III. Proper Forcing

§0. Introduction

In Sect. 1 we introduce the property "proper" of forcing notions: preserving stationarity not only of subsets of ω_1 but even of any $S \subseteq \mathcal{S}_{\leq\aleph_0}(\lambda)$. We then prove its equivalence to another formulation.

In Sect. 2 we give more equivalent formulations of properness, and show that c.c.c. forcing notions and $\aleph_1$-complete ones are proper.

In Sect. 3 we prove that countable support iteration preserves properness (another proof, for a related iteration, found about the same time is given in IX 2.1; others are given in X §2 (with revised support) and XII §1 (by games)). Also we give a proof by Martin Goldstern (in §3).

In Sect. 4 it is proved that starting with V with one inaccessible κ, for some forcing notion P: P is proper of cardinality κ, do satisfy the κ-c.c. and $\Vdash_P$ "*if* Q is a forcing notion of cardinality $\aleph_1$, not destroying stationarity of subsets of ω_1 and $\mathcal{I}_i \subseteq Q$ is dense for $i < \omega_1$, *then* for some directed $G \subseteq Q$, $\bigwedge_{i<\omega_1} G \cap \mathcal{I}_i \neq \emptyset$". For this we need to give a sufficient condition for $\mathrm{Lim}\bar{Q}$ to satisfy the κ-c.c. (where $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \kappa\rangle$ is a CS iteration of proper forcing such that for each $i < \kappa$ we have $\Vdash_{P_i}$ "$|\underset{\sim}{Q}_i| < \kappa$"). For this we show that the family of hereditarily countable conditions is dense in each P, so $i < \kappa \Rightarrow P_i$ has density $< \kappa$.

In sections 5, 6 we present known theorems on speciality of Aronszajn trees.

In Sect. 7 we prove: for V satisfying CH there is an $\aleph_2$-c.c. proper P such that $\Vdash_P$ " *if* for $i < \omega_1$, the set $A_i \subseteq \omega_1$ is countable with no last element and $\mathrm{otp}(A_i) < \sup(A_i)$, *then* for some club C of ω_1 we have $i < \omega_1 \Rightarrow \sup(C \cap A_i) < \sup(A_i)$".

In Sect. 8 we prove the consistency of the Kurepa hypothesis (first proved by Silver [Si67] and see more Devlin). This is a proof from the author's lecture in 1987.

§1. Introducing Properness

1.1 Discussion. When we iterate we are faced with the problem of obtaining for the iteration the good properties of the single steps of iteration. Usually, in our context, the worst possible vice of a forcing notion is that it collapses $\aleph_1$. The virtue of not collapsing $\aleph_1$ is not inherited by the iteration from its single components. As we saw, the virtue of the c.c.c. is inherited by the FS iteration from its components. However in many cases the c.c.c. is too strong a requirement. We shall look for a weaker requirement which is more naturally connected to the property of not collapsing $\aleph_1$, and which is inherited by suitable iterations.

We shall now study a certain generalization of the concepts of a closed unbounded and a stationary subset of ω_1. They were introduced and investigated by Jech and Kueker.

1.2 Definition. For A uncountable let $\mathcal{S}_{\aleph_0}(A) = \{s : s \subseteq A, |s| \leq \aleph_0\}$. Let $W \subseteq \mathcal{S}_{\aleph_0}(A)$ be called closed if it is closed under unions of increasing (by $\subseteq$ of course) ω-sequences. $W \subseteq \mathcal{S}_{\aleph_0}(A)$ is called unbounded (in $\mathcal{S}_{\aleph_0}(A)$) if for every $s \in \mathcal{S}_{\aleph_0}(A)$ there is a $t \in W$ such that $t \supseteq s$. If $W \subseteq \mathcal{S}_{\aleph_0}(A)$, the closure of W is $\mathrm{cl}(W) = \{\bigcup_{n<\omega} s_n : s_n \in W \text{ and } s_n \subseteq s_{n+1} \text{ for } n < \omega\}$, (clearly $W \subseteq \mathrm{cl}(W)$ and $\mathrm{cl}(W)$ is closed).

1.3 Lemma. The intersection of $\aleph_0$ closed unbounded subsets W_i, $i < \omega$, of $\mathcal{S}_{\aleph_0}(A)$ is a closed unbounded subset of $\mathcal{S}_{\aleph_0}(A)$.

