IV. On Oracle-c.c., the Lifting Problem of the Measure Algebra, and "$\mathcal{P}(\omega)$/finite Has No Non-trivial Automorphism"

§0. Introduction

We present here the oracle chain condition and two applications: the lifting problem for the measure algebra, and the automorphism group of $\mathcal{P}(\omega)$/finite.

Let $\mathcal{B}$ be the family of the Borel subsets of $(0,1)$ (i.e. sets of reals which are > 0 but < 1). Let I_{mz} be the family of $A \in \mathcal{B}$ which have Lebesgue measure zero. Clearly I_{mz} is an ideal. The lifting problem is: "Can the natural homomorphism from $\mathcal{B}$ to $\mathcal{B}/I_{mz}$ be lifted ($\equiv$ split), i.e. does it have a right inverse? Equivalently, define on $\mathcal{B}$ an equivalence relation: $A_1, A_2 \in \mathcal{B}$ are equivalent if $(A_1 \setminus A_2) \cup (A_2 \setminus A_1)$ has Lebesgue measure zero: is there a set of representatives which forms a Boolean algebra? If CH holds the answer is positive (see Oxtoby [Ox]). (This holds for any $\aleph_1$-complete ideal). We will show, in §4, that a negative answer is also consistent with ZFC.

Since the problem of splitting the measure algebra is simpler, we will consider it first, but in the introduction we use the second problem to describe the main idea of our technique.

It is well known that if CH holds then $\mathcal{P}(\omega)$/finite is a saturated (model-theoretically) atomless Boolean algebra of power $\aleph_1$, hence has $2^{2^{\aleph_0}}$-many automorphisms, as any isomorphism from one countable subalgebra to another can be extended to two different automorphisms. It was not clear what the situation is if CH fails.

On the other hand any one-to-one function f from one co-finite subset of ω onto another co-finite subset of ω induces an automorphism of $\mathcal{P}(\omega)/\text{finite}$: A/finite is mapped to $f(A)/\text{finite}$. We call such an automorphism trivial. A priori it is not clear whether $\mathcal{P}(\omega)/\text{finite}$ has nontrivial automorphisms. Our main conclusion is that possibly all automorphisms are trivial (i.e. this holds in some model of ZFC); In fact in some generic extension of V. Hence there is e.g.-no Borel definition of such automorphism.

In this chapter our main aim is to present "oracle chain condition forcing". Iterating forcing satisfying the $\aleph_1$-chain condition, introduced by Solovay and Tennenbaum [ST], is well known (see Ch II). We use mainly the same framework: we start, e.g., with the constructible universe $V = L$, and use (finite support) iteration of length ω_2 of forcing notions, $\langle P_i, \underset{\sim}{Q}_i : i < \omega_2\rangle$, each Q_i satisfies the $\aleph_1$-c.c. and is of power $\aleph_1$. At stage α we guess an automorphism f_α of $\mathcal{P}(\omega)/\text{finite}$ in V^{P_α}, more exactly a P_α-name of such an automorphism, such that for any P_{ω_2}-name $\underset{\sim}{f}$ of an automorphism of $\mathcal{P}(\omega)/\text{finite}$, there are stationarily many $\alpha < \aleph_2$, for which we have: $\text{cf}(\alpha) = \aleph_1$, and $\underset{\sim}{f}_\alpha \subseteq \underset{\sim}{f}$ (this is possible if we have a diamond sequence on $\{\delta < \omega_2 : \text{cf}(\delta) = \aleph_1\}$ and CH holds, by some simple considerations). Our means of killing the automorphism, i.e., guaranteeing f_α cannot be extended to an automorphism of $\mathcal{P}(\omega)/\text{finite}$ in $V^{P_{\omega_2}}$, is to add a set $X \subseteq \omega$ so that for no $Y \subseteq \omega$:

$$(*)\ X \cap A \text{ is finite} \Leftrightarrow Y \cap f_\alpha(A) \text{ is finite for every } A \in [\mathcal{P}(\omega)]^{V^{P_\alpha}}.$$

The demand $(*)$ helps us since if $f \supseteq f_\alpha$ is an automorphism of the Boolean algebra $[\mathcal{P}(\omega)]^{V^{P_{\omega_2}}}/\text{finite}$ then $Y/\text{finite} = f(X/\text{finite})$ satisfies $(*)$.

This looks very reasonable, but even if we succeed to show that when we add X (by forcing Q_α) we do not add a Y satisfying $(*)$, how can we know that such a Y is not added later on during the iteration?

This is the role of the oracle chain condition. The best way we can explain the construction is as follows. We can look at iterated forcing as a construction of the continuum in $\aleph_2$ steps, i.e., at each successor step we add more reals, and even at limit steps of cofinality $\aleph_0$ we do so (but not at limit steps of cofinality $\aleph_1$). Now promising not to add a Y as in $(*)$ is an omitting type obligation, and building a model of power $\aleph_2$ by a chain of ω_2 approximations, promising to omit types along the way, is a widely used method in model theory (mainly for $\aleph_1$ instead of $\aleph_2$). See e.g., Keisler's work on $L(Q)$, for $\aleph_1$; the $\aleph_2$ case like ours is somewhat more difficult, see [Sh:82], [Sh:107], [HLSh:162].

The oracle chain condition is an effective version of the $\aleph_1$-c.c. Assume that we want to construct an $\aleph_1$-c.c. forcing notion P with an underlying set ω_1. If $\bar{M} = \langle M_\delta : \delta < \omega_1 \rangle$ (the $\aleph_1$-oracle) is a $\diamondsuit$-sequence on ω_1; i.e. for each $A \subseteq \omega_1$ the set $\{\delta : A \cap \delta \in M_\delta\}$ is stationary, and we demand:

(†) If $A \in M_\delta$ is pre-dense in $P{\restriction}\delta$, then A is pre-dense in P, then, as we will see in 1.6(1), P satisfies $\aleph_1$-c.c.

We call (a variant of) this the $\bar{M}$-c.c., (assuming each M_δ is closed enough). The connection between the $\bar{M}$-c.c. and the omitting type argument discussed above is the following.

Suppose B_i are Borel sets of reals ($i < \aleph_1$) such that their intersection is empty even if we add a Cohen real. (Note: finite support iteration always adds Cohen reals at limit stages of cofinality $\aleph_0$. Hence we have to deal with Cohen extensions.) If $\diamondsuit$ holds then, as we will see in §2, there is an $\aleph_1$-oracle $\bar{M}$ such that in any generic extension of V by a forcing satisfying the $\bar{M}$-chain condition no real is in all the B_i's (note that in a bigger universe we reinterpret each B_i using the same definition).

Clearly in order to apply oracle chain condition forcings we have to prove relevant lemmas on composition and preservation by direct limit. It is also very helpful to replace a sequence $\langle \bar{M}_i : i < \aleph_1 \rangle$ of oracles by a single oracle $\bar{M}$ such that any P satisfying the $\bar{M}$-c.c. will satisfy all $\bar{M}_i$-c.c.'s. For this we have to choose the right variant of the definition. Altogether the situation is that at step α we are given an $\aleph_1$-oracle which Q_α has to satisfy, and are allowed to demand, for $\aleph_1$ Borel types, that $V^{P_{\omega_2}}$ will omit them provided that not only our universe V^{P_α} omits them, but even forcing by Cohen forcing does not change this. This is done in §1, 2, 3, and of course does not depend on any understanding of the automorphism of $\mathcal{P}(\omega)/\text{finite}$ or of homomorphism from $\mathcal{B}/I_{mz}$ into $\mathcal{B}$. So it is sufficient to read §1-3 if you want to apply the oracle chain condition; though you may want to read 4.6 (which is the proof of Theorem 4.3) to see how all the threads are put together (modulo specific lemma, in that case 4.5), and it is commonly believed that seeing examples helps to understand a method.

Of course, to apply this we have to look at the specific application at some point; this is done in the induction step, i.e., working in V^{P_α} and being given an automorphism f_α, and an ω_1-oracle $\bar{M}$ we have to find $\underset{\sim}{Q}_\alpha$ adding a real $\underset{\sim}{X}$ and satisfying the $\bar{M}$-c.c. so that $(*)$ holds for no $\underset{\sim}{Y}$ even if we add a Cohen real.

Of course if f_α is trivial this cannot be done. We shall construct Q_α, assume that there is a ($Q_\alpha\times$Cohen)-name $\underset{\sim}{Y}$ forced to satisfy $(*)$, and analyzing this situation we shall eventually prove f_α is trivial. We have to note that if $\underset{\sim}{f}$ is a P_{ω_2}-name of a nontrivial automorphism of $\mathcal{P}(\omega)$/finite, then for a closed unbounded $C \subseteq \omega_2$, for every $\delta \in C$ with $\mathrm{cf}(\delta) = \omega_1$, $\underset{\sim}{f}\restriction[\mathcal{P}(\omega)/\text{finite}]^{V^{P_\delta}}$ has a P_δ-name forced (for P_δ) to be a nontrivial automorphism of $\mathcal{P}(\omega)$/finite in V^{P_δ}, hence a diamond on $\{\delta < \omega_2 : \mathrm{cf}(\delta) = \omega_1\}$ can guess it stationarily often, hence we have "killed" it somewhere in the process. We can have the results of §4 and §5 together.

A natural question is:

Question. Find a parallel to $\bar{M}$-c.c., replacing Cohen forcing (in the assumptions of the omitting type theorem) by random real forcing, or any other reasonable forcing adding a real.
In [Sh-b, IV], only the case "every automorphism of $\mathcal{P}(\omega)$/finite is trivial" appear. The other is from [Sh:185]. On the first use of the method see [Sh:100]. On further works see [BuSh:437], [Ju92], [Ve86], [Ve93], [ShSr:296], [ShSr:315], [ShSr:427].

Note that if $P \subseteq Q$ then $P \lessdot Q$ iff every pre-dense subset of P is pre-dense in Q (to see that this is a necessary condition, assume $\mathcal{I} \subseteq P$ is pre-dense; let $\mathcal{I}^* = \{p \in P : (\exists q \in \mathcal{I}) p \geq q\}$ and let $\mathcal{I}^{**}$ be a maximal set of pairwise incompatible conditions contained in $\mathcal{I}^*$; then $\mathcal{I}^{**}$ is a maximal antichain of P, hence of Q, therefore it is pre-dense in Q and so is $\mathcal{I}$, see I 5.4(3)).

§1. On Oracle Chain Conditions

1.1 Definition. An $\aleph_1$-*oracle* is a sequence $\bar{M} = \langle M_\delta : \delta$ is a limit ordinal $< \omega_1\rangle$, where M_δ is a countable transitive model of ZFC^- (or a large enough portion of ZFC) such that $\delta + 1 \subseteq M_\delta$, $M_\delta \models$ "δ is countable" and $\bar{M}$ satisfies: $(\forall A \subseteq \omega_1)[\{\delta : A \cap \delta \in M_\delta\}$ is stationary]. (ZFC$^-$ is ZFC without the power set axiom).

Remark. Note that the existence of an $\aleph_1$-oracle is equivalent to the holding of $\diamondsuit_{\aleph_1}$ (by a theorem of Kunen, we shall use only the trivial implication: $\diamondsuit_{\aleph_1}$ implies the existence of an $\aleph_1$-oracle), and that $\diamondsuit_{\aleph_1}$ implies CH.

To help the reader to understand our intention to associate a condition on forcing notions with each $\aleph_1$-oracle $\bar{M}$, we give here a tentative definition (which we will modify later):

1.2 First try. A forcing P with universe ω_1 satisfies the $\bar{M}$-*c.c.* if for every $\delta < \omega_1$:

(†) $(\forall A \in M_\delta)$ $[A \subseteq \delta \& A$ is pre-dense in $P{\restriction}\delta \Rightarrow A$ is pre-dense in $P]$.

But if we adopted this definition, it could happen that we have two isomorphic forcing notions, one satisfying the definition and the other not. The solution will be to require the property (†) not for all δ, but only for a large enough set of δ's.

1.3 Definition. With each $\aleph_1$-oracle $\bar{M}$ we associate the filter $D_{\bar{M}}$ over ω_1 generated by the sets $I_{\bar{M}}(A) = \{\delta < \omega_1 : A \cap \delta \in M_\delta\}$ for $A \subseteq \omega_1$. Let $\mathcal{P}(\delta)$ be $\{A : A \subseteq \delta\}$.

1.3A Remark. Recall that $\langle S_\delta : \delta < \omega_1 \rangle$ is a $\diamondsuit^*$-sequence if $|S_\delta| \le \aleph_0$ and for every $A \subseteq \omega_1$, $\{\delta : A \cap \delta \in S_\delta\}$ contains a closed unbounded subset of ω_1; and if $V = L$, then a $\diamondsuit^*$-sequence exists (by Jensen's work).

1.4 Claim. 1) If $\langle M_\delta \cap \mathcal{P}(\delta) : \delta < \omega_1 \rangle$ is a $\diamondsuit^*$-sequence (see 1.3A) then every generator of $D_{\bar{M}}$, $I_{\bar{M}}(A)$, contains a closed unbounded set;
2) For every $A, B \subseteq \omega_1$ there is $C \subseteq \omega_1$ such that $I_{\bar{M}}(C) = I_{\bar{M}}(A) \cap I_{\bar{M}}(B)$. So: for a set $Z \subseteq \omega_1$, $Z \in D_{\bar{M}} \Leftrightarrow (\exists A)[I_{\bar{M}}(A) \subseteq Z]$.
3) $D_{\bar{M}}$ is a proper normal filter, containing every closed unbounded set of limit ordinals $< \omega_1$.
4) For every $A \subseteq \omega_1 \times \omega_1$ or even $A \subseteq H(\aleph_1)$ the set $I_{\bar{M}}(A)$ belongs to $D_{\bar{M}}$.

Proof. 1) By the definitions of a $\diamondsuit^*$-sequence and of $I_{\bar{M}}(A)$.
2) Let $g : \omega_1 \to \omega_1$ be the map defined by $g(\alpha) = 2\alpha$, and let $f : \omega_1 \to \omega_1$ be the map defined by $f(\alpha) = 2\alpha + 1$. If $\delta < \omega_1$ is a limit ordinal, then δ is closed under g, f, and $g{\restriction}\delta, f{\restriction}\delta \in M_\delta$.

Now, taking $C = g(A) \cup f(B)$ we obtain what we need (remembering M_δ is a model of ZFC^-).

3) a) *$D_{\bar{M}}$ contains every closed unbounded set of limit ordinals $< \omega_1$.*

For this, we have to construct, for a given $\bar{M}$ and an increasing continuous sequence $\langle \delta_i : i < \omega_1 \rangle$ of limit ordinals, a subset A of ω_1 such that if $\delta < \omega_1$, $\delta \neq \delta_i$ for all $i < \omega_1$, then $A \cap \delta \notin M_\delta$. We construct such A piece by piece, namely determining $A \cap [\delta_i, \delta_i + \omega)$ by induction on i (we begin with $i = -1$ taking $\delta_{-1} = 0$; outside these intervals all ordinals are in A). Having determined $A \cap \delta_i$, we have $2^{\aleph_0}$ possibilities for $A \cap (\delta_i + \omega)$; we choose one which does not belong to the countable set $\{B \cap (\delta_i + \omega) : B \in M_\delta,\ \delta_i < \delta < \delta_{i+1}\}$. It is easy to check that A satisfies our requirement.

b) *$D_{\bar{M}}$ is a proper filter.*

By 2) every set in $D_{\bar{M}}$ contains some $I_{\bar{M}}(A)$ which is stationary by the assumption "$\bar{M}$ is an $\aleph_1$-oracle", hence it is nonempty.

c) *$D_{\bar{M}}$ is normal.*

It suffices to show that if $A_i \subseteq \omega_1 (i < \omega_1)$ then there is $A \subseteq \omega_1$ such that $A \cap \delta \in M_\delta$ implies $A_i \cap \delta \in M_\delta$ for all $i < \delta$. By a) and 2) it suffices if the implication holds for a closed unbounded set of δ's.

Let $\langle -, - \rangle$ be a nicely defined pairing function from ordinals to ordinals, (preserving countability), and let $C = \{\delta < \omega_1 : \delta \text{ is closed under } \langle -, - \rangle\}$. For $\delta \in C$, the restriction of $\langle -, - \rangle$ to $\delta \times \delta$ belongs to M_δ because the nice definition of $\langle -, - \rangle$ is absolute for M_δ, hence C is closed unbounded in ω_1.

Let $A = \{\langle i, \alpha \rangle : \alpha \in A_i \text{ and } i < \omega_1\}$, assume $\delta \in C$ and $A \cap \delta \in M_\delta$, and let $i < \delta$. Then also $A_i \cap \delta \in M_\delta$, which completes the proof.

4) Similar to the proof of 3). $\square_{1.4}$

1.5 Definition (the real one). We define when a forcing notion P *satisfies the $\bar{M}$-c.c.* by cases

a) If $|P| \leq \aleph_0$, always.

b) If $|P| = \aleph_1$ and for some (every) $f : P \to \omega_1$ which is one-to-one,

$$\{\delta < \omega_1 : \text{for } A \in M_\delta, A \subseteq \delta, f^{-1}(A) \text{ is pre} - \text{dense in } f^{-1}\{i : i < \delta\}$$
$$\text{implies } f^{-1}(A) \text{ is pre} - \text{dense in } P\} \in D_{\bar{M}}$$

To see the equivalence of the "some" and "every" versions let $f : P \to \omega_1$ witness the "some" version and let $g : P \to \omega_1$ be another one-to-one function. Then the set

$$B \stackrel{\text{def}}{=} \{\delta < \omega_1 : f^{-1}(\{i : i < \delta\}) = g^{-1}(\{i : i < \delta\})$$
$$\text{and for every } A \in M_\delta$$
$$[f^{-1}(A) \text{ is pre-dense in } f^{-1}(\{i : i < \delta\}) \Rightarrow f^{-1}(A) \text{ is pre-dense in } P]$$
$$\text{and } (f \circ g^{-1}) \restriction A \in M_\delta\}$$

belongs to $D_{\bar{M}}$ and witnesses that g also satisfies clause b).

c) If $|P| > \aleph_1$ and for every $P^\dagger \subseteq P$: if $|P^\dagger| \leq \aleph_1$ then there are P'' such that $|P''| \leq \aleph_1$, $P^\dagger \subseteq P'' \subseteq P$ and $f : P'' \to \omega_1$, as in b) such that: $P'' \subseteq_{ic} P$, which means, if $p, q \in P''$ then: $P \models p \leq q \Leftrightarrow P'' \models p \leq q$, and: p, q are compatible in P *iff* p, q are compatible in P''.

Notation. $P \models \bar{M}$-c.c. denotes that P satisfies $\bar{M}$-c.c.

Remark. In this chapter we use iterations of length ω_2 starting with $V = L$ and every iterand is of size ω_1, so we work explicitly only with the case $|P| = \aleph_1$.

1.6 Claim. 0) If P_1, P_2 are isomorphic forcing notions, then: P_1 satisfies the $\bar{M}$-c.c. if P_2 satisfies the $\bar{M}$-c.c.

1) If P satisfies the $\bar{M}$-c.c. for some $\aleph_1$-oracle $\bar{M}$, then P satisfies the $\aleph_1$-c.c.

2) Let $\bar{M}$ be an $\aleph_1$-oracle, $P \lessdot Q$. If Q satisfies the $\bar{M}$-c.c., then P satisfies the $\bar{M}$-c.c.

3) In Definition 1.5 for the case $|P| > \aleph_1$, we can demand $P'' \lessdot P$ and get an equivalent definition (remember we are assuming CH as we are assuming $\diamondsuit_{\aleph_1}$).

Proof. 0) Trivial (by the equivalence proved in Definition 1.5 clause (b)).

1) Clearly, it suffices to prove it for P with universe ω_1. So, assume that $\mathcal{J}$ is a maximal antichain in P of power $\aleph_1$. Then there is a closed unbounded subset C of ω_1 such that: if $\delta \in C$, $q < \delta$ then there is $p \in \mathcal{J} \cap \delta$ compatible with q, and also if $p, q < \delta$ are compatible then they have a common upper bound in $P \restriction \delta$.

