XVI. Large Ideals on $\aleph_1$ from Smaller Cardinals

§0. Introduction

We give here better consistency strength than in XIII for having some large ideal on ω_1; possibly without adding a real using e.g. a Woodin cardinal. By this we keep old promises from 84 – 85, mentioned in [Sh:253], Shelah and Woodin [ShWd:241], (part of the delay was because it was originally intended to be part of [ShWd:241] which later was splitted to three). This will be continued elsewhere - getting suitable axioms in 2.4, 2.5, 2.6+2.10. Woodin told the author that the results (in 2.4 – 2.6(+2.7)) threw some light on the structure of universes of set theory satisfying AD. In §2 we use from §1 only 1.2(1),(2), 1.3(1), 1.8 for 2.1; weakening somewhat the results in §2, we can use 2.8, 2.9 instead of 2.1 (so replace $(*)^a_{ab}[\lambda]$ by "λ is a Woodin cardinal" in 2.4, 2.4A, 2.5, 2.6 thus using only 1.14, 1.15, 2.2 – 2.10).

The large cardinals from [ShWd:241] are defined in 1.14, 1.15.

§1. Bigness of Stationary $T \subseteq \mathcal{S}_{\leq\aleph_0}(\lambda)$

1.1 Notation. 1) λ a fixed regular cardinal $> \aleph_0$.

2) For sets a, b let $a \leq_\kappa b$ mean: $a \cap \kappa = b \cap \kappa$ and $a \subseteq b$ and let $a <_\kappa b$ means:

$a \subseteq b$ and $a \cap \kappa = b \cap \sup(a \cap \kappa)$ (i.e. $\alpha \in a \cap \kappa \Rightarrow a \cap (\alpha+1) = b \cap (\alpha+1)$), so $a <_\kappa b \not\Rightarrow a \leq_\kappa b$! And $a <_\kappa a$ holds!

3) $H(\alpha)$ is the family of sets x whose transitive closure has cardinality $< \alpha$, and if α is not cardinal we add: x of rank $< \alpha$. Let $<^*_\alpha$ be some well ordering of $H(\alpha)$ increasing with α. We let N denote a model (usually $N \prec (H(\chi), \in, <^*_\chi), N$ countable), $|N|$ its universe, and $\|N\|$ its cardinality. We write $N_1 \leq_\kappa N_2$ instead $|N_1| \leq_\kappa |N_2|$, similarly for $<_\kappa$.

4) $\mathcal{S}_{\leq\mu}(A) = \{b:\ b \subseteq B,\ |b| \leq \mu\}$

$\mathcal{D}_{\leq\mu}(B)$ is the filter on $\mathcal{S}_{\leq\mu}(B)$ generated by the closed unbounded subsets of $\mathcal{S}_{\leq\mu}(B)$ (similarly $\mathcal{D}_{<\mu}(A)$ for μ regular uncountable).

5) S, T denote subsets of some $\mathcal{S}_{\leq\mu}(A)$. We concentrate on $\mu = \aleph_0$.

1.2 Definition. 1) $T \subseteq \mathcal{S}_{\leq\aleph_0}(\lambda)$ is (θ, C^*)-big (where $\aleph_0 < \theta = \mathrm{cf}(\theta) \leq \lambda$ and $C^* \subseteq \lambda$ closed unbounded) *if*:

for every $\alpha < \lambda$ there is β, $\alpha \leq \beta < \mathrm{Min}(C^* \setminus (\alpha+1))$ such that for every $C \in \mathcal{D}_{\leq\aleph_0}(\beta)$ the set $\{a \in \mathcal{S}_{\leq\aleph_0}(\alpha) : (\exists b \in C \cap T)[a <_\theta b]\}$ belongs to $\mathcal{D}_{\leq\aleph_0}(\alpha)$.

We say T is $(<\sigma, C^*)$-big if for each $\theta < \sigma$ we have T is (θ^+, C^*)-big.

We define T is (θ, f)-big where $f : \lambda \to \lambda$ similarly only "$\beta < f(\alpha)$" replace "$\beta < \mathrm{Min}(C^* \setminus (\alpha+1))$". If $\neg(\aleph_0 < \theta = \mathrm{cf}\theta)$ we mean the first such $\theta^1 > \theta$.

2) $T \subseteq \mathcal{S}_{\leq\aleph_0}(B)$, is $\theta-$*big (where $\theta \subseteq B$) *if*: for every χ regular large enough and countable $N \prec (H(\chi), \in, <^*_\chi)$ to which T, B, θ belong there is $N', N <_\theta N' \prec (H(\chi), \in, <^*_\chi)$ such that $N' \cap B \in T$.

3) Let $\lambda \subseteq B$ and $\theta \subseteq B$. We say $T \subseteq \mathcal{S}_{\leq\aleph_0}(B)$ is θ-big (for B; if the identity of λ not clear we add "in λ") *if*: for every $C \in \mathcal{D}_{\leq\aleph_0}(B)$ and $\alpha < \lambda$ such that $[\theta < \lambda \Rightarrow \alpha \geq \theta]$ we have: $\{a$: $a \in \mathcal{S}_{\leq\aleph_0}(\alpha)$, and for some $b \in T \cap C$ we have $a <_\theta b\} \in \mathcal{D}_{\leq\aleph_0}(\alpha)$.

If $\theta = \lambda$ we may omit θ (remember that λ is fixed (see 1.1(1))). If $B = \lambda$ we may omit it.

4) We say T is $(\theta, *)$-big if $T \subseteq \mathcal{S}_{\leq\aleph_0}(\lambda)$ is (θ, f)-big for some $f : \lambda \to \lambda$; equivalently (θ, C^*)-big for some club C^* of λ.

If not said otherwise, and λ is strongly inaccessible, then we assume: (here as well as in 1.11, 1.13) for $\beta \in C^*$, we have $f(\beta)$ (or $\mathrm{Min}(C^* \setminus (\beta+1))$) is a strong limit of cofinality $> \beta$.

1.3 Definition.

(1) For cardinals $\mu \geq \lambda \geq \theta$ we say $T \subseteq \mathcal{S}_{\leq\aleph_0}(\lambda)$ is (μ,θ)-big or big for (μ,θ) *if*: for every $C^1 \in \mathcal{D}_{\leq\aleph_0}(\mu)$ for some $C \in \mathcal{D}_{\leq\aleph_0}(\mu)$:

$$(\forall a \in C)(\exists b)[a \subseteq b \in C^1 \ \&\ a <_\theta b \ \&\ b \cap \lambda \in T]$$

(2) We say T is $(<\mu,\theta)$-big if it is (μ_1,θ)-big for every $\mu_1, \lambda \leq \mu_1 < \mu$.

1.4 Definition. Suppose $\lambda \subseteq B$. We say $T \subseteq \mathcal{S}_{\leq\aleph_0}(B)$ is θ-essentially end extension closed set (for B) *if* for some $E \in \mathcal{D}_{\leq\aleph_0}(B)$:

$$[a \in E \ \&\ b \in E \ \&\ a <_\theta b \ \&\ a \in T \Rightarrow b \in T]$$

In short we write θ-EEEC and we call E a *witness* for T. If $E = \mathcal{S}_{\leq\aleph_1}(B)$ then we say T is θ-end extension closed set (for B), in short θ-EEC.

1.5 Definition. 1) $\mathrm{Pr}^0_\theta(\lambda)$ means: every θ-EEEC θ-big (see Definition 1.2(3)) set $T \subseteq \mathcal{S}_{\leq\aleph_0}(\lambda)$ is also $(2^\lambda,\theta)$-big (see Definition 1.3(1)).
2) $\mathrm{Pr}^1_\theta(\lambda)$ means:

for every semiproper forcing notion P of cardinality $< \lambda$,

$$\Vdash_P \text{“}\mathrm{Pr}^0_\theta(\lambda)\text{”}$$

3) $\mathrm{Pr}_\ell(\lambda)$ means $\mathrm{Pr}^\ell_\lambda(\lambda)$.

1.6 Fact. 1) In Definition 1.3(1) we can replace μ by any set A satisfying $\lambda \subseteq A$, $|A| = \mu$.
2) If $\theta_1 \leq \theta_2 \leq \lambda \leq \mu_1 \leq \mu_2$ and $T \subseteq \mathcal{S}_{\leq\aleph_0}(\lambda)$ is (μ_2,θ_2)-big (see Definition 1.3(1)) *then* T is (μ_1,θ_1)-big.

3) If $\lambda \subseteq B, \theta_1 \leq \theta_2$ and $T \subseteq \mathcal{S}_{\leq\aleph_0}(B)$ is θ_2-big (see Definition 1.2(3) so for B, in λ) *then* it is θ_1-big.

1.7 Fact. 1) If λ is weakly compact, $T \subseteq \mathcal{S}_{\leq\aleph_0}(\lambda)$ is θ-big (see Definition 1.2(3)) and $\theta < \lambda$ *then* T is (λ, θ)-big (see Definition 1.3(1)).
2) If λ is weakly compact, $T \subseteq \mathcal{S}_{\leq\aleph_0}(\lambda)$ is big (i.e. λ-big) *then* T is $(\lambda, *)$-big.

Proof. 1) Let $C^1 \in \mathcal{D}_{\aleph_0}(\lambda)$ be given. Let

$$E = \{a \in C^1 : \neg(\exists a' \in C^1)[a <_\theta a' \in C^1 \& a' \cap \lambda \in T]\}$$

If $E = \emptyset \bmod \mathcal{D}_{\leq\aleph_0}(\lambda)$ we finish.

Otherwise by weak compactness, for some $\lambda^* < \lambda$ (inaccessible, $\theta < \lambda \Rightarrow \lambda^* \Rightarrow \theta$) we have $E \cap \mathcal{S}_{\leq\aleph_0}(\lambda^*) \neq \emptyset \bmod \mathcal{D}_{\leq\aleph_0}(\lambda^*)$. As T is θ-big we get a contradiction.

2) Easy too. $\square_{1.7}$

1.8 Fact. 1) If $T \subseteq \mathcal{S}_{\leq\aleph_0}(\lambda)$ is big for $(2^\lambda, \theta)$ (see definition 1.3(1)) *then* T is $\theta-{}^*$big (see Definition 1.2(2)).
2) If $T \subseteq \mathcal{S}_{\leq\aleph_0}(\lambda)$ is $\theta-{}^*$big, and $\mu \geq \lambda$ then T is (μ, θ)-big.

1.8A Remark. So the two conditions in 1.8(1) are equivalent.

Proof. 1) We check definition 1.2(2), so say $\chi > 2^\lambda$. Clearly $H(\lambda^+) \in N$, $|H(\lambda^+)| = 2^\lambda$, and $Sb \stackrel{\text{def}}{=} \{M \prec (H(\lambda^+), \in, <^*_{\lambda^+}) : \|M\| = \aleph_0$ and T, λ, θ belong to $M\} \in N$ and $Sb \in \mathcal{D}_{\leq\aleph_0}(H(\lambda^+))$. By the assumption of 1.8(1) for some $C \in \mathcal{D}_{\aleph_0}(H(\lambda^+))$:

$$(\forall a \in C)(\exists a')[a <_\theta a' \in Sb]$$

As all the parameters in the requirements on C belong to N, without loss of generality $C \in N$. As $C \in N$ is a club of $\mathcal{S}_{\leq\aleph_0}(H(\lambda^+))$, clearly $N \cap H(\lambda^+) \in C$. So there is $N' \in Sb, N \cap H(\lambda^+) <_\theta N'$.