Proof. Since each set W_i is closed, the intersection $\cap_{i<\omega} W_i$ is obviously closed too. Let us prove now that $\cap_{i<\omega} W_i$ is unbounded too. Let $s \in \mathcal{S}_{\aleph_0}(A)$; we have to prove the existence of a set $t \supseteq s$ such that $t \in \cap_{i<\omega} W_i$. We shall define a sequence $\langle s_\alpha : \alpha < \omega^2 \rangle$ of members of $\mathcal{S}_{\aleph_0}(A)$ as follows. Let $s_0 = s$, for $\alpha > 0$, $\alpha = \omega \cdot k + \ell$ choose s_α as an arbitrary member of W_ℓ which includes $\cup_{\beta<\alpha} s_\beta \in \mathcal{S}_{\aleph_0}(A)$; it exists as W_ℓ is unbounded. We take now $t = \cup_{\alpha<\omega^2} s_\alpha$. Obviously $t \supseteq s_0 = s$. For a fixed $i < \omega$ and every $\alpha < \omega^2$, let $\alpha = \omega \cdot k + \ell$ then $\alpha < \omega(k+1) + i$, hence, by the definition of $s_{\omega(k+1)+i}$ we have $s_\alpha \subseteq s_{\omega(k+1)+i}$. Therefore $t = \cup_{\alpha<\omega^2} s_\alpha = \cup_{k<\omega} s_{\omega \cdot k+i}$. The sequence $\langle s_{\omega \cdot k+i} : k < \omega \rangle$ is a $\subseteq$-increasing ω-sequence of members of W_i, and since W_i is closed also its union t is in W_i. Thus $t \in \cap_{i<\omega} W_i$, which is what we had to prove. $\square_{1.3}$

1.4 Definition. By the last lemma we know that the closed unbounded subset of $\mathcal{S}_{\aleph_0}(A)$ generate an $\aleph_1$-complete filter, namely the filter of all subsets of $\mathcal{S}_{\aleph_0}(A)$ which include a closed unbounded set. We denote this filter with $\mathcal{D}_{\aleph_0}(A)$ or $\mathcal{D}_{<\aleph_1}(A)$ or $\mathcal{D}(A)$. A subset of $\mathcal{S}_{\aleph_0}(A)$ is called *stationary* if it meets every closed unbounded subset of $\mathcal{S}_{\aleph_0}(A)$, i.e., if it meets every member of $\mathcal{D}_{\aleph_0}(A)$.

We shall now present the lemma which says that for $|A| = \aleph_1$, $\mathcal{S}_{\aleph_0}(A)$ and $\mathcal{D}_{\aleph_0}(A)$ do not differ significantly from ω_1 and the filter $\mathcal{D}_{\omega_1}$ generated by the closed unbounded subsets of ω_1.

1.5 Lemma. A subset of ω_1 is a closed unbounded subset of ω_1 (in the usual sense of a closed unbounded subset of an ordinal) iff it is a closed unbounded subset of $\mathcal{S}_{\aleph_0}(\omega_1)$.

Proof. Easy. $\square_{1.5}$

We shall now introduce a more restricted set of generators for $\mathcal{D}(A)$.

1.6 Definition. M will denote an algebra, with universe A, and with countably many functions. Let

$Sm(M) = \{s : s \subseteq A, |s| \leq \aleph_0, s \text{ is closed under the operations of } M\}$, i.e., $Sm(M)$ is the set of countable subalgebras of M. Now $Sm(M)$ is obviously a closed unbounded subset of $\mathcal{S}_{\aleph_0}(A)$ (even if M is a partial algebra).

A subset of $\mathcal{S}_{\leq\aleph_0}(A)$ of the form $Sm(M)$, is called an Sm-generator of $\mathcal{D}(A)$.

1.7 Lemma. For every closed unbounded subset W of $\mathcal{S}_{\aleph_0}(A)$ there is an algebra M on A such that $Sm(M) \subseteq W$.

Proof. We shall define, for every finite sequence $\bar{a} = \langle a_0, \ldots, a_{n-1}\rangle$ of members of A, by induction on the length n, a set $s(\bar{a}) \in W$ such that $s(\bar{a}) \supseteq \{a_0, \ldots, a_{n-1}\}$ and $s(\bar{a}) \supseteq s(\langle a_0, \ldots, a_{n-2}\rangle)$ when $n \geq 1$ (of course if $n = 1$ $\langle a_0, \ldots, a_{n-2}\rangle$ is the empty sequence). This is obviously possible because W is unbounded. We define now n-place functions $F^n_\ell, \ell < \omega$, for all $n < \omega$ such that $s(\langle a_0, \ldots, a_{n-1}\rangle) = \{F^n_\ell(a_0, \ldots, a_{n-1}) : \ell < \omega\}$. Let $M = (A, F^n_\ell)_{n<\omega,\ell<\omega}$. Let $s = \{a_0, a_1, \ldots\}$ be a subalgebra of M. Denote $s_n = s(\langle a_0, \ldots, a_{n-1}\rangle) = \{F^n_\ell(a_0, \ldots, a_{n-1}) : \ell < \omega\}$. We have:

a) $s_n \subseteq s$, since s is a subalgebra.
b) $s_n \subseteq s_{n+1}$, by definition of $s(\langle a_0, \ldots, a_n\rangle)$.
c) $a_n \in s_{n+1}$, also the choice of $s(\langle a_0, \ldots, a_{n-1}\rangle)$
d) $s_n \in W$.