As P satisfies the $\bar{M}$-c.c., there is $\delta \in C \cap I_{\bar{M}}(\mathcal{J})$ with the property: $[A \in M_\delta$ & A is a pre-dense subset of $P \restriction \delta] \Rightarrow [A$ is pre-dense in $P]$ (actually, the set of these δ's is in $D_{\bar{M}}$). But $\mathcal{J} \cap \delta$ is a counterexample as: $\mathcal{J} \cap \delta \in M_\delta$ (because $\delta \in I_{\bar{M}}(\mathcal{J})$), and $\mathcal{J} \cap \delta$ is a pre-dense subset of $P \restriction \delta$ (because $\delta \in C$ and the definition of C) hence (by the previous-sentence) any $p \in \mathcal{J} \setminus \delta$ is compatible with some $q \in \mathcal{J} \cap \delta$; contradiction.

2) Assume first that $|P| = |Q| = \aleph_1$. W.l.o.g. the universe of Q is ω_1. We have to show that for a set in $D_{\bar{M}}$ of δ's, if A is pre-dense in $P \restriction \delta$ and $A \in M_\delta$ then A is pre-dense in P; since $P \lessdot Q$ we know $P \subseteq_{ic} Q$ hence it suffices to

show that A is pre-dense in Q. Since Q satisfies the $\bar{M}$-c.c., it suffices to show that A is pre-dense in $Q{\restriction}\delta$.

For $q \in Q$ define $\mathcal{I}_q \stackrel{\text{def}}{=} \{r \in P : r \text{ is incompatible with } q, \text{ or } (\forall r^\dagger \geq r)(r^\dagger \in P \Rightarrow r^\dagger \text{ is compatible with } q)\}$.

Let $\mathcal{J}_q$ be a maximal set of pairwise incompatible elements in $\mathcal{I}_q$. As clearly $\mathcal{I}_q$ is dense in P, $\mathcal{J}_q$ is a maximal antichain in P, hence in Q (since $P \lessdot Q$). Pick $r_q \in \mathcal{J}_q$ such that $(\forall r^\dagger \geq r_q)(r^\dagger \in P \Rightarrow r^\dagger$ is compatible with $q)$; such r_q exists since otherwise q would contradict the fact that $\mathcal{J}_q$ is a maximal antichain in Q. We may assume that δ is such that: $q < \delta$ implies $r_q < \delta$, and any $p_1, p_2 \in P{\restriction}\delta$ compatible in P are compatible in $P{\restriction}\delta$, and the same holds for Q.

Now let A be pre-dense in $P{\restriction}\delta$ and let $q \in Q{\restriction}\delta$. Since $r_q \in P{\restriction}\delta$, there is $p \in A$ compatible with r_q. Let $r^\dagger \in P{\restriction}\delta$ be above p and r_q. By the choice of r_q we know that $r^\dagger$ is compatible with q, hence p is compatible with q, so A is pre-dense in $Q{\restriction}\delta$. Hence we have finished the case $|P| = |Q| = \aleph_1$.

Before proving the claim for all P, Q we present the following simple fact on forcing notions.

1.6A Fact. (CH). If $P_1 \subseteq P_2$, $|P_1| \leq \aleph_1$, and P_2 satisfies the $\aleph_1$-c.c., *then* there is P_3 such that $|P_3| \leq \aleph_1$ and $P_1 \subseteq P_3 \lessdot P_2$.

Proof of Fact. We define, by induction, an increasing continuous sequence $\langle P^{(\alpha)} : \alpha < \omega_1 \rangle$ of subsets of P of cardinality $\leq \aleph_1$ as follows:

$P^{(0)} = P_1$,

$P^{(\alpha)} = \cup_{\beta<\alpha} P^{(\beta)}$ for limit α.

For $\alpha = \beta + 1$, for every countable subset $\mathcal{I}$ of $P^{(\beta)}$ which is not pre-dense in P_2, pick $p_{\mathcal{I}} \in P_2$ exemplifying this. Let $P_0^{(\alpha)}$ be the set obtained by adding the $p_{\mathcal{I}}$'s to $P^{(\beta)}$. Let $P^{(\alpha)}$ be the set obtained from $P_0^{(\alpha)}$ by adding a common upper bound for every pair in $P_0^{(\alpha)}$ having one in P_2.

Now, $P_3 = \cup_{\alpha<\omega_1} P^{(\alpha)}$ is as required, proving the Fact. $\square_{1.6A}$

Continuation of the proof of 1.6

Returning to our claim, we distinguish the following cases:

i) $|P| \leq \aleph_0$. Then trivially P satisfies the $\bar{M}$-c.c.

ii) $|P| = \aleph_1$. Then since $Q \models \bar{M}$-c.c. there is $P^\dagger \subseteq_{\rm ic} Q$, $P \subseteq P^\dagger$, such that $P^\dagger$ satisfies the $\bar{M}$-c.c. and $|P^\dagger| = \aleph_1$. Since $P \lessdot Q$, clearly $P \lessdot P^\dagger$, and now by what we have already proved P satisfies the $\bar{M}$-c.c.

iii) $|P| > \aleph_1$. Let $P^\dagger \subseteq P$, $|P^\dagger| \leq \aleph_1$. Using the Fact 1.6A, we obtain P'', such that $|P''| \leq \aleph_1$ and $P^\dagger \subseteq P'' \lessdot P$. Since $P \lessdot Q$ it follows that $P'' \lessdot Q$, so by i) or ii) we have: P'' satisfies the $\bar{M}$-c.c. Hence P satisfies the $\bar{M}$-c.c.

3) Let P satisfy the $\bar{M}$-c.c. (by Definition 1.5), $|P| > \aleph_1$, and let $P^\dagger \subseteq P$, $|P^\dagger| \leq \aleph_1$ and (by 1.6A) choose P'', $|P''| \leq \aleph_1$ such that $P^\dagger \subseteq P'' \lessdot P$. From 2) we know that P'' satisfies the $\bar{M}$-c.c., hence is as required. $\square_{1.6}$

1.7 Fact.

1) If $P \lessdot R$, $P \subseteq Q \subseteq_{ic} R$ then $P \lessdot Q$

2) P_1, $P_2 \lessdot Q$, $P_1 \subseteq P_2$ then $P_1 \lessdot P_2$; if P_1, $P_2 \subseteq_{ic} Q$, $P_1 \subseteq P_2$ then $P_1 \subseteq_{ic} P_2$

3) $\subseteq_{ic}$ is a partial order. $\square_{1.7}$

§2. The Omitting Type Theorem

2.1 Lemma. ($\diamondsuit_{\aleph_1}$). Suppose $\psi_i(x)(i < \omega_1)$ are Π^1_2 (i.e., $\forall$ real $\exists$ real ...) formulas on reals (with a real parameter, possibly). Suppose further that there is no solution to $\bigwedge_{i<\omega_1} \psi_i(x)$ in V; and moreover even if we add a Cohen real to V there will be none. *Then* there is an $\aleph_1$-oracle $\bar{M}$ such that:

(*) $P \vDash \bar{M}$-c.c. implies: in V^P there is no solution to $\bigwedge_i \psi_i(x)$.

Proof. Let $n < \omega$ be large enough so that the forcing theorems can be proved from $\sum_n$ sentences in ZFC, and the assumption of the lemma can be formulated as a $\sum_n$ statement.

Now, for a given countable forcing notion P and a given P-name for a real $\underset{\sim}{\tau}$ (w.l.o.g. canonical), let $M(P, \underset{\sim}{\tau})$ be a countable $\sum_n$-elementary submodel of V containing P, $\underset{\sim}{\tau}$ and $\langle \psi_i : i < \omega_1 \rangle$. Let $\mathcal{I}(P, \underset{\sim}{\tau})$ be the collection of all pre-dense subsets of P lying in $M(P, \underset{\sim}{\tau})$.

2.1A Claim.

1) If $P \subseteq_{ic} P^\dagger$ and every $A \in \mathcal{I}(P, \underset{\sim}{\tau})$ is pre-dense in $P^\dagger$, then

$$V^{P^\dagger} \models \neg \bigwedge_i \psi_i(\underset{\sim}{\tau}[G])$$

(note that $\underset{\sim}{\tau}$ is also a $P^\dagger$-name; we can restrict ourselves to $i \in M(P, \underset{\sim}{\tau})$).

2) Assume that P is a countable forcing, $\underset{\sim}{\tau}$ a P-name of a real, a canonical one, φ a Π^1_1-formula, $p \Vdash_P$ "$\varphi(\underset{\sim}{\tau})$". Then for some pre-dense $\mathcal{I}_n \subseteq P$ for $n < \omega$, *if* $P \subseteq_{ic} P^\dagger$, each $\mathcal{I}_n$ is pre-dense in $P^\dagger$ *then* $p \Vdash_{P^\dagger}$ "$\varphi(\underset{\sim}{\tau})$".

Proof of Claim. 1) Let G be $P^\dagger$-generic over V. Since every $A \in \mathcal{I}(P, \underset{\sim}{\tau})$ is pre-dense in $P^\dagger$ and therefore intersects G, $G \cap P$ is P-generic over $M(P, \underset{\sim}{\tau})$. (Why is $G \cap P$ directed? If $p, q \in G \cap P$ then $\mathcal{J} = \{r \in P : r$ incompatible with p (in P or equivalently in $P^\dagger$), *or* r incompatible with q *or* $p \le r$ & $q \le r\} \in M(P, \underset{\sim}{\tau})$ is predense in P hence in $P^\dagger$, hence not disjoint to $G \cap P$.) Also $\underset{\sim}{\tau}[G] = \underset{\sim}{\tau}[G \cap P]$. Since $M(P, \underset{\sim}{\tau})$ is a $\sum_n$-elementary submodel of V, it satisfies the assumption of the lemma, hence $M(P, \underset{\sim}{\tau})[G \cap P] \models$"$\neg \psi_i(\underset{\sim}{\tau}[G])$ for some $i < \aleph_1$". So for some $i < \aleph_1$, $i \in M(P, \underset{\sim}{\tau})$ we have $M(P, \underset{\sim}{\tau})[G \cap P] \models \neg \psi_i(\underset{\sim}{\tau}[G])$ (remember that P is countable (even in the sense of M) and therefore adding a generic subset of P is equivalent to adding a Cohen real). Since $\sum^1_2$ statements are upward absolute from $M(P, \underset{\sim}{\tau})[G \cap P]$ to $V[G]$ by Schoenfields's absoluteness theorem, it follows that $V[G] \models$ "$\neg \psi_i(\underset{\sim}{\tau}[G])$", proving the claim.

2) Proved above. $\square_{2.1A}$

Continuation of the proof of 2.1: Now, using $\diamondsuit_{\aleph_1}$ and a closing process, we can obtain an $\aleph_1$-oracle $\bar{M} = \langle M_\delta : \delta < \omega_1 \rangle$ satisfying: if P is a forcing notion, has universe $\delta < \omega_1$ and $P, \underset{\sim}{\tau} \in M_\delta$ then $\mathcal{I}(P, \underset{\sim}{\tau}) \subseteq M_\delta$ (remember that $P, \mathcal{I}(P, \underset{\sim}{\tau})$ are countable). We will show that $\bar{M}$ satisfies the requirement of the lemma.

Assume, on the contrary, that $P^\dagger \models \bar{M}$-c.c. but $\underset{\sim}{\tau}$ is a $P^\dagger$-name of a real so that $V^{P^\dagger} \models \bigwedge_i \psi_i(\underset{\sim}{\tau})$. We may assume that $|P^\dagger| = \aleph_1$ because:

a) $|P^\dagger| \leq \aleph_0$ is impossible by the assumption of the lemma.

b) if $|P^\dagger| > \aleph_1$ then by 1.6(1) and 1.6(3) we can find P'' such that $\underset{\sim}{\tau}$ is a P''-name, $|P''| = \aleph_1$, $P'' \lessdot P^\dagger$ and $P'' \models \bar{M}$-c.c. Since $P'' \lessdot P^\dagger$, $V^{P''} \models \neg\psi_i(\underset{\sim}{\tau})$ would imply $V^{P^\dagger} \models \neg\psi_i(\underset{\sim}{\tau})$(again by an absoluteness argument), hence $V^{P''} \models \bigwedge_i \psi_i(\underset{\sim}{\tau})$. So P'' can replace $P^\dagger$ in the sequel.

W.l.o.g. the universe of $P^\dagger$ is ω_1. We can find $\delta < \omega_1$ such that letting $P = P^\dagger \restriction \delta$ the following will hold:

i) $\underset{\sim}{\tau}$ is a P-name.

ii) $P, \underset{\sim}{\tau} \in M_\delta$.

iii) $P \subseteq_{ic} P^\dagger$.

iv) $A \in M_\delta, A$ is a pre-dense subset of $P \Rightarrow A$ is pre-dense in $P^\dagger$.

From these facts, the claim and the construction of $\bar{M}$, it follows that $V^{P^\dagger} \models \neg \bigwedge_i \psi_i(\underset{\sim}{\tau})$, a contradiction. $\square_{2.1}$

2.2 Example. If $V \models [\diamondsuit_{\aleph_1} \& A \subseteq \mathbb{R}$ & A is of the second category] *then* there is an ω_1-oracle $\bar{M}$ such that:

$P \models \bar{M}$-c.c. $\Rightarrow V^P \models$ "A is of the second category".

Proof. Let $A = \{r_i : i < \omega_1\}$, $\psi_i(x) = (r_i \in B_x)$ where $B_x = \cup_{n<\omega} B_x^n, B_x^n$ closed nowhere dense, where B_x^n is "simply" described by x such that every set of the first category is a subset of B_x for some x and absolutely, for every real x and $n < \omega$, B_x^n is nowhere dense. Each ψ_i is trivially Π^1_2. We still have to show:

$(*)$ if P is countable, the following fails: $\Vdash_P$ " for some real x, for every $i < \omega_1$ we have $r_i \in B_x$".

Suppose, on the contrary, that there are $r \in P$ and a P-name $\underset{\sim}{x}$, so that $r \Vdash_P$ "$\underset{\sim}{x}$ is a counterexample". Then for every i, for some $p \in P$ and n we have $p \Vdash_P$ "$r_i \in B^n_{\underset{\sim}{x}}$". Let $A^n_p = \{r_i : p \Vdash_P$ "$r_i \in B^n_{\underset{\sim}{x}}$"$\}$; we have shown $A \subseteq \cup_{n,p} A^n_p$. We shall show that every A^n_p is nowhere dense and get a contradiction.

Let $n < \omega$, $p \in P$, (c_1, c_2) a rational interval. It suffices to find a rational subinterval disjoint from A^n_p. But $\Vdash_P$ "$\underset{\sim}{B}^n_x$ is nowhere dense". So there are $p^\dagger \geq p$, $(c^\dagger_1, c^\dagger_2) \subseteq (c_1, c_2)$ such that $p^\dagger \Vdash_P$ "$\underset{\sim}{B}^n_x \cap (c^\dagger_1, c^\dagger_2) = \emptyset$". Clearly:

$$r_i \in (c^\dagger_1, c^\dagger_2) \Rightarrow p^\dagger \Vdash_P \text{“} r_i \notin \underset{\sim}{B}^n_x \text{”} \Rightarrow$$

$$\Rightarrow p \nVdash_P \text{“} r_i \in \underset{\sim}{B}^n_x \text{”} \Rightarrow r_i \notin A^n_p.$$

$\square_{2.2}$

The arguments above give the following fact, too:

2.2A Fact. If $A \subseteq \mathbb{R}$ is of the second category and P is the forcing notion of adding α Cohen reals to V, *then* $V^P \models$ "A is of the second category". $\square_{2.2}$

It simplifies to note:

2.3 Fact. Let P be a countable forcing notion, Q Cohen forcing

1) If $\mathcal{J}$ is a pre-dense subset of $P \times Q$, then we can find pre-dense subsets $\mathcal{I}_n$ of P, for $n < \omega$ such that:

(*) if P' is a forcing notion, $P \subseteq P'$ and each $\mathcal{I}_n$ is a pre-dense subset of P' too, *then* $\mathcal{J}$ is a pre-dense subset of $P' \times Q$.

2) The same applies to $P \times \underset{\sim}{Q}$, $P' \times \underset{\sim}{Q}$ as for any forcing R, $R \times Q$ is a dense subset of $R \times \underset{\sim}{Q}$.

Proof. 1) Let $\{q_n : n < \omega\}$ list the members of Q and let $\mathcal{I}_n = \{p \in P :$ for some q we have: $q_n \leq q \in Q$ and (p, q) is above some member of $\mathcal{J}\}$.

2) Should be clear. $\square_{2.3}$

§3. Iterations of $\bar{M}$-c.c. Forcings

3.1 Claim. If $\bar{M}_i$ (for $i < \omega_1$) are $\aleph_1$-oracles then there is an $\aleph_1$-oracle $\bar{M}$ such that:

$$(P \models \bar{M}\text{-c.c.}) \Rightarrow \Big(\bigwedge_{i<\omega_1} P \models \bar{M}_i\text{-c.c.}\Big).$$

Proof. Let $\bar{M}_i = \langle M^i_\delta : \delta < \omega_1 \rangle$.

Choose M_δ such that $\cup_{i<\delta} M^i_\delta \subseteq M_\delta$, M_δ a countable transitive model of ZFC$^-$. Clearly $\bar{M} = \langle M_\delta : \delta < \omega_1 \rangle$ is an $\aleph_1$-oracle. It suffices to prove that:

$(*)$ if $\bar{M}_1, \bar{M}_2$ are $\aleph_1$-oracles, $\{\delta : M^1_\delta \subseteq M^2_\delta\}$ is co-bounded (i.e., with a bounded complement) then $\bar{M}_2$-c.c. $\Rightarrow \bar{M}_1$-c.c.

Since for each $A \subseteq \omega_1$, every sufficiently large $\delta \in I_{\bar{M}_1}(A)$ is in $I_{\bar{M}_2}(A)$, it suffices to show that if δ has the required property with respect to $\bar{M}_2$ (and δ is sufficiently large) then it has the required property with respect to $\bar{M}_1$. But this follows trivially from $M^1_\delta \subseteq M^2_\delta$. (Of course in $(*)$ we can require e.g. just "$\{\delta : M^1_\delta \not\subseteq M^2_\delta\}$ is not stationary or just $= \emptyset \mod D_{\bar{M}^2}$"). $\square_{3.1}$

3.2 Claim. If $P_i (i < \alpha)$ is the result of a finite support iteration, each P_i satisfying the $\bar{M}$-c.c. and $P = \cup_{i<\alpha} P_i$, *then* P satisfies the $\bar{M}$-c.c.

Proof. We demand superficially less on the P_i's: $P_i \lessdot P_j$ for $i < j$, and $P_\delta = \cup_{i<\delta} P_i$ for limit $\delta < \alpha$, and each P_i (for $i < \alpha$) satisfies the $\bar{M}$-c.c.

Case I: α successor.

Clearly $P \in \{P_i : i < \alpha\}$ (it is $P_{\alpha-1}$) so by the assumption P satisfies the $\bar{M}$-c.c.

Case II: $\mathrm{cf}(\alpha) > \aleph_1$.

If $P^\dagger \subseteq P$, $|P^\dagger| = \aleph_1$, then there is $i < \alpha$ with $P^\dagger \subseteq P_i$ because $\mathrm{cf}(\alpha) > \aleph_1$. Since $P_i \models \bar{M}$-c.c. we can find an $\bar{M}$-c.c. $P'' \subseteq_{\mathrm{ic}} P_i$ of size $\aleph_1$ such that $P^\dagger \subseteq P''$. So $P \models \bar{M}$-c.c.

Case III: $\mathrm{cf}(\alpha) = \aleph_1$.

Obviously it is sufficient to deal with the case $\alpha = \omega_1$. We will now show that it is sufficient to deal with the case $|P| = \aleph_1$.

Indeed, if $|P| < \aleph_1$ then the conclusion of our claim holds. So assume $|P| > \aleph_1$, and P does not satisfy the $\bar{M}$-c.c. Then there is $P^\dagger \subseteq P$ of power $\aleph_1$ so that $P^\dagger \subseteq P'' \subseteq_{\mathrm{ic}} P$, $|P''| = \aleph_1$ imply that P'' does not satisfy the $\bar{M}$-c.c. We note that P satisfies the $\aleph_1$-c.c. (See Claim 1.6(1) and the proof of Lemma II 2.8).