Now $\lambda \cap$ Skolem Hull $[N \cup (N' \cap \lambda)] = N \cap \lambda$ (Skolem Hull - in $(H(\chi), \in, <^*_\chi)$) and this implies the conclusion. $\Big[$Why the equality holds? Enough to look at $\tau(x, y)$ for τ a term, $x \in N, y \in N' \cap \lambda$ such that $\forall xy[\tau(x, y) \in \lambda)]$. In N there are $x' \in N \cap H(\lambda^+)$ and a term τ' such that $(\forall y \in \lambda)[\tau(x, y) = \tau'(x', y)]$. Now $x' \in N'$ and we finish.$\Big]$

2) Easy. $\square_{1.8}$

1.9 Fact. Suppose T is big for $(\lambda, \theta), 2^\lambda = \lambda^+$ and

(*) *for every* $u \subseteq \mathcal{S}_{\leq\aleph_0}(\lambda^+)$ such that $u \neq \emptyset \bmod \mathcal{D}_{\leq\aleph_0}(\lambda^+)$ *for some* $B \subseteq \lambda^+$, $u \cap \mathcal{S}_{\leq\aleph_0}(B) \neq \emptyset \bmod \mathcal{D}_{\leq\aleph_0}(B)$ and $|B| < \lambda$.

Then T is $\theta-{}^*$big.

Proof. Like the proof of 1.7 (remembering 1.8).

Similarly we can prove

1.9A Fact. Suppose for every μ, such that $\lambda \leq \mu \leq 2^\lambda$ we have:

$(*)_1$ $(\forall \text{ stat } E \subseteq S_{\leq\aleph_0}(\mu))(\exists A \subseteq \mu)\Big[|A| < \mu \ \& \ [\theta < \lambda \Rightarrow \theta \subseteq A] \ \& \ E \cap S_{\leq\aleph_0}(A) \neq \emptyset \bmod \mathcal{D}_{\leq\aleph_0}(A)\Big]$.

Then every θ-big T is $(2^\lambda, \theta)$-big (equivalently, $\theta-{}^*$big.)

1.10 Fact. If $Pr_0(\lambda)$ (see Definition 1.5(3) and 1.5(1)) and $\lambda = \kappa^+ = 2^\kappa$ *then* $\mathcal{D}_{\leq\aleph_0}(\kappa)$ is precipitous; moreover, semiproper (see below).

Proof. Let χ be regular large enough, $N \prec (H(\chi), \in, <^*_\chi)$ countable.

It suffices to prove $\mathcal{D}_{\leq\aleph_0}(\kappa)$ is semiproper; i.e.:

1.10A Definition. $\mathcal{D}_{\leq\aleph_0}(\kappa)$ is semiproper provided that the following holds. If $\langle B_i : i < \lambda \rangle$ is a maximal antichain of stationary subsets of $\mathcal{S}_{\leq\aleph_0}(\kappa)$ which belongs to N where $N \prec (H(\chi), \in, <^*_\chi)$ and χ large enough *then* there is a countable $M, M \prec (H(\chi), \in <^*_\chi), N \prec M, N <_{\kappa^+} M$ and $M \cap \kappa \in \bigcup_{i \in M} B_i$ [i.e. sealing forcing is semiproper].

(Hence by repeating one such M works for every such $\langle B_i : i < \lambda\rangle$ which belongs to it; this definition is what we need; from this precipitousness follows).

Continuation of the proof of 1.10: So let $N \prec (H(\chi), \in, <^*_\chi)$ and $\langle B_i : i < \lambda\rangle \in N$ be as in Definition 1.10A. Let $T = \{a \in \mathcal{S}_{\leq\aleph_0}(\lambda) : a \cap \kappa \in \bigcup_{i\in a} B_i\}$. We shall first prove that for α in the interval $[\kappa, \lambda)$ the set $E_\alpha = \{N \cap \alpha : N \prec (H(\chi), \in, <^*_\chi), N \cap \kappa \in \bigcap_{i\in N} B_i\}$ belongs to $\mathcal{D}_{\leq\aleph_0}(\alpha)$.

If $E_\alpha \notin \mathcal{D}_{\leq\aleph_0}(\alpha)$ let $f : \alpha \to \kappa$ be one to one onto, let

$C' = \{a \subseteq \alpha : f''(a) = a \cap \alpha \text{ and } a = f^{-1\prime\prime}(a \cap \alpha)\}$.

Clearly $C' \in \mathcal{D}_{\leq\aleph_0}(\alpha)$ and

$[(\mathcal{S}_{\leq\aleph_0}(\alpha) \setminus E_\alpha) \cap C')]\restriction\kappa$ is stationary (i.e. $\neq \emptyset \bmod \mathcal{D}_{\leq\aleph_0}(\kappa)$)

(where $E^*\restriction\kappa \overset{\text{def}}{=} \{a \cap \kappa : a \in E^*\}$), hence this set is not disjoint to some $B_{i(*)}$ and then we get an easy contradiction. So $E_\alpha \in \mathcal{D}_{\leq\aleph_0}(\alpha)$ for $\alpha \in [\kappa, \kappa^+)$. Let $\theta \overset{\text{def}}{=} \lambda(= \kappa^+)$, clearly T is $\theta - EEEC$ (see Definition 1.4, use as witness $C = \mathcal{S}_{\leq\aleph_0}(\lambda)$, noting that $B = \lambda$ here). Also as $E_\alpha \in \mathcal{D}_{\leq\aleph_0}(\alpha)$ for $\alpha \in [\kappa, \kappa^+)$, clearly T is θ-big (see Definition 1.2(3)). But by an assumption $\mathrm{Pr}_0(\lambda) = \mathrm{Pr}^0_\theta(\lambda)$ (as $\theta = \lambda$) hence we can deduce T is $(2^\lambda, \theta)$-big (Definition 1.3(1)), hence by Fact 1.8(1), T is $\theta-^*$big. So by Definition 1.2(2) there is N', $N <_\theta N' \prec (H(\chi), \in, <^*_\chi)$ such that $N' \cap \lambda \in T$. By the choice of T there is $i \in N' \cap \lambda$, such that $N' \cap \kappa \in B_i$, as required in Definition 1.10A.

So we have proved semiproperness. $\square_{1.10}$

1.11 Definition. 1) $Pr^2_\theta(\lambda, D, C^*)$ means: C^* a club of λ, D is a normal filter on λ concentrating on regular cardinals and for every θ-EEEC $(\theta,\ C^*)$-big $T \subseteq \mathcal{S}_{\leq\aleph_0}(\lambda)$ we have:

$$\{\kappa < \lambda :\ T \cap \mathcal{S}_{\leq\aleph_0}(\kappa) \text{ is } (2^\kappa, \theta \cap \kappa)\text{−big}\} \in D$$

(so here we use Def. 1.3(1) with λ replaced by κ.)

We may replace C^* by a function $f : \lambda \to \lambda$ as in Definition 1.2(1).

1.11A Remark. for "θ-EEEC" see Definition 1.4.

1.12 Definition. 1) $\mathrm{Pr}^3_\theta(\lambda, D, C^*)$ means that: for every semiproper forcing P of power $< \lambda$, we have $\Vdash$ "$\mathrm{Pr}^2_\theta(\lambda, D, C^*)$".

(D generates a normal filter in V^P and we do not strictly distinguish between the two).

1.13 Definition. 1) $\mathrm{Pr}^2_\theta(\lambda)$ means $(\exists D)(\forall C^*)\mathrm{Pr}^2_\theta(\lambda, D, C^*)$.

2) $Pr_3(\lambda, C)$ means: for some fixed D, for every semiproper P of power $< \lambda$, $\Vdash_P \mathrm{Pr}^2_\theta(\lambda, D, C)$.

From Shelah and Woodin [ShWd:241]:

1.14 Definition. (Shelah) 1) $\mathrm{Pr}_a(\kappa)$ means: $\mathrm{Pr}_a(\kappa, f)$ for every $f : \kappa \to \kappa$, where

2) $\mathrm{Pr}_a(\kappa, f)$ means: $f : \kappa \to \kappa$ and there is $\mathbf{j} : V \to M$ (elementary embedding into a transitive class) with critical point κ (i.e. $\mathbf{j}$ is the identity on κ hence on $H(\kappa)$) such that $H\Big(\mathbf{j}(f)(\kappa)\Big) \subseteq M$ and $M^{<\kappa} \subseteq M$. Let $\mathrm{Pr}_a(\kappa, f, D)$ means $\mathrm{Pr}_a(\kappa, f)$ is witnessed by $\mathbf{j}$ and $D = \{A \subseteq \kappa : \kappa \in \mathbf{j}(A)\}$. Note κ is necessarily measurable in all those cases.

1.15 Definition. (Woodin) $\mathrm{Pr}_b(\kappa)$, now called "κ is a Woodin cardinal" means:

for every $f : \kappa \to \kappa$ there is $\lambda < \kappa$ such that $\mathrm{Pr}_a(\lambda, f{\restriction}\lambda)$; equivalently

for every $f : \kappa \to \kappa$, there is an elementary embedding $\mathbf{j} : V \to M$ with critical point $\lambda < \kappa$, such that $H\Big(\mathbf{j}(f)(\lambda)\Big) \subseteq M$ and $M^{<\kappa} = M$.

So κ is a Mahlo cardinal, but not necessarily a weakly compact cardinal.

We can add

1.16 Definition. For $W \subseteq \kappa$, we can add:

1) $\mathrm{Pr}_a(\kappa, W)$ means $\mathrm{Pr}_a(\kappa, W, f)$ for every $f : \kappa \to \kappa$, which means $\mathrm{Pr}_a(\kappa, f, D)$ for some D to which W belongs.

2) Let $\mathrm{Pr}_b(\kappa, W)$ mean for every $f : \kappa \to \kappa$ there is $\lambda < \kappa$ such that $\mathrm{Pr}_a(\lambda, W \cap \lambda, f{\restriction}\lambda)$ (so in particular $\mathrm{Rang}(f{\restriction}\lambda) \subseteq \lambda$)

§2. Getting Large Ideals on $\aleph_1$

Note Ξ is a maximal antichain of $\mathcal{D}_{\omega_1}$ if $\Xi \subseteq \mathcal{P}(\omega_1)$ and for no stationary $S \subseteq \omega_1$ do we have $(\forall A \in \Xi)(A \cap S = \emptyset \mod \mathcal{D}_{\omega_1})$; we do not strictly distinguish $A \in \Xi$ and $A/\mathcal{D}_{\omega_1}$ or Ξ and $\{A/\mathcal{D}_{\omega_1} : A \in \Xi\}$.

