.5.4. *Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory, A a set and X, Y infinite loci over A such that* $\deg(X) = \deg(Y) = 1$. *Let g, h be generic maps of X into Y. We say g is* generically equal to h on X *if for all $a \in X$ generic over $\{g, h\}$, $g(a) = h(a)$. The set X is omitted from the term when it is clear from context.*

Lemma 4.5.1. *Let X and Y be infinite loci over $\emptyset$, g a generic map of X into Y and B a set. If $a \in X$ is generic over $B \cup \{g\}$ then $g(a)$ is generic over $B \cup \{g\}$.*

Proof. See Exercise 4.5.3.

Lemma 4.5.2. *Let X, Y and Z be degree 1 infinite loci over A. Let g map X generically to Y and h map Y generically to Z. Then $h \circ g$ maps X generically to Z.*

Proof. Without loss of generality, $A = \emptyset$. Let $a \in X$ be generic over $\{g, h\}$.

Claim. (i) $g(a)$ is generic over h.
(ii) $\{a, h \circ g, (h \circ g)(a)\}$ is pairwise independent.

(i) By Lemma 4.5.1.
(ii) By (i), $h(g(a))$ exists and is an element of Z such that $\{g(a), h, h(g(a))\}$ is pairwise independent. Again by Lemma 4.5.1, $h(g(a))$ is independent from $\{g, h\}$ and independent from a. Thus, $\{a, h \circ g, (h \circ g)(a)\}$ is pairwise independent, proving the claim.

Let $a \in X$ be generic over $\{g, h\}$ and $b \in X$ be generic over $h \circ g$. Then a is also generic over $h \circ g$, so a and b have the same type over $h \circ g$. Thus, $h \circ g(b)$ is defined, an element of Z and $\{b, h \circ g, (h \circ g)(b)\}$ is pairwise independent. In other words, $h \circ g$ maps X generically to Z.

Generic equality is not generally an equivalence relation, however

Lemma 4.5.3. *Let A be a set and X, Y infinite loci over A such that* $\deg(X) = \deg(Y) = 1$. *Let g be a generic map of X into Y and Z a $B-$definable set of generic maps of X into Y which contains g, where $B \supset A$. Then there is a $B-$definable equivalence relation $\sim$ such that for all h, $k \in Z$, h is generically equal to k if and only if $h \sim k$.*

Proof. Let $p \in S(\mathfrak{C})$ be the unique free extension of the type over A realized by the elements of X. Since $\deg(X) = 1$, p is definable over B. Since Z is B−definable there is a formula $f(x, z)$ over B such that for all $c \in Z$, $f(x, c)$ is a generic map of X into Y with name c. Let $\zeta(v)$ be the formula over B defining Z and $\varphi(x, y, z)$ the formula $\zeta(y) \wedge \zeta(z) \wedge (f(x, y) = f(x, z))$. Let $\theta(y, z)$ be the formula over B defining p_φ (where p is a type in x). Then, for all c, $d \in Z$, the following are equivalent:

- c is generically equal to d,
- $\forall a \in X$ generic over $\{c, d\}$, $f(a, c) = f(a, d)$,
- $\varphi(x, c, d) \in p$,
- $\models \theta(c, d)$.

This proves the lemma.

In algebraic geometry a germ of morphisms is an equivalence class X of generically equal morphisms. This germ is identified with a generic map g by defining $g(a)$ to be $h(a)$ for any h in X such that a is in the domain of h and a is generic over h. In other words, g is a canonical representative of X. We define a germ similarly except we must be sensitive to the fact that generic equality is definable only when restricted to a definable family of generic maps.

Definition 4.5.5. *Let A be a set and X, Y infinite loci over A such that* $\deg(X) = \deg(Y) = 1$. *A generic map g of X into Y is called a* germ *if*

(∗) *for all h realizing $tp(g/A)$, if h is generically equal to g on X then $h = g$.*

The next two lemmas give the existence of germs and useful tools for working with them.

Lemma 4.5.4. *Let A be a set and X, Y infinite loci over A such that* $\deg(X) = \deg(Y) = 1$. *Let h be a generic map from X to Y, $a \in X$ generic over h and $p \in S(\mathfrak{C})$ the unique free extension of $tp(ah(a)/A \cup \{h\})$.*

(i) Given a generic map g from X into Y, g is generically equal to h if and only if for any $b \in X$ generic over g, p is also the unique free extension of $tp(bg(b)/A \cup \{g\})$.

(ii) The canonical parameter c of p is interdefinable with a generic map from X into Y which is generically equal to h.

Proof. Without loss of generality, $A = \emptyset$. That $tp(ah(a)/h)$ is stationary and hence has a unique free extension in $S(\mathfrak{C})$ is Remark 4.5.4(ii).

(i) See Exercise 4.5.5.

(ii) Note, $c \in dcl(h)$ so a is generic over $\{c, h\}$. The restriction of p to c is stationary and $\{a, h(a)\}$ is independent from h over c.

Claim. $h(a) \in dcl(a, c)$.

Assuming the claim to fail there is a $b \neq h(a)$ realizing $tp(h(a)/c, a)$ which is independent from h over $\{c, a\}$, hence ab is independent from h over c. Thus, $tp(ab/c, h) = tp(ah(a)/c, h)$, so $b = h(a)$. This contradiction proves the claim.

By Remark 4.5.6(iii) there is a $g \in dcl(c)$ which is a generic map from X to Y with $g(a) = h(a)$. Since a is generic over $\{g, h\}$ we conclude that g is generically equal to h. By (i) p is definable over g, hence $c \in dcl(g)$. This proves the lemma.

Lemma 4.5.5. *Let A be a set and X, Y infinite loci over A such that* $\deg(X) = \deg(Y) = 1$. *Given a generic map g of X into Y the following are equivalent.*

(1) *g is a germ.*
(2) *For any generic map h of X into Y which is generically equal to g there is an $a \in X$ generic over h such that g is a canonical parameter of $tp(ah(a)/A \cup \{h\})$.*
(3) *For some set I of generics of X such that $I \cup \{g\}$ is A-independent,*

$$g \in dcl(\{(a, g(a)) : a \in I\}).$$

Proof. Without loss of generality $A = \emptyset$.

(2) $\Longrightarrow$ (1) Let $a \in X$ be generic over g. Let $p \in S(\mathfrak{C})$ be the unique free extension of $tp(ag(a)/g)$. Let g' be a realization of $tp(g)$ which is generically equal to g. By Lemma 4.5.4(i) p is also the unique free extension of $tp(bg'(b)/g')$, for any $b \in X$ generic over g'. Thus, any $\alpha \in \mathrm{Aut}(\mathfrak{C})$ that maps g to g' also maps p to itself. By (2) g is a canonical parameter of p, hence $\alpha(g) = g$. Thus, $g' = g$; i.e., g is a germ.

(1) $\Longrightarrow$ (3) Assume g is a germ and $I \subset X$ is an infinite g-independent set of elements generic over g. Let $J = \{\, (a, g(a)) : a \in I \,\}$ and suppose g' realizes $tp(g/J)$. Since I is infinite there is an $a \in I$ generic over $\{g, g'\}$. Since g' and g have the same type over J, $g'(a) = g(a)$. Then $g' = g$ (because g is a germ), hence $g \in dcl(J)$.