Let N be an elementary submodel of $(H(\chi), \in, <^*_\chi)$ for a large enough regular χ such that $P^\dagger$, $\langle P_i : i \le \omega_1 \rangle \in N$, $N^\omega \subseteq N$ and $\|N\| = \aleph_1$. Take $P_i'' \stackrel{\text{def}}{=} P_i \cap N$. Then $P_i'' \lessdot P_i$ (as in the proof of 1.6A) and $\langle P_i'' : i \le \omega_1 \rangle$ is increasing continuous and $P^\dagger \subseteq P''_{\omega_1}$. But $|P''_{\omega_1}| = \aleph_1$ and it does not satisfy the $\bar{M}$-c.c., so we have reduced our counterexample to one of power $\aleph_1$.

We assume now, w.l.o.g., that $P_i \setminus \cup_{j<i} P_j \subseteq \omega_1 \times \{i\}$. We let $A_i \stackrel{\text{def}}{=} \{\delta :$ if $C \in M_\delta$ is a pre-dense subset of $P_i \restriction (\delta \times \delta)$ then C is pre-dense in $P_i\}$.

For each $i < \omega_1, A_i \in D_{\bar{M}}$ (as $P_i \models \bar{M}$-c.c.). Since $P_i \lessdot P$ and $P_i \models \aleph_1$-c.c., by the proof of claim 1.6(2), we can find a function pr_i from $P \times \omega$ into P_i such that: $p \in P_i$ is compatible with $q \in P$ in P iff for some n, $pr_i(q, n), p$ are compatible in P_i. We let $B_i = \{\delta : pr_i''((\delta \times \delta) \times \omega) \subseteq \delta \times \delta$ and $pr_i \cap (\delta \times \delta) = pr_i \restriction ((\delta \times \delta) \times \omega) \in M_\delta\}$.

For each $i < \omega_1$, $B_i \in D_{\bar{M}}$.

We let $A \stackrel{\text{def}}{=} \nabla_i A_i = \{\delta :$ for every $i < \delta, \delta \in A_i\}$ and $B = \nabla_i B_i \stackrel{\text{def}}{=} \{\delta :$ for every $i < \delta, \delta \in B_i\}$. As $D_{\bar{M}}$ is a normal filter, clearly A, B, $A \cap B \in D_{\bar{M}}$. Assume now that $\delta \in A \cap B$, $Y \in M_\delta$ is a pre-dense subset of $P \restriction (\delta \times \delta)$. Notice that

(*) $p \in P_i$ is compatible with some $q \in Y$ iff it is compatible with some $r \in pr_i(Y \times \omega)$.

We know that for all $i < \delta$ we have $pr_i(Y \times \omega) \in M_\delta$ (as $\delta \in B_i$) and it is a pre-dense subset of $P_i \restriction (\delta \times \delta)$ by (*). Therefore, as $\delta \in A_i$, $pr_i(Y \times \omega)$ is pre-dense in P_i. Using (*) again, (as $P_\delta = \cup_{i<\delta} P_i, P \restriction (\delta \times \delta) \subseteq P_\delta$), Y is pre-dense in P_δ, and since $P_\delta \lessdot P$, Y is pre-dense in P. Thus we have proved that P satisfies the $\bar{M}$-c.c. (Note that for some/any bijection $f : \omega_1 \times \omega_1 \to \omega_1$, $\{\delta : f''(\delta \times \delta) = \delta\}$ is closed unbounded).

Case IV: $\mathrm{cf}(\alpha) = \aleph_0$.

One can deal with this case as with Case III (the diagonal intersections are replaced by countable intersections). $\square_{3.2}$

We shall deal now with the "real" successor case:

3.3 Claim. Assume $V \models \diamondsuit_{\aleph_1}$, $\bar{M}$ is an $\aleph_1$-oracle, P is a forcing notion of power $\aleph_1$ satisfying the $\bar{M}$-c.c. *Then* in V^P there is an $\aleph_1$-oracle $\bar{M}^*$ such that: if $Q \in V^P$ satisfies the $\bar{M}^*$-c.c. (in V^P) *then* $P * \underset{\sim}{Q}$ (which is in V) satisfies the $\bar{M}$-c.c.

Proof. First we observe that if $\bar{M}^*$ in V^P is good for all Q of power $\leq \aleph_1$ then it is good for all Q. Hence we shall treat $\underset{\sim}{Q}$ as a P-name of a binary relation on ω_1 (but we shall define $\bar{M}^*$ without depending on a given $\underset{\sim}{Q}$). We also assume that P has universe ω_1, and we fix for the rest of the proof a generic $G \subseteq P$ (which has a canonical P-name in V). Note that $\omega_1 \times \omega_1$ is a dense subset of $P * \underset{\sim}{Q}$, so we can use it as the set of members.

For $A \subseteq \omega_1 \times \omega_1$ and $G \subseteq P$ we define $A[G] = A^G = \{\alpha : (\exists p \in G)[(p, \alpha) \in A]\}$.

Let $\delta < \omega_1$ be a limit ordinal. In $V[G]$ we let M^*_δ be a countable transitive model of ZFC$^-$ containing δ, M_δ and $\{A^G : A \subseteq \delta \times \delta, A \in M_\delta\}$. We will first show that $\bar{M}^*$ is a $\aleph_1$-oracle in $V[G]$.

The following fact implies this statement.

3.3A Fact. $(\forall A \in D^{V[G]}_{\bar{M}^*})(\exists B \in D_{\bar{M}})[B \subseteq A]$.

Proof of the Fact. Let $\underset{\sim}{S}$ be a P-name of a subset of ω_1. Let $T = \{(p, \alpha) : p \in P$ and $p \Vdash_P$ "$\alpha \in \underset{\sim}{S}$"$\}$. Clearly $\underset{\sim}{S}[G] = T[G]$. Since P satisfies $\aleph_1$-c.c. and so $(\forall \alpha < \omega_1)(\exists \beta < \omega_1)\left[T[G] \cap \alpha = T[G \cap \beta] \cap \alpha\right]$, we have in V a club $C \subseteq \omega_1$ such that $\Vdash_P$ "$T[\underset{\sim}{G}] \cap \delta = T[\underset{\sim}{G} \cap \delta]$" for each $\delta \in C$. So $\delta \in C$ and $A = T \cap (\delta \times \delta) \in M_\delta$ implies $A^G = T[G \cap \delta] \cap \delta = T[G] \cap \delta \in M^*_\delta$. But these restrictions give an element of $D_{\bar{M}}$ by 1.4(3). So we have proved that for every P-name $\underset{\sim}{S} \subseteq \omega_1$, for some $T \subseteq \omega_1 \times \omega_1$ we have $\Vdash$ "$I_{\bar{M}^*}(S)^{V^P} \supseteq I_{\bar{M}}(T)^V$" (well, $T \subseteq \omega_1 \times \omega_1$, and not $T \subseteq \omega_1$, so use a pairing function, see 1.4(4)).

$\square_{3.3A}$

Thus we have proved that $\bar{M}^*$ is an $\aleph_1$-oracle in $V[G]$.

We now assume that $A \subseteq \delta \times \delta$, $A \in M_\delta$ and A is pre-dense in $(P * \underset{\sim}{Q}) \restriction (\delta \times \delta)$. We want to show that A is pre-dense in $P * \underset{\sim}{Q}$. The proof will be broken into three steps, and as we proceed we will impose restrictions on δ, the reader can easily verify that these restrictions are allowable (in the sense of $D_{\bar{M}}$) and do not depend on A and G.

First Step. If $G \subseteq P$ is generic over V, then $A^G \subseteq \delta$, $A^G \in M^*_\delta$ and A^G is pre-dense in $\underset{\sim}{Q}[G]{\restriction}\delta$.

Proof. The first two facts are immediate.

Let $(p,q),(\alpha,\beta) \in P * \underset{\sim}{Q}$. We define $f_{(p,q)}(\alpha,\beta)$, whenever (p,q) and (α,β) are compatible, as the least ordinal which is the possible P-part of a condition above (p,q) and (α,β) in $P * \underset{\sim}{Q}$. We may assume that for each p, $q < \delta$ $f''_{(p,q)}(\delta \times \delta) \subseteq \delta$ and for each $q < \delta$ the mapping $p \mapsto f_{(p,q)}{\restriction}(\delta \times \delta)$ for $p < \delta$ belongs to M_δ.

Let $q \in \underset{\sim}{Q}[G]{\restriction}\delta$, and define $\mathcal{I}_q = \cup_{p<\delta} f_{(p,q)}(A)$. Then $\mathcal{I}_q \subseteq \delta$, $\mathcal{I}_q \in M_\delta$ and $\mathcal{I}_q$ is dense in $P{\restriction}\delta$ (as A is pre-dense in $(P * \underset{\sim}{Q}){\restriction}(\delta \times \delta)$ and by the definition of $f_{(p,q)}$). Hence we may assume that $\mathcal{I}_q$ is pre-dense in P (by our assumption that P satisfies the $\bar{M}$-c.c.).

Let $r \in \mathcal{I}_q$. Then for some $p < \delta$, $s \in Q$ and $(\alpha,\beta) \in A$ we have in $P * \underset{\sim}{Q}$ $(r,s) \geq (p,q),(\alpha,\beta)$. As $P \models$ "$r \geq \alpha$", $r \Vdash_P$ "$\alpha \in \underset{\sim}{G}_P$", so $r \Vdash_P$ "$\beta \in A^{\underset{\sim}{G}_P}$"; as $r \Vdash_P$ "$\beta \leq_{\underset{\sim}{Q}} s$ and $q \leq_{\underset{\sim}{Q}} s$" it follows that $r \Vdash_P$ "q is compatible with some element of A^G". As this holds for all $r \in \mathcal{I}_q$ and $\mathcal{I}_q$ is pre-dense in P, it is true in $V[G]$, and as this holds in $V[G]$ for all $q \in \underset{\sim}{Q}[G]{\restriction}\delta$, we obtain that A^G is pre-dense in $\underset{\sim}{Q}[G]{\restriction}\delta$.

Second Step. If $G \subseteq P$ is generic over V, then A^G is pre-dense in $\underset{\sim}{Q}[G]$.

Proof. This follows from the assumption that in $V[G]$, the forcing notion $\underset{\sim}{Q}[G]$ satisfies the $\bar{M}^*$-c.c. We only have to observe here that the restriction to a set in $D_{\bar{M}^*}$ can be replaced by a restriction to a set in $D_{\bar{M}}$, by Fact 3.3A.

Third Step. A is pre-dense in $P * \underset{\sim}{Q}$.

Proof. Let $(p, q) \in P * \underset{\sim}{Q}$. Since for any $G \subseteq P$ generic over V, in $V[G]$ the set A^G is pre-dense in $\underset{\sim}{Q}[G]$, there exist $r \geq p$, $(\alpha, \beta) \in A$ and $s \in Q$ such that $r \Vdash_P$ "$\alpha \in \underset{\sim}{G}_P$ and $\beta \leq_{\underset{\sim}{Q}} s$, and $q \leq_{\underset{\sim}{Q}} s$". As $r \Vdash_P$ "$\alpha \in \underset{\sim}{G}_P$", $r \geq \alpha$, so $(r, s) \geq (p, q), (\alpha, \beta)$, in $P * \underset{\sim}{Q}$, proving what we need.

$\square_{3.3}$

§4. The Lifting Problem of the Measure Algebra

4.1 Notation. Let $\mathcal{B}$ be the family of Borel subsets of $(0, 1)$. Every Borel set $\subseteq (0, 1)$ has a definition φ (in the propositional calculus $L_{\omega_1,\omega}$), i.e. let φ be a sentence in the $L_{\omega_1,\omega}$ propositional calculus, with vocabulary $\{t_q : q \in \mathbb{Q}\}$, where $\mathbb{Q}$ denotes the rational numbers, (t_q stands for the statement $r < q$). We let $A = \mathrm{Bo}[\varphi]$ be the Borel set correspondending to this definition, i.e. $A = \{r : r \in (0, 1)$, and if we assign to the propositional variables t_q the truth value of "$r < q$", the sentence φ becomes true$\}$. Notice that the answer to "$r \in \mathrm{Bo}[\varphi]$" is absolute.

If $B \in \mathcal{B}$ in V, and $V[G]$ is a generic extension of V, then let $B^{V[G]}$ be the unique B_1 such that for some φ, $V \models$"$B = \mathrm{Bo}(\varphi)$", and $V[G] \models$"$B_1 = \mathrm{Bo}[\varphi]$". Note that the choice of φ is immaterial.

Let I_{mz} be the family of $A \in \mathcal{B}$ of measure zero and I_{fc} be the family of $A \in \mathcal{B}$ which are of the first category. If a, b are reals, $0 \leq a$, $b \leq 1$, let

$$(a, b) = \{x : a < x < b \ \text{ or } \ b < x < a\}$$

$[a, b]$ is defined similarly. Again this is absolute. We can use $\mathbb{R}$ or $(0, 1)$, does not matter.

4.2 Definition. If B is a Boolean algebra, I is an ideal, we say that B/I splits if there is a homomorphism $h : B/I \to B$ such that $h(x/I)/I = x/I$.

Equivalently there is a homomorphism $h : B \to B$ with kernel I such that $h(x) = x \mod I$.

4.3 Theorem. It is consistent with ZFC that $\mathcal{B}/\mathcal{B} \cap I_{mz}$ does not split (if ZFC is consistent).

4.4 Discussion. (1) If CH holds then $\mathcal{B}/I_{mz}$ splits (see Oxtoby [Ox]); in fact this holds for any $\aleph_1$-complete ideal.

(2) Note that $\mathfrak{B}/I_{mz}$ has a natural set of representatives:

$$h^m(X) = \{a \in R : 1 = \lim_{\varepsilon \to 0}[\mathrm{Leb}(X \cap (a - \varepsilon, a + \varepsilon))/2|\varepsilon|]\}$$

where Leb(X) is the Lebesgue measure (for a set of reals). Unfortunately h^m is not a homomorphism.

Now for $X \in \mathcal{B}$, $h^m(X) \in \mathcal{B}$ and $h^m(X) = X \mod I_{mz}$ (see Oxtoby [Ox]).

(3) Note also that $(\mathcal{B} + I^{\oplus}_{mz})/I^{\oplus}_{mz}$ splits where $\mathcal{B} + I^{\oplus}_{mz}$ is the Boolean algebra of subsets of $(0, 1)$ generated by $\mathcal{B}$ and $I^{\oplus}_{mz}$ and $I^{\oplus}_{mz} = \{A : A$ is a subset of some member of $I_{mz}\}$. A function exemplifying this can be defined as follows: for each real r let E_r be an ultrafilter on $(0, 1)$ such that if $A \subseteq (0, 1)$ and $1 = \lim_{\varepsilon \to 0}[\mathrm{Leb}(A \cap (r - \varepsilon, r + \varepsilon))/2|\varepsilon|]$, then $A \in E_r$ and for every $X \in \mathcal{B} + I_{mz}$ let

$$h(X) = \{r : X \in E_r, r \in (0, 1)\}$$

Clearly $h(X) = h^m(X) \mod I_{mz}$ hence $h(X) = X \mod I_{mz}$.

(4) The dual problem, replacing "measure zero" by "meager" can be solved too. I.e.

(∗) it is consistent with ZFC that $\mathcal{B}/\mathcal{B} \cap I_{\mathrm{fc}}$ does not split (if ZFC is consistent).

The proof is like the proof of 4.3, replacing "measure zero" by "meager", it is done explicitly in [Sh:185] [were also Theorem 4.3 was proved].

(5) The author was asked this problem by Fremlin and Talagrand during the Kent Conference, summer 79, who said it is very important for measure theory; Talagrand even promised me flowers on my grave from measure theorists. I did not check that yet ...

4.5 Main Lemma. Let $\bar{M}$ be an $\aleph_1$-oracle (so CH and even $\diamondsuit_{\aleph_1}$ hold) and h be a homomorphism from $\mathcal{B}$ to $\mathcal{B}$ with kernel I_{mz}, such that $h(X) = X \mod I_{mz}$ for every $X \in \mathcal{B}$.

Then there is a forcing notion P of cardinality $\aleph_1$ satisfying the $\bar{M}$-chain condition, and a P-name $\underset{\sim}{X}$ (of a Borel set) such that for every generic $G \subseteq P \times Q$ over V (where Q is the Cohen forcing) there is no Borel set A in $V[G]$ satisfying:

α) $A = \underset{\sim}{X}[G] \mod I_{mz}^{V[G]}$

β) for every $B \in \mathcal{B}^V$, if $B^{V[G]} \subseteq \underset{\sim}{X}[G] \mod I_{mz}$ then $(h(B))^{V[G]} \subseteq A$,

γ) for every $B \in \mathcal{B}^V$, if $B^{V[G]} \cap \underset{\sim}{X}[G] = \emptyset \mod I_{mz}$ then $(h(B))^{V[G]} \cap A = \emptyset$.

4.6. *Proof of the Theorem from the Main Lemma.* For simplicity our ground model is a model of $V = L$. We intend to define a finite support iteration $\langle P_i, Q_i : i \leq \omega_2\rangle$ such that $|P_i| < \aleph_2$ and $P_i \models \aleph_1$-c.c. for $i < \omega_2$.

We also want to define a sequence $\langle F_i : i < \omega_2\rangle$, $F_i \in V^{P_i}$, $F_i \subseteq [\mathcal{B} \times \mathcal{B}]^{V^{P_i}}$ (more exactly a P_i-name of such set), such that for any $F \subseteq [\mathcal{B} \times \mathcal{B}]^{V^{P_{\omega_2}}}$, the set

$$\{i < \omega_2 : cf(i) = \aleph_1, F\restriction[\mathcal{B}]^{V^{P_i}} = F_i\}$$

is stationary in ω_2.

This is possible, using $\diamondsuit_{\{\delta<\omega_2:cf(\delta)=\aleph_1\}}$.

(F_i is a P_i-name. So it becomes meaningful after we have defined P_i, but using $\diamondsuit_{\{\delta<\omega_2:cf(\delta)=\omega_1\}}$ we obtain a schematic definition of the F_i-s which does not depend on the actual construction of the P_i's or just guess $P\restriction i$ and $\underset{\sim}{F}_i\restriction i$. See more detatils for the second case in Claim 5.3).

So how do we define P_i for $i \leq \omega_2$? As we are going to iterate with finite support, $P_i (i \leq \omega_2)$ will be determined as soon as we define $\langle \underset{\sim}{Q}_i : i < \omega_2\rangle$ by the relations $P_0 = \{\emptyset\}, P_{i+1} = P_i * \underset{\sim}{Q}_i$ and $P_\delta = \cup_{i<\delta} P_i$ for limit δ and $P_{j,i} = P_i/P_j$. Actually, we will define by induction $\underset{\sim}{Q}_i$, $P_{j,i} \in V^{P_j}$ and $\aleph_1$-oracles $\bar{M}_i \in V^{P_{i+1}}$ (actually a P_{i+1}-name) and $\bar{M}_i^j \in V^{P_i}$ for $j < i < \omega_2$ (remember that having defined $\underset{\sim}{Q}_i$ we have determined P_{i+1} so we can go about our next task). We define by induction demanding that:

$(*)_i$ for each $j < i$ in the universe $V^{P_{j+1}}$, the forcing notion $P_{j+1,i}$ satisfies the $\bar{M}_j$-c.c (if $j+1=i$ this is an empty demand).