Remember $\mathfrak{B} = \mathcal{P}(\omega_1)/\mathcal{D}_{\omega_1}$; on seal($\Xi$) and variations see XIII 2.4(2).

2.1 Lemma. *A Sealing is a Semiproper Criterion:* Let λ be strongly inaccessible, $C \subseteq \lambda$ closed unbounded, $[\delta \in C \Rightarrow (H(\delta), \in, <^*_\delta) \prec (H(\lambda), \in, <^*_\lambda)]$ (so each $\delta \in C$ is a strong limit cardinal). The following conditions satisfy $(B)^+ \Rightarrow (C) \Rightarrow (A) \Rightarrow (B)^-$.

(A) Let C be an end segment of C^*. For every Levy$(\aleph_1, < \lambda)$- name $\underset{\sim}{\Xi} = \{\underset{\sim}{A}_i : i < \lambda\}$ of a maximal antichain of $\mathcal{D}_{\omega_1}^{V[\text{Levy}(\aleph_1, <\lambda)]}$, the forcing notion Levy$(\aleph_1, < \lambda) *$ seal$(\underset{\sim}{\Xi})$ is semiproper, provided that:

$$\oplus_A \text{ for } \delta \in C, \Big(H(\delta), \in, <^*_\delta, \underset{\sim}{\Xi} \cap H(\delta)\Big) \prec \Big(H(\lambda), \in, <^*_\lambda, \underset{\sim}{\Xi}\Big).$$

$(B)^-$ Let C be an end segment of C^*. If $T \subseteq \mathcal{S}_{\leq \aleph_0}(\lambda)$ is $(< \lambda, C)$-big (see Definition 1.2(1)) and λ - EEEC (see Definition 1.4) *then* T is $(2^\lambda, \omega_2)$-big (see Definition 1.3(1)) provided that:

$$\oplus_B \text{ for } \delta \in C, \Big(H(\delta), \in, <^*_\delta, T \cap (\bigcup_{\alpha < \delta} \mathcal{S}_{\leq \aleph_0}(\alpha))\Big) \prec \Big(H(\lambda), \in, <^*_\lambda, T\Big).$$

$(B)^+$ Let C be an end segment of C^*. If $T \subseteq \mathcal{S}_{\leq \aleph_0}(\lambda)$ is $(\aleph_2, C)$-big (a weaker assumption see Definition 1.2(1)) λ-EEEC and *then* T is $(2^\lambda, \omega_2)$-big provided that $\oplus_B$ holds.

(C) Let C be an end segment of C^*. Suppose $\bar{P} = \langle P_i : i < \lambda \rangle$ is $\lessdot$-increasing, for $i < \lambda$, $P_i \in H(\lambda), P_i \lessdot P_\lambda$ where $P_\lambda \stackrel{\text{def}}{=} \bigcup_{\alpha < \lambda} P_\alpha$, and the forcing notions $P_i, P_\lambda / P_i$ are semiproper, P_λ satisfies the λ - c.c., $\Vdash_{P_\lambda}$ "$\lambda = \aleph_2$", and $\underset{\sim}{\Xi} = \{\underset{\sim}{A}_i / \mathcal{D}_{\omega_1} : i < \lambda\}$ a P_λ - name of a maximal antichain of $\mathcal{D}_{\omega_1}$.

Then $P_\lambda *$ seal(Ξ) *is semiproper provided that:*

$$\oplus_C \text{ for } \delta \in C, \ \left(H(\delta), \in, <^*_\delta, \bar{P}{\restriction}\delta, \underset{\sim}{\Xi} \cap H(\delta)\right) \prec \left(H(\lambda), \in, <^*_\lambda, \bar{P}, \underset{\sim}{\Xi}\right)$$

2.1A Definition. Assume λ is strongly inacessible, $\bar{P} = \langle P_i : i < \lambda \rangle$, P are as in clause (C) of 2.1 or $P_i = \text{Levy}(\aleph_1, < i)$, $P = \bigcup_{i<\lambda} P_i$ (for some closed unbounded $C \subseteq \lambda$). If $\underset{\sim}{A}$ is a P-name of a subset of ω_1 let

$$i(\underset{\sim}{A}/\mathcal{D}_{\omega_1}) = \min\{i : \text{ for some } P_i\text{-name } \underset{\sim}{A}', \Vdash_P \text{“}\underset{\sim}{A} = \underset{\sim}{A}'_i\text{” mod } \mathcal{D}_{\omega_1}\}$$

(note $i(A) < \lambda$ as P satisfies the λ-c.c.). Let us redefine

$$\underset{\sim}{A}/\mathcal{D}_{\omega_1} = \{\underset{\sim}{B} : \underset{\sim}{B} \text{ is a } P_{i(\underset{\sim}{A})}\text{-name of a subset of } \omega_1 \text{ such that } \Vdash_{P_\lambda} \text{“}\underset{\sim}{B} = \underset{\sim}{A}\text{”}\}.$$

Proof. Clearly $(B)^+ \Rightarrow (B)^-$, just read Definition 1.2(1).

<u>$\neg(A) \Rightarrow \neg(C)$</u>

Immediate: use $P = \text{Levy}(\aleph_1, < \lambda)$, and $P_i = \text{Levy}(\aleph_1, < i)$.

<u>$\neg(B)^- \rightarrow \neg(A)$</u>

Let T, C be a counterexample to (B)$^-$, in particular $\oplus_B$ holds and we can choose a club $E \subseteq \mathcal{S}_{<\aleph_1}(\lambda)$ witnessing T is EEEC i.e. $a \in E$ & $b \in E$ & $a <_\lambda b$ & $a \in T \Rightarrow b \in T$.

Let

$$W \overset{\text{def}}{=} \Big\{\delta < \lambda : \ (\forall \alpha < \delta)(\forall a \in \mathcal{S}_{\le\aleph_0}(\alpha)) \\ \Big[(\exists b)[b \in T \ \& \ a <_\lambda b] \Rightarrow (\exists b)[b \in T \ \& \ \sup(b) < \delta \ \& \ a <_\lambda b]\Big]\Big\}.$$

So W is a club of λ, definable in $(H(\lambda), \in, <^*_\lambda, T)$ hence by $\oplus_B$ for $\delta \in C$ we have $\sup(W \cap \delta) = \delta$, and $W \supseteq C$.

For $\delta < \lambda$ after forcing with $\text{Levy}(\aleph_1, < |\delta|^+)$ we have $\langle a^\delta_\zeta : \zeta < \omega_1 \rangle$ increasing continuous, each a^δ_ζ countable, $\bigcup_{\zeta<\omega_1} a^\delta_\zeta = \delta$. Let $\langle \underset{\sim}{a}^\delta_\zeta : \zeta < \omega_1 \rangle$ be a $\text{Levy}(\aleph_1, < |\delta|^+)$-name for such a sequence, and $\underset{\sim}{B}_\delta \overset{\text{def}}{=} \{\zeta : \underset{\sim}{a}^\delta_\zeta \in T\}$, this is a

Levy$(\aleph_1, < |\delta|^+)$-name; and then let (again a Levy$(\aleph_1, < |\delta|^+)$-name):
$\underset{\sim}{A}_\delta \stackrel{\text{def}}{=} \underset{\sim}{B}_\delta - \nabla_{\alpha<\delta} \underset{\sim}{B}_\alpha$ (∇-diagonal union, actually well defined only mod $\mathcal{D}_{\omega_1}$).

As T is λ-EEEC, clearly in $V^{\text{Levy}(\aleph_1,<\lambda)}$, $B_\delta/\mathcal{D}_{\omega_1}$ $(\delta \in W)$ is increasing and is the least upper bound of $\{A_\alpha/\mathcal{D}_{\omega_1} : \alpha \in (\delta+1) \cap W\}$ (in $(\mathcal{P}(\omega_1)/\mathcal{D}_{\omega_1})^{\text{Levy}(\aleph_1,<\lambda)}$). Let $\underset{\sim}{W}^* = \{\alpha : \alpha \in W$, and $\underset{\sim}{A}_\alpha \neq \emptyset \mod \mathcal{D}_{\omega_1}\}$ (it is a Levy$(\aleph_1, < \lambda)$-name)

Clearly $\underset{\sim}{\Xi} = \{\underset{\sim}{A}_\alpha/\mathcal{D}_{\omega_1} : \alpha \in \underset{\sim}{W}^*\}$ is an antichain (we should not mind the $\emptyset/\mathcal{D}_{\omega_1}$'s, i.e. some A_α's are not stationary).

Clearly $\underset{\sim}{\Xi}$ is a Levy$(\aleph_1, < \lambda)$-name satisfying $\oplus_A$.

2.1B Fact. $\underset{\sim}{\Xi}$ is a maximal antichain.

Suppose toward contradiction that $\underset{\sim}{A}$ is a Levy$(\aleph_1, < \lambda)$-name of a stationary subset of ω_1, but $p \in$ Levy$(\aleph_1, < \lambda)$ force it is a counterexample. So for some $\theta < \lambda$, $\underset{\sim}{A}$ is a Levy$(\aleph_1, < \theta)$-name, and $p \in$ Levy$(\aleph_1, < \theta)$. Let $\theta_1 = (2^\theta)^+$, $\mu = 2^{\theta_1}$, and

$$Y_p^\mu \stackrel{\text{def}}{=} \{a \in \mathcal{S}_{\leq\aleph_0}(H(\mu)) : \text{there is } q \in \text{Levy}(\aleph_1, < \theta) \text{ such that: } p \leq q,$$
$$q \text{ is an } (a, \text{Levy}(\aleph_1, < \theta))\text{-generic condition,}$$
$$\text{and } q \Vdash \text{“} a \cap \omega_1 \in \underset{\sim}{A}\text{”.}\}$$

Clearly $Y_q^\mu \neq \emptyset \mod \mathcal{D}_{\leq\aleph_0}(H(\mu))$.