(3) $\Longrightarrow$ (2) Let $I \subset X$ be a set of elements generic over g such that $g \in dcl(\{\, (a, g(a)) : a \in I \,\}$. Without loss of generality, I is finite. By Lemma 4.5.4(ii) there is a map h generically equal to g which is a canonical parameter of $p_0 = tp(ag(a)/g)$, for some $a \in I$. It remains to show that $g \in dcl(h)$. Let $J = \{\, (b, g(b)) : b \in I \,\}$. Since every element of J realizes p_0 and J is g-independent, J is a Morley sequence in p_0 over g. Since h is a canonical parameter of p_0, J is also a Morley sequence in p_0 over h. In particular, $tp(J/h)$ is stationary and J is independent from g over h. From $g \in dcl(J)$ we conclude that $g \in dcl(h)$.

Corollary 4.5.1. *Let X and Y be infinite loci over $\emptyset$ such that* $\deg(X) = \deg(Y) = 1$.

(i) Given a generic map g of X into Y there is a germ $h \in dcl(g)$ which is generically equal to g.

(ii) If R is an $\bigwedge-$definable set (over $\emptyset$) of generic maps there is an $\emptyset-$definable function γ such that for any $f \in R$, $\gamma(f)$ is a germ generically equal to f. Let R' be the $\bigwedge-$definable set $\gamma(R)$. We can choose γ so that for all h, $k \in R'$, if h is generically equal to k, then $h = k$.

Proof. (i) Combine Lemmas 4.5.4 and 4.5.5.
(ii) See Exercise 4.5.6.

Corollary 4.5.2. *Let X be an infinite loci over $\emptyset$ of degree 1. If g is a germ defined generically on X there is an $\emptyset-$definable set Z such that Z is a set of generic maps on X and for all h, $k \in Z$, if h and k are generically equal then $h = k$.*

Proof. Let $q = tp(g/A)$. There is a formula $\theta \in q$ such that any h satisfying θ is a map defined generically on X. By the Definability Lemma there is a formula $\sigma(y, z)$ over A such that whenever $\models \theta(h) \wedge \theta(k)$, $\models \sigma(h, k)$ if and only if

$$\forall a \in X \text{ generic over } \{h, k\},\ h(a) = k(a) \iff h = k.$$

Any pair of realizations of q satisfies $\sigma(y, z)$. The existence of Z now follows by compactness.

Remark 4.5.7. Given X and Y as usual, there may well be distinct germs g, h mapping X generically to Y which are generically equal. By the same token, when k maps X generically to Y there may be more than one germ in $dcl(k)$ which is generically equal to k.

Definition 4.5.6. *Let $\mathfrak{C}$ be the universe of an uncountably categorical theory, A a set and X, Y loci over A such that* $\deg(X) = \deg(Y) = 1$. *The set of germs from X into Y is denoted $\mathcal{O}(X, Y)$. Let $\mathcal{O}(X, X) = \mathcal{O}(X)$ and $\mathcal{O}^i(X, Y)$ the set of invertible elements of $\mathcal{O}(X, Y)$.*

Let Z be a degree 1 locus over A. With notation as in the definition, composition maps $\mathcal{O}(X, Y) \times \mathcal{O}(Y, Z)$ into $\mathcal{O}(X, Z)$ in that, given $g \in \mathcal{O}(X, Y)$ and $h \in \mathcal{O}(Y, Z)$, there is a germ in $\mathcal{O}(X, Z)$ generically equal to $h \circ g$. (So, in fact, the composition of g and h followed by the operation of taking a germ generically equal to the $h \circ g$ maps (g, h) to an equivalence class of generically equal germs.) In this sense $\mathcal{O}(X)$ is closed under composition. From hereon, when dealing with germs, composition will be denoted by $\cdot$ instead of $\circ$.

Our goal is to find a definable group $G \subset \mathcal{O}(X)$ acting on a definable set $X_0 \supset X$. As an intermediate step we find an $\bigwedge-$definable group contained in $\mathcal{O}(X)$. The naive way to find a group contained in $\mathcal{O}(X)$ which is at least $\mathrm{Aut}(\mathfrak{C})-$invariant is to close some locus $R = r(\mathfrak{C}) \subset \mathcal{O}^i(X)$ under inversion and composition. While this will yield a group H there is no reasons to think it is $\bigwedge-$definable unless there is a finite k such that each element of H is $h_1^{\epsilon_1} \cdot \ldots \cdot h_k^{\epsilon_k}$, where $h_i \in R$ and $\epsilon_i = \pm 1$, for $i = 1, \ldots, k$. We will show that there is such a bound k (and H is $\bigwedge-$definable) if R has generic composition (see Definition 4.5.7).

While not every germ is invertible we do have right cancelation:

Lemma 4.5.6. *Let X, Y and Z be degree 1 infinite loci over $\emptyset$, $h \in \mathcal{O}(X,Y)$ and g_1, $g_2 \in \mathcal{O}(Y,Z)$. If $g_1 \cdot h$ is generically equal to $g_2 \cdot h$ then g_1 is generically equal to g_2. Thus $g_1 \in acl(g_1 \cdot h, h)$.*

Proof. Let $a \in X$ be generic over $\{g_1, g_2, h\}$. Since $g_1(h(a)) = g_2(h(a))$ and $h(a)$ is generic over $\{g_1, g_2\}$ (by Lemma 4.5.1), g_1 is generically equal to g_2.

Lemma 4.5.7. *Let H be an $\bigwedge$-definable semigroup in an uncountably categorical theory which has right cancelation. Then H is a group.*

Proof. Recall from Exercise 3.3.15 that $(*)$ given $\varphi(u,v)$ a formula and $A = \{ a_i : i < \omega \}$ a set such that $\models \varphi(a_i, a_j)$ if and only if $i \leq j$, A must be finite. Given an $a \in H$ we must find a $b \in H$ such that $ba = 1$. By compactness it suffices to show that for any definable $X \supset H$ (on which $\cdot$ is defined) there is a $b \in X$ such that $ba = 1$. Pick an arbitrary definable $X \supset H$. Without loss of generality, $\cdot$ is defined on $X \times X$ and satisfies the right cancelation law on X. Let $X_1 \subset X$ be a definable set such that $H \subset X_1$ and for all x, $y \in X_1$, $x \cdot y \in X$. Let $u|v$ denote the formula $(\exists w \in X_1)(w \cdot u = v)$. For $m \leq n < \omega$, $a^m | a^n$. By $(*)$ there are $m < n$ such that $a^n | a^m$. Using right cancelation on X we get a $b \in X$ such that $b \cdot a = 1$, completing the proof.

Definition 4.5.7. *Let X be a degree 1 infinite locus over A and $R \subset \mathcal{O}(X)$ a degree 1 infinite locus over A. We say R has* generic composition *if for g, $h \in R$ independent, $\{g \cdot h, g, h\}$ is pairwise independent and $g \cdot h \in R$.*

One preliminary lemma before getting to the main result involving generic composition (which is essential to the proof of Theorem 4.5.1).