For $i = 0$ we have nothing to check. For i limit use the claims on FS iteration from II and Claim 3.2. So to carry the induction assume that $i < \omega_2$ and $\underset{\sim}{Q}_j$ $(j < i)$ and $\bar{M}_j \in V^{P_{j+1}}$ (for $j < i$) have been defined (so also P_j for $j \leq i$, and $(*)_i$ holds). For each $j < i$, let $P_{j,i} \in V^{P_j}$ be the (P_j-name of the) forcing notion (of power $\aleph_1$) such that $P_i = P_j * \underset{\sim}{P}_{j,i}$ i.e. $P_i/\underset{\sim}{G}_{P_j}$. We assume that $F_i \in V^{P_i}$ is a splitting homomorphism; otherwise we let $\underset{\sim}{Q}_i$ be a P_i-name of the Cohen forcing and $\bar{M}_i$ be any $\aleph_1$-oracle in $V^{P_{i+1}}$ (remember that $\diamondsuit_{\aleph_1}$ holds in the ground model and the P_i's satisfy the $\aleph_1$-c.c. and have power $< \aleph_2$ and Cohen forcing satisfies every $\aleph_1$-oracle). Let $\bar{M}_i^j \in V^{P_i}$ (for $j < i$) be an $\aleph_1$-oracle with the property: if $Q \vDash \bar{M}_i^j$-c.c. in V^{P_i} *then* $P_{j+1,i} * \underset{\sim}{Q} \vDash \bar{M}_j$-c.c. in $V^{P_{j+1}}$ (note that $\bar{M}_i^j$ exists by Claim 3.3., by the induction hypothesis). Now, let $\bar{M}^* \in V^{P_i}$ be an $\aleph_1$-oracle such that $\bar{M}^*$-c.c. implies $\bar{M}_i^j$-c.c. for $j < i$ (existing by Claim 3.1). Apply now the main lemma 4.5 in V^{P_i} (for $h = F_i$) to obtain $P = Q_i \in V^{P_i}$ and $\underset{\sim}{X}_i$ (a Q_i-name so actually a P_{i+1}-name) satisfying the conclusions of 4.5, so by the choice of $\bar{M}^*$, $j < i$ implies in $V^{P_{j+1}}$, the forcing notion $P_{j+1,i+1} = P_{j+1,i} * \underset{\sim}{Q}_i$ satisfies the $\bar{M}_j$-c.c. Finally apply Lemma 2.1 in $V^{P_{i+1}} = V^{P_i * \underset{\sim}{Q}_i}$ for the type appearing in $(\alpha) - (\gamma)$ of 4.5, to obtain an $\aleph_1$-oracle $\bar{M}_i \in V^{P_{i+1}}$ such that the above mentioned type is omitted in any generic extensions of $V^{P_{i+1}}$ with forcing notions satisfying the $\bar{M}_i$-c.c. and such that: if $\underset{\sim}{R}$ is a (P-name of a) forcing notion satisfying the $\bar{M}_i$-c.c. in $V^{P_{i+1}}$ then $Q_i * \underset{\sim}{R}$ satisfies the $\bar{M}^*$-c.c. in V^{P_i}. Now we can check $(*)_{i+1}$.

As P_{ω_2} satisfies the $\aleph_1$-c.c. (by 1.6(1)) and has power $\aleph_2$, there are at most $\aleph_2^{\aleph_0} = \aleph_2$ P_{ω_2}-names of reals; on the other hand we have added reals $\aleph_2$ times, so $V^{P_{\omega_2}} \vDash$"$2^{\aleph_0} = \aleph_2$".

To conclude the proof of 4.3, assume that $F \in V^{P_{\omega_2}}$ is a splitting homomorphism. Then there is some $i < \omega_2$ such that $F \restriction [\mathcal{B}]^{V^{P_i}} = F_i$ is a splitting homomorphism in V^{P_i} (and $\mathrm{cf}(i) = \aleph_1$). So at stage i we had $\underset{\sim}{Q}_i$ introducing $\underset{\sim}{X} \in \mathcal{B}$ and an $\aleph_1$-oracle $\bar{M}_i \in V^{P_{i+1}}$ such that $F(\underset{\sim}{X})$ cannot exist in generic extensions of $V^{P_{i+1}}$ with forcing notions satisfying the $\bar{M}_i$-c.c. Examining our construction and remembering Claim 3.2 it is easy to prove by induction on $\alpha \leq \omega_2$ that for $\alpha > i$ we have $V^{P_{i+1}} \models$ "$P_{i+1,\alpha}$ satisfies the $\bar{M}_i$-c.c.". Hence $V^{P_{\omega_2}} = (V^{P_{i+1}})^{P_{i+1,\omega_2}}$ is such an extension of $V^{P_{i+1}}$, a contradiction. $\square_{4.3}$

4.7. *Proof of the Main Lemma 4.5.* In this proof let Se denote the set of sequences $\bar{a} = \langle a_i : i \leq \omega \rangle$ such that the sequence is monotone, $a_i \neq a_{i+1}$, for $i < \omega, a_i$ is rational, but a_ω is irrational, and $\langle a_i : i < \omega \rangle$ converges to a_ω (and they are from the interval $(0,1)$).

4.7A Definition. Let $P = P(\langle \bar{a}^\alpha : \alpha < \beta \rangle)$ where $\beta \leq \omega_1, \bar{a}^\alpha \in Se$ and a^α_ω for $\alpha < \beta$ are pairwise distinct, denote the following forcing notion: $p \in P$ iff the following three conditions hold:

(a) $p = (U_p, f_p)$, where U_p is an open subset of $(0,1)$, $\mathrm{cl}(U_p)$ is of measure $< 1/2$, and f_p is a function from U_p to $\{0,1\}$;

(b) there are n and b_l, (for $l \leq n$), I_l (for $l < n$) such that $0 = b_0 < b_1 < \cdots < b_{n-1} < b_n = 1$ and $U_p = \cup_{l=0}^{n-1} I_l$, I_l is an open subset of (b_l, b_{l+1}) and even $\mathrm{cl}(I_l) \subseteq (b_l, b_{l+1})$.

(c) I_l is either a rational interval, $f_p \restriction I_l$ constant, *or* for some $\alpha < \beta$ and $k_l < \omega$, $I_l = \cup_{k_l \leq m < \omega}(a^\alpha_{2m}, a^\alpha_{2m+1})$, $f_p \restriction (a^\alpha_{4m+2i}, a^\alpha_{4m+2i+1})$ is constantly i when $k_l \leq 2m$, $m < \omega$, $i \in \{0,1\}$.

The order on P is: $p \leq q$ iff $U_p \subseteq U_q, f_p \subseteq f_q$, and $\mathrm{cl}(U_p) \cap U_q = U_p$.

Finally we let $\underset{\sim}{X}_P = \cup\{f_p^{-1}\{0\} : p \in G_P\}$.

4.7B Stage 1. We will define here a statement, in the next stage prove that it suffices for proving the main lemma, and later prove it.

(St) Let $P_\delta = P(\langle \bar{a}^\alpha : \alpha < \delta \rangle), \delta < \omega_1$ be given, $\delta \geq \omega$ as well as a countable model M^*_δ of ZFC$^-$ such that $P_\delta \in M^*_\delta$, a condition $(p^*, r^*) \in P_\delta \times Q$ and a $(P_\delta \times Q)$-name $\underset{\sim}{\varphi}$ of a definition of a Borel set (this is a candidate for $h(\underset{\sim}{X}_P)$ i.e. the A in 4.5; Q is of course Cohen forcing).

Then we can find $\bar{a}^\delta \in Se$ such that $(\forall \alpha < \delta)[a^\delta_\omega \neq a^\alpha_\omega]$ and letting $P_{\delta+1} = P(\langle \bar{a}^\alpha : \alpha \leq \delta \rangle)$, the following conditions hold:

(A) Every pre-dense subset of P_δ which belongs to M_δ is a pre-dense subset of $P_{\delta+1}$ (note that if M_δ is quite closed this implies the same for $(P_\delta \times Q, P_{\delta+1} \times Q)$, see 2.3(1)).

(B) There is $(p', r') \in P_{\delta+1} \times Q$, such that $(p^*, r^*) \leq (p', r')$ and one of the following holds, for some n:

(B1) $(p', r') \Vdash_{P_{\delta+1} \times Q}$ "$a^\delta_\omega \in \mathrm{Bo}[\underset{\sim}{\varphi}]$, and $\cup_{n<m<\omega}(a^\delta_{4m+2}, a^\delta_{4m+3}) \cap \underset{\sim}{X}_{P_{\delta+1}} = \emptyset$" and "$a^\delta_\omega \in h(\cup_{n<m<\omega}(a^\delta_{4m+2}, a^\delta_{4m+3}))$";

or

(B2) $(p', r') \Vdash_{P_{\delta+1} \times Q}$"$a^\delta_\omega \notin \mathrm{Bo}(\underset{\sim}{\varphi})$ and $\bigcup_{n<m<\omega}(a^\delta_{4m}, a^\delta_{4m+1}) \subseteq \underset{\sim}{X}_{P_{\delta+1}}$" and "$a^\delta_\omega \in h(\bigcup_{n<m<\omega}(a^\delta_{4m}, a^\delta_{4m+1}))$".

4.7C Stage 2. It is enough to prove the statement (St).

We choose M^α and $\bar{a}^\alpha \in Se$ by induction on $\alpha < \omega_1$ such that $\beta < \alpha \Rightarrow a^\alpha_\omega \neq a^\beta_\omega$, and we also choose pre-dense subsets $\mathcal{I}_\gamma$ of $P(\langle \bar{a}^\beta : \beta < \alpha \rangle)$ for $\gamma < \omega\alpha + \omega$ such that: $\gamma < \omega\alpha_1 + \omega$, $\alpha_1 < \alpha_2$ implies $\mathcal{I}_\gamma$ is a pre-dense subset of $P(\langle \bar{a}^\beta : \beta < \alpha_2 \rangle)$ (of course $\alpha_1 < \alpha_2$ implies $P(\langle \bar{a}^\alpha : \alpha < \alpha_1 \rangle) \subseteq P(\langle \bar{a}^\alpha : \alpha < \alpha_2 \rangle)$, moreover $P(\langle \bar{a}^\alpha : \alpha < \alpha_1 \rangle) \subseteq_{\mathrm{ic}} P(\langle \bar{a}^\alpha : \alpha < \alpha_2 \rangle)$, read Definition 4.7A). For $\alpha \leq \omega$ choose any $\bar{a}^\alpha \in Se$ with $\mathcal{I}_\gamma = \{(\emptyset, \emptyset)\}$. Generally make sure $\{\mathcal{I}_{\omega\alpha+n} : n < \omega\}$ include $\{\mathcal{I} : \mathcal{I} \in \bigcup_{\beta \leq \alpha} M_\beta, \mathcal{I}$ a pre-dense subset of $P(\langle \bar{a}^\beta : \beta < \alpha \rangle)\}$. Now in stage $\alpha \geq \omega$ a bookkeeping gives us a $(P_\delta \times Q)$-name $\underset{\sim}{\varphi}_\alpha$ as in (St). Choose M^α including $\bigcup_{\gamma \leq \alpha} M_\gamma$, $\underset{\sim}{\varphi}_\alpha$, $\langle \bar{a}^\beta : \beta < \alpha \rangle$, choose $\bar{a}^\alpha$ as in (St) and choose $\{\mathcal{I}_{\omega\alpha+n} : n < \omega\}$ such that all forcing statement mentioned in clause (B) of (St) continue to hold if we replace $P_{\delta+1}$ by any P such that $P_{\delta+1} \subseteq_{ic} P$ and each $\mathcal{I}_{\omega\alpha+n}$ is pre-dense in P for $n < \omega$ and each predense subset of P_δ which belongs to M_δ appears in $\{\mathcal{I}_{\omega\alpha+n} : n < \omega\}$ (possible by 2.1(A)(2)). Clearly $P = P(\langle \bar{a}^\alpha : \alpha < \omega_1 \rangle)$ satisfies the $\bar{M}$-c.c. For the conclusion of 4.5 let $\underset{\sim}{X} = \underset{\sim}{X}_P$ and $(p^*, r^*) \in P \times Q$ force $\underset{\sim}{A}$ is a counterexample, (so $(\alpha), (\beta), (\gamma)$ there hold) so for some $\delta \in [\omega, \omega_1]$, $\underset{\sim}{A} = \underset{\sim}{\varphi}_\delta$.

Remember that if $(p', r') \Vdash_{P_{\delta+1} \times Q}$ "$a^\delta \in \mathrm{Bo}(\underset{\sim}{\varphi})$" then this remains valid if we replace $P_{\delta+1}$ by any forcing notion $P, P_{\delta+1} \subseteq_{\mathrm{ic}} P$ provided that certain (countably many) maximal antichains of $P_{\delta+1}$ remain maximal antichains of P

which we have guaranteed.
If clause (B1) holds then $B = \bigcup_{n<m<\omega}(a^\delta_{4m+2}, a^\delta_{4m+3})$ contradicts clause (γ) of 4.5 (for $\underset{\sim}{A}$ =Bo($\underset{\sim}{\varphi}$)) and if clause (B2) holds then $B = \bigcup_{n<m<\omega}(a^\delta_{4m}, a^\delta_{4m+1})$ contradicts clause (β) of 4.5 (for $\underset{\sim}{A}$ =Bo($\underset{\sim}{\varphi}$)). We get a contradiction, so P, $\underset{\sim}{X}$ from above proves the main lemma 4.5 (which suffices for proving theorem 4.3).

So from now on we concentrate on the proof of (St).

4.7D Stage 3. Choosing $\bar{a}^\delta$. So let P_δ, $\langle \bar{a}^\alpha : \alpha < \delta \rangle$, $\underset{\sim}{\varphi}$, M^*_δ and (p^*, r^*) be given, as in the assumption of (St), choose λ big enough (i.e. $\lambda = \beth^+_8$), N a countable elementary submodel of $(H(\lambda), \in)$ containing $P_\delta, \langle \bar{a}^\alpha : a < \delta \rangle, \underset{\sim}{\varphi}, M^*_\delta, h$.

Choose a real a^δ_ω, which belongs to $(0,1) \setminus \mathrm{cl}(U_{p^*})$ but does not belong to any Borel set of measure zero which belongs to N. This is possible as by demand (a) in the definition of $P(\langle \bar{a}^\alpha : \alpha < \delta \rangle), \mathrm{cl}(U_{p^*})$ has measure $< 1/2$. So $(0,1) \setminus \mathrm{cl}(U_{p^*})$ has positive measure, whereas the union of all measure zero Borel sets in N is a countable union hence has measure zero. So a^δ_ω is a random real over N and $N[a^\delta_\omega]$ is a model of enough set theory: ZFC$^-$+ "$\mathcal{P}^8(\omega)$ exists" (where $\mathcal{P}(A)$ is the power set, $\mathcal{P}^{n+1}(A) = \mathcal{P}(\mathcal{P}^n(A))$) (all those facts are well known; see e.g. Jech [J]). Clearly $a^\delta_\omega \in h((0, a^\delta_\omega))$ or $a^\delta_\omega \in h((a^\delta_\omega, 1))$, so w.l.o.g. the former occurs. It is also clear that for every $\varepsilon > 0$, $a^\delta_\omega \in h((a^\delta_\omega - \varepsilon, a^\delta_\omega))$. [Otherwise, choose a rational b, $a^\delta_\omega - \varepsilon < b < a^\delta_\omega$, then $a^\delta_\omega \notin h((a^\delta_\omega - \varepsilon, a^\delta_\omega)) \Rightarrow a^\delta_\omega \in h((0,b))$. Hence $a^\delta_\omega \in h((0,b)) \setminus (0,b)$, but this set has measure zero (by the property of h) and obviously belongs to N].

Now let $\langle b_n : n < \omega \rangle \in N[a^\delta_\omega]$ be a strictly increasing sequence of rationals converging to a^δ_ω. Next in $N[a^\delta_\omega]$ we define a forcing notion R (the well known dominating function forcing):

$R = \{(f,g) : f$ a function from some $n < \omega$ to ω, satisfying $(\forall i < n) f(i) > i$, and g is a function from ω to $\omega\}$.

The order is

$(f,g) \le (f',g')$ *iff* $f \subseteq f'$, $(\forall l) g(l) \le g'(l)$ and

$$(\forall i)[i \in \mathrm{Dom}(f') \& i \notin \mathrm{Dom}(f) \Rightarrow g(i) \le f'(i)].$$

Choose a subset G of R (remember that $R \in N[a^\delta_\omega]$) generic over $N[a^\delta_\omega]$, next define a function $f^* = \underset{\sim}{f}^*[G] = \cup\{f : (f,g) \in G\} \in (N[a^\delta_\omega])[G]$. So it is known that $(N[a^\delta_\omega])[G] = (N[a^\delta_\omega])[f^*]$ and a finite change in f^* preserve genericity, i.e. if $f' : \omega \to \omega$, $(\forall i < \omega) f(i) > i$ and $\{n : f^*(n) \neq f'(n)\}$ is finite then for some subset G' of R generic over $N[a^\delta_\omega]$ we have $f' = \underset{\sim}{f}[G']$ and $(N[a^\delta_\omega])[G] = (N[a^\delta_\omega])[f^*] = (N[a^\delta_\omega])[f'] = (N[a^\delta_\omega])[G'])$. We shall work for a while with the model $N[a^\delta_\omega][f^*]$. We define (in this model) a sequence of natural numbers $\langle n(l) : l < \omega\rangle$, defining $n(l)$ by induction on l. Let $n(0) = 0$, and $n(l+1) = f^*(n(l))$. Now we define for $m < 4$ and $k < \omega$ a set A^k_m, $A^k_m = \bigcup_{k \le l < \omega} (b_{n(4l+m)}, b_{n(4l+m+1)})$.

So $A^0_m (m = 0,1,2,3)$ is a partition of (b_0, a^δ_ω) modulo measure zero as $\{b_n : n < \omega\}$ has measure zero, but remember $a^\delta_\omega \in h((b_0, a^\delta_\omega))$ hence for some unique $m(*), a^\delta_\omega \in h(A^0_{m(*)})$. Note that $A^k_m \in N[a^\delta_\omega][f^*]$, but $h \restriction N[a^\delta_\omega][f^*]$ does not necessarily belong to this model, so we determine $m(*)$ in V. As we could have made a finite change in f^* (replacing $f^*(0)$ by $f^*(n(m(*)))$, legal as $f^*(n(m(*))) > n(m(*)) > m(*) \ge 0$) we can assume $a^\delta_\omega \in h(A^0_0)$.

As $a^\delta_\omega \in h((a^\delta_\omega - \varepsilon, a^\delta_\omega))$ for every ε, and as h is a homomorphism, $a^\delta_\omega \in h(A^k_0)$ for every k.

First let us try to choose $\bar{a}^\delta = \langle b_{n(l)} : l < \omega\rangle \hat{}\langle a^\delta_\omega\rangle$.

4.7E Stage 4. Condition (A) of (St) holds.

This means:

4.7E1 Subclaim. *Every pre-dense subset $\mathcal{J}$ of P_δ which belongs to M_δ is pre-dense in $P_{\delta+1}$.*

Proof of the Subclaim. As $M_\delta \in N$ clearly $\mathcal{J} \in N$. Let $p \in P_{\delta+1}$, $p \notin P_\delta$, so by the definition of $P_{\delta+1}$ there are $q \in P_\delta$ and rational numbers c_0, c_1 and a natural number $l(0)$ such that

$$0 < c_0 < a^\delta_\omega < c_1 < 1, \quad c_0 < b_{n(4l(0))}, \quad c_0 > b_{n(4l(0))-1},$$
$$\mathrm{cl}(U_q) \cap [c_0, c_1] = \emptyset, \quad U_p = U_q \cup A^{l(0)}_0 \cup A^{l(0)}_2, \quad f_p = f_q \cup 0_{A^{l(0)}_0} \cup 1_{A^{l(0)}_2}.$$

(0_A is the function with domain A which has constant value 0; similarly 1_A.)

Before we continue we prove:

4.7E2 Fact. Let $r \in P_\delta, \mathcal{J} \subseteq P_\delta$ be dense, $(c_0, c_1) \subseteq (0,1)$ be an open interval disjoint from U_r. Then

$$C = \{x \in (c_0, c_1) : \text{ there is } r_1 \in \mathcal{J} \text{ such that } r_1 \geq r \text{ and } x \notin \mathrm{cl}(U_{r_1}\}$$

has measure $|c_1 - c_0|$ (subtraction as reals).