Now, as λ is strongly inaccessible, $2^\mu < \lambda$, and as T is (θ_1, C)-big (as T exemplifies $\neg(B)$), there is β satisfying $2^\mu < \beta < \lambda$ (and moreover $2^\mu < \beta < \min(C \setminus (\beta+1))$), such that: for every $E \in \mathcal{D}_{\leq\aleph_0}(\beta)$ we have:

$$\{a \in \mathcal{S}_{\leq\aleph_0}(2^\mu) : \text{ there is } b \text{ such that } a <_{\theta_1} b \text{ and } b \in E \cap T\} \in \mathcal{D}_{\leq\aleph_0}(2^\mu).$$

Hence, as $|H(\mu)| \leq 2^\mu$, for every $E \in \mathcal{D}_{\leq\aleph_0}(\beta)$

$$\{a \in \mathcal{S}(H(\mu)) : \text{there is } b \text{ such that } a <_{\theta_1} b \text{ and } b \in E,$$
$$\text{and } b \cap \beta \in T\} \in \mathcal{D}_{\leq\aleph_0}(H(\mu)).$$

Let

$$E_1 \stackrel{\text{def}}{=} \{N : N \text{ is a countable elementary submodel of } (H(\beth_7(\lambda)^+), \in, <^*) \text{ to which } p,\ \lambda,\ \theta,\ \mu,\ \beta,\ \underset{\sim}{A},\ \langle(\underset{\sim}{B}_\alpha, \underset{\sim}{A}_\alpha) : \alpha < \lambda\rangle \text{ and } \langle\langle \underset{\sim}{a}^\alpha_\zeta : \zeta < \omega_1\rangle : \alpha < \lambda\rangle \text{ belong}\}.$$

Clearly it is a club of $\mathcal{S}_{<\aleph_1}(H(\beth_7(\lambda)^+))$, and let

$$E_2 \stackrel{\text{def}}{=} \{N \cap \beta :\ N \in E_1\},$$

clearly it belongs to $\mathcal{D}_{<\aleph_1}(H(\beta))$. So we can use E_2 as E above hence

$$E_3 \stackrel{\text{def}}{=} \{N : N \text{ is a countable, elementary submodel of } (H(\mu), \in), \text{ such that } p, \theta, \underset{\sim}{A}, \beta \text{ belong to it and for some } M_N \in E_1 \text{ we have } M_N \cap \beta \in T \text{ and } N <_{\theta_1} M_N \in T\}$$

belongs to $\mathcal{D}_{<\aleph_1}(H(\mu))$. Hence we can find $N \in E_3 \cap Y^\mu_p$, hence by the definition of Y^μ_p there is a condition $q \in \text{Levy}(\aleph_1, <\theta)$ such that $p \leq q \in \text{Levy}(\aleph_1, <\theta)$ and q is $(N, \text{Levy}(\aleph_1, <\theta))$-generic and $q \Vdash$ "$N \cap \omega_1 \in \underset{\sim}{A}$". As $N \in E_3$ clearly M_N is well defined (see the definition of E_3), so $M_N \in E_1$, $N \prec M_N \in E_1$ and $N <_{\theta_1} M_N$, hence $N \cap 2^\theta = M_N \cap 2^\theta$, hence $N \cap \text{Levy}(\aleph_1, <\theta) = M_N \cap \text{Levy}(\aleph_1, <\theta)$ and moreover $N \cap \mathcal{P}(\text{Levy}(\aleph_1, <\theta)) = M_N \cap \mathcal{P}(\text{Levy}(\aleph_1, <\theta))$; hence as q is $(N, \text{Levy}(\aleph_1, <\theta))$-generic we know that q is $(M_N, \text{Levy}(\aleph_1, <\theta))$-generic. As $\text{Levy}(\aleph_1, <\lambda)/\text{Levy}(\aleph_1, <\theta)$ is $\aleph_1$-complete there is $q_1 \in \text{Levy}(\aleph_1, <\lambda)$ such that $q_1 \restriction \theta = q$ and q_1 is $(M_N, \text{Levy}(\aleph_1, <\mu))$-generic, hence clearly $q_1 \Vdash$ "$M_N \cap \beta = \underset{\sim}{a}^\beta_{M_N \cap \omega_1} = a^\beta_{N \cap \omega_1}$" but $M_N \cap \beta \in T$ (see the definition of E_3) hence $q_1 \Vdash$ "$N \cap \omega_1 \in \underset{\sim}{B}_\mu$".

There is $\underset{\sim}{C}'$ such that $\Vdash_{\text{Levy}(\aleph_1, <\lambda)}$ "if $\underset{\sim}{B}_\beta \cap \underset{\sim}{A}$ is not stationary then $\underset{\sim}{B}_\beta \cap \underset{\sim}{A} \cap \underset{\sim}{C}' = \emptyset$, and $\underset{\sim}{C}'$ is a club of ω_1". So as $\underset{\sim}{B}_\beta$, $\underset{\sim}{A} \in N \subseteq M_N$, clearly w.l.o.g. $\underset{\sim}{C}' \in M_N$ hence $q_1 \Vdash$ "$N \cap \omega_1 \in \underset{\sim}{C}'$" hence $q_1 \Vdash$ "$\underset{\sim}{B}_\beta \cap \underset{\sim}{A} \cap \underset{\sim}{C}' \neq \emptyset$ hence $q_1 \Vdash_{\text{Levy}(\aleph_1, <\lambda)}$ "$\underset{\sim}{B}_\beta \cap \underset{\sim}{A}$ is stationary" which is enough for the fact 2.1B as $B_\beta = \underset{\alpha \leq \delta}{\nabla} A_\alpha \mod \mathcal{D}_{\omega_1}$. $\square_{2.1B}$

Continuation of the proof of 2.1.

Lastly to show that $\neg(A)$ holds, we still have to show that: the forcing notion $Q \stackrel{\text{def}}{=} \text{Levy}(\aleph_1, <\lambda) * \text{seal}(\underset{\sim}{\Xi})$ is not semi proper.

Suppose it is semiproper, χ large enough. Let $N \prec (H(\chi), \in, <^*_\chi)$ be countable, $Q, T, \underset{\sim}{\Xi}, C \in N$. Let $\delta = N \cap \omega_1$.

So there is $p \in Q$ which is (N, Q)-semi generic. So for some q satisfying $p \leq q \in Q$, and α, we have $q \Vdash_Q$ "$\alpha \in \underset{\sim}{W}^*$, $\delta \in \underset{\sim}{A}_\alpha$, $\alpha \in N[\underset{\sim}{G}_Q]$", ($\underset{\sim}{W}^*$ was defined just before $\underset{\sim}{\Xi}$); so $\underset{\sim}{A}_\alpha$, $\underset{\sim}{B}_\alpha \in N[\underset{\sim}{G}_Q]$ and clearly $q \Vdash_Q$ "$\underset{\sim}{a}^\alpha_\zeta = N[\underset{\sim}{G}_Q] \cap \alpha$", hence necessarily also $q \Vdash_Q$ "$\delta \in \underset{\sim}{B}_\alpha$".

Hence $\quad q \Vdash_Q$ "$N[\underset{\sim}{G}_Q] \cap \alpha \in T$" (read the definition of $\underset{\sim}{B}_\alpha$).

Hence w.l.o.g. for some $b \in T$ we have $\quad q \Vdash$ "$N[\underset{\sim}{G}_Q] \cap \alpha = b$".

Let N_1 be the Skolem Hull of $|N| \cup b$ in $(H(\chi), \in, <^*_\chi)$. Clearly $N_1 \cap \alpha = b \in T$ and $N_1 \cap \omega_1 = b \cap \omega_1 = \delta$ so $N <_{\aleph_2} N_1$. This shows T is a $\aleph_2 -$ *big (see Definition 1.2(2)), which by 1.8 is equivalent to "T is $(2^\lambda, \aleph_2)$-big"; but this is a contradiction to our assumption "T exemplifies $\neg(B)^-$".

<u>$\neg(C) \Rightarrow \neg(B)^+$</u>

We also prove $\neg(A) \rightarrow \neg(B)^+$

Let $\bar{P}, \underset{\sim}{\Xi} = \{\underset{\sim}{A}_i : i < \lambda\}$ and C contradict (C) or (A) (in the later case $P_i = \text{Levy}(\aleph_1, < i)$). Let, for each $p \in P_\lambda$:

$$\begin{aligned} T_p \stackrel{\text{def}}{=} \{N \cap \lambda : \; & p \in N, \text{ for some strong limit cardinal } \sigma < \lambda, \\ & N \prec (H(\sigma), \in, <^*_\lambda, \bar{P}\restriction\sigma, \underset{\sim}{\Xi}\restriction\sigma, \underset{\sim}{\bar{A}}\restriction\sigma) \text{ so we consider} \\ & \bar{P}, \underset{\sim}{\Xi}, \underset{\sim}{\bar{A}} \text{ as predicates, and } N \text{ is countable,} \\ & \{\bar{P}, \underset{\sim}{\Xi}, \langle \underset{\sim}{A}_i : i < \lambda\rangle\} \text{ belongs to } N, \\ & \text{and there are } j, i \in N \cap \sigma \text{ and } q \in P_i, \text{ such that} \\ & p \leq q, q \text{ is } (N, P_i)\text{-semi-generic}, \underset{\sim}{A}_j \text{ is a } P_i\text{-name}, \\ & \text{and } q \Vdash_{P_i} \text{"} N \cap \omega_1 \in \underset{\sim}{A}_j \text{ and } j \in N[\underset{\sim}{G}_{P_i}]\text{"}\}. \end{aligned}$$

$$T^+_p \stackrel{\text{def}}{=} \{b \in \mathcal{S}_{\leq\aleph_1}(\lambda) : \text{ for some } a \in T_p, a <_\lambda b\}.$$

Assume first that every T_p^+ is $(2^\lambda, \aleph_2)$-big. So for every $\chi > 2^\lambda$ and countable $N \prec (H(\chi), \in, <^*_\chi)$, to which $\bar{P}$, $\underset{\sim}{\Xi}$, $\langle \underset{\sim}{A}_i : i < \lambda \rangle$ belong, and $p \in N \cap P_\lambda$, we know $\lambda \in N$ hence $H(\lambda) \in N$ and $<^*_\lambda \in N$ hence $T_p \in N$. By 1.8(1) we know T_p hence T_p^+ is $\aleph_2-{}^*$big, hence (Definition 1.2(2)) we can find M, $N <_{\aleph_2} M \prec (H(\chi), \in, <^*_\chi)$, M countable, $M \cap \lambda \in T_p^+$, hence for some $M_1 \in T_p$ we have $M_1 \cap \lambda <_\lambda M \cap \lambda$. Clearly for α, $|beta \in M_1 \cap \lambda$ we have

$$M_1 \models \text{“}\alpha \text{ is cardinal”} \Leftrightarrow M \models \text{“}\alpha \text{ is a cardinal,”}$$

$$M_1 \models \text{“}2^\alpha = \beta\text{”} \Leftrightarrow M \models \text{“}2^\alpha = \beta\text{“};$$

and so $(\forall \sigma \in M_1 \cap \lambda)(2^\sigma \in M_1 \cap \lambda)$; as $[\sigma \in M_1 \cap \lambda \Rightarrow H(\sigma)$ is an initial segment by $<^*_\lambda$ of $H(\lambda)]$, easily $\sigma \in M_1 \cap \lambda \Rightarrow M_1 \cap H(\sigma) = M \cap H(\sigma)$, hence $M - 1 <_\lambda M$. Let q, σ, i, j witness $M_1 \in T_p$ (see the definition of T_p), and easily we can deduce what semiproperness would have required. But $P_\lambda * \text{seal}(\underset{\sim}{\Xi})$ is not semiproper (as $\bar{P}, \underset{\sim}{\Xi}$ contradict (C)). So the assumption above was wrong, i.e., for some $p \in P_\lambda$, T_p^+ is not $(2^\lambda, \aleph_2)$-big. Let $j(*) = \min\{j \in C : p \in H(j)\}$, and let $C' = C \setminus j(*)$, we shall prove that T_p^+, C' exemplify $\neg(B)^+$, renaming $C' = C$ i.e. $p \in H(\min(C))$. Also $\oplus_B$ holds for T_p^+ easily. Let $j(*) = \min\{j \in C : p \in H(j)\}$, and let $C^* = C \setminus j(*)$, we shall prove that T_p^+, C^* exemplify $\neg(B)^+$, renaming $C^* = C$ i.e. $p \in H(\min(C))$. Also T_p^+ is λ-EEEC by its definition. To complete the proof of “T_p^+ exemplifies $\neg(B)^+$” we need only to prove “T_p^+ is $(< \lambda, C)$-big” (see Definition 1.2(1)). So let $\theta < \lambda$, $\theta \geq \aleph_2$, and we shall prove that T_p is (θ, C)-big; this suffices. We can find $i(*)$ such that $\Vdash_{P_{i(*)}}$ “$|\theta| = \aleph_1$”. So let $\alpha < \lambda$ be given such that $\alpha > i(*)$, θ. We define in $(H(\lambda), \in, <^*_\lambda, \bar{P}, \underset{\sim}{\Xi})$ a function g from $\mathcal{P}(\mathcal{S}_{\leq\aleph_0}(\alpha))$ to λ, $g(X)$ is: the first strong limit cardinal of uncountable cofinality $\beta < \lambda$ such that $\beta > i(*)$, $\beta > \alpha$, $\beta > \theta$ and:

$(*)^{\alpha,\beta}_X$ for every $\mathcal{U} \in \mathcal{D}_{\leq\aleph_0}(\beta)$, the set $\{a \in X : (\exists b \in \mathcal{U} \cap T_p)[a <_\theta b]\}$ is $\neq \emptyset \bmod \mathcal{D}_{\leq\aleph_0}(\alpha)$ *if* there is such β, and $\alpha + 1$ *otherwise.*

So g is definable in $(H(\lambda), \in, <^*_\lambda, \bar{P}, \underset{\sim}{\Xi})$ with the parameters θ, α, $i(*)$ hence $\beta^* = \sup\text{Rang}(g) < \text{Min}(C \setminus (\alpha + 1))$ (remember $\oplus_C$ is assumed). If β^* is not as required in Definition 1.2(1), then there is $\mathcal{U}^* \in \mathcal{D}_{\leq\aleph_0}(\beta^*)$, such that $X \stackrel{\text{def}}{=} \{a \in \mathcal{S}_{\leq\aleph_0}(\alpha) : \neg(\exists b \in \mathcal{U}^* \cap T_p)[a <_\theta b]\}$ is $\neq \emptyset \bmod \mathcal{D}_{\leq\aleph_0}(\alpha)$. Now we know $\Vdash_P$ "$|\alpha| \leq \aleph_1$", and as P satisfies the λ-c.c., there is a name for a function exemplifying this mentioning only members of some $P_i (i < \lambda)$, but $P_i \lessdot P$, so $\Vdash_{P_i}$ "$|\alpha| \leq \aleph_1$", say $\underset{\sim}{h}$ is P_i-name of a function from ω_1 onto α. As P_i is semiproper, by the assumption on X we have $\Vdash_{P_i}$ "$\underset{\sim}{Y} \stackrel{\text{def}}{=} \{\varepsilon < \omega_1 :$ there is $a \in X$, $\varepsilon \subseteq a \subseteq h''(\varepsilon), \varepsilon = a \cap \omega_1\}$ is a stationary subset of ω_1". Hence $\Vdash_P$ "$\underset{\sim}{Y} \subseteq \omega_1$ is stationary" hence $\Vdash_P$ " for some $\xi < \lambda$, $\underset{\sim}{Y} \cap \underset{\sim}{A}_\xi \subseteq \omega_1$ is stationary". Hence for some $j \in (i, \lambda)$ and P_j-name $\underset{\sim}{\xi}$ of an ordinal $< j$ we have: $\underset{\sim}{Y}$, $\underset{\sim}{\xi}$ and $\underset{\sim}{A}_{\underset{\sim}{\xi}}$ are P_j-names and $\Vdash_{P_j}$ "$\underset{\sim}{Y} \cap \underset{\sim}{A}_{\underset{\sim}{\xi}} \subseteq \omega_1$ is stationary".

Hence there is a strong limit $j_1 \in (j, \lambda)$ such that

$$(H(j_1), \in, <^*_\lambda \restriction H(j_1), \bar{P}\restriction j_1, \Xi \restriction H(j_1)) \prec (H(\lambda), \in, <^*_\lambda, \lessdot \bar{P}, \underset{\sim}{\Xi}),$$

$\mathrm{cf}(j_1) > \aleph_0$, and $\underset{\sim}{Y}$, $\underset{\sim}{\xi}$, i, j, α, β^*, $\mathcal{U}^* \in H(j_1)$. Now there are $\delta < \omega_1$ and countable $N \prec (H(\chi), \in, <^*_\chi, \bar{P}, \underset{\sim}{\Xi})$ and q such that: $\{\underset{\sim}{Y}, \underset{\sim}{\xi}, i, j, \alpha, j_1\} \in N$, $p \leq q \in P_j$, q is (N, P_j)-semi generic, $N \cap \omega_1 = \delta$, $q \Vdash_{P_j}$ "$\delta \in \underset{\sim}{Y} \cap \underset{\sim}{A}_{\underset{\sim}{\xi}}$", and (remember the definition of $\underset{\sim}{Y}$) there is $a^* \in X$, $\delta \subseteq a^* \subseteq N$. Clearly $j_1 \in N$, $N \restriction H(j_1) \prec (H(\chi), \in, <^*_\chi, \bar{P}, \underset{\sim}{\Xi})$ and $N \restriction H(j_1) \in T_p$ (see the definition of T_p). As $j_1 \in N$, $X \in N$ this implies that for every $\mathcal{U} \in \mathcal{D}_{\leq\aleph_0}(j_1)$ we have $\{a \in X : (\exists b \in \mathcal{U} \cap T_P)[a <_{\aleph_2} b]\} \neq \emptyset \bmod \mathcal{D}_{\leq\aleph_0}(\alpha)$. So $(*)^{\alpha, j_1}_X$ holds; hence by the definition of g and β^* without loss of generality $j_1 \leq \beta^*$, hence (check definition) $(*)^{\alpha,\beta^*}_X$ hold, but this contradicts the choice of of X. So together we have gotten a counterexample to $(\mathrm{B})^+$. $\square_{2.1}$

2.2 Definition. 1) $(*)^a[\lambda, C]$ means condition (C) of 2.1 holds for $\bar{P}$ and C (so C satisfies $\otimes_C$) such that

$$\{\delta < \lambda : \text{ if } \delta \text{ is strongly inaccessible then } P_\delta = \bigcup_{i<\delta} P_i\}$$

contains a club of λ (so for many C's this is empty demand).

2) $(*)^a_{ab}[\lambda, C]$ means that for every semiproper forcing Q from $H(\mathrm{Min}C)$ we have $\Vdash_Q$ "$(*)^a[\lambda, C]$".

3) We omit C if this holds for every club C of λ.

2.3 Conclusion. Suppose $(*)^a_{ab}[\lambda, C]$, λ strongly inaccessible. If $\bar{P}$, C and $\underset{\sim}{\Xi}$ are as in 2.1(C) and $i < \lambda$, *then* in V^{P_i} the forcing notion $(P_\lambda/P_i)*$ seal $(\underset{\sim}{\Xi})$ is semiproper.

2.3A Remark. So if $\bar{Q}$ is a semiproper iteration, $\langle P_{i+1} : i < \lambda\rangle, C$, $\underset{\sim}{\Xi}$ as in 2.1(C) *then* $\bar{Q}\hat{\ }\langle \text{Rlim}\bar{Q}, \text{seal}(\underset{\sim}{\Xi})\rangle$ is a semi proper iteration.

2.4 Theorem. Suppose κ is strongly inaccessible, and:

$(*)^b_{ab}[\kappa]$ for every closed unbounded $C \subseteq \kappa$, for some $\lambda \in C$ (strongly inaccessible) we have $\lambda = \sup(C \cap \lambda)$, and $(*)^a_{ab}[\lambda,\ C \cap \lambda]$.

Let $S \subseteq \omega_1$ be stationary.

Then for some semiproper forcing P of cardinality λ satisfying the λ-c.c., we have $\Vdash_P$ "$\mathcal{D}_{\omega_1} + S$ is $\aleph_2$-saturated".

Also P is $(\omega_1 \setminus S)$-complete hence if $\omega_1 \setminus S$ is stationary it does not add ω-sequences of ordinals.

Moreover

2.4A Lemma. 1) The following homogeneous forcing can serve in 2.4. We define by induction on α a semiproper iteration $\bar{Q}^\alpha = \langle P_i, \underset{\sim}{Q}_i : i < \alpha\rangle$ with $|P_i| < \lambda$ (and, for simplicity, $\bar{Q}^i \in H(\lambda)$) for $i < \alpha$ (see XIII 1.8) as follows. If P_i is defined, i strongly inaccessible and $j < i \Rightarrow |P_j|$, then let, in V^{P_i}, Q_i be the product with countable support of $\{\text{seal}(\Xi) : \Xi \in \mathbf{\Xi}_i\} \cup \text{Levy}(\aleph_1, 2^{\aleph_2})^{V^{P_i}}$ where $\mathbf{\Xi}_i$ is $\{\Xi : \Xi$ (in V^{P_i}) is a maximal antichain of $\mathcal{D}_{\omega_1}$ and for every $j < i$, $\Vdash_{P_{j+1}}$ "$P_i/P_{j+1} * \text{seal}(\underset{\sim}{\Xi})$ is semiproper", such that $\omega_1 \setminus S \in \Xi$ if it is stationary $\}$. Otherwise Q_i is $\text{Levy}(\aleph_1, 2^{\aleph_2})^{V^{P_i}}$.

2) Moreover we can replace Ξ_i by

$$\Xi_i' = \{\Xi : \Xi \text{ (in } V^{P_i}) \text{ is a maximal antichain of } \mathcal{D}_{\omega_1} \\ \text{which is semi proper (that is seal}(\Xi) \text{ is semi proper)} \\ \text{such that } \omega_1 \setminus S \in \Xi \text{ if it is stationary}\}$$

provided that λ is Woodin.

2.4B Remark. 1) We can e.g. use Levy$(\aleph_1, 2^{\aleph_2})^{V^{P_i}}$ when i is not strongly inaccessible and the CS product of $\{\text{seal}(\Xi) : \Xi \in \Xi_i\}$ otherwise.
2) By 2.7(3) below if κ is Woodin then it satisfies the assumption of Theorem 2.4. Similarly in 2.5 and 2.6 concerning the μ in the definition of W^*.
3) If $\omega_1 \setminus S$ is stationary, the iteration is essentially CS (as the condition with a "real" support are dense).
4) Homogeneity is actually gotten also in the other proofs, in particular 2.5, 2.6 (and results in Chapter XIII).

Proof of 2.4. Follow by 2.4A.