Lemma 4.5.8. *Let X be a degree 1 infinite locus over A and $R \subset \mathcal{O}(X)$ a degree 1 infinite locus over A with generic composition. Then for all $f, g, h \in R$ there are $j, k \in R$ such that $f \cdot g \cdot h = j \cdot k$.*

Proof. Let g_2 be an element of R independent from $\{f, g, h\}$. Since R has generic composition there is a $g_1 \in R$ such that $g = g_1 \cdot g_2$ and $\{g, g_1, g_2\}$ is pairwise independent. By Lemma 4.5.6 $g_1 \in acl(g, g_2)$. Thus, f is independent from g_1 over $\{g, g_2\}$. Since g_2 is independent from $\{f, g\}$, f is independent from g_1 over g. From the independence of g and g_1 we derive the independence of f and g_1. Let $j = f \cdot g_1$ and $k = g_2 \cdot h$. Since R has generic composition both j and k are in R. The equation $f \cdot g \cdot h = j \cdot k$ completes the proof.

Theorem 4.5.4. *Let X be an infinite locus of degree 1 over A in an uncountably categorical theory and $R \subset \mathcal{O}(X)$ an infinite locus of degree 1 over A with generic composition. There is an $A-$definable group $H \subset \mathcal{O}(X)$ which is connected and has R as its set of generic elements.*

Proof. Let $H_0 = R \cup \{1\}$ and $A = \emptyset$. Let

$$H_0' = \{ f : f \text{ is a generic map on } X \text{ and } f = g \cdot h \text{ for some } g, h \in H_0 \}.$$

Since H_0' is $\bigwedge$−definable Corollary 4.5.1 can be applied to find an $\emptyset$−definable function γ and an $\bigwedge$−definable set $H = \gamma(H_0')$ such that

– for all $f \in H_0'$, $\gamma(f)$ is a germ generically equal to f and
– $h = k$ whenever h, $k \in H$ are generically equal.

Without loss of generality, $\gamma(f) = f$ whenever $f \in H_0$; i.e., $R \subset H$. The key properties of H are highlighted in

Claim. (i) If f, $g \in H$ agree generically on X then $f = g$.
(ii) H is closed under multiplication.
(iii) H has right cancelation.

(i) is part of the definition of H. For (ii) let f, $g \in H$. There are f_i, $g_i \in H_0$, for $i = 1, 2$, such that f is a germ generically equal to $f_1 \cdot f_2$ and g is a germ generically equal to $g_1 \cdot g_2$. By Lemma 4.5.8 there are h_1, $h_2 \in H_0$ such that $f_1 f_2 g_1 g_2 = h_1 h_2$. There is a unique $h \in H$ generically equal to $h_1 h_2$, which we set equal to $f \cdot g$. H has right cancelation by Lemma 4.5.6, completing the proof of the claim.

From Lemma 4.5.7 we conclude that H is a group. By Theorem 3.5.3, H is not only $\bigwedge$−definable, but definable. Since H is a group each element of H is invertible. As a consequence

(∗) whenever $A \subset H$, $a \in A$ and $b \in R$ is independent from A, $b \cdot a$ is interdefinable with b over A, hence $MR(b \cdot a/A) = MR(b/A) = MR(R)$.

It remains to show that H is connected and R is the set of generics of H.

Claim. If $a \in H$ and $b \in R$ is independent from a, then $b \cdot a$ is in R.

Let c and d be elements of R such that $a = c \cdot d$ and $\{c, d\}$ is independent from b. Since R has generic composition, $b \cdot c$ is an element of R. By (∗), $b \cdot c$ is generic over $\{c, d\}$. Thus, $(b \cdot c) \cdot d$ is an element of R; i.e., $b \cdot a \in R$.

Let a be a generic of H and $b \in R$ generic over a. Then $b \cdot a$ is generic; it is also an element of R by the claim, hence the elements of R are generic. For b, $c \in R$ independent, $b \cdot c^{-1}$ is a generic in the connected component of H by basic facts about generics. Moreover, $b \cdot c^{-1} \in R$ by the claim. Thus, R is the set of generics in the connected component of H. Since H^o is closed under multiplication and every element of H is a product of elements of $R \cup \{1\}$, $H^o = H$. This proves the theorem.

We now make the jump from a definable group of generic maps on an $\bigwedge$−definable set to a definable group action.

Proposition 4.5.1. *Let X be a degree 1 locus over $\emptyset$ in an uncountably categorical theory and $G \subset \mathcal{O}^i(X)$ a connected $\emptyset$−definable group. Then there is a definable group action $(G, X_0, \star)$ for some definable X_0 such that*

– *the action of G on X_0 is faithful and transitive;*

– *X can be identified with a subset of X_0;*
– *for any $g \in G$ and $x \in X$ generic over g, $g \star x = g(x)$.*

Proof. We begin by addressing the problem of the elements of G only being defined generically on X.

Claim. There is an $\bigwedge$−definable set Y and a definable operation $\star$ such that

– $(G, Y, \star)$ is a faithful transitive group action,
– X can be identified with a subset of Y, $GX = Y$, and
– for $g \in G$ and $x \in X$ generic over g, $g \star x = g(x)$.

Consider the set Z of pairs (g, a), where $g \in G$ and $a \in X$. Define an equivalence relation $\sim$ on Z by: $(g, a) \sim (g', a')$ if and only if for every (some) $h \in G$ generic over $\{g, a, g', a'\}$, $(hg)a = (hg')a'$. ($hg \in \mathcal{O}(X)$ and a is generic over hg, so $(hg)a$ is defined.) As usual, by the Definability Lemma, $\sim$ is the restriction to Z of an $\emptyset$−definable equivalence relation. Let $[g, a]$ denote the $\sim$ −class of $(g, a) \in Z$ and Y the set of equivalence classes. We claim that given $g_0 \in G$ and $(g, a) \in Z$, if $(g, a) \sim (g', a')$, then $(g_0 g, a) \sim (g_0 g', a')$. (Given $h \in G$ generic over $\{g_0, g, g', a, a'\}$, hg_0 is generic over $\{g, g', a, a'\}$, hence $(hg_0)ga = (hg_0)g'a'$.) Thus, the operation $\star$ given by $g' \star [g, a] = [g'g, a]$ defines an action of G on Y. We may take $\star$ to be the restriction to $G \times Y$ of a definable operation.

That the map $a \mapsto [1, a]$ is an embedding of X into Y is clear since the elements of G are invertible germs. The definition of Y shows that any $y \in Y$ is $g \star x$ for some $x \in X$ and $g \in G$. It follows quickly that $(G, Y, \star)$ is a faithful transitive group action, completing the proof of the claim.

Let Y_0 be an $\emptyset$−definable set containing Y such that

– $\star$ is defined on $G \times Y_0$,
– for all $x \in Y_0$ and $g, h \in G$, $g \star (h \star x) = (gh) \star x$, and
– $g \star x = x \implies g = 1$.

Let $\theta(v)$ be the formula such that $\models \theta(a)$ if and only if for $x \in X$ generic over a, $(\exists g \in G)(g \star x = a)$. Since all elements of X are in the same orbit under the action of G, $\theta(\mathfrak{C}) \supset X$. Let $Y_1 = Y_0 \cap \theta(\mathfrak{C})$ and $X_0 = GY_1$. The reader can verify that $(G, X_0, \star)$ satisfies all of the conditions of the proposition.