Proof of the Fact. The condition is equivalent to "$(c_0, c_1) \setminus C$ has measure zero", so we can partition (c_0, c_1) into finitely many intervals and prove the conclusion for each of them. So w.l.o.g. the measure of (c_0, c_1) is $< 1/2$. Now for every $\varepsilon > 0$ we can find r_0, $r \leq r_0 \in P_\delta$ such that $U_{r_0} \cap (c_0, c_1) = \emptyset$ and U_{r_0} has measure $\geq 1/2 - \varepsilon$ (but of course $< 1/2$). As $\mathcal{J} \subseteq P_\delta$ is dense, there is $r_2 \in \mathcal{J}, r_0 \leq r_2 \in P_\delta$. So (ignoring the sets $\mathrm{cl}(U_{r_l}) \setminus U_{r_l}$, $l = 0, 2$ which have measure zero):

(i) $(c_0, c_1) \setminus U_{r_2} \subseteq C$ (by C's definition);

(ii) $(c_0, c_1) \cap U_{r_2} \subseteq U_{r_2} \setminus U_{r_0}$.

Hence

(iii) $\mathrm{Leb}((c_0, c_1) \setminus C) \leq \mathrm{Leb}((c_0, c_1) \cap U_{r_2}) \leq \mathrm{Leb}(U_{r_2} \setminus U_{r_0}) \leq \mathrm{Leb}(U_{r_2}) - \mathrm{Leb}(U_{r_0})$
$\leq 1/2 - (1/2 - \varepsilon) = \varepsilon$.

As this holds for every ε we finished the proof of the fact. $\square_{4.7E2}$

4.7E3. *Continuation of the proof of the subclaim.* Let $\mathcal{J}_1 = \{r \in P_\delta : (\exists q_1 \in \mathcal{J})(q_1 \leq r)\}$, so $\mathcal{J}_1$ is a dense open subset of P_δ and $\mathcal{J}_1 \in N$. Now for every $k \geq n(4l(0))$ let

$T_k = \{t : t \in P_\delta, U_t$ is the union of finitely many intervals whose endpoints are from $\{b_l : n(4l(0)) \leq l < k\}$ and $\mathrm{Leb}(U_q \cup U_t) < 1/2\}$.

So T_k is finite, and for every $t \in T_k$, $q \leq q \cup t \in P_\delta$, and $a^\delta_\omega \notin \mathrm{cl}(U_t)$ (where $q \cup t = (U_q \cup U_t,\ f_q \cup f_t)$, of course; on q see beginning of the proof of 4.7E).

Now in the model N (as $M_\delta \in N$ hence $\mathcal{J}_1 \in N$) we can define, for each k and $t \in T_k$:

$D_t \stackrel{\text{def}}{=} \{x \in (c_0, c_1) : \text{there is } r \in \mathcal{J}_1, r \geq q \cup t, x \notin \mathrm{cl}(U_r)\}$.

By the fact 4.7E2 we have proved, we know that $(c_0, c_1) \setminus D_t \setminus U_t$ has measure zero (note, $\mathrm{cl}(U_t) \setminus U_t$ has measure zero). As $a^\delta_\omega \in (0,1) \setminus U_t$ and a^δ_ω does not belong to any Borel set of measure zero which belongs to N, clearly,

$$(*) \qquad \text{for every } k \geq n(4l(0)) \text{ and } t \in T_k \text{ we have } a^\delta_\omega \in D_t.$$

So for each $k \geq n(4l(0))$ and $t \in T_k$, there is $r_t \in P_\delta, r_t \in \mathcal{J}_1$, $t \cup q \leq r_t$ such that $a^\delta_\omega \notin \mathrm{cl}(U_{r_t})$. Hence for some $g(t) < \omega$, $[b_{g(t)}, a^\delta_\omega] \cap \mathrm{cl}(U_{r_t}) = \emptyset$ and $(a^\delta_\omega - b_{g(t)}) < 1/2 - \mathrm{Leb}(U_{r_t})$. As T_k is finite, we can define a function $g : \omega \to \omega$, by $g(k) = \max\{g(t) : t \in T_k\}$. Clearly $g \in N[a^\delta_\omega]$, because, in defining g we have used $\mathcal{J}_1$, $n(4l(0))$, $\langle b_l : l < \omega \rangle$, a^δ_ω, but not $\langle b_{n(l)} : l < \omega \rangle$, $\langle n(l) : i < \omega \rangle$ or f^*. Hence for every large enough $l, g(l) < f^*(l)$, because f^* dominates $N[a^\delta_\omega] \cap {}^\omega\omega$. So by the choice of the $n(l)$'s, for every large enough $l, g(n(l)) < n(l+1)$. Choose a large enough l, let $k = n(4l)+1, U_t = U_p \restriction [b_{n(4l(0))}, b_k], f_t = f_p \restriction U_t, t = (U_t, f_t)$ and it belongs to T_k. Now r_t and p are compatible, (the "$(a^\delta_\omega - b_{g(t)}) < 1/2 - \mathrm{Leb}(U_{r_t})$" guarantees that $\mathrm{Leb}(U_p \cup U_{r_t}) < 1/2$: other parts should be clear too). So we finish the proof of the subclaim 4.7E1. $\square_{4.7E1}$

This really proves part (A) of (St).

4.7F Stage 5. Condition (B) of (St). Remember $\underset{\sim}{\varphi}$ is a $P_\delta \times Q$-name of a (code of a) Borel set (and it belongs to N, hence the $\aleph_0$ maximal antichains involved in its definition are, by 4.7E1 maximal antichains of $P_{\delta+1} \times Q$), so it is also a $(P_{\delta+1} \times Q)$-name. Pick a large enough k such that:

$p^*_1 = (U_{p^*} \cup A^k_0 \cup A^k_2, f_{p^*} \cup 0_{A^k_0} \cup 1_{A^k_2})$ belongs to $P_{\delta+1}$.

So $p^*_1 \in N[a^\delta_\omega][f^*]$ and $(p^*_1, q^*) \geq (p^*, q^*)$ in $P_{\delta+1} \times Q$, so there is $(p', r') \geq (p^*_1, q^*)$ forcing an answer to "$a^\delta_\omega \in \mathrm{Bo}[\underset{\sim}{\varphi}]$", i.e. $(p', r') \Vdash_{P_{\delta+1} \times Q}$ "$a^\delta_\omega \in \mathrm{Bo}[\underset{\sim}{\varphi}]$" or $(p', r') \Vdash_{P_{\delta+1} \times Q}$ "$a^\delta_\omega \notin \mathrm{Bo}[\underset{\sim}{\varphi}]$". If the second possibility holds, then (B2) holds, so condition (B) of (St) holds (remember we have made $a^\delta_\omega \in h(A^k_0)$ for every k in stage 3).

So suppose $(p', r') \Vdash_{P_{\delta+1} \times Q}$ "$\underset{\sim}{a}^{\delta}_{\omega} \in \text{Bo}[\underset{\sim}{\varphi}]$". Observe that the truth value of such a statement can be computed in $N[a^{\delta}_{\omega}][f^*]$ (i.e., we get the same result in the universe and in this countable model). But f^* is R-generic over $N[a^{\delta}_{\omega}]$. So if something holds, then there is $(f_0, g_0) \in G$ is such that:

(α) $(f_0, g_0) \Vdash_R$ "$(p', r') \Vdash_{P_{\delta+1} \times Q}$ "$\underset{\sim}{a}^{\delta}_{\omega} \in \text{Bo}[\underset{\sim}{\varphi}]$" ",

(β) $f_0 \subseteq f^*, (\forall i < \omega)[i \notin \text{Dom}(f_0) \Rightarrow g_0(i) \leq f^*(i)]$.

(Note that the definition of $P_{\delta+1}$ depends on f^*.) So if we change f^* in finitely many places, maintaining (β), it will still be true that $(p', r') \Vdash_{P_{\delta+1} \times Q}$ "$\underset{\sim}{a}^{\delta}_{\omega} \in \text{Bo}[\underset{\sim}{\varphi}]$". But we can do it in such a way that only finitely many $n(l)$'s are changed and for some n the old A^k_0 becomes A^n_2, and the old A^k_2 becomes A^{n+1}_0, so now clause (B1) of (St) will hold.

In any case (B) of (St) holds, hence we have finished proving (St) hence by 4.7C we have finished the proof of the main lemma and by 4.6 we have finished the proof of the theorem. $\square_{4.5}$

$\square_{4.3}$

§5. Automorphisms of $\mathcal{P}(\omega)$/finite

General topologists have been interested in what $AUT(\mathcal{P}(\omega)/\text{finite})$ can be. ($\mathcal{P}(\omega)$ is viewed as a Boolean algebra of sets, $\mathcal{P}(\omega)$/finite is the quotient when dividing by the ideal of finite subsets of ω). Or equivalently: what can be the group of autohomeomorphisms of $\beta(\mathbb{N}) \setminus \mathbb{N}$?

Note that $AUT(\mathcal{P}(\omega))$ is isomorphic to the set of permutations of ω ($Per(\omega)$). It is well known that:

5.1 Claim. (CH) $\mathcal{P}(\omega)$/finite is a saturated atomless BA (= Boolean algebra) of power $\aleph_1$, so it has $2^{\aleph_1}$ automorphisms (saturated in the model theoretic sense). $\square_{5.1}$

5.1A Remark. This claim will not be used here.

It is also easy to note that:

(*) If $f \in Per(\omega)$ the set of permutations of ω, then $A/\text{finite} \to f(A)/\text{finite}$ is an automorphism of $\mathcal{P}(\omega)/\text{finite}$.

Moreover:

5.2 Claim. If f is a one-to-one function, $\mathrm{Dom}(f), \mathrm{Rang}(f)$ are cofinite subsets of ω then $A/\text{finite} \to f(A)/\text{finite}$ is an automorphism of $\mathcal{P}(\omega)/\text{finite}$. (Such an f is called an almost permutation of ω, and the induced automorphism is called trivial, and we say it is induced by f). $\square_{5.2}$

5.3 Claim. If $V = L$, or just $V \models$ "$2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$ and $\Diamond_{S^2_1}$ holds where $S^2_1 = \{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_1\}$" *then* we can define for $i \in S^2_1$, P^*_i, F_i such that:

(a) P^*_i is a c.c.c. forcing of cardinality $\aleph_1$ and for simplicity the set of elements of P^*_i is i.

(b) $\underset{\sim}{F}_i$ is a P^*_i-name of a subset of $[\mathcal{P}(\omega) \times \mathcal{P}(\omega)]^{V^{P^*_i}}$

(c) if $\langle P_i : i < \omega_2 \rangle$ is $\lessdot$-increasing continuous sequence of c.c.c. forcing notions of cardinality $\aleph_1$, the set of elements of P_i is $\subseteq \omega_2$ and $P_{\omega_2} = \bigcup_{i<\omega_2} P_i$, $\underset{\sim}{F}$ is a P_{ω_2}-name and $\underset{\sim}{F} \subseteq [\mathcal{P}(\omega) \times \mathcal{P}(\omega)]^{V^{P_{\omega_2}}}$, i.e. this is forced *then* the set $\{i : S^2_1 : P^*_i = P^*_i \text{ and } \underset{\sim}{F} \restriction [\mathcal{P}(\omega)]^{V^{P_i}} = \underset{\sim}{F}_i\}$ is stationary in ω_2.

Proof. Translate everything to P_{ω_2}-names and apply $\Diamond_{\{\delta<\omega_2:\mathrm{cf}(\delta)=\aleph_1\}}$ (remember the $\aleph_1$-chain condition). $\square_{5.3}$

5.4 Claim.

Suppose P_i $(i \le \omega_2)$ are as in Claim 5.3, $F \in [AUT(\mathcal{P}(\omega)/\text{finite})]^{V^{P_{\omega_2}}}$ and

$\{i < \omega_2 : \mathrm{cf}(i) = \aleph_1$ and $F \restriction [\mathcal{P}(\omega)/\text{finite}]^{V^{P_i}}$ is

(i) not in V^{P_i} or

(ii) not an automorphism of $[\mathcal{P}(\omega)/\text{finite}]^{V^{P_i}}$ or

(iii) induced by some almost permutation f of ω

($f \in V^{P_i}$ i.e. a P_i-name, of course)$\}$

is stationary.

Then F is trivial.

Proof. Since the sets defined by (i) and (ii) are not stationary,

$$S = \{i < \omega_2 : \mathrm{cf}(i) = \aleph_1, F{\restriction}[\mathcal{P}(\omega)/\text{finite}]^{V^{P_i}} \text{ is in } V^{P_i},$$
$$\text{and is a trivial automorphism }\}$$

is stationary.

For $i \in S$, let $f_i \in V^{P_i}$ be an almost permutation of ω inducing $F{\restriction}[P(\omega)/\text{finite}]^{V^{P_i}}$. By a suitable version of Fodor's lemma we can find $f^* \in V^{P_{\omega_2}}$, $S^* \subseteq S$ stationary, such that $(\forall i \in S^*)$ $[f_i = f^*]$. Clearly f^* induces $F{\restriction}[P(\omega)/\text{finite}]^{V^{P_{\omega_2}}}$. $\square_{5.4}$

We want to show:

5.5 Main Theorem. CON(ZFC) implies CON(ZFC $+2^{\aleph_0} = \aleph_2+$ every automorphism of $\mathcal{P}(\omega)$/finite is trivial). In fact, if $\diamondsuit_{\aleph_1}$ and $\diamondsuit_{S^2_1}$, where $S^2_1 = \{\delta < \omega_2 : \mathrm{cf}(\delta) = \aleph_1\}$, then for some c.c.c. forcing notion P of cardinality $\aleph_2$, the conclusion holds in V^P.

We postpone the work concerning specifically automorphisms of $\mathcal{P}(\omega)$/finite to the next section. To be precise we will prove there the following.

5.6 Main Lemma. Suppose

$V \models$ "$2^{\aleph_0} = \aleph_1$, $\bar{M}^*$ is an $\aleph_1$-oracle, $F \in AUT(\mathcal{P}(\omega)/\text{finite})$ is not trivial".

Then there is a forcing notion P such that

1) $|P| = \aleph_1$,
2) P satisfies the $\bar{M}^*$-c.c.
3) P introduces a subset X of ω, such that if Q is the Cohen forcing, then in $V^{P*\underset{\sim}{Q}}$ there is no $Y \subseteq \omega$ satisfying $\{Y \cap F(C) =_{ae} F(B) \Leftrightarrow X \cap C =_{ae} B : C, B \in [\mathcal{P}(\omega)]^V\}$ (where $A =_{ae} A^\dagger$ means $A \subseteq_{ae} A^\dagger$ and $A^\dagger \subseteq_{ae} A$, where $A \subseteq_{ae} A^\dagger$ iff $A \setminus A^\dagger$ is finite) or even just satisfying $\{Y \cap F(C) =_{ae} F(B) : X \cap C =_{ae} B \text{ where } B, C \in [\mathcal{P}(\omega)]^V\}$.

5.6A Remark. Note we really replace F by some $F^\dagger$ from $\mathcal{P}(\omega)$ to $\mathcal{P}(\omega)$ such that $F(A/\text{finite}) = F^\dagger(A)/\text{finite}$; but we do not distinguish.

Proof of 5.5 (assuming 5.6). The proof of 5.5 from 5.6 is very similar to the proof of 4.6 (just replace "splitting homomorphism" by "nontrivial automorphism". Note that 5.6(3) corresponds to 4.5 (α)–(γ)). $\square_{5.5}$

5.7 Claim. In our scheme (of oracle c.c., as used in 4.6) we can demand:

1) for any second category $A \subseteq [\mathcal{P}(\omega)]^{V^{P_{\omega_2}}}$ we have $\{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \aleph_1 \Rightarrow A \cap [\mathcal{P}(\omega)]^{V^{P_i}} \in V^{P_i}$ and is of 2nd category$\}$ contains a closed unbounded set.

2) if $A \subseteq [\mathcal{P}(\omega)]^{V^{P_i}}$, $A \in V^{P_i}$, A is of 2nd category, *then* there is an oracle $\bar{M}$ (in V^{P_i}) such that for any forcing notion $P \in V^{P_i}$ satisfying the $\bar{M}$-c.c., in $(V^{P_i})^P$ the set A remains of the 2nd category.

Proof. For 1) It follows by 2). For 2) see 2.2.

5.8 Conclusion. 1) We can add in 5.5 that in V^P any second category set contains such a subset of cardinality $\aleph_1$.

2) We can find $\aleph_1$-dense sets of reals, some of the first category, some of the second category.

3) Baumgartner's construction [B4] cannot be carried out here as his conclusion fails.

Proof. 1), 2) Left to the reader.

3) Baumgartner [B4] proved the consistency of: if for $\ell = 1, 2$, $A_\ell \subseteq \mathbb{R}$, $|A_\ell \cap (c, d)| = \aleph_1$ for any reals $c < d$ then $(A_1, <) \cong (A_2, <)$; but this fails in V^P by part (2). His proof was: for given A_1, A_2 in a universe satisfying CH, builds a c.c.c. forcing notion P, such that $\Vdash_P$ "$(A_1, <) \cong (A_2, <)$". So for an $\aleph_1$-oracle $\bar{M}$, and A_1, A_2 from V (where $V \models \diamondsuit_{\aleph_1}$) there is no such P satisfying the $\bar{M}$-c.c.

§6. Proof of Main Lemma 5.6

Since the proof is quite long it will be divided into stages.

Stage A. Preliminaries. For a sequence $\bar{A} = \langle A_i : i < \alpha \rangle$ $(\alpha \leq \omega_1)$ of almost disjoint infinite subsets of ω, define

$$P(\bar{A}) = \{f : f \text{ is a partial function from } \omega \text{ to } 2 = \{0,1\},$$
$$f \text{ is a finite union of functions of the form:}$$
$$(1) \text{ finite functions,}$$
$$(2)\ 1_{A_i} \setminus \text{ finite,}$$
$$(3)\ 0_{A_i} \setminus \text{ finite}\}$$

where ℓ_A $(l \in \{0,1\})$ is the function $f(\alpha) = \ell$ if $\alpha \in A$, undefined otherwise, and $f\backslash$finite means $f{\restriction}(\mathrm{Dom}(f) \setminus A)$ for some finite A.

The idea of the proof is as follows: we try to define $\bar{A} = \langle A_i : i < \omega_1 \rangle$ such that: $P(\bar{A})$ will satisfy the requirements of the lemma for X whose characteristic function is $\cup\{p : p \in G_{P(\bar{A})}\}$. We will have a general pattern for defining $\bar{A}$. Our work will be to show that if all instances of this pattern fail then F must be trivial.

Our first worry is how to make $P(\bar{A})$ satisfy the $\bar{M}^*$-c.c. The following $(*1)$ takes care of this while leaving us with a sufficient degree of freedom in the construction of $\bar{A}$.

$(*1)$ If $\bar{A} = \langle A_j : j < \alpha \rangle$, $\alpha < \omega_1$, M_α is a countable collection of pre-dense subsets of $P(\bar{A})$, $B \subseteq \omega$ is almost disjoint from each $A_j (j < \alpha)$ and is infinite, *then* we can find disjoint, infinite C_1, C_2, $C_1 \cup C_2 = B$ such that for every $\ell \in \{1,2\}$ and every $A \subseteq C_\ell$, if $|A| = |C_\ell \setminus A| = \aleph_0$, then every $\mathcal{I} \in M_\alpha$ is pre-dense in $P(\bar{A}\,\hat{}\,\langle A, C_\ell \setminus A \rangle)$.

Remark. Note that by 2.3, the same statement holds if we replace $P(\bar{A})$ by $P(\bar{A}) \times Q$, where Q denotes the Cohen forcing.

Proof of $(*1)$. We define by induction on $n < \omega$ a number $k_n < \omega$ and disjoint finite subsets C^1_n, C^2_n of B such that $C^1_n \cup C^2_n = B \cap k_n$ and

$$m < n \Rightarrow C^1_m = C^1_n \cap k_n \ \& \ C^2_m = C^2_n \cap k_n.$$

In the end we will let $C_l = \bigcup_{n<\omega} C^l_n$.