Proof of 2.4A. By XIII 2.13(1) clearly $\bar{Q}^i = \langle P_j, \underset{\sim}{Q}_j \ : j < i\rangle$ is a semiproper iteration ($P_i = \text{Rlim}\bar{Q}^i$) and if $j < i$ then $\Vdash_{P_{j+1}}$ "$(P_i/P_{j+1}) * \underset{\sim}{Q}_i$ is semiproper". Also the $(\omega_1 \setminus S)$-completeness and λ-c.c. are clear. Why $\Vdash_{P_\lambda}$ "$\mathcal{D}_{\omega_1} + S$ is $\aleph_2$-saturated"? Let $\underset{\sim}{\Xi}$ be a P_λ-name of a maximal antichain of $\mathcal{D}_{\omega_1}$ (to which $\omega_1 \setminus S$ belongs if stationary), so let $\underset{\sim}{\Xi} = \{\underset{\sim}{A}_i : i < \lambda\}$. Let

$$C = \{\mu < \lambda : (H(\mu), \in, <^*_\mu, \{(\bar{Q}^i, \underset{\sim}{A}_i/\mathcal{D}_{\omega_1}) : \ i < \mu\}) \\ \prec (H(\lambda), \in, <^*_\lambda, \ \{(\bar{Q}^i, \underset{\sim}{A}_i/\mathcal{D}_{\omega_1}) : \ i < \lambda\}) \\ \text{and } \mu \text{ is strong limit}\}$$

So by the assumption of 2.4 for some regular (hence strongly inaccessible) $\mu \in C$ we have $\mu = \sup(\mu \cap C)$, and $(*)^a_{ab}[\mu, C \cap \mu]$. For part (1), by 2.3, $\{\underset{\sim}{A}_i : i \in I\} \in \Xi_\mu$, and the rest is easy. For part (2) similarly using 2.8. $\square_{2.4}$

2.5 Theorem. Suppose λ is strongly inaccessible, $\bar{S} = \langle S_1, S_2, S_3\rangle$ a partition of ω_1, S_1 stationary and

$W^* \stackrel{\rm def}{=} \{\mu < \lambda :\ \ \mu$ strongly inaccessible and $(*)^b_{ab}[\mu]\}$ is a stationary subset of λ.

1) *Then* for some forcing notion P:

(a) $|P| = \lambda$, P satisfies the λ-c.c.

(b) P is semiproper.

(c) $\Vdash_P$ "$\mathcal{P}(\omega_1)/(\mathcal{D}_{\omega_1} + S_1)$ is W^*-layered" (see XIII 3.1A(4), (5)).

(d) P is S_3-complete hence if S_3 is stationary, then P adds no new ω-sequences of ordinals.

(e) $\Vdash_P$ "W^* is a stationary subset of $\{\delta < \aleph_2 = \lambda : \mathrm{cf}(\delta) = \aleph_1\}$".

2) Hence, if Q is the forcing notion of shooting a club through $\{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_0\} \cup W^*$ in the universe V^P, then in $V^{P*\underset{\sim}{Q}}$ we have: $\mathcal{P}(\omega_1)/(\mathcal{D}_{\omega_1} + S_1)$ is layered (see XIII 3.1A(4),(5)) (and hence e.g. there is a uniform ultrafilter E on ω_1 such that $\aleph_0^{\omega_1}/E = \aleph_1$ so E not regular; by [FMSh:252]).

Proof. 1) Similar to XIII 3.1 (see on history there).

We define by induction on $i < \kappa, P_i, \underset{\sim}{Q}_i, \mathbf{t}_i$ such that:

(A) $\bar{Q}^\alpha = \langle P_i, \underset{\sim}{Q}_j, \mathbf{t}_j : \mathbf{t}_j : i \le \alpha, j < \alpha\rangle$ is an S_1-suitable iteration (see Definition XIII 2.1).

(B) $\mathbf{t}_\alpha$ is 1 iff: α is strongly inaccessible, $[i < \alpha \Rightarrow |P_i| < \alpha]$ and $\Vdash_{P_\alpha}$ "$\mathfrak{B}^{V^{P_\alpha}} \restriction S_1$ satisfies the α-c.c. i.e. $\aleph_2$-c.c.".

(C) $\underset{\sim}{Q}_\alpha$ is defined , in V^{P_α}, as (where $\kappa_{\alpha+1}$ is the first strongly inaccessible $> |P_\alpha|$) $Q^0_\alpha *$ SSeal$(\langle \mathfrak{B}^{P_i} : i \le \alpha, \mathbf{t}_i = 1\rangle, S_1, \kappa_{\alpha+1})$ (see Definition XIII 2.4(5)) where Q^0_α is the product with countable support of $\{\mathrm{seal}(\Xi) : \Xi \in \Xi_\alpha\}$ (defined as in the proof of 2.4(1); or use Ξ'' from 2.4A(2)).

We should prove by induction on α that $\bar{Q}^\alpha$ is an S_1-suitable iteration.

first case for $\alpha = 0$ - this is trivial.

second case for α limit - this holds by XIII 2.3(1).

third case for $\alpha = \beta + 1, \mathbf{t}_\beta = 0$.

We should repeat the proof of XIII 2.14(1); we do this case in details.

Let χ be regular large enough, $i < \beta, G_{i+1} \subseteq P_{i+1}$ generic over V, in $V[G_{i+1}]$, N is a countable elementary submodel of $(H(\chi)[G_{i+1}], \in, <^*_\chi)$ such that $\bar{Q}^\alpha \in N$, $p \in P_\alpha/G_{i+1}, p \in N$.

We should find $q, p \leq q \in P_\alpha/G_{i+1}$, and q is $(N[G_{i+1}], P_\alpha/G_{i+1})$-generic. By repeating the use XIII 2.12 ω times, we can find $q_0 \in P_\beta/G_{i+1}, p\restriction\beta \leq q_0$ such that if $G_\beta \subseteq P_\beta$ is generic over $V, G_{i+1} \cup \{q_0\} \subseteq G_\beta$ then:

(*) in $V[G_\beta]$, there is $N', N \subseteq N' \prec (H(\chi)[G_\beta], \in, <^*_\chi), N'$ countable, $N' \cap \omega_1 = N \cap \omega_1$, and: for every $\Xi \in N' \cap H(\kappa)$ a dense subset of $\mathfrak{B}_\gamma$ for some $\gamma \in N' \cap \beta$, such that $\mathbf{t}_\gamma = 1$ we have $N' \cap \omega_1 \in \bigcup_{A \in N' \cap \Xi} A$.

In $V[G_\beta]$ we can find $p_n \in \underset{\sim}{Q}^0_\beta[G_\beta], p_n \in N', p_n \leq p_{n+1}, p_0$ the $\underset{\sim}{Q}^0_\beta[G]$-component of $p(\beta)$, such that

(a) *if* $\mathcal{I} \in N'$ is a dense subset of $\underset{\sim}{Q}^0_\beta[G_\beta]$ *then* for some $n, p_n \in \mathcal{I}$.

(b) if $\underset{\sim}{\Xi}$ is a $\underset{\sim}{Q}^0_\beta[G_\beta]$-name of a pre-dense subset of $\mathfrak{B}^{P_\gamma}, \gamma \in \beta \cap N, \mathbf{t}_\gamma = 1$, *then* for some n and A, $p_n \Vdash_{\underset{\sim}{Q}^0_\beta[G_\beta]}$ "$A \in \underset{\sim}{\Xi}$" and $N' \cap \omega_1 \in A$.

By standard bookkeeping there are no problem; taking care of an instance of (b) is just like the proof of XIII 2.9, as

(**) *if* $\gamma \in \beta \cap N', \mathbf{t}_\gamma = 1, \Xi \in N'$ is a pre-dense subset of $\mathfrak{B}^{P_\gamma}, \omega_1 \setminus S \in \Xi$ *then* $N' \cap \omega_1 \in \bigcup_{A \in \Xi \cap N'} A$.

Why does this hold? As β is strongly inaccessible $\bigwedge_{\gamma<\beta} |P_\gamma| < \beta$, we know $\mathfrak{B}^{P_\beta} = \bigcup_{\gamma<\beta} \mathfrak{B}^{P_{\gamma+1}}$, hence $[\mathbf{t}_\gamma = 1 \Rightarrow \mathfrak{B}^{P_\gamma} \lessdot \mathfrak{B}^{P_\beta}]$ and $|\mathfrak{B}^{P_\gamma}| = \aleph_1$ in V^{P_β}.

<u>fourth case</u> $\alpha = \beta + 1, \mathbf{t}_\beta = 1$.
$\underset{\sim}{Q}^0_\beta$ is semiproper by XIII 2.8(3) and SSeal$(\langle \mathfrak{B}^{P_\gamma} : \gamma \leq \beta, \mathbf{t}_\gamma = 1\rangle, S_1, \kappa_{\beta+1})$ is the same as SSeal$(\mathfrak{B}^{P_\beta}, S_1, \kappa_{\beta+1})$ which is semiproper by XIII 2.14(1).

* * *

Now if $\lambda \in W^*$, $\bigwedge_{\gamma<\lambda} [|P_\gamma| < \lambda]$, then exactly as in the proof of Theorem 2.4, $\Vdash_{P_\lambda}$ "$\mathfrak{B}^{P_\lambda}$ satisfies the λ-c.c.", hence $\mathbf{t}_\lambda = 1$, hence $\mathfrak{B}^{P_\lambda} \lessdot \mathfrak{B}^{P_\kappa}$.

As $\mathfrak{B}^{P_\lambda} = \mathfrak{B}^{\bar{Q}\restriction\lambda}$ and $\langle \bar{\mathfrak{B}}^{\bar{Q}\restriction\alpha} : \alpha < \kappa\rangle$ is increasing continuous with limit $\mathcal{B}^{P_\kappa}$, clearly P_κ is as required.

2) No problem (or see proof of XIII 3.1). $\square_{2.5}$

2.5A Remark. Of course, we know $|P_i| \leq$ first strongly inaccessible $\geq |P_i|$ (by a variant could have gotten $|P_i| \leq \beth_{i+1}$).

2.6 Theorem. Suppose λ strongly inaccessible and the set
$W^* = \{\mu < \lambda : \quad \mu$ measurable and $(*)^b_{ab}[\mu].\}$
is not only stationary, but for stationarity many $\kappa < \lambda$, $W^* \cap \kappa$ is stationary.

Let $\langle S_1, S_2, S_3 \rangle$ be a partition of ω_1, S_1 is stationary.

Then for some forcing notion P

(a) $|P| = \lambda$, P satisfies the λ- cc.,

(b) P is semiproper.

(c) $\Vdash_P$ "$\mathcal{P}(\omega_1)/(\mathcal{D}_{\omega_1} + S_1)$ is the Levy algebra" (i.e. as isomorphic to the complete Boolean algebra which Levy $(\aleph_0, < \aleph_2)$ generate).

(d) P is pseudo $(*, S_3)$-complete hence if S_3 is stationary then P adds no reals.

Proof. Similar to XIII 3.7.