Corollary 4.5.3. *Let X be a degree 1 locus over $\emptyset$ in an uncountably categorical theory and $R \subset \mathcal{O}(X)$ a degree 1 locus with generic composition. Then there is an $\emptyset$−definable group action $(H, X_0, \star)$ such that*

– *$H \subset \mathcal{O}(X)$ which is connected and has R as its set of generic elements;*
– *the action of H on X_0 is faithful and transitive;*
– *X can be identified with a subset of X_0;*
– *for any $g \in H$ and $x \in X$ generic over g, $g \star x = g(x)$.*

Proof. Simply combine Theorem 4.5.4 and Proposition 4.5.1.

Of course, this corollary is useless unless we can find a locus of germs with generic composition. Any instance in which we can prove such a locus exists is a special case of the next proposition.

Proposition 4.5.2. *Let X, Y and Z be loci over $acl(\emptyset)$ in an uncountably categorical theory and suppose there are $f \in \mathcal{O}(X,Y)$ and $g \in \mathcal{O}(Y,Z)$, both invertible, such that $\{f, g, g\cdot f\}$ is pairwise independent and $MR(f)$, $MR(g) < \omega$. Then there is a locus (over $acl(\emptyset)$) $R \subset \mathcal{O}(X)$ of invertible germs such that R has generic composition and $MR(R) = MR(f)$.*

Proof. Since $\{f, g, g \cdot f\}$ is pairwise independent and each element of the set is algebraic in the other two (by the invertibility of f and g) $MR(f) = MR(g) = MR(g \cdot f) = \alpha$. Let F be the locus of f over $acl(\emptyset)$, G the locus of g over $acl(\emptyset)$ and H the locus of $g \cdot f$ over $acl(\emptyset)$. Let f_0 be an element of F generic over f and R the locus of $f_0^{-1} \cdot f$ over $acl(\emptyset)$. One preliminary claim before showing that R has generic composition:

Claim. For independent $k, l \in R$ there is an independent $\{m_0, m_1, m_2, m_3\} \subset F$ such that $k = m_0^{-1} \cdot m_1$ and $l = m_2^{-1} \cdot m_3$.

If $\{f_0, f_1, f_2, f_3\} \subset F$ is independent then $k' = f_0^{-1} \cdot f_1$ and $l' = f_2^{-1} \cdot f_3$ are independent elements of R. The claim follows from the conjugacy over $acl(\emptyset)$ of all independent pairs in R.

Thus, to prove generic composition in R it suffices to show

Claim. Given $\{f_0, f_1, f_2, f_3\} \subset F$ independent there are $f_4, f_5 \in F$ such that $f_0^{-1} \cdot f_1 \cdot f_2^{-1} \cdot f_3 = f_4^{-1} \cdot f_5$ and $\{f_0^{-1} \cdot f_1, f_2^{-1} \cdot f_3, f_4^{-1} \cdot f_5\}$ is pairwise independent.

Let $g \in G$ be generic over $\{f_0, f_1, f_2, f_3\}$. As a first observation:

$$\{g \cdot f_0, g \cdot f_1, f_0^{-1} \cdot f_1\} \text{ is pairwise independent.} \tag{4.6}$$

(Since $\{g, f_0, f_1\}$ is independent $MR(g\cdot f_0/f_0^{-1}\cdot f_1) \geq MR(g\cdot f_0/\{g, f_0, f_1\}) = MR(g \cdot f_0/\{g, f_0\}) = MR(g \cdot f_0)$. That is, $g \cdot f_0$ is independent from $f_0^{-1} \cdot f_1$. Similarly $g \cdot f_1$ is independent from $f_0^{-1} \cdot f_1$ and $g \cdot f_0$ is independent from $g \cdot f_1$.)

Write $(f_0^{-1} \cdot f_1)$ as $(g \cdot f_0)^{-1} \cdot (g \cdot f_1)$, where $(g \cdot f_0)^{-1}$ is independent from $(g \cdot f_1)$ by (4.6). From

- $(f_0^{-1} \cdot f_1)$, $(f_2^{-1} \cdot f_3) \in R$;
- $(g \cdot f_0)$, $(g \cdot f_1) \in H$;
- $\bar{a}_1 = \{f_0^{-1} \cdot f_1, g \cdot f_0\}$ is independent and
- $\bar{a}_2 = \{f_2^{-1} \cdot f_3, g \cdot f_1\}$ is independent;

we conclude that $tp(\bar{a}_1/acl(\emptyset)) = tp(\bar{a}_2/acl(\emptyset))$. Thus there is an $h \in H$ independent from $g \cdot f_1$ such that $f_2^{-1} \cdot f_3 = (g \cdot f_1)^{-1} \cdot h$. It is routine to verify the independence of $g \cdot f_0$ from $\{g \cdot f_1, f_2^{-1} \cdot f_3\}$, hence $(g \cdot f_0)^{-1}$ is independent from h. So, by the conjugacy over $acl(\emptyset)$ of $\{(g \cdot f_1)^{-1}, h\}$ and $\{(g \cdot f_0)^{-1}, h\}$,

$(g \cdot f_0)^{-1} \cdot h$ is equal to $f_4^{-1} \cdot f_5$, for some independent $f_4, f_5 \in F$. The pairwise independence of $\{f_0^{-1} \cdot f_1, f_2^{-1} \cdot f_3, f_4^{-1} \cdot f_5\}$ follows from a rank calculation like that done at the beginning of the proof. This proves the claim and completes the proof that R is a locus of invertible germs with generic composition. The reader should show that any $f_0^{-1} \cdot f_1 \in R$ can be written as $l^{-1} \cdot m$ for some $l, m \in H$ with $\{l, m, f_1\}$ independent. Thus, $MR(f_0^{-1} \cdot f_1) = \alpha$. This proves the proposition.

Corollary 4.5.4. *Let X and Y be infinite loci of degree 1 over $\emptyset$. Suppose there is an invertible germ in $\mathcal{O}(X,Y)$ and there is an $n < \omega$ such that $MR(f) \leq n$ for all invertible $f \in \mathcal{O}(X,Y)$. Then there is a locus (over $acl(\emptyset)$) $R \subset \mathcal{O}(X)$ of invertible germs which has generic composition.*

Proof. Let $g \in \mathcal{O}(X,Y)$ be an invertible germ whose type over $\emptyset$ has maximal Morley rank, C the locus of g over $acl(\emptyset)$ and $m = MR(C)$. Note: any invertible $f \in \mathcal{O}(X)$ (or $\mathcal{O}(Y)$) realizes a type of Morley rank $\leq m$ over $\emptyset$. (Without loss of generality, g is independent from f. Then $g \cdot f \in \mathcal{O}(X,Y)$ is interalgebraic with f over g, hence $n = MR(g) \geq MR(g \cdot f)$.) Let $h \in C$ be generic over g. Since $h^{-1} \cdot g$ is an invertible germ in $\mathcal{O}(X)$, $MR(h^{-1} \cdot g) \leq m$. A rank calculation shows that $\{g, h^{-1}, h^{-1} \cdot g\}$ is pairwise independent. Proposition 4.5.2 can be applied to find the locus R.