In each stage we deal with an obligation of the following kind:

$\mathcal{I} \in M_\alpha$ is a pre-dense subset of $P(\bar{A})$, τ is a term for a member of $P(\bar{A}\,\hat{}\,\langle A, C_1 \setminus A\rangle)$ (or C_2 in the other case); note: A, C_1, are not known yet, still $A \subseteq C_1$; our task is to ensure τ will be compatible with some member of $\mathcal{I}$ no matter how A and C_1 are defined as long as $C_1 \subseteq B$, $C^1_n \subseteq C_1$ and $C^2_n \cap C_1 = \emptyset$.

τ has the form $f \cup i_{A\setminus k} \cup j_{C_1\setminus A\setminus k}$ where $i, j \in \{0,1\}$, $k < \omega$, $f \in P(\bar{A})$ (if this union is not a function, then we don't need to do anything.) We don't know what $A \cap C^1_n$ is. There are $2^{|C^1_n|}$ subsets of C^1_n : $A_1, \ldots, A_\ell, \ldots$, for $1 \le \ell \le 2^{|C^1_n|}$; look at $f_\ell \stackrel{\text{def}}{=} f \cup i_{A_\ell\setminus k} \cup j_{C^1_n\setminus A_\ell\setminus k}$. Suppose this is a function, then it belongs to $P(\bar{A})$, hence is compatible with some $g_\ell \in \mathcal{I}$. Now $\mathrm{Dom}(g_\ell) \cap B$ is finite (for B being almost disjoint from every A_j and $g_\ell \in \mathcal{I} \subseteq P(\bar{A})$).

Let : $k_{n,\ell} = \mathrm{Max}(\mathrm{Dom}(g_\ell) \cap B)$ for $1 \le \ell \le 2^{|C^1_n|}$, and $k_{n,0} = k_n$;

$$k_{n+1} \stackrel{\text{def}}{=} \mathrm{Max}\{(k_{n,\ell}+1) : 0 \le \ell \le 2^{|C^1_n|}\};$$

$$C^1_{n+1} = C^1_n;$$

$$C^2_{n+1} = C^2_n \cup [(k_{n+1} \setminus C^1_n) \cap B].$$

Taking additional care to make $C_1 = \cup_{n<\omega} C^1_n$ and $C_2 = \cup_{n<\omega} C^2_n$ infinite, we obtain what we need. $\square_{(*1)}$

In view of $(*1)$, it is convenient to introduce the following notation:

Q will denote Cohen forcing. For $\alpha \le \omega_1$, P_α will denote the forcing notion corresponding to $\bar{A} = \langle A_j : j < \alpha\rangle$ (we delete α from our further notation when α is fixed).

$P_\ell[A]$ will denote $P(\bar{A}\,\hat{}\,\langle A, C_\ell \setminus A\rangle)$ for $\ell \in \{1,2\}$, $A \subseteq C_\ell$ as in $(*1)$.

$P^Q_\ell[A]$ will denote $P_\ell[A] \times Q$.

If $p, q \in P_{\omega_1} \times Q$ then we let $p = ((p)_0, (p)_1)$ and $q = ((q)_0, (q)_1)$, we will write $p \cup q$ for $((p)_0 \cup (q)_0, (p)_1 \cup (q)_1)$. For $p \in P_{\omega_1}$, $q \in P_{\omega_1} \times Q$, $p \cup q$ means $(p \cup (q)_0, (q)_1)$, etc. We identify P_{ω_1} with $P_{\omega_1} \times \{\emptyset\} \subseteq P_{\omega_1} \times Q$.

For $A \subseteq B$, Ch^A_B will denote $1_A \cup 0_{B \setminus A}$.

Stage B: The outline of the proof. We say that F is trivial on $A \subseteq \omega$ iff there are $A' \equiv_{\mathrm{ae}} A$ and $f : A' \overset{1-1}{\to} \omega$ such that $F(B) \equiv_{\mathrm{ae}} f(B)$ for each $B \subseteq A'$. Let $J = \{A \subseteq \omega : F \text{ is trivial on } A\}$.

We should distinguish two cases:

Case 1: J is dense, i.e. for every $A \in [\omega]^{\aleph_0}$ we have $J \cap [A]^{\aleph_0} \neq \emptyset$

Case 2: there is $A^+ \in [\omega]^{\aleph_0}$ such that $[A^+]^{\aleph_0} \cap J = \emptyset$.

We will construct the sequence $\bar{A} = \langle A_\alpha : \alpha < \omega_1 \rangle$ by induction. During the induction we will require $\oplus_1$, where

$\oplus_1$ $A_\alpha \in J$ in Case 1 for each $\alpha < \omega_1$ and $\omega \setminus A_0 \subseteq A^+$ in Case 2.

Note that in Case 2 $A_\alpha \subseteq A^+$ guarantees $A_\alpha \notin J$ for $0 < \alpha < \omega_1$.

In stage C below we shall describe the inductive construction of $\langle A_\alpha : \alpha < \omega_1 \rangle$ and $\langle M_\alpha : \alpha < \omega_1 \rangle$ and we show that either $P(\langle A_\alpha : \alpha < \omega_1 \rangle)$ satisfies our requirements or in some step α the statement $(*2)$ defined in stage C holds.

In stage D we shall show that if $(*2)$ holds, then F is trivial on any B provided that B is almost disjoint from $\{A_i : i < \alpha\}$, so $B \in J$, hence Case 1 holds.

In stage E we shall construct a function g which generates F on ω, which proves that F is trivial. This concludes the proof of the main Lemma.

Stage C: Note that $(*1)$ enables us to construct $\bar{A}$ as follows: We let $\langle A_n : n < \omega \rangle$ be any pairwise disjoint infinite subsets of ω satisfying $\oplus_1$ from Stage B. At each later stage, we can use $(*1)$ for any infinite B which is almost disjoint from each earlier set (we can always find such B: arrange the earlier sets in an ω-sequence $\langle B_n : n < \omega \rangle$, and let $B = \{b_n : n < \omega\}$ where $b_n \in B_n \setminus \cup_{m<n} B_m$). Then we choose by induction on $\alpha \in [\omega, \omega_1)$ the set A_α.

What about M_α? It includes M^*_α (in order to prepare the satisfaction of the $\bar{M}^*$-c.c., $\bar{M}^*$ is given in the assumption of 5.6), as well as additional pre-dense subsets which will be specified in the sequel, when their necessity becomes apparent; and of course M_α increases with α and is a model of ZFC$^-$. Sometimes, these will be subsets of $P_\alpha \times Q$ which are to remain pre-dense in $P^Q_\ell[A]$, but we can deal with such subsets in the framework of $(*1)$ without difficulty by 2.3(1). Also, we will preserve pre-density above a given condition, but all these pre-dense sets are in M_α provided M_α is closed enough.

Altogether our induction hypothesis on $\alpha \in [\omega, \omega_1)$ are $\otimes_1$ and

$\oplus_2$ $\langle A_\beta : \beta < \alpha \rangle$ is a sequence of infinite, pairwise almost disjoint subsets of ω and $P_\alpha \stackrel{\text{def}}{=} P(\langle A_i : i < \alpha \rangle)$ (and P_{ω_1} will be $P(\langle A_i : i < \omega_1 \rangle)$).

$\oplus_3$ N_α is a countable set of pre-dense subsets of $P(\langle A_\beta : \beta < \alpha \rangle)$ increasing (but not continuous) in α, $M^*_\alpha \subseteq M_\alpha$, and N_α is equal to $M_\alpha \cap \{\mathcal{I} : \mathcal{I}$ a pre-dense subset of $P(\langle A_\beta : \beta < \alpha \rangle)\}$ where M_α is the Skolem hull of N_α in $(H(\beth_8^+), \in, <^*_{\beth_8^+})$.

Now, in using $(*1)$ we can choose between various candidates for B, and for each of them we can choose between various continuations of $\bar{A}$ (namely, we can choose $\ell \in \{1, 2\}$ and $A \subseteq C_\ell$). How do we choose?

In the verification of 3) of the lemma, we have to consider all possible $\underset{\sim}{Y}$ which are $P_{\omega_1} \times Q$-names of subsets of ω. By CH and the $\aleph_1$-c.c., we can find $P_\alpha \times Q$-names $\underset{\sim}{Y}_\alpha$, which are determined schematically before the actual construction of $\bar{A}$, so that the sequence

$$\langle \underset{\sim}{Y}_\alpha : \omega \leq \alpha < \omega_1, \alpha \text{ an even ordinal } \rangle$$

will contain all possible $\underset{\sim}{Y}$, each appearing $\aleph_1$-times. [More explicitly, we can identify a $P_{\omega_1} \times Q$-name of a $\underset{\sim}{Y} \subseteq \omega$ with $\langle q_{i,j}, \mathbf{t}_{i,j} : i, j < \omega \rangle$, $q_{i,j} \in P_{\omega_1} \times Q$, $\mathbf{t}_{i,j}$ a truth value, for each $i < \omega$ the sequence $\langle q_{i,j} : j < \omega \rangle$ is a maximal antichain, $q_{i,j} \Vdash$ "$[i \in \underset{\sim}{Y}] \equiv \mathbf{t}_{i,j}$". We consider $\underset{\sim}{Y}_\alpha$ only if M_α already contains each $\langle q_{i,j} : j < \omega \rangle$, so they remain maximal antichains. Of course we do not know apriori P_{ω_1} but we can make a list of length ω_1 including all possible $P_i (i < \omega_1)$ and $\underset{\sim}{Y}_\alpha$. As there are $\leq \aleph_1$ candidates, and each candidate appears $\aleph_1$ times everything is clear]. Our choice of B (hence C_1, C_2 by $(*1)$), ℓ, A will be as follows: whenever possible, we choose them so that $\oplus_1$ from Stage B holds and

$$[Ch^A_{C_\ell} \Vdash_{P^Q_\ell[A]} \text{“}\underset{\sim}{Y}_\alpha \cap F(C_\ell) =_{ae} F(A)\text{”}] \text{ is false .}$$

Assume now that P_{ω_1} was obtained by such a procedure fails to satisfy the requirements of the lemma. As 5.6(1) is clearly satisfied and we ensured the satisfaction of 5.6(2), only 5.6(3) may be false. We will show that this is exemplified by $\underset{\sim}{X} = \cup\{f^{-1}(\{1\}) : f \in \underset{\sim}{G}\}$ where $\underset{\sim}{G} \subseteq P_{\omega_1}$ is the name of the generic subset of P_{ω_1}. If the existence of an appropriate Y is not forced already by $(\emptyset, \emptyset)$ the minimal condition in $P_{\omega_1} \times Q$, then some $r \in P_{\omega_1} \times Q$ forces that there is such Y and in this case $P_{\omega_1} \restriction \{p : p \geq (r)_0\}$ satisfies the requirements of the lemma (as the M_α's take care of the satisfaction of the $\bar{M}^*$-c.c. for such restrictions of P_{ω_1} as well). Hence we may assume that $(\emptyset, \emptyset)$ forces the existence of such $\underset{\sim}{Y}$. So, there is $\underset{\sim}{Y} = \underset{\sim}{Y}_\alpha(\omega \leq \alpha < \omega_1, \alpha \text{ even})$ which realizes the relevant type, and w.l.o.g. α is large relative to $\underset{\sim}{Y}$ (remember each $\underset{\sim}{Y}$ appears $\aleph_1$ times.) Since $Ch^A_{C_\ell} \Vdash_{P_{\omega_1} \times Q}$ “$\underset{\sim}{X} \cap C_\ell = A$” (referring to the C_ℓ and A with which we actually choose at stage α), it follows that $Ch^A_{C_\ell} \Vdash_{P_{\omega_1} \times Q}$ “$\underset{\sim}{Y} \cap F(C_\ell) =_{ae} F(A)$”. ($F$ from the assumption of 5.6.)

Now, when we are at stage α we know $\underset{\sim}{Y}_\alpha, P_\alpha$ but not P_{ω_1}. So we want to show that this is already true if we consider forcing in $P^Q_\ell[A]$. As the forcing relation for each formula $n \in \underset{\sim}{Y}$ is not affected by this shift, our only problem arises from the fact that equality is forced to hold only *ae*. This difficulty is overcome as follows: if, on the contrary, there is $p \geq Ch^A_{C_\ell}$ in $P^Q_\ell[A]$ forcing “$\underset{\sim}{Y} \cap F(C_\ell) \neq_{ae} F(A)$”, then for each $k < \omega$ the set of those conditions “knowing” some counterexample to the equality above k is pre-dense above p in $P^Q_\ell[A]$; so, as we remarked above, we can assume this pre-density is still true in $P_{\omega_1} \times Q$, but (as $p \geq Ch^A_{C_\ell}$) $p \Vdash_{P_{\omega_1} \times Q}$ “$\underset{\sim}{Y} \cap F(C_\ell) =_{ae} F(A)$”, a contradiction (i.e. we can discard this case instead taking care of it).

So, $Ch^A_{C_\ell} \Vdash_{P^Q_\ell[A]}$ “$\underset{\sim}{Y} \cap F(C_\ell) =_{ae} F(A)$”. But we chose B, ℓ and A doing our best to avoid this, so we must have had no better choice; thus:

Fact. If P_{ω_1} does not satisfy our requirements *then* there is $\alpha < \omega_1$, $\alpha \geq \omega$ such that $(*2)$ below holds:

(*2) Assume $A_j (j < \alpha)$, $M_i (i < \alpha)$ were determined as well as a P_α-name $\underset{\sim}{Y} = \underset{\sim}{Y}_\alpha$ of a subset of ω. Let B be any infinite subset of ω almost disjoint from each $A_j (j < \alpha)$ such that $B \in J$ if Case 1 holds, (in Stage B), and let $B = C_1 \cup C_2$ be some partition obtained as (*1). *Then* for every $\ell \in \{1, 2\}$ and every $A \subseteq C_\ell$ such that $|A| = |C_\ell \setminus A| = \aleph_0$, we have:

$$Ch^A_{C_\ell} \Vdash_{P^Q_\ell [A]} \text{“}\underset{\sim}{Y} \cap F(C_\ell) =_{ae} F(A)\text{”}.$$

Note that (*2) is a kind of definition of F, not so nice though; so our aim is to show that if F is definable in this weak sense then it is very nicely definable: is trivial.

Stage D: Under (*2) we have F is trivial on B (i.e. for some one to one $g : B \to \omega$, for every $B' \subseteq B$, $F(B') =_{\text{ae}} \{g(m) : m \in B'\}$.

We will eventually show that (*2) implies that F is trivial but first we want to prove that $F \restriction \mathcal{P}(B)$ is trivial (this will be accomplished in (*13) below; however those proofs will be used later, too).

We now assume B is as in (*2) and concentrate on C_1; C_2 can be treated similarly, putting together we can get the result on B. So now we shall use $P^Q_1[A]$ for $A \subseteq C_1$.

Let $P_\alpha \times Q = \{p_\ell : \ell < \omega\}$, and let for $n < \omega$, $\mathcal{I}_n$ be a dense subset of $P_\alpha \times Q$, $\mathcal{I}_n = \{q^n_i : i < \omega\}$, such that $q^n_i \Vdash_{P_\alpha \times Q}$ “$[n \in \underset{\sim}{Y}] \equiv \mathbf{t}^n_i$” ($\mathbf{t}^n_i$ denotes a truth value, 1 is true, 0 is false).

We want to show that $F \restriction \mathcal{P}(C_1)$ is trivial (i.e. for some $f : C_1 \to \omega$ for every $C' \subseteq C_1$, $F(C') =_{\text{ae}} \{f(n) : n \in C_1\}$ (so Rang$(f) =_{\text{ae}} F(C_1)$). To do this we partition C_1 into three pieces, $C_1 = A^*_0 \cup A^*_1 \cup A^*_2$ and we shall prove (step by step):

(a) $F \restriction A^*_i$ is nicely defined by (*6) on a set of 2^{nd} category.

(b) $F \restriction A^*_i$ is continuous on a set of 2^{nd} category.

(c) $F \restriction A^*_i$ is trivial apart from a set of first category.

(d) F is trivial on A^*_i.

We shall now define by induction on $k < \omega$ a natural number n_k and a function $f_k : a_k \to \{0,1\}$ where $a_k = [n_k, n_{k+1}) \cap C_1$. We will take $A_i^* \stackrel{\text{def}}{=} \bigcup_{m<\omega} a_{3m+i}$ for $i < 3$. We want our n_k, f_k to satisfy (∗3) and (∗4) below:

Let us denote by $IF(k)$ the set of the functions f, $\mathrm{Dom}(f)$ a proper initial segment of $C_1 \setminus [0, n_k), \mathrm{Rang}(f) \subseteq \{0,1\}$.

(∗3) (a) $n_0 = 0$

(b) $n_{k+1} > n_k, \mathrm{Rang}(f_k) = \{0,1\}$

(c) For any $\ell < k, m < n_k$, a function f from $C_1 \cap [0, n_k)$ to $\{0,1\}$ and $\mathbf{t}$ denoting a truth value, the following holds:
if for some i, $\mathbf{t}_i^m = \mathbf{t}$ and $f \cup f_k \cup p_\ell \cup q_i^m \in P_\alpha \times Q$
then for some j, $\mathbf{t}_j^m = \mathbf{t}$ and for every $h \in IF(k+1)$,
$f \cup f_k \cup h \cup p_\ell \cup q_j^m \in P_\alpha \times Q$.

(∗4) For every $\ell < k$ and g_0, g_1 functions from $[0, n_k) \cap C_1$ to $\{0,1\}$, one of the following holds: (F is from the hypothesis of the Main Lemma 5.6):

Case (d_α): for *no* $m \in F(C_1) \setminus [0, n_{k+1})$ are there $i(0) < \omega$, $i(1) < \omega$, $\mathbf{t}_{i(0)}^m \neq \mathbf{t}_{i(1)}^m$, and $h \in IF(k+1)$, *such that* for every $h^\dagger$, $h \subseteq h^\dagger \in IF(k+1)$ implies
$g_0 \cup f_k \cup h^\dagger \cup p_\ell \cup q_{i(0)}^m$ and $g_1 \cup f_k \cup h^\dagger \cup p_\ell \cup q_{i(1)}^m$
are both in $P_\alpha \times Q$.

Case (d_β): for every $k(*) > k$ and function h from $[n_{k+1}, n_{k(*)}) \cap C_1$ to $\{0,1\}$ there are $m \in F(C_1) \cap [n_{k(*)}, n_{k(*)+1})$, $i(0) < \omega$, $i(1) < \omega$ *such that* $\mathbf{t}_{i(0)}^m \neq \mathbf{t}_{i(1)}^m$, and for every $h^\dagger \in IF(k(*)+1)$,
$g_0 \cup f_k \cup h \cup f_{k(*)} \cup h^\dagger \cup p_\ell \cup q_{i(0)}^m$ and $g_1 \cup f_k \cup h \cup f_{k(*)} \cup h^\dagger \cup p_\ell \cup q_{i(1)}^m$
are both in $P_\alpha \times Q$.

We have now to convince the reader that we can define n_k, f_k that satisfy (∗3), (∗4). Assume that we have defined $n_0, \ldots, n_{k-1}, n_k$ and $f_0, \ldots, f_{k-1}$ and we want to define n_{k+1}, f_k. We have finitely many tasks, of three types:

(i) instances of (∗3), (c) namely: given $\ell < k$, $m < n_k$, $f : C_1 \cap [0, n_k) \to \{0,1\}$ and $\mathbf{t}$, we have to satisfy "*if* for some $i \ldots$"

(ii) instances of $(*4)$ (d_α), namely: given $\ell < k$, $g_0, g_1 \in \{0,1\}^{[0,n_k)\cap C_1}$, we try to satisfy "for no $m \dots$" (but sometimes we fail – then we will remember this failure in (iii), at all later stages).

(iii) failures of $(*4)(d_\alpha)$ in the past: given a task of type (ii) which we did not fulfill at a previous step, we have to satisfy (d_β) for $k(*) =$ our present k.

We will define f_k by taking approximations of it in $IF(k)$, each one of them intends to fulfill one task and contains its predecessor – this ensures that once we have constructed an approximation fulfilling a given task, the f_k that we will eventually obtain will also fulfill it. Our initial approximation is an arbitrary element of $IF(k)$ assuming the values 0 and 1.