* * *

Of course we can translate our assumptions to a standard large cardinal hierarchy, essentially by Shelah and Woodin [ShWd:241], i.e. we note:

2.7 Fact. 1) Suppose $\mathrm{Pr}_a(\lambda, f)$ (see Definition 1.14), C a club of λ, $[\delta \in C \Rightarrow (H(\mu), \in, <^*_\mu) \prec (H(\lambda), \in, <^*_\lambda)]$ and $f(i) \leq \mathrm{Min}(C \setminus (i+1))$. *Then* $(*)^a[\lambda, C]$ (see Definition 2.2(1)).

2) As $\mathrm{Pr}_a(\lambda, f)$ is preserved by forcing of cardinality $< \lambda$, we can deduce in (1) also $(*)^a_{ab}[\lambda, C]$ (see Definition 2.2(2)).

3) If λ is a Woodin cardinal i.e. $\mathrm{Pr}_b(\lambda)$ (see Definition 1.15) *then* $(*)^b_{ab}[\lambda]$ (see definition in Theorem 2.4).

Proof. 1) By 2.8 below, condition (C) of 2.1 holds in the cases refered to in Definition 2.2, hence (see Definition 2.2(1)) we get $(*)^a[\lambda, C]$.

2) Easy.

3) See Definition 1.15 and part (2) of 2.7. $\square_{2.7}$

2.7A Remark. If you want to get versions of 2.4, 2.5, 2.6 without §1 + 2.1, you can use 2.8 below (+2.9).

2.8 Claim. Sealing is Semiproper Criterion.

Suppose

(i) $\bar{P} = \langle P_i : i < \lambda \rangle$ is $\lessdot$-increasing sequence of forcing notion, $P_i \in H(\lambda)$ and $\Vdash_{P_i}$ "$\aleph_1^V$ is a cardinal", and for any $j < \lambda$ for some i, $j < i < \lambda$ and $2^{\aleph_2}$ of V^{P_j} is collapsed to $\aleph_1$ in V^{P_i}.

(ii) $\mathrm{Pr}_a(\lambda, f, D)$ (defined in Definition 1.14).

(iii) $\{\delta < \lambda : P_\delta = \bigcup_{i<\delta} P_i\} \in D$ [hence $P_i \lessdot P_\lambda$ where $P_\lambda \stackrel{\text{def}}{=} \bigcup_{i<\lambda} P_i$ and P_λ satisfies the λ-c.c.]

Hence

$$B_0 \stackrel{\text{def}}{=} \{\mu < \lambda : \text{(a) } \mu \text{ is a strong limit}$$

$$\text{(b) } P_\mu = \bigcup_{i<\mu} P_i$$

$$\text{(c) } P_\mu \text{ satisfies the } \mu\text{-c.c.}$$

$$\text{(d) } \Vdash_{P_\mu} \text{"}\mu = \aleph_2\text{" and}$$

$$\text{(e) for } A \in \mathcal{P}(\omega_1)^{V^{P_\mu}} \text{ the statement}$$

$$\text{"}A \subseteq \omega_1 \text{ is stationary"}$$

$$\text{is preserved by } P/P_\mu\} \in D$$

hence

$$B_1 \stackrel{\text{def}}{=} \{\delta < \lambda : P/P_\delta \text{ preserves the stationarity of } A \in \mathcal{P}(\omega_1)^{V^{P_\delta}}\}$$

is unbounded in λ.

(iv) $B = \{\alpha < \lambda : P_\lambda/P_\alpha \text{ is semiproper}\}$ is unbounded in λ.

(v) $\{\delta < \lambda : P_\lambda/P_\delta$ does not destroy semi stationarity (see Definition XIII 1.1(3)) of subsets of $\mathcal{S}_{\leq\aleph_0}(2^{\aleph_2})$) (where $2^{\aleph_2}$ is computed in V^{P_δ})$\} \in D$. By Claim XIII 1.4, (P_λ/P_δ) being semiproper is enough.

(vi) $\Vdash_{P_\lambda}$ "$\bar{\underset{\sim}{A}} = \langle \underset{\sim}{A}_i : i < \lambda \rangle$ is a maximal antichain of $\mathcal{D}_{\omega_1}$" and

(vii) The following set belongs to D:

$\{\delta < \lambda : f(\delta)$ is a strong limit and for some β satisfying $\delta < \beta < f(\delta)$ we have $\Vdash_{P_\beta}$ "$(2^{\aleph_2})^{V^{P_\delta}}$ is collapsed to $\aleph_1$", $P_\beta \in H(f(\delta))$ and for every P_β-name $\underset{\sim}{A}$ of a subset of ω_1 stationary in V^{P_λ}, for some $\alpha(*) \in B$ (see clause (iv)) and $\underset{\sim}{i}$ we have: $\underset{\sim}{A}_{\underset{\sim}{i}} \cap \underset{\sim}{A}$ is forced to be stationary, $\underset{\sim}{i}$ and $\underset{\sim}{A}_{\underset{\sim}{i}}$ are $P_{\alpha(*)}$-names, $\mu < \alpha(*) < f(\delta)$, and $\{\underset{\sim}{A}, P_{\alpha(*)}, \alpha(*), \underset{\sim}{i}, \underset{\sim}{A}_{\underset{\sim}{i}}\} \in H(f(\delta))\}$.

Then $\Vdash_{P_\lambda}$ "$\bar{\underset{\sim}{A}}$ is semiproper" (see XIII 2.4(6)).

Proof. Assume the conclusion fails. Let $\mathbf{j} : V \to M$ be an elementary embedding, M a transitive class, and
$\Big[H\Big((\mathbf{j}(f))(\lambda)\Big)\Big]^V \subseteq M$ and $D = \{A \subseteq \lambda : \ \lambda \in \mathbf{j}(A)\}$ and $M^{<\lambda} \subseteq M$
(exists as $\mathrm{Pr}_a(\lambda, f, D)$ holds by assumption (ii)).

By assumption (iii) we have $M \models$ "$P_\lambda = (\mathbf{j}(\bar{P}))\ (\lambda)$", let $\mathbf{j}(\bar{P}_\lambda) = \langle P_i : i < \mathbf{j}(\lambda)\rangle$ and $P_{\mathbf{j}(\lambda)} = \mathbf{j}(P_\lambda) = \bigcup_{i<\mathbf{j}(\lambda)} P_i$ (Note: the two definitions of P_i for $i \leq \lambda$ are compatible by the beginning of this sentence). Similarly let $\mathbf{j}(\bar{A}) = \langle \underset{\sim}{A}_i : i < \mathbf{j}(\lambda)\rangle$.

By (vii) we have $M \models$ "$(\mathbf{j}(f))(\lambda)$ is strong limit", so as $[H((\mathbf{j}(f))(\lambda))]^V \subseteq M$, really $\mathbf{j}(f)(\lambda)$ is strong limit in V so for statements in $H(\mathbf{j}(f)(\lambda))$ we can move freely between V and M. Let $G_{\mathbf{j}(\lambda)} \subseteq P_{\mathbf{j}(\lambda)}$ be generic over M, so we let $G_i \stackrel{\text{def}}{=} G_{\mathbf{j}(\lambda)} \cap P_i$.

Clearly $G_{\lambda+1} \subseteq P_{\lambda+1}$ is generic over V because, generally $P_i \in H\big(\mathbf{j}(f)(\lambda)\big)$ implies G_i is generic over V and $P_{\lambda+1} \in H((\mathbf{j}(f))(\lambda))$ by (vii). Until almost the end we shall use G_λ only. Note: in $V[G_\lambda]$ we have $M[G_\lambda]^{<\lambda} \subseteq M[G_\lambda]$ because P_λ satisfies the λ-c.c. (see (iii)).

Remembering (vi), in $V[G_\lambda]$, $\bar{\underset{\sim}{A}}[G_\lambda] = \langle \underset{\sim}{A}_i[G_\lambda] : i < \lambda\rangle$ is a maximal antichain of $\mathcal{D}_{\omega_1}$ and seal$(\bar{A})$ has cardinality $(2^{\aleph_1})^{V[G_\lambda]} = \lambda = \aleph_2^{V[G_\lambda]}$ (remember (i)). Let

$S \stackrel{\text{def}}{=} \{N : N \prec (H(\lambda^+)^{V[G_\lambda]}, \in, <^*),$ N countable, and there is no N_1, $N \prec N_1 \prec (H(\lambda^+)^{V[G_\lambda]}, \in, <^*), N_1$ countable and $N_1 \cap \omega_1 = N \cap \omega_1 \in \bigcup_{i \in N_1} \underset{\sim}{A}_i[G_\lambda]\}$

In $V[G_\lambda]$ the set S is semi-stationary (subset of $\mathcal{S}_{\leq\aleph_0}(H(\lambda^+)^{V[G_\lambda]})$, as we are assuming that the conclusion failed — by XIII 1.3; we note: $H(\lambda^+)^{V[G_\lambda]} = H(\lambda^+)^{M[G_\lambda]}$. Clearly λ belongs to the set defined in assumption (vii), so in M there is β as there, so $\lambda < \beta < f(\lambda)$, $\Vdash_{P_\beta}$ "$(2^{\aleph_2})^{V^{P_\lambda}}$ is collapsed to $\aleph_1$", $P_\beta \in H((\mathbf{j}(f))(\lambda))$ and the last condition there hold.

So there is a P_β-name $\langle \underset{\sim}{a}_\zeta : \zeta < \omega_1\rangle$ such that:

$\Vdash_{P_\beta}$ "$\langle \underset{\sim}{a}_\zeta : \zeta < \omega_1\rangle$ is increasing continuous, each $\underset{\sim}{a}_\zeta$ countable,

$\bigcup_{\zeta<\omega_1} \underset{\sim}{a}_\zeta = H(\lambda^+)^{V[G_\lambda]}$".

Let $\underset{\sim}{A} = \{\zeta : \quad (\exists N \in S)[\omega_1 \cap \underset{\sim}{a}_\zeta \subseteq |N| \subseteq \underset{\sim}{a}_\zeta]\}$, clearly it is a P_β-name.

By assumption (v), $\Vdash_{P_\beta}$ "$\underset{\sim}{A}$ is a stationary subset of ω_1 " hence by the last condition in (vii) for some $\underset{\sim}{i}$, $\alpha(*)$ we have: $\alpha(*) \in \mathbf{j}(B)$, $\underset{\sim}{i}$ and $\underset{\sim}{A}_{\underset{\sim}{i}}$ are $P_{\alpha(*)}$-names, $\beta < \alpha(*) < \mathbf{j}(f)(\lambda)$ and $\{\underset{\sim}{i}, \alpha(*), \underset{\sim}{A}, \underset{\sim}{A}_{\underset{\sim}{i}}, \quad P_{\alpha(*)}\} \in H\Big((\mathbf{j}(f))(\lambda)\Big)$. So for some regular μ (in M and in V) we have $\mu < (\mathbf{j}(f))(\lambda)$ and this set $\in H(\mu) = H(\mu)^M$ moreover $\mathcal{P}(P_{\alpha(*)}) \in H(\mu)$. So in $M[G_{\alpha(*)}]$, we have $\underset{\sim}{A}_{\underset{\sim}{i}}[G_{\alpha(*)}] \cap \underset{\sim}{A}[G_\beta]$ is a stationary subset of ω_1. Again this holds in $V[G_{\alpha(*)}]$ too, (and of course in $V[G_{\alpha(*)}]$ $\aleph_1$ is not collapsed). Let, in M, $w = \{i < \alpha(*) : \underset{\sim}{A}_i$ is a $P_{\alpha(*)}$-name$\}$, so $w \in H(\mu)^M = H(\mu)$.