The main application of Proposition 4.5.2 is

Proposition 4.5.3. *Let X and Y be loci of degree 1 over $\emptyset$ in a 1−based uncountably categorical theory and suppose there is an invertible germ in $\mathcal{O}(X,Y)$. Then $\mathcal{O}(X)$ contains a connected group $\bigwedge$−definable over $acl(\emptyset)$ and having Morley rank $MR(X)$.*

Using existing results the proof will follow quickly from

Lemma 4.5.9. *Let X and Y be infinite loci of degree 1 over $\emptyset$ in a 1−based uncountably categorical theory and $g \in \mathcal{O}(X,Y)$. Then $MR(g) = MR(Y)$.*

Proof. Let $a \in X$ be generic over g, $b = g(a)$ and recall that $\{g, a, b\}$ is pairwise independent simply because g is a generic map. By Lemma 4.5.4(ii) and the 1−basedness of the theory $g \in acl(a, b)$, hence g is interalgebraic with b over a. Thus, $MR(g) = MR(g/a) = MR(b/a) = MR(b) = MR(Y)$, proving the lemma.

Proof of Proposition 4.5.3. Since there is an invertible germ in $\mathcal{O}(X,Y)$, $MR(X) = MR(Y)$. By Lemma 4.5.9 any invertible germ in $\mathcal{O}(X,Y)$, $\mathcal{O}(Y,X)$ or $\mathcal{O}(X)$ realizes a type of Morley rank $MR(Y)$. Then, given invertible f, $g \in \mathcal{O}(X,Y)$ independent, a standard rank calculation shows that $\{f^{-1}, g, f^{-1} \cdot g\}$ is pairwise independent. By Proposition 4.5.2, $\mathcal{O}(X)$ contains a locus R over $acl(\emptyset)$ with generic composition with $MR(R) = MR(f)$. There is an $acl(\emptyset)$−definable connected group $G \subset \mathcal{O}(X)$ which has R as its set of generic elements (by Theorem 4.5.4). Noting that $MR(G) = MR(X)$ completes the proof.

4.5.2 Getting a Group from an Algebraic Quadrangle

In this section Theorem 4.5.1 and its corollaries are proved. Theorem 4.5.4 reduces the problem to finding in $\mathcal{O}(X)$ for some X a locus of germs (with a special relationship to a_1) which has generic composition. The theme is to successively replace the original algebraic quadrangle by a "nicer" quadrangle until (many of) the algebraic closure relations in the quadrangle are instances of definable closure. A definition is needed to state the key result. Remember that every theory in this section is assumed to be uncountably categorical.

The following illustrates the relationship between algebraic quadrangles and group actions.

Remark 4.5.8. Let K be an algebraically closed field and G the group of affine transformations on K (see Example 3.5.3). Let $h,\ g \in G$ be independent generics and $a \in X$ generic over $\{h, g\}$. Then $(a, h(a), g^{-1}h(a), h, g^{-1}, g^{-1}h)$ is an algebraic quadrangle. (The verification is left to the reader.)

Definition 4.5.8. *Let A be a set and $Q = (a_1, a_2, a_3, b_1, b_2, b_3)$, $Q' = (a'_1, a'_2, a'_3, b'_1, b'_2, b'_3)$ algebraic quadrangles over A. Then Q is* interalgebraic with Q' over A *if for all $1 \leq i \leq 3$, a_i is interalgebraic with a'_i over A and b_i is interalgebraic with b'_i over A.*

Proposition 4.5.4. *Given an algebraic quadrangle $Q = (a_1, a_2, a_3, b_1, b_2, b_3)$ there is a finite set A independent from Q and $Q' = (a'_1, a'_2, a'_3, b'_1, b'_2, b'_3)$ an algebraic quadrangle over A such that*

(1) *Q and Q' are interalgebraic over A,*
(2) *a'_1 and a'_3 are interdefinable over $A \cup \{b'_2\}$, and*
(3) *a'_2 and a'_3 are interdefinable over $A \cup \{b'_1\}$.*

The proposition will be proved in several stages, finding progressively "nicer" algebraic quadrangles over increasingly large sets of parameters. To simplify the notation we will replace at each stage the original algebraic quadrangle Q by the nicer one and absorb the parameters into the language.

Lemma 4.5.10. *If $Q = (a_1, a_2, a_3, b_1, b_2, b_3)$ is an algebraic quadrangle and for each $1 \leq i \leq 3$, a'_i is interalgebraic with a_i over $\emptyset$ and b'_i is interalgebraic with b_i over $\emptyset$, then $Q' = (a'_1, a'_2, a'_3, b'_1, b'_2, b'_3)$ is an algebraic quadrangle.*

Proof. The proof quickly reduces to showing that, for instance, b'_3 is interalgebraic with the canonical parameter of $tp(a'_1 a'_2/acl(b'_3))$. This is not difficult using that $\{a'_1, a'_2, b_3\}$ is pairwise independent, a'_1 is interalgebraic with a'_2 over b'_3, and the corresponding fact is true in Q. See Exercise 4.5.7.

Part of the definition of an algebraic quadrangle is that the ℓ_{ijk}'s are independent over their intersections. Using the independence of other sets we can show in addition

Lemma 4.5.11. *Let* $Q = (a_1, a_2, a_3, b_1, b_2, b_3)$ *be an algebraic quadrangle and* $\{i, j, k\} = \{1, 2, 3\}$*. Then* $\{b_j, b_k\}$ *is independent from* $\{a_j, a_k\}$ *over* b_i*.*

Proof. The proof is a two line exercise left to the reader.

The next lemma will see extensive use.

Lemma 4.5.12. *Let* $Q = (a_1, a_2, a_3, b_1, b_2, b_3)$ *be an algebraic quadrangle and* $1 \leq i \leq 3$*. A realization* a *of* $tp(a_i/Q \setminus \{a_i\})$ *is interalgebraic with* a_i *over* $\emptyset$*. Thus, letting* e *be a name for the (finite) set of realizations of* $tp(a_i/Q \setminus \{a_i\})$*,* e *is interalgebraic with* a_i *and* $e \in dcl(Q \setminus \{a_i\})$*.*

Proof. Without loss of generality, $i = 3$.

Claim. a and a_3 are interalgebraic over b_1 and interalgebraic over b_2.

Since a_2 and a_3 are interalgebraic over b_1, a_2 and a are interalgebraic over b_1. Thus, a and a_3 are interalgebraic over b_1. Similarly, a is interalgebraic with a_3 over b_2.

Let c be the canonical parameter of $tp(aa_3/acl(b_1b_2))$. Since a_3 is independent from $\{b_1, b_2\}$ and $a \in acl(a_3, b_1)$, aa_3 is independent from $\{b_1, b_2\}$ over b_1. Thus, $c \in acl(b_1)$. Similarly, $c \in acl(b_2)$. Since b_1 is independent from b_2, $c \in acl(\emptyset)$, hence aa_3 are interalgebraic over $\emptyset$.

It is clear from the first part of the lemma that e is interalgebraic with a_i. Since e is the name of a set definable over $Q \setminus \{a_i\}$, $e \in dcl(Q \setminus \{a_i\})$, completing the proof.