So now we deal separately with each type:

(i) Assume such i exists (with f_k interpreted as our current approximation of it). The addition of h can create a contradiction only inside C_1 (by the definition of $IF(k+1)$), and since C_1 is almost disjoint from the domain of any condition in P_α (in this case $(p_\ell \cup q_i^m)_0$), by defining f_k in the dangerous finite portion $C_1 \cap \mathrm{Dom}(p_\ell \cup q_i^m)_0$, as $(p_\ell \cup q_i^m)\restriction C_1$ (and extending for making its domain a (proper) initial segment of $C_1 \setminus [0, n_k)$) we fulfill our task.

(ii) Let $IF_c(k)$ denote the subset of $IF(k)$ consisting of those functions containing our current approximation of f_k. Assume that:

$\otimes_1$ There exists $f \in IF_c(k)$ such that for every $m \in F(C_1)$, $m > \mathrm{Sup}(\mathrm{Dom}(f))$, and for every $f^\dagger \in IF_c(k)$, $f^\dagger \supseteq f$, and for every $i(0), i(1) < \omega$, *if* for every $f'' \in IF_c(k)$, $f'' \supseteq f^\dagger$ both $g_0 \cup f'' \cup p_\ell \cup q^m_{i(0)}$ and $g_1 \cup f'' \cup p_\ell \cup q^m_{i(1)}$ are in $P_\alpha \times Q$ *then* $\mathbf{t}^m_{i(0)} = \mathbf{t}^m_{i(1)}$.

Assuming this, we take such f as our next approximation of f_k, which will satisfy this instance of (d_α). So if we fail, we put burden on the future cases of (iii) but we know that:

$\otimes_2$ For every $f \in IF_c(k)$ there are $m \in F(C_1)$, $m > \mathrm{Sup}(\mathrm{Dom}(f))$ and $f^\dagger \in IF_c(k)$, $f^\dagger \supseteq f$ and $i(0), i(1) < \omega$ *such that* $\mathbf{t}^m_{i(0)} \neq t^m_{i(1)}$ and for every $f'' \in IF_c(k)$, $f'' \supseteq f^\dagger$ both $g_0 \cup f'' \cup p_\ell \cup q^m_{i(0)}$ and $g_1 \cup f'' \cup p_\ell \cup q^m_{i(1)}$ are in $P_\alpha \times Q$.

(iii) Adapting our notation to that of (ii), we denote our present k as $k(*)$ and consider some $k < k(*)$ and assume that the property $\otimes_2$ (written at the end of (ii)) held for $\ell < k$ and g_0, $g_1 \in {}^{[0,n_k)\cap C_1}\{0,1\}$. (This we have found in stage k, note there are for each k only finitely many triples (ℓ, g_0, g_1).)

By the formulation of case (d_β) we are given $h \in {}^{[n_{k+1},n_{k(*)})\cap C_1}\{0,1\}$; let $f^0_{k(*)}$ be our current approximation of $f_{k(*)}$ and use the property $\otimes_2$ for $f = f_k \cup h \cup f^0_{k(*)}$ to obtain m, $f^\dagger$, $i(0), i(1)$ as described there. Letting $f^1_{k(*)} = f^\dagger \setminus (f_k \cup h) = f^\dagger \restriction (\mathrm{Dom}(f^\dagger) \setminus [n_k, n_{k(*)}])$ and extending it if necessary to ensure that $m < n_{k(*)+1}$, we obtain the next approximation to $f_{k(*)}$ fulfilling our task.

So the definition of n_k, f_k can be carried out, so we have gotten $(*3)$ and $(*4)$.

$\square_{(*3),(*4)}$

Our treatment from here on concentrates on $A^* = \cup_{k<\omega} a_{3k+1}$ (remember $a_k = [n_k, n_{k+1}) \cap C_1$). The other "two thirds" of C_1 can be treated similarly. Let $f^* = \cup_{k<\omega}(f_{3k} \cup f_{3k+2})$, $A_1 = f^{*-1}(\{1\})$. Notice that by $(*3)$ (b), both A_1 and its complement in $\mathrm{Dom}(f^*)$ are infinite.

Thus, from $(*2)$ it follows that:

$(*5)$ For every $A \subseteq A^*$, $Ch^A_{A^*} \cup f^* \Vdash_{P^Q_1[A\cup A_1]}$ "$\underset{\sim}{Y} \cap F(C_1) =_{ae} F(A \cup A_1)$".
Hence, for every $A \subseteq A^*, Ch^A_{A^*} \cup f^*$ can be extended to a condition (in $P^Q_1[A \cup A_1]$) forcing equality above a certain n and in particular deciding for each $m \geq n$ in $F(C_1)$ whether it belongs to $\underset{\sim}{Y}$ (as no other names are involved in the equality). Looking at $\mathcal{P}(A^*)$ as a topological space, we conclude (as $|P_\alpha \times Q| = \aleph_0$):

$(*6)$ There exists a condition $p^* \in P_\alpha \times Q$, a finite function h^* satisfying $\mathrm{Dom}((p^*)_0) \cap A^* \subseteq \mathrm{Dom}(h^*) \subseteq A^*$, and $n^* < \omega$ such that

$$\mathcal{A} \stackrel{\text{def}}{=} \{A \subseteq A^* : h^* \subseteq Ch^A_{A^*} \text{ and for each } m \geq n^* \text{ in } F(C_1) \text{ we have: } [p^* \cup Ch^A_{A^*} \cup f^* \Vdash_{P^Q_1[A\cup A_1]} \text{“}m \in \underset{\sim}{Y}\text{” or } p^* \cup Ch^A_{A^*} \cup f^* \Vdash_{P^Q_1[A\cup A_1]} \text{“}m \notin \underset{\sim}{Y}\text{”}]\}$$

is of the second category everywhere "above" h^* (i.e., its intersection with any open set determined by a finite extension of h^* is of the second category).

We now define, for $A \in \mathcal{A}$,

$$F_1(A) = \{m \in F(C_1) : p^* \cup Ch^A_{A^*} \cup f^* \Vdash_{P_1^Q[A \cup A_1]} \text{“}m \in \underset{\sim}{Y}\text{”}\} .$$

It follows from $(*5)$ that

$(*6\text{A})$ $F(A \cup A_1) =_{ae} F_1(A)$ for all $A \in \mathcal{A}$.

We will verify next that $\mathcal{A}$ is a Borel set. The demand $h^* \subseteq Ch^A_{A^*}$ defines an open set, and then we have a conjunction over all $m \geq n^*$ in $F(C_1)$ of conditions each requiring that $p^* \cup Ch^A_{A^*} \cup f^*$ decide whether m belongs to $\underset{\sim}{Y}$. Denoting $\mathcal{A}_q = \{A \subseteq A^* : q \text{ is incompatible with } p^* \cup Ch^A_{A^*} \cup f^*\}$ for $q \in P_\alpha \times Q$, and remembering that $\mathcal{I}_m = \{q^m_i : i < \omega\}$ is dense in $P_\alpha \times Q$ and we can preserve its pre-density in $P_1^Q[A \cup A_1]$, we see that $A \subseteq A^*$ satisfies the requirement for m iff it belongs to $[\cap\{\mathcal{A}_{q^m_i} : i < \omega, \mathbf{t}^m_i \text{ is true }\}] \cup [\cap\{\mathcal{A}_{q^m_i} : i < \omega, \mathbf{t}^m_i \text{ is false}\}]$. As each $\mathcal{A}_q$ is open, we have shown that $\mathcal{A}$ is Borel.

As $\mathcal{A}$ is a Borel set, we know by a theorem of Baire that (by extending h^*) we may assume that $\{A \subseteq A^* : A \notin \mathcal{A}, \text{ and } h^* \subseteq Ch^A_{A^*}\}$ is of the first category.

Let k be large enough so that $n_{3k} \geq n^*$, $\max\mathrm{Dom}(h^*)$ and $\ell < 3k - 1$ where $p^* = p_\ell$.

For $A \in \mathcal{A}$, let $\mathbf{t}^m_A$ denote the truth value of $m \in F_1(A)$. Then for every $m \in [n_{3k}, n_{3k+3}) \cap F(C_1)$ we know $p^* \cup Ch^A_{A^*} \cup f^*$ is compatible with some q^m_i but is not compatible with any q^m_i such that $\mathbf{t}^m_i \neq \mathbf{t}^m_A$ (remember the definitions of $F_1(A)$ and $\mathcal{A}$ and observe that $m \geq n^*$, $m \in F(C_1)$).

Hence by (c) of $(*3)$, $\mathbf{t}^m_A$ is determined, for $A \in \mathcal{A}$ and $m \in [n_{3k}, n_{3k+3}) \cap F(C_1)$, by $Ch^A_{A^*}{\restriction}[0, n_{3k+4}) = Ch^A_{A^*}{\restriction}[0, n_{3k+2})$; that is: $\mathbf{t}^m_A = \mathbf{t}$ iff for some i, $\mathbf{t}^m_i = \mathbf{t}$ and for every $h \in IF(3k+4)$ we have

$$Ch^A_{A^*}{\restriction}[0, n_{3k+2}) \cup f^*{\restriction}[0, n_{3k+4}) \cup h \cup p^* \cup q^m_i \in P_\alpha \times Q.$$

Now we want to show that $\mathbf{t}^m_A$ is determined (for A, m, k as above) by $Ch^A_{A^*}\restriction a_{3k+1}$ alone. So let us assume that $A^{(0)}, A^{(1)} \in \mathcal{A}$ and $Ch^{A^{(0)}}_{A^*}\restriction a_{3k+1} = Ch^{A^{(1)}}_{A^*}\restriction a_{3k+1}$, and for some $m \in [n_{3k}, n_{3k+3}) \cap F(C_1)$ we have $\mathbf{t}^m_{A^{(0)}} \neq \mathbf{t}^m_{A^{(1)}}$. Let for $t = 0, 1$, $g_t = (Ch^{A^{(t)}}_{A^*} \cup f^*)\restriction([0, n_{3k-1}) \cap C_1)$. From the way $\mathbf{t}^m_{A^{(0)}}$, $\mathbf{t}^m_{A^{(1)}}$ are determined (see end of previous paragraph) and from $\mathbf{t}^m_{A^{(0)}} \neq \mathbf{t}^m_{A^{(1)}}$ it follows that at stage $3k-1$ of (*4), for the above g_0, g_1 and $p^* = p_\ell$, clause (d_α) fails (take $h = f^*\restriction(a_{3k} \cup a_{3k+2} \cup a_{3k+3}) \cup Ch^{A^{(t)}}_{A^*}\restriction a_{3k+1}$, which does not depend on t). Hence (d_β) holds in this case. Using this, we want to construct $B^{(0)}, B^{(1)} \in \mathcal{A}$ such that $A^{(i)} \cap n_{3k-1} = B^{(i)} \cap n_{3k-1}$ for $i = 0, 1$, and $B^{(0)} \setminus n_{3k-1} = B^{(1)} \setminus n_{3k-1}$ and we shall choose them in such a way that for infinitely many $m \in F(C_1)$, $Ch^{B^{(0)}}_{A^*}$ and $Ch^{B^{(1)}}_{A^*}$ determine $\mathbf{t}^m_{B^{(0)}}, \mathbf{t}^m_{B^{(1)}}$, respectively, to be distinct. Since $B^{(0)} =_{ae} B^{(1)}$, also by (*6A) we know $F_1(B^{(0)}) =_{ae} F(B^{(0)} \cup A_1) =_{ae} F(B^{(1)} \cup A_1) =_{ae} F_1(B^{(1)})$, so this will be a contradiction. Thus, it remains to show how to carry out the construction of $B^{(0)}, B^{(1)}$. We do it by an ω-sequence of finite approximations, the set of finite approximations is $\{(e_1, e_2) :$ for some n, e_1, e_2 are finite functions from $A^* \cap [0, n)$ to $\{0, 1\}$ e_i, includes $Ch^{A(i) \cap n_{3k+1}}_{A^* \cap n_{3k+1}}$ and $e_1\restriction[n_{3k+1}, n) = e_2\restriction[n_{3k+1}, n)\}$.

We start with the respective characteristic functions of $A^{(0)}, A^{(1)}$ up to n_{3k-1}. As $A^{(0)}, A^{(1)} \in \mathcal{A}$, both characteristic functions contain h^*.

As $\{A \subseteq A^* : A \notin \mathcal{A}$ and $h^* \subseteq Ch^A_{A^*}\}$ is of the first category, we have a countable family of nowhere dense subsets of $\mathcal{P}(A^*)$ which are to be avoided by $B^{(0)}, B^{(1)}$, to ensure $B^{(0)}, B^{(1)} \in \mathcal{A}$. So we arrange all these tasks (each time dealing with either $B^{(0)}$ or $B^{(1)}$ and one nowhere dense subset) in an ω sequence, and by the definition of "nowhere dense" we obtain each time a finite extension of our former approximations which ensures the implementation of our task. Say $\langle \mathcal{A}_n : n < \omega \rangle$ is this sequence of nowhere dense sets, and we choose (e^i_1, e^i_2) for $i < \omega$ approximations such that for $i = 3j$, (e^{i+1}_1, e^{i+1}_2) ensure $B^{(0)} \notin \mathcal{A}_j$ and $\max\mathrm{Dom}(e^i_1) > j$, for $i = 3j+1$ (e^{i+1}_1, e^{i+1}_2) ensure $B^{(1)} \notin \mathcal{A}_j$, and for $i = 3j+2$ (e^{i+1}_1, e^{i+1}_2) ensure that for some $k < j$ and $m \in [n_k, n_{k+1})$ is as required (i.e. $Ch^{B^{(0)}}_{A^*}$, $Ch^{(1)}_{A^*}$ determine $\mathbf{t}^m_{B^{(0)}}$, $\mathbf{t}^m_{B^{(1)}}$ are distinct). If we succeed, $B^{(0)} = \bigcup_{i<\omega} e^i_1$, $B^{(1)} = \bigcup_{i<\omega} e^i_2$ are as required. Why can we carry the construction? The least trivial case is the last.

We will show that our construction is all right by proving: if $k(*) > 3k-1$ and $k(*) \equiv 1 \pmod 3$ then there is $m \in F(C_1) \cap [n_{k(*)}, n_{k(*)+1})$ such that for $t = 0,1$ there are $i(t) < \omega$ such that $\mathbf{t}^m_{i(0)} \neq \mathbf{t}^m_{i(1)}$ and for every $h \in IF(k(*)+3)$ we have $Ch^{B^{(t)}}_{A^*} \restriction [0, n_{k(*)}) \cup f_{k(*)} \cup f^* \restriction [0, n_{k(*)+3}) \cup h \cup p^* \cup q^m_{i(t)} \in P_\alpha \times Q$. To prove this, use (d_β) for such $k(*)$ and $h = (Ch^{B^{(t)}}_{A^*} \cup f^*) \restriction [n_{3k}, n_{k(*)})$. We thus get the desired contradiction.

So, by the last four paragraphs and (*6A), as F is an automorphism and A_1 is fixed, we have proven (for the last equality: we can just subtract $F(A_1)$):

(*7) There are functions $G'_k : \mathcal{P}(a_{3k+1}) \to \mathcal{P}(F(A^*) \cap [n_{3k}, n_{3k+3}))$ such that for every $A \in \mathcal{A}$ we have $F(A) =_{ae} F(A \cup A_1) \setminus F(A_1) =_{ae} F_1(A) \setminus F(A_1) =_{ae} \cup_{k<\omega} G'_k(A \cap a_{3k+1})$.

Now, (*7) would be good enough (as we shall see later) if it were not restricted to $\mathcal{A}$. So we want to achieve a similar result without this restriction.

We define by induction a sequence of pairs $\langle u_\ell, g_\ell \rangle$ $(\ell < \omega)$ such that $\langle u_\ell : \ell < \omega \rangle$ is a partition of $A^* \setminus \mathrm{Dom}(h^*)$ into finite subsets and $g_\ell : u_\ell \to \{0,1\}$, as follows: for $r < \omega$, let $\mathcal{A}_r$ be the r-th element in a sequence of nowhere dense subsets of $\mathcal{P}(A^*)$ showing that $\{A \subseteq A^* : A \notin \mathcal{A}, h^* \subseteq Ch^A_{A^*}\}$ is of the first category. Each $\langle u_\ell, g_\ell \rangle$ is defined so that:

(i) u_ℓ is disjoint from u_m for all $m < \ell$.

(ii) if $\ell \in (A^* \setminus \mathrm{Dom}(h^*)) \setminus \cup_{m<\ell} u_m$ then $\ell \in u_\ell$.

(iii) if $A \subseteq A^*$, $g_\ell \subseteq Ch^A_{A^*}$ and $\ell = 2k$ or $\ell = 2k+1$ then $A \notin \mathcal{A}_k$.

To see that this is possible, use the fact that $\mathcal{A}_\ell$ is nowhere dense taking into account one after another all the elements of ${}^v\{0,1\}$ where $v = (\cup_{m<\ell} u_m) \cup \mathrm{Dom}(h^*)$.

Now, for $t = 0,1$ let $B^*_t = \cup_{r<\omega} u_{2r+t}$ and $g^*_t = \cup_{r<\omega} g_{2r+t}$, $B_t = g^{*-1}_t(\{1\})$. Then $\mathrm{Dom}(g^*_t) = B^*_t$ and $\langle B^*_0, B^*_1, \mathrm{Dom}(h^*) \rangle$ is a partition of A^*. From (iii) it follows that if $A \subseteq A^*$, $h^* \cup g^*_t \subseteq Ch^A_{A^*}$ then $A \in \mathcal{A}$.

From $(*7)$ we can thus derive a definition for $F(A\cap B_0^*)$ working uniformly for all $A \subseteq A^*$ extending G'_k to G''_k

$$G''_k(A \cap B_0^* \cap a_{3k+1}) = G'_k([(A \cap B_0^*) \cup B_1] \cap a_{3k+1}) \setminus F(B_1)$$

(where $F(B_1) \in \mathcal{P}(\omega)$ denotes one fixed representative of the equivalence class $F(B_1)$). Similarly we can derive a definition for $F(A \cap B_1^*)$. From these two definitions (we can use $F(\bigcup_{n<\omega} u_{2n+1})$, $F(\bigcup_{n<\omega} u_{2n}),$) remembering that $\mathrm{Dom}(h^*)$ is finite, we obtain:

$(*8)$ There are functions $G_k : \mathcal{P}(a_{3k+1}) \to \mathcal{P}(F(A^*) \cap [n_{3k}, n_{3k+3}))$ such that for every $A \subseteq A^*$ we have $F(A) =_{ae} \cup_{k<\omega} G_k(A \cap a_{3k+1})$.

With this result in hand, we have good control of $F \restriction \mathcal{P}(A^*)$. We prove first:

$(*9)$ For every large enough k and disjoint $b_1, b_2 \subseteq a_{3k+1}$,

$$G_k(b_1) \cap G_k(b_2) = \emptyset.$$

Otherwise there is an infinite $S \subseteq \omega$, and $b_1^k, b_2^k \subseteq a_{3k+1}$ for $k \in S$, $b_1^k \cap b_2^k = \emptyset$, $G_k(b_1^k) \cap G_k(b_2^k) \neq \emptyset$. Then let $A_1 = \cup_{k \in S} b_1^k$, $A_2 = \cup_{k \in S} b_2^k$. Clearly $A_1 \cap A_2 = \emptyset$ and $A_1, A_2 \subseteq A^*$, and $F(A_1) \cap F(A_2) \neq_{ae} \emptyset$. This contradicts that F/finite commutes with $\cap$.

As F commutes with $\cup$ we can get similarly:

$(*10)$ For every large enough k and disjoint $b_1, b_2 \subseteq a_{3k+1}$,

$$G_k(b_1 \cup b_2) = G_k(b_1) \cup G_k(b_2),$$

and as F is onto (and monotonic) we have:

$(*11)$ For every large enough k, for any singleton $b \subseteq a_{3k+1}$ we have: $G_k(b)$ is a singleton (and also G_k is onto $F(A^*) \cap [n_{3k}, n_{3k+3})$.