So in $M[G_\lambda]$

$$S_1 \stackrel{\text{def}}{=} \{N \prec \big(H(\mu)^{M[G_\lambda]}, \in, <^*, M, G_\lambda\big) : N \text{ is countable}$$
$$\{\mathbf{j}(\bar{Q}){\restriction}\alpha(*), \bar{\underset{\sim}{A}}{\restriction}w, \underset{\sim}{A}_{\underset{\sim}{i}}\} \in N$$
$$\text{and for some } p \in P_{\alpha(*)}/G_\lambda,$$
$$p \text{ is } (N, P_{\alpha(*)}/G_\lambda)\text{-semi-generic}$$
$$\text{and } p \Vdash \text{“}N \cap \omega_1 \in \underset{\sim}{A} \cap \underset{\sim}{A}_{\underset{\sim}{i}}\text{”}\}$$

is stationary subset of $\Big[\mathcal{S}_{\leq\aleph_0}\big(H(\mu)\big)\Big]^{M[G_\lambda]}$ in $M[G_\lambda]$, hence in $V[G_\lambda]$ too. Note that $\mathbf{j}$ induces a unique elementary embedding $\mathbf{j}^+$ from $V[G_\lambda]$ into $M[G_{\mathbf{j}(\lambda)}]$, $\mathbf{j}^+$ is really $\underset{\sim}{\mathbf{j}}^+$, a $P_{\mathbf{j}(\lambda)}$-name, and if $x \in H(\lambda^+)^{M[G_\lambda]}$ then $\underset{\sim}{\mathbf{j}}^+(x) \in M[G_\lambda]$, that is the name belong, and it can be considered a $P_{\mathbf{j}(\lambda)}/G_\lambda$-name (but $\underset{\sim}{\mathbf{j}}^+ \notin M[G_\lambda]$).

In $V[G_\lambda]$,

$$S_2 \stackrel{\text{def}}{=} \{N \prec \Big(H[\beth_3(\mathbf{j}(\lambda^+))^+]^{M[G_\lambda]}, \in, <^*, M, G_\lambda\Big) : N \text{ countable, and } \langle \underset{\sim}{a}_\zeta : \zeta < \omega_1\rangle,\ \mathbf{j}(\bar{P}),\ \alpha(*),\ \underset{\sim}{A}, \underset{\sim}{A}_i, \text{ and } \mathbf{j}^+ \restriction H(\lambda^+), H[\beth_2(\mathbf{j}(\lambda^+))], \bar{P}, G_\lambda \text{ belong to } N\}.$$

is in $\Big[\mathcal{D}_{\leq\aleph_0}\Big(H(\beth_3(\mathbf{j}(\lambda^+)))^+)^{M[G_\lambda]}\Big)\Big]^{V[G_\lambda]}$ and is a subset of $M[G_\lambda]$ (though not a member) as $V[G_\lambda] \models$ "$M[G_\lambda]^{<\lambda} \subseteq M[G_\lambda]$".

So there are $N_1 \in S_1$, $N_2 \in S_2$, such that $N_2 \restriction H(\mu)^{M[G_\lambda]} = N_1$ and $p \in P_{\alpha(*)}/G_\lambda$ witnessing $N_1 \in S_1$ (see the definition of S_1). Let $\delta \stackrel{\text{def}}{=} N_1 \cap \omega_1$. Note: $N_1, N_2 \in M[G_\lambda]$.

Now as $p \Vdash_{P_{\alpha(*)}/G_\lambda}$ "$\delta \in \underset{\sim}{A}$", by the definition of $\underset{\sim}{A}$ there are q and b satisfying $p \leq q \in P_{\alpha(*)}/G_\lambda$, and $N \in S$, such that letting $b \stackrel{\text{def}}{=} |N|$, we have $q \Vdash$ "$\delta \subseteq b \subseteq \underset{\sim}{a}_\delta$" so $b \in M[G_\lambda]$ as $b \in S$.

Also as q is $(N_1, P_{\alpha(*)}/G_\lambda)$-semi-generic (being above p, as p witness $N_1 \in S_1$) and $\langle \underset{\sim}{a}_\zeta : \zeta < \omega_1\rangle \in N_1$ (as it belongs to N_2 and to $H(\mu)^{M[G_\lambda]}$) clearly

$$q \Vdash \text{"}\underset{\sim}{a}_\delta = N_1[\underset{\sim}{G}_{P_{\alpha(*)}/G_\lambda}] \cap H(\lambda^+)^{M[G_\lambda]}\text{"}.$$

[Why? As $\langle \underset{\sim}{a}_\zeta : \zeta < \omega\rangle \in N_1$ and $H(\lambda^+)^{M[G_\lambda]} \in N_1$, clearly the function $h_1 : H(\lambda^+)^{M[G_\lambda]} \to \omega_1$, $h_1(x) = \min\{\zeta < \omega_1 : x \in \underset{\sim}{a}_\zeta\}$ belongs to $N_1[\underset{\sim}{G}_{P_{\alpha(*)}/G_\lambda}]$ and also some function $h_2 : \omega_1 \times \omega \to H(\lambda^+)^{M[G_\lambda]}$ such that $\underset{\sim}{a}_\zeta = \{h_2(\zeta, n) : n < \omega\}$ belongs to $N_1[G_{P_{\alpha(*)}/G_\lambda}]$.]

Hence

$$q \Vdash_{P_{\alpha(*)}/G_\lambda} \text{"}b \subseteq N_1[\underset{\sim}{G}_{P_{\alpha(*)}/G_\lambda}] \cap H(\lambda^+)^{M[G_\lambda]}\text{"}.$$

As $N_1 \cap \mathcal{P}(P_{\alpha(*)}/G_\lambda) = N_2 \cap \mathcal{P}(P_{\alpha(*)}/G_\lambda)$ (power set in $M[G_\lambda]$), we can also replace in those statements N_1 by N_2. As $P_{\mathbf{j}(\lambda)}/P_{\alpha(*)}$ is semiproper in $M[G_\lambda]$ ($\alpha(*)$ being in $\mathbf{j}(B)$) there is q', $q \leq q' \in P_{\mathbf{j}(\lambda)}$ such that q' is $(N_2, P_{\mathbf{j}(\lambda)}/G_\lambda)$ semi-generic in $M[G_\lambda]$.

W.l.o.g. $q' \in G_{\mathbf{j}(\lambda)}$ as only G_λ was used. Work in $M[G_{\mathbf{j}(\lambda)}]$, remember $N_2 \in M[G_\lambda]$. So really $b \subseteq N_2[G_{\mathbf{j}(\lambda)}]$, now as $\mathbf{j}^+$ maps $N_2[G_{P_{\mathbf{j}(\lambda)}/G_\lambda}] \cap H(\lambda^+)^{M[G_\lambda]}$

into $N_2[G_{P_{\mathbf{j}(\lambda)}/G_\lambda}]$ (see the definition of S_2) clearly $b_1 = \mathbf{j}^{+\prime\prime}(b) = \{\mathbf{j}^+(x) : x \in b\} \subseteq N_2[G_{\mathbf{j}(\lambda)}]$. Now as $M[G_\lambda] \models$ "b is countable", necessarily $b_1 = \mathbf{j}^+(b)$. By the properties of $\mathbf{j}^+$, $b_1 = \mathbf{j}^+(b) \in M[G_{\mathbf{j}(\lambda)}]$; remember $\mathbf{j}^+$ is the elementary embedding $\mathbf{j}$ induces from $V[G_\lambda]$ into $M[G_{\mathbf{j}(\lambda)}]$, so as $b \in S$ we have: $M[G_{\mathbf{j}(\lambda)}] \models$ "$b_1 \in \mathbf{j}^+(S)$",

But as $q \leq q' \in G_{\mathbf{j}(\lambda)}$, $N_2[G_{\mathbf{j}(\lambda)}/G_\lambda]$, $\underset{\sim}{i}[G_{\mathbf{j}(\lambda)}/G_\lambda]$ contradicts this. So we have finished proving 2.8. $\square_{2.8}$

When you want to accomplish other things by forcing remember XIII 1.10 (2):

2.9 Conclusion. 1) Assume

(i) $\bar{P} = \langle P_i : i < \lambda \rangle$ is $\lessdot$-increasing sequence of forcing notion, $P_i \in H(\lambda)$ and $\Vdash_{P_i}$ "$\aleph_1^V$ is a cardinal", and for any $j < \lambda$ for some i, $j < i < \lambda$ and $2^{\aleph_2}$ of V^{P_j} is collapsed to $\aleph_1$ in V^{P_i}, let $P_\lambda = \bigcup_{i<\lambda} P_i$,

(ii) $\mathrm{Pr}_b(\lambda)$, i.e. λ is a Woodin cardinal,

(iii) for a club of cardinals $\mu < \lambda$, if μ is strongly inaccessible then $P_\mu = \bigcup_{i<\mu} P_i$, P_λ/P_μ is semiproper.

Then in V^{P_λ}, every maximal antichain Ξ of $\mathcal{D}_{\omega_1}$ is semiproper i.e. seal(Ξ) is a semiproper forcing.

2) We can above replace (ii), (iii) by

(ii)′ $\mathrm{Pr}_b(\lambda, W)$,

(iii)′ $W = \{\delta < \lambda : P_\delta = \bigcup_{i<\delta} P_i$ and P_λ/P_δ is semiproper$\}$.

2.10 Concluding Remarks. Can we improve 2.6?

Note: we do not know imitate XIII 3.9 (on the Ulam property) as the supercompactness was used more deeply. But even trying to imitate XIII 3.7 (getting the Levy algebra, that is weakening the assumption of 2.6 to "for stationary many $\mu_0 < \lambda$, for stationary many $\mu_1 < \mu_0$ we have $(*)^a_{ab}[\mu_1]$) we have a problem: Is Nm semi proper? In 2.6 the measurability demand in the definition of W^* solves the problem. But it is natural and better to use $W^* = \{\mu < \lambda : \mu$ strongly inaccessible and $(*)^b_{[ab]}[\mu]\}$ or $W^{**} = \{\mu < \lambda : \mathrm{Pr}_b(\mu)\}$

To get such a theorem it is natural to use XV §3 to prove that the forcing does not collapse $\aleph_1$ and does not destroy stationary subsets of ω_1. If $S_3 = \emptyset$ we finish. To prove (d) - relativize Chapter XI to S_1 (as done in XI §8, or see XV). Still we have to check the parallel of 2.8. We intend to continue in [Sh:311].