Lemma 4.5.13. *Let* $Q = (a_1, a_2, a_3, b_1, b_2, b_3)$ *be an algebraic quadrangle. There are* b_1', a_2', a_3' *and a finite set* A *such that*

(1) *A is independent from Q;*
(2) $Q' = (a_1, a_2', a_3', b_1', b_2, b_3)$ *is an algebraic quadrangle interalgebraic with Q over A; and*
(3) $a_2' \in dcl(a_3', b_1', A)$.

Proof. Let d_2 be a realization of $tp(b_2/acl(\emptyset))$ which is independent from Q. Since b_2 is independent from $\{b_1, a_2, a_3\}$,

$$tp(d_2/\{b_1, a_2, a_3\}) = tp(b_2/\{b_1, a_2, a_3\}),$$

hence there are c_1, d_3 so that $Q_0 = (c_1, a_2, a_3, b_1, d_2, d_3)$ realizes $tp(Q/acl(\emptyset))$. Let a_2' be a name for the finite set of realizations of $tp(a_2/Q_0 \setminus \{a_2\})$. Then $a_2' \in dcl(Q_0 \setminus \{a_2\})$ and a_2' is interalgebraic with a_2 over $\emptyset$ by Lemma 4.5.12.

Now fix $A = \{d_2\}$ as a set of parameters, $b_1' = \{b_1, d_3\}$ and $a_3' = \{a_3, c_1\}$. Let $Q' = (a_1, a_2', a_3', b_1', b_2, b_3)$. Then Q' is an algebraic quadrangle over A, interalgebraic with Q over A, and $a_2' \in dcl(A \cup \{a_3', b_1'\})$ as desired.

Lemma 4.5.14. *Let $Q = (a_1, a_2, a_3, b_1, b_2, b_3)$ be an algebraic quadrangle in which $a_2 \in dcl(a_3, b_1)$. Then there are b_2' interalgebraic with b_2 and a_1' interalgebraic with a_1 such that $a_1' \in dcl(a_3, b_2')$.*

Proof. Since $tp(a_1/a_3b_2)$ is algebraic there is a b_2' interalgebraic with b_2 so that $tp(a_1/a_3b_2')$ implies $tp(a_1/\{a_3\} \cup acl(b_2))$. Using Lemma 4.5.11 it follows that

$$tp(a_1/\{a_3, b_2'\}) \text{ implies } tp(a_1/\{a_3, b_1, b_2', b_3\}). \qquad (4.7)$$

Claim. If a realizes $tp(a_1/\{a_3, b_2'\})$ then a_1 and a are interalgebraic.

Given a realizing $tp(a_1/\{a_3, b_2'\})$, a also realizes $tp(a_1/\{a_3, b_1, b_2', b_3\})$, by (4.7). Since $a_2 \in dcl(a_3, b_1)$, a realizes $tp(a_1/\{a_2, a_3, b_1, b_2', b_3\})$. The 6–tuple $(a_1, a_2, a_3, b_1, b_2', b_3)$ forms an algebraic quadrangle, so Lemma 4.5.12 forces a_1 and a to be interalgebraic as claimed.

Let a_1' be the (finite) set of realizations of $tp(a_1/\{a_3, b_2'\})$, which is hence in $dcl(a_3, b_2')$. By the claim a_1' is interalgebraic with a_1, proving the lemma.

Lemma 4.5.15. *Let $Q = (a_1, a_2, a_3, b_1, b_2, b_3)$ be an algebraic quadrangle in which $a_2 \in dcl(a_3, b_1)$ and $a_1 \in dcl(a_3, b_2)$. Then there is a d_3 independent from Q and there is $Q' = (a_1', a_2', a_3', b_1', b_2', b_3')$ an algebraic quadrangle over d_3, interalgebraic with Q over d_3 such that*

(1) *a_1' and a_3' are interdefinable over $\{b_2', d_3\}$, and*
(2) *a_2' and a_3' are interdefinable over $\{b_1', d_3\}$.*

Proof. First let $a_3' \in dcl(Q \setminus \{a_3\})$ be interalgebraic with a_3 (which exists by Lemma 4.5.12). Let d_3 be a realization of $tp(b_3/acl(\emptyset))$ which is independent from Q. Find d_1 and c_2 so that $tp(c_2d_1d_3/acl(b_2, a_1, a_3)) = tp(a_2b_1b_3/acl(b_2, a_1, a_3))$. Note that $a_3' \in dcl(a_1, c_2, d_1, b_2, d_3)$. Let $a_1' = (a_1, c_2)$ and $b_2' = (b_2, d_1)$. Summarizing, we have a_1', a_3', b_2' and d_3 so that

- $Q_0 = (a_1', a_2, a_3', b_1, b_2', b_3)$ is an algebraic quadrangle over d_3 interalgebraic with Q over d_3,
- $a_3' \in dcl(a_1', b_2', d_3)$, and
- $a_1' \in dcl(a_3', b_2', d_3)$.

Similarly we find elements d_2 and c_1 so that $tp(c_1d_2d_3/acl(b_1a_2a_3')) = tp(a_1b_2b_3/acl(b_1a_2a_3'))$. Let $a_2' = (a_2, c_1)$ and $b_1' = (b_1, d_2)$. Drawing together the accumulated properties:

- $Q' = (a_1', a_2', a_3', b_1', b_2', b_3)$ is an algebraic quadrangle over d_3 interalgebraic with Q over d_3,
- $a_3' \in dcl(a_1', b_2', d_3)$,
- $a_1' \in dcl(a_3', b_2', d_3)$,
- $a_3' \in dcl(a_2', b_1', d_3)$, and
- $a_2' \in dcl(a_3', b_1', d_3)$.

This proves the lemma.

Proof of Proposition 4.5.4. Combine Lemmas 4.5.13, 4.5.14 and 4.5.15.

A quadrangle with this amount of definable closure produces a group of germs acting generically on the locus of any of the a_i's over $acl(\emptyset)$:

Proposition 4.5.5. *Let $Q = (a_1, a_2, a_3, b_1, b_2, b_3)$ be an algebraic quadrangle in which a_2 is interdefinable with a_3 over b_1 and a_1 is interdefinable with a_3 over b_2. Let X be the locus of a_1 over $acl(\emptyset)$. Then there is a connected group $G \subset \mathcal{O}(X)$, definable over $acl(\emptyset)$, such that $MR(G) = MR(b_i)$.*

Proof. Let X, Y and Z be the loci over $acl(\emptyset)$ of a_1, a_2 and a_3, respectively. Since a_1 and a_3 are interdefinable over b_2 and $\{a_1, a_3, b_2\}$ is pairwise independent there is $f \in acl(b_2)$ which is the germ of an invertible generic map from X into Z with $f(a_1) = a_3$. By Lemma 4.5.5 f is the canonical parameter of $tp(a_1a_3/f)$, which is also the canonical parameter of $tp(a_1a_3/acl(b_2))$. From one clause in the definition of an algebraic quadrangle f is interalgebraic with b_2.

Similarly let g be an invertible germ in $\mathcal{O}(Z, Y)$ such that $g(a_3) = a_2$ and g is interalgebraic with b_1. Then $g \cdot f$ is an invertible germ from X to Y, a_1 is generic over $g \cdot f$ and $g \cdot f(a_1) = a_2$.

Claim. $g \cdot f$ is interalgebraic with b_3.