Define g, $\mathrm{Dom}(g) = A^*$, $G_k(\{\ell\}) = \{g(\ell)\}$ for $\ell \in a_{3k+1}$, k large enough. So clearly by $(*8)$–$(*11)$:

$(*12)$ For every $A \subseteq A^*$, $F(A) =_{ae} g(A) = \{g(i) : i \in A\}$ and g is one-to-one.

Remember that A^* is one third of C_1. Doing this separately for each third and then the same for C_2, we obtain for every B which is almost disjoint from each $A_j (j < \alpha)$: (that we can get a one-to-one g is proved like $(*9)$):

$(*13)$ There is a one-to-one function g, $\mathrm{Dom}(g) =_{ae} B$, such that $(\forall A \subseteq B)$ $[g(A) =_{ae} F(A)]$.

Stage E: F is trivial on ω.

Let I be the ideal generated by $\{A_j : j < \alpha\}$ and the finite subsets of ω. Let $I_1 \stackrel{\text{def}}{=} \{B \subseteq \omega : (\forall A \in I)|B \cap A| < \aleph_0\}$. Clearly, I_1 is also an ideal.

Remember $J = \{A \subseteq \omega : F \text{ is trivial on } A\}$. In stage D we have shown that F is trivial on sets in I_1 (if $(*2)$ holds). If Case 2 from stage B holds then there is $A^+ \in [\omega]^{\aleph_0}$ such that $[A^+]^{\aleph_0} \cap J = \emptyset$, hence $\omega \setminus A_0 \subseteq A^+$ but there are B's as in stage B and by it $(*2)$ holds hence by stage D we know that $B \in J$ and obviously $B \cap A_0$ is finite. So B gives contradiction. Hence Case 1 of Stage B holds i.e. J is dense. We now outline the rest of the proof, and then shall give it in details. Using the knowledge we gathered on F in the previous stage we construct countably many functions $\{g_p : p \in P_\alpha \times Q\}$ such that for each $B \in I_1 \cap J$ there is $p \in P_\alpha \times Q$ such that $g_p \restriction B$ induces F on B.

Next we shall put these functions together to get a single function g such that g induces F on every $A \in I_1$. Finally we shall show that in this case g defines F.

Notice:

$(*14)$ Every A which is not in I contains an infinite subset in I_1.

To see this, let $\langle B_n : n < \omega\rangle$ be an ω-enumeration of the generators of I (i.e. $\{A_j : j < \alpha\}$). We try to define a sequence $\langle a_n : n < \omega\rangle$ such that $a_m \neq a_n$ for $m < n$ and $a_n \in A \setminus \cup_{m<n} B_n$. If we succeed we obtain a subset as required.

If we cannot define a_n, then $A \subseteq (\cup_{m<n} B_m) \cup \{a_m : m < n\}$, hence $A \in I$, contrary to our assumption.

Let $B \in J \cap I_1$ be infinite, $B = C_1 \cup C_2$ as in (*1).

(*15) Let g induce $F \restriction \mathcal{P}(B)$ (in the sense of (*13)), and let $\ell \in \{1, 2\}$. Then for every $p \in P_\alpha \times Q$ there are q, $p \leq q \in P_\alpha \times Q$ and n such that:

$(\forall m \in C_\ell)(m \geq n \Rightarrow [q \cup \{\langle m, 0\rangle\} \Vdash_{P_\alpha \times Q}$ "$g(m) \notin \underset{\sim}{Y}$"$])$

and:

$(\forall m \in C_\ell)(m \geq n \Rightarrow [q \cup \{\langle m, 1\rangle\} \Vdash_{P_\alpha \times Q}$ "$g(m) \in \underset{\sim}{Y}$"$])$

Proof of (*15). It is enough to prove it for $m \in C_1$, so $\ell = 1$. Suppose it fails for g, p. Let $\{q_\ell : \ell < \omega\}$ be the set of all conditions in $P_\alpha \times Q$ above p. Let $n \mapsto ((n)_0, (n)_1)$ be a mapping of ω onto $\omega \times \omega$.

We will define an increasing sequence of finite functions functions $h_n (n < \omega)$, $\mathrm{Dom}(h_n) \subseteq C_1$, $\mathrm{Rang}(h_{n+1} \setminus h_n) = \{0, 1\}$, and if $n \in C_1$ then $n \in \mathrm{Dom}(h_{n+1})$; we will also define $m_n \in C_1 (n < \omega)$, as follows:

We let $h_0 = (p)_0 \restriction (\mathrm{Dom}((p)_0) \cap C_1)$.

Assume that we have defined h_n, and we will see how we define m_n and h_{n+1}. If h_n is not compatible with $q_{(n)_0}$ then m_n is chosen arbitrarily and $h_{n+1} \supseteq h_n$ is chosen in accordance with our requirements above. If h_n is compatible with $q_{(n)_0}$, we use our assumption for $q = h_n \cup q_{(n)_0}$. Notice that if m is large enough then $m \notin \mathrm{Dom}((q)_0) \cap C_1$ and $[m \in C_1 \Rightarrow g(m)$ is defined, $g(m) \in F(C_1)$, $g(m) \geq (n)_1]$. So we can find $m = m_n \in C_1$ satisfying all these, and $k_n \in \{0, 1\}$ such that $h_n \cup q_{(n)_0} \cup \{\langle m_n, k_n\rangle\} \nVdash_{P_\alpha \times Q}$ "$g(m_n) \notin \underset{\sim}{Y}$ iff $k_n = 0$". Note: $k_n = 0$ is a failure of the first conclusion of (*15), $k_n = 1$ is a failure of the second conclusion of (*15) . We will choose $k_n = 0$ if possible.

Let $q_n^\dagger \geq h_n \cup q_{(n)_0} \cup \{\langle m_n, k_n\rangle\}$ force "$g(m_n) \in \underset{\sim}{Y}$ iff $k_n = 0$". Let $h_{n+1}^\dagger = (q_n^\dagger)_0 \restriction (\mathrm{Dom}((q_n^\dagger)_0) \cap C_1)$, and let $h_{n+1} \supseteq h_{n+1}^\dagger$ satisfy $\mathrm{Rang}(h_{n+1} \setminus h_n) = \{0, 1\}$ and $n \in \mathrm{Dom}(h_{n+1})$ if $n \in C_1 \setminus \bigcup_{m \leq n} \mathrm{Dom}(h_m)$.

Now, $\mathrm{Dom}(\cup_{n<\omega} h_n) = C_1$. Let $A \subseteq C_1$ be the subset with characteristic function $\cup_{n<\omega} h_n$. So A and $C_1 \setminus A$ are infinite, so by (*2) we know $Ch^A_{C_1} \Vdash_{P_1^Q[A]}$ "$\underset{\sim}{Y} \cap F(C_1) =_{ae} F(A)$". By our choice of h_0, $Ch^A_{C_1} = \cup_{n<\omega} h_n$ is compatible with p, so some extension of $p \cup Ch^A_{C_1}$ in $P_1^Q[A]$ forces the equality above some integer. So there is n such that:

$$q_{(n)_0} \cup Ch^A_{C_1} \Vdash_{P_1^Q[A]} \text{“}\underset{\sim}{Y} \cap F(C_1) \setminus (n)_1 = F(A) \setminus (n)_1\text{”}$$

and $F(A) \setminus (n)_1 = g(A) \setminus (n)_1$.

We refer now to the definition of m_n, h_{n+1}, k_n, suppose that $k_n = 0$, the other case is similar. Since $h_n \subseteq Ch^A_{C_1}$ is compatible with $q_{(n)_0}$, we had $q^\dagger_n \Vdash_{P_\alpha \times Q}$ “$g(m_n) \in \underset{\sim}{Y}$” where $q^\dagger_n$ was above $h_n \cup q_{(n)_0} \cup \{\langle m_n, 0\rangle\}$. Now, $m_n \notin A$ since h_{n+1} says so, hence $g(m_n) \notin g(A)$. As $g(m_n) \geq (n)_1$, $g(m_n) \notin F(A)$; but $g(m_n) \in F(C_1)$, hence $q_{(n)_0} \cup Ch^A_{C_1} \Vdash_{P_1^Q[A]}$ “$g(m_n) \notin \underset{\sim}{Y}$”. But $q^\dagger_n$ forces the opposite (we know it in $P_\alpha \times Q$, but it does not matter), and they are compatible (because $q_{(n)_0} \subseteq q^\dagger_n$ and $Ch^A_{C_1}$ and $q^\dagger_n$ are compatible by our choice of $h^\dagger_{n+1}$), a contradiction. So we have finished the proof of $(*15)$.

We define now, for every $p \in P_\alpha \times Q$, a partial a function from ω to ω as follows: $g_p(m) = k$ iff $[p \cup \{\langle m, 0\rangle\} \Vdash_{P_\alpha \times Q}$ “$k \notin \underset{\sim}{Y}$” and $p \cup \{\langle m, 1\rangle\} \Vdash_{P_\alpha \times Q}$ “$k \in \underset{\sim}{Y}$” and k is the only one satisfying this$]$.

Our intention in defining g_p is to have countably many functions which induce $F{\restriction}\mathcal{P}(B)$ uniformly for all $B \in I_1$.

$(*16)$ For every $B \in J \cap I_1$ and every $p \in P_\alpha \times Q$ there is q, $p \leq q \in P_\alpha \times Q$, such that $B \subseteq_{ae} \mathrm{Dom}(g_q)$ and $(\forall A \subseteq B)[F(A) =_{ae} g_q(A)]$.

Proof of $(*16)$. If B is finite, then there is nothing to prove, so we assume that B is infinite. Let g exemplify $B \in J$ (remember $I_1 \subseteq J$).

Let us call k a candidate to be $g_p(m)$, if it satisfies the first two parts of the definition (but maybe not the uniqueness requirement). We notice that if k is a candidate to be $g_p(m)$ and $k^\dagger$ is a candidate to be $g_p(m^\dagger)$ and $m \neq m^\dagger$, then $k \neq k^\dagger$ (consider $p \cup \{\langle m, 0\rangle, \langle m^\dagger, 1\rangle\}$, it forces both $k \notin \underset{\sim}{Y}$ and $k^\dagger \in \underset{\sim}{Y}$). Notice also that in $(*15)$ we have shown that for every large enough $m \in C_\ell$ (and appropriate q) $g(m)$ is a candidate to be $g_q(m)$.

Thus, given B and p, we apply $(*15)$ first for $\ell = 1$ and p, and then for $\ell = 2$ and the condition obtained for $\ell = 1$, obtaining $q \geq p$ such that for every large enough $m \in B$, $g(m)$ is a candidate to be $g_q(m)$. Now, it suffices to show that $B \subseteq_{ae} \mathrm{Dom}(g_q)$, since whenever $g_q(m)$ is defined there is only one candidate, but $g(m)$ is one, so for almost all $m \in B$ we have $g_q(m) = g(m)$ and we know that g induces $F{\restriction}\mathcal{P}(B)$.

Assume that $B \not\subseteq_{ae} \mathrm{Dom}(g_q)$. Then w.l.o.g. B is disjoint from $\mathrm{Dom}(g_q)$ and we have k_m $(m \in B)$ such that k_m and $g(m)$ are distinct candidates to be $g_q(m)$. Let $C \stackrel{\text{def}}{=} \{k_m : m \in B\}$. Then C is infinite and disjoint from $g(B)$ (as no k is a candidate for two m's). We let $D' = F^{-1}(C)$; more exactly $F(D') =_{\mathrm{ae}} C$ (possible as F is onto). Choose an infinite $D \subseteq D'$ such that $D \in J$. Then as $g(B) =_{ae} F(B)$, D' is almost disjoint from B.

Case I: $D \in I_1$.

In this case, we apply (*15) for D and q twice (as we did above for B and p) to obtain $q^\dagger \geq q$ such that for large enough $m^\dagger \in D$, $g^\dagger(m^\dagger)$ is a candidate to be $g_{q^\dagger}(m^\dagger)$ (where $g^\dagger$ induces $F{\restriction}\mathcal{P}(D)$). But for large enough $m^\dagger \in D$, $m^\dagger \notin B$ and $g^\dagger(m^\dagger)$ is a candidate to be $g_{q^\dagger}(m)$ for some $m \in B$ (as $g^\dagger(D) =_{ae} F(D)$, $F(D) \subseteq_{\mathrm{ae}} \{k_m : m \in B\}$, and $q \leq q^\dagger \in P_\alpha \times Q$), a contradiction (as in the beginning of the proof of (*16) we observe that no k is a candidate to be $g_{q^\dagger}(n)$ for two n's).

Case II: $D \notin I_1$.

Then w.l.o.g. $D \subseteq A_j$ for some $j < \alpha$. By strengthening q we can assume w.l.o.g. that $1_{A_j}\backslash\text{finite} \leq q$ or $0_{A_j}\backslash\text{finite} \leq q$. Assume that the former is the case (the second case is dealt with similarly). Then $q \Vdash_{P_{\omega_1} \times Q}$ "$A_j \subseteq_{ae} \underset{\sim}{X}$", hence $q \Vdash_{P_{\omega_1} \times Q}$ "$F(A_j) \subseteq_{ae} \underset{\sim}{Y}$". As we have seen in the proof of (*2), by preserving the pre-density of (countably many) appropriate subsets above corresponding conditions, we can conclude that $q \Vdash_{P_\alpha \times Q}$ "$F(A_j) \subseteq_{ae} \underset{\sim}{Y}$". Hence there are n and $q^\dagger \geq q$ in $P_\alpha \times Q$ such that $q^\dagger \Vdash_{P_\alpha \times Q}$ "$n^\dagger \in \underset{\sim}{Y}$" for all $n^\dagger \in F(A_j)$, $n^\dagger \geq n$. But $F(D) \subseteq_{ae} C \cap F(A_j)$, so taking $n^\dagger = k_m$ large enough we know that $n^\dagger$ is a candidate to be $g_{q^\dagger}(m)$ – a contradiction.

So we have finished the proof of (*16).

(*17) For $p_1, p_2 \in P_\alpha \times Q$, $B = \{n : g_{p_1}(n), g_{p_2}(n) \text{ are defined and distinct}\} \in I$.

Proof of (*17): If not, then by (*14) there exists an infinite $B' \subseteq B$ which belongs to I_1, but as we are in Case 1 (from stage B, as said in the begining of stage D) there is an infinite $B_1 \subseteq B'$ which belongs to J. Together $B_1 \in I_1 \cap J$

is such that g_{p_1}, g_{p_2} are defined everywhere in B_1 but they never agree there. By (*16) we can find for $i = 1, 2$ condition $q_i \geq p_i$ such that $B_1 \subseteq_{ae} \mathrm{Dom}(g_{q_i})$ and $(\forall A \subseteq B_1) F(A) =_{ae} g_{q_i}(A)$. It follows that $g_{q_1} \restriction B_1 =_{ae} g_{q_2} \restriction B_1$; otherwise, we have an infinite $B_2 \subseteq B_1$ on which the functions never agree, and we can divide B_2 into three parts each having disjoint images under the two functions (decide inductively for $n \in B_2$ to which part is belongs, a good decision always exists as the functions are one-to-one); one part at least is infinite, call it B_3, then $F(B_3)$ is almost equal to both $g_{q_1}(B_3)$ and $g_{q_2}(B_3)$ which are disjoint. But, whenever g_{p_1}, g_{q_1} are both defined they must agree (since $q_1 \geq p_1$), and the same holds for p_2, q_2, so we obtain that $g_{p_1} \restriction B_1 =_{ae} g_{p_2} \restriction B_1$, contradicting our assumption. So (*17) holds.

By arranging the conditions in $P_\alpha \times Q$ in an ω-sequence and using (*17) we obtain that there is a partial function g^0 from ω to ω such that for every $p \in P_\alpha \times Q$ we have

$$\{n : g_p(n) \text{ is defined but } g^0(n) \text{ is not defined or is } \neq g_p(n)\} \in I.$$

Hence, by (*16), for every $B \in J \cap I_1$, $B \subseteq_{ae} \mathrm{Dom}(g^0)$ and $(\forall A \subseteq B) F(A) =_{ae} g^0(A)$.

We have almost achieved our goal of inducing $F \restriction \mathcal{P}(B)$ uniformly for all $B \in J \cap I_1$ – the missing point is that we want g^0 to be one-to-one. But this must be true after discarding from $\mathrm{Dom}(g^0)$ a set from I, because otherwise we can construct $\langle b_n : n < \omega \rangle$ with $b_n \neq b_m$ for $n \neq m$, $g^0(b_{2k}) = g^0(b_{2k+1})$ and $B = \{b_n : n < \omega\} \in J \cap I_1$ (see the proof of (*14) noting $J \cap I_1$ is an ideal); then for large enough n and an appropriate g_q (inducing $F \restriction \mathcal{P}(B)$) $g^0(b_n) = g_q(b_n)$, so g_q is not one-to-one, a contradiction. Since discarding a set in I does not affect the other properties of g^0, we have a one-to-one g^0 inducing $F \restriction \mathcal{P}(B)$ uniformly for all $B \in J \cap I_1$. But any element of I_1 contains an element of $J \cap I_1$, so g^0 induces $F \restriction \mathcal{P}(B)$ for all $B \in I_1$. [Why? Assume $B \in I_1$, $\neg[F(B) =_{ae} g^0(B)]$, then one of the following occurs:

(a) $F(B) \setminus g^0(B)$ is infinite, so as F is onto for some infinite $B_1 \subseteq B$, $F(B_1) \cap g^0(B) = \emptyset$ and we can find $B_2 \subseteq B_1$ which belongs to J, so $F(B_2) =_{ac}$

$g^0(B_2)$ but $g^0(B_2) \subseteq g^0(B)$ whereas $F(B_2) \subseteq_{ae} F(B_1) \leq_{ae} \omega \setminus g^0(B)$ contradiction,

(b) $g^0(B) \setminus F(B)$ is infinite so there is an infinite $B_1 \subseteq B$ such that $g^0(B_1) \subseteq g^0(B) \setminus F(B)$ and we get a similar contradiction.]

As we are in Case 1 (from Stage B, see beginning of Stage E) $(\forall j < \alpha) A_j \in J$, and they are almost disjoint, there is g^1, one-to-one partial function from ω to ω, so that for every $B \in I$, $F(B) =_{ae} g^1(B)$ and $B \subseteq_{ae} \mathrm{Dom}(g^1)$. Clearly $\mathrm{Dom}(g^1) \cup \mathrm{Dom}(g^0)$ is co-finite (use $(*14)$).

Let I^* be the ideal that $I \cup I_1$ generates. Let $D \stackrel{\text{def}}{=} \{n : g^1(n) \neq g^0(n)$ and both are defined$\}$.

If $D \notin I^*$ we can find (as we found $B_3 \subseteq B_2$ in the proof of $(*17)$) $D_1 \subseteq D$ such that $D_1 \notin I^*$ and $g^1(D_1) \cap g^0(D_1) = \emptyset$.

As F is onto, for some $D_1^\dagger \subseteq D_1$, $F(D_1^\dagger) =_{ae} F(D_1) \cap g^0(D_1)$. Then $D_1^\dagger \in I_1$, otherwise it has an infinite subset $D_1'' \in I$, so $F(D_1'') =_{ae} g^1(D_1'')$ hence $F(D_1'') \subseteq_{ae} g^1(D_1)$, but $F(D_1'') \subseteq_{ae} F(D_1^\dagger) \subseteq_{ae} g^0(D_1)$, a contradiction. Similarly, $D_1 \setminus D_1^\dagger \in I$; so $D_1 \in I^*$, a contradiction.

So $D \in I^*$, hence by trivial changes in g^0, g^1 we get $D = \emptyset$. Let $g = g^1 \cup g^0$. For every $A \in I^*$, $F(A) =_{ae} g(A)$, so as I^* is dense, F is an automorphism, this holds for any A.

As $F(\omega) =_{ae} g(\mathrm{Dom}(g))$, $\mathrm{Rang}(g)$ is co-finite.

Why can we assume g is one-to-one? No integer can have an infinite origin set A, since then $F(A)$ is infinite while $g(A)$ is a singleton. Only finitely many integers can have a non-singleton origin set, otherwise we would have two disjoint infinite sets with the same infinite image under g. So we can throw out the problematic finite part of g.

Thus F is trivial. $\square_{5.6,5.5}$