The germ $g{\cdot}f$ is definable over $\{b_1, b_2\}$ and a_1 is generic over $\{b_1, b_2\}$, hence $g{\cdot}f$ is interdefinable with the canonical parameter c of $p = tp(a_1a_2/acl(b_1, b_2))$ by Lemma 4.5.5. Since $b_3 \in acl(b_1, b_2)$ p is also $tp(a_1a_2/acl(b_1, b_2, b_3))$. Since Q is an algebraic quadrangle a_1 is interalgebraic with a_2 over b_3, thus p is the unique free extension of $tp(a_1a_2/acl(b_3))$. Hence both c and $g \cdot f$ are not only algebraic in b_3 but interalgebraic with b_3 as claimed.

By the claim and the pairwise independence of $\{b_1, b_2, b_3\}$ $\{f, g, g \cdot f\}$ is pairwise independent. By Proposition 4.5.2 and Theorem 4.5.4 there is a connected group $G \subset \mathcal{O}(X)$, definable over $acl(\emptyset)$, with $MR(G) = MR(f) = MR(b_i)$.

Proof of Theorem 4.5.1. Let $Q = (a_1, a_2, a_3, b_1, b_2, b_3)$ be the hypothesized algebraic quadrangle. By Proposition 4.5.4 there is a finite set A' independent from Q and an algebraic quadrangle $Q' = (a'_1, a'_2, a'_3, b'_1, b'_2, b'_3)$ over A' such that

(1) Q and Q' are interalgebraic over A',
(2) a'_1 and a'_3 are interdefinable over $A' \cup \{b'_2\}$, and
(3) a'_2 and a'_3 are interdefinable over $A' \cup \{b'_1\}$.

Proposition 4.5.5 yields a connected group G of germs acting generically on X' = the locus of a'_1 over $acl(A')$ which is definable over $acl(A')$ and has Morley rank $= MR(b'_2)$. By Proposition 4.5.1 there are:

– a finite $A \subset acl(A')$;

– an $A-$definable degree 1 set $X \supset X'$;
– an $A-$definable transitive group action of G on X.

This proves the theorem.

Theorem 4.5.2 will follow from a slightly more general result stated momentarily. An uncountably categorical theory with universal domain $\mathfrak{C}$ is *trivial* if for all A there is no set $\{A_0, A_1, A_2\}$ which is pairwise $A-$independent but not $A-$independent. Note: When $\mathfrak{C}$ is strongly minimal this definition agrees with the earlier definition of a trivial strongly minimal set. The set of elements $X = \{a_0, a_1, a_2\}$ is an *algebraic triangle over A* if X is pairwise $A-$independent and for each $i \leq 2$, $a_i \in acl(A \cup X \setminus \{a_i\}) \setminus acl(A)$.

Theorem 4.5.5. *An nontrivial* $1-$*based uncountably categorical theory contains an infinite definable group.*

This theorem follows from the next two results.

Lemma 4.5.16. *Let* $\mathfrak{C}$ *be the universal domain of a* $1-$*based uncountably categorical theory and* A, A_0, A_1 *and* A_2 *sets such that* $\{A_0, A_1, A_2\}$ *is pairwise* $A-$*independent but not* $A-$*independent. Then there are* $a_i \in acl(A_i \cup A)$, *for* $i \leq 2$, *such that* $\{a_0, a_1, a_2\}$ *is an algebraic triangle over* A.

Proof. Without loss of generality each A_i is finite and $A = \emptyset$. Find $a_0 \in acl(A_0) \cap acl(A_1 \cup A_2)$ so that A_0 is independent from $A_1 \cup A_2$ over a_0. Also choose $a_1 \in acl(A_1) \cap acl(A_0 \cup A_2)$ with A_1 independent from $A_0 \cup A_2$ over a_1 and $a_2 \in acl(A_2) \cap acl(A_0 \cup A_1)$ with A_2 independent from $A_0 \cup A_1$ over a_2. The pairwise independence of $\{A_0, A_1, A_2\}$ forces $\{a_0, a_1, a_2\}$ to be pairwise independent. Since $a_0 \in acl(A_1 \cup A_2)$ and A_1 is independent from $\{a_0\} \cup A_2$ over a_1, $a_0 \in acl(\{a_1\} \cup A_2)$. Continuing this reasoning $a_0 \in acl(a_1, a_2)$. By the symmetric roles of the a_i's in this proof, $a_1 \in acl(a_0, a_2)$ and $a_2 \in acl(a_0, a_1)$, proving the lemma.

Proposition 4.5.6. *Let* $\mathfrak{C}$ *be the universal domain of a* $1-$*based uncountably categorical theory containing an algebraic triangle* $P = \{c_0, c_1, c_2\}$. *Then there is a finite set* A, *independent from* P, *an* $A-$*definable connected group* G, *an* $A-$*definable set* X *and an* $A-$*definable transitive action of* G *on* X *such that* c_1 *is interalgebraic over* A *with a generic of* X *and* $MR(G) = MR(X)$.

Proof. An algebraic quadrangle containing P is found as follows. First rename the elements of P as $b_2 = c_0$, $a_1 = c_1$ and $a_3 = c_2$. Let $b_1 a_2$ be a realization of $tp(b_2 a_1/a_3)$ independent from $b_2 a_1$ over a_3. Let b_3 be the canonical parameter of $tp(a_1 a_2/acl(b_1, b_2))$. We will show that $Q = (a_1, a_2, a_3, b_1, b_2, b_3)$ is an algebraic quadrangle.

Claim. b_2 is interalgebraic with the canonical parameter of $tp(a_1 a_3/acl(b_2))$.

The canonical parameter c of $tp(a_1a_3/acl(b_2))$ is in $acl(b_2)$ and b_2 is independent from a_1a_3 over c. Since $b_2 \in acl(a_1, a_3)$, $b_2 \in acl(c)$ as claimed.

Since $tp(b_1a_2/a_3) = tp(b_2a_1/a_3)$, b_1 is interalgebraic with the canonical parameter of $tp(a_2a_3/acl(b_1))$. The element b_3 was chosen as the canonical parameter of $tp(a_1a_2/acl(b_3))$. The remaining steps in the verification that Q is an algebraic quadrangle are organized in

(a) $\{a_1, a_2, b_3\}$ is an algebraic triangle.
(b) $\{b_1, b_2, b_3\}$ is pairwise independent.
(c) $\{b_1, b_2, b_3\}$ is an algebraic triangle.
(d) For $\{i, j, k\} = \{i', j', k'\} = \{1, 2, 3\}$, ℓ_{ijk} is independent from $\ell_{i'j'k'}$ over $\ell_{ijk} \cap \ell_{i'j'k'}$ (where $\ell_{ijk} = \{b_i, a_j, a_k\}$).

(a) The a_3-independence of a_1b_2 and a_2b_1 forces a_1 and a_2 to be independent from b_1b_2. Since $b_3 \in acl(b_1, b_2)$, $\{a_1, a_2, b_3\}$ is pairwise independent. a_1 and a_2 are interalgebraic over b_3 because these elements are interalgebraic over b_1b_2. The theory is 1-based so $b_3 \in acl(a_1, a_2)$, proving (a).

(b) Again the selection of the elements $\{a_1, a_2, a_3, b_1, b_2\}$ yields the independence of b_1 from a_1a_2 and b_2 from a_1a_2. Thus $\{b_1, b_2, b_3\}$ is pairwise independent.

(c) a_2 is independent from $b_1b_2b_3$ and $b_1 \in acl(b_2, b_3, a_2)$, so $b_1 \in acl(b_2, b_3)$. Similarly $b_2 \in acl(b_1, b_3)$. It has already been noted that $b_3 \in acl(b_1, b_2)$, hence $\{b_1, b_2, b_3\}$ is an algebraic triangle.

(d) The cases not explicitly verified above are left to the reader.

Thus, Q is an algebraic quadrangle. By Theorem 4.5.1 there are:

- a finite set A independent from Q;
- an A-definable set X of degree 1 containing a generic interalgebraic with a_1 over A;
- an A-definable connected group G and A-definable transitive action of G on X with $MR(G) = MR(b_2)$.

Since $MR(b_2) = MR(c_0) = MR(c_1) = MR(a_1) = MR(X)$, $MR(G) = MR(X)$. This proves the proposition.

Proof of Theorem 4.5.5. This follows immediately from Lemma 4.5.16 and Proposition 4.5.6.

Completing our applications to 1-based theories we have:

Proof of Theorem 4.5.2. Being nontrivial D contains a finite B and $\{c_0, c_1, c_2\}$ which is an algebraic triangle over B. By Proposition 4.5.6 there is a finite $A \supset B$ and a connected A-definable group G of Morley rank 1. A connected group of Morley rank 1 is strongly minimal. Since the theory is uncountably categorical we can choose A large enough so that an element of $G \setminus acl(A)$ is interalgebraic over A with an element of $D \setminus acl(A)$.

Corollary 4.5.5. *Given a nontrivial locally modular strongly minimal set D there is a finite set A and a strongly minimal group G over A such that a generic of G is interalgebraic over A with an element of $D \setminus acl(A)$ and G is a $*-$vector space over some division ring F. Thus the geometry associated to D_A is projective geometry over F.*

Proof. The existence of G and its relationship to D is simply by Theorem 4.5.2. By Theorem 4.3.4 G is a $*-$vector space over the division ring $F = \text{End}^*(G)$. The geometry associated to G is a projective geometry over F. The relation of being interalgebraic over A defines a one-to-one correspondence between the elements of the geometry associated to G and the geometry associated to D_A. In other words the geometry associated to G is isomorphic to the geometry associated to D_A, completing the proof.

Remark 4.5.9. A more sophisticated series of arguments shows that when D is locally modular and nonmodular there is a strongly minimal group definable over $acl(\emptyset)$, an $\emptyset-$definable equivalence relation E with finite classes and an $acl(\emptyset)-$definable regular action of G on the strongly minimal set $D' = \{ a/E : a \in D \}$. Thus the geometry associated to D' (which is also the geometry associated to D) is affine geometry over the vector space G/G^-. See [Hru87].

Our final installment in this study of defining groups is Theorem 4.5.3. This is proved by assuming to the contrary the theory contains a pseudomodular strongly minimal set which is not locally modular, proving the theory contains an infinite definable field, and that this leads directly to a contradiction.

Lemma 4.5.17. *Let D be a strongly minimal set such that for some A there are $A-$definable operations $+$ and $\cdot$ under which D is a field. Then D is not pseudomodular.*

Proof. This follows quickly from Example 4.2.2(iii).

Lemma 4.5.18. *Let D be a strongly minimal set, A a finite set, $a \in D \setminus acl(A)$ and $a' \in acl(A \cup \{a\}) \cap D'$, for D' an $A-$definable strongly minimal set. Then D is pseudomodular if and only if D' is pseudomodular.*

Proof. See Exercise 4.5.8.

Proof of Theorem 4.5.3. Suppose to the contrary that D is pseudomodular, not locally modular, and $k > 1$ is the maximum Morley rank of a plane curve in D. Let $a_1, a_3 \in D$ and b_2 be such that $tp(a_1a_3/b_2)$ is strongly minimal, b_2 is the canonical parameter of this type and $MR(b_2) = k$. Let a_2b_1 be a realization of $tp(a_1b_2/a_3)$ independent from a_1b_2 over a_3. Let b_3 be the canonical parameter of $p = tp(a_1a_2/acl(b_1, b_2))$. Since p is strongly minimal b_3 is the name for a plane curve in D hence $MR(b_3) \leq k$.

Claim. $Q = (a_1, a_2, a_3, b_1, b_2, b_3)$ is an algebraic quadrangle.

As a first step $b_3 \in acl(b_1, b_2)$ because it is the canonical parameter of a type over $acl(b_1, b_2)$. From the a_3-independence of $a_1 b_2$ and $a_2 b_1$, $a_1 a_3$ is independent from $b_1 b_2 b_3$ over b_2. Since $a_3 \in acl(b_1, b_3, a_1)$, $a_1 a_3$ is independent from $b_1 b_2 b_3$ over $b_1 b_3$. Thus $b_2 =$ the canonical parameter of $tp(a_1 a_3/b_2)$ is in $acl(b_1, b_3)$. In other words, b_2 and b_3 are interalgebraic over b_1. From this relation and $MR(b_3) \leq k$ we conclude that $MR(b_3) = k$ and $\{b_1, b_2, b_3\}$ is pairwise independent. Similarly $b_1 \in acl(b_2, b_3)$. The remaining steps in showing that Q is an algebraic quadrangle are left to the reader.

By Theorem 4.5.1 there is a finite set A, an $a' \in acl(A \cup \{a_2\})$ which is a generic of an $A-$definable strongly minimal set D' and an $A-$definable group G acting transitively on D' such that $MR(G) = MR(b_2)$. By Lemma 4.5.18 D' is pseudomodular, while there is a definable field structure on D' (perhaps with extra parameters) by Theorem 3.5.2. This contradicts Lemma 4.5.17 to prove the theorem.

Historical Notes. Algebraic quadrangles were developed by Zil'ber in his proof that a totally categorical theory is not finitely axiomatizable. His most up to date treatment is found in [Zil93]. The proof given here is based on the more general results proved by Hrushovski. One source for this material is Bouscaren's article in [NP89]. It is also found in [BH]. A more complete set of results can be found in [Pil]. Theorem 4.5.3 was first proved (using methods different from those here) by Buechler and Hrushovski [Bue91].

Exercise 4.5.1. Prove Remark 4.5.2

Exercise 4.5.2. Prove Remark 4.5.4

Exercise 4.5.3. Prove Lemma 4.5.1

Exercise 4.5.4. Prove Remark 4.5.6(iii)

Exercise 4.5.5. Prove Lemma 4.5.4(i).

Exercise 4.5.6. Prove Corollary 4.5.1.

Exercise 4.5.7. Let $\mathfrak{C}$ be the universal domain of an uncountably categorical theory and $\{a_1, a_2, b\}$ a pairwise independent set such that $a_1 \in acl(a_2, b)$ is and b is interalgebraic with the canonical parameter of $tp(a_1 a_2/acl(b))$. Suppose a_i' is interalgebraic with a_i, for $i = 1, 2$. Prove that b is interalgebraic with the canonical parameter of $tp(a_1' a_2'/acl(b))$.

Exercise 4.5.8. Prove (ii) of Lemma 4.5.18