By (a) and (c) we have $s = \cup_{n<\omega} s_n$; by (b) and (d) we get $s \in W$. Thus we have shown $Sm(M) \subseteq W$. $\square_{1.7}$

We have now seen that the filter $\mathcal{D}_{\aleph_0}(A)$ is generated by the family of sets $Sm(M)$ where M is an algebra on A as above. We shall now see one use of this fact.

1.8 Theorem. Let $P \in V$ be a forcing notion which satisfies the c.c.c., let λ be an uncountable cardinal, and let G be a generic subset of P over V. Every closed unbounded subset B of $\mathcal{S}_{\aleph_0}(\lambda)^{(V[G])}$ in $V[G]$ includes the closure of a set which is a closed unbounded subset of $\mathcal{S}_{\aleph_0}(\lambda)^{(V)}$ in V. In fact $\mathcal{D}_{\aleph_0}(\lambda)^{(V[G])}$ is generated by the closures of the Sm-generators of $\mathcal{D}_{\aleph_0}(\lambda)^{(V)}$ (for any Sm-generator $(Sm(M)^V$ of $\mathcal{D}(\lambda)^{(V)}$ in V, its closure in $V[G]$, is an Sm-generator of $\mathcal{D}(\lambda)^{(V[G])}$ in $V[G]$, as it is $(Sm(M))^{(V[G])})$.

Proof. By what we have proved above we have the following in $V[G]$. There is an algebra $M = (\lambda, F^n_\ell)_{n,\ell<\omega}$ such that $Sm(M) \subseteq B$. In V the function F^n_ℓ has a name $\underset{\sim}{F}^n_\ell$ (moreover we have in V the sequence $\langle \underset{\sim}{F}^n_\ell : n < \omega, \ell < \omega\rangle$). W.l.o.g. $\Vdash$ "$\underset{\sim}{F}^n_\ell$ is an n-place function from λ to λ". Because of the c.c.c., by Lemma I. 3.6 (ii) for all $\alpha_0, \dots, \alpha_{n-1} < \lambda$ we know in V that the set of possible values of $\underset{\sim}{F}^n_\ell(\alpha_0, \dots, \alpha_{n-1})$ is countable and not empty. We define the functions $F^n_{\ell,k}$ for $k < \omega$ so that these $\leq \aleph_0$ values are $\{F^n_{\ell,k}(\alpha_0, \dots, \alpha_{n-1}) : k < \omega\}$. So we know, for all $n, \ell < \omega$ and $\alpha_0, \dots, \alpha_{n-1} < \lambda$ that in $V[G]$ we have $F^n_\ell(\alpha_0, \dots, \alpha_{n-1}) \in \{F^n_{\ell,k}(\alpha_0, \dots, \alpha_{n-1}) : k < \omega\}$. So $N = (\lambda, F^n_{\ell,k})_{n,\ell,k<\omega}$ is an algebra in V and (in $V[G]$) every subalgebra of N is clearly a subalgebra of M. We have $Sm(N)^{(V)} \subseteq Sm(N)^{V[G]} \subseteq Sm(M)^{(V[G])} \subseteq B$ and $Sm(N)^{(V)}$ is a closed unbounded subset of $\mathcal{S}_{\aleph_0}(\lambda)^{(V)}$ in V, and the closure $Sm(N)^V$ in $V[G]$ is $Sm(N)^{V[G]} \subseteq B$. $\square_{1.8}$

A consequence of this theorem is that in a c.c.c. extension $V[G]$ of V every stationary subset of $\mathcal{S}_{\aleph_0}(\lambda)^{(V)}$ in V is also a stationary subset of $\mathcal{S}_{\aleph_0}(\lambda)^{(V[G])}$ in $V[G]$; in short, the extension does not destroy the stationarity of stationary subsets of $\mathcal{S}_{\aleph_0}(\lambda)$. We shall use this property to define the concept of proper forcing. While it is a consequence of the fact that $\mathcal{D}_{\aleph_0}(\lambda)^{V[G]}$ is generated by the closures of the members of $\mathcal{D}_{\aleph_0}(\lambda)^V$ is does not seem to require as much.

1.9 Definition. A forcing notion P is called *proper* if for every (uncountable) cardinal λ, forcing with P preserves stationarity modulo $\mathcal{D}_{\aleph_0}(\lambda)$. We shall denote this condition for λ with $Con_1(\lambda)$ (more exactly $\mathrm{Con}_1(\lambda, P)$ but we omit P when, as usual, P is clear from the context). Note that $Con_1(\aleph_0)$ is meaningless, or trivially true if you like.

Note that properness is preserved by equivalence of forcing notions.

1.10 Theorem. 1) P is proper *iff* the following condition holds for each cardinal λ.

$Con_2(\lambda) = Con_2(\lambda, P)$: Assume that $\{p_{i,j} : j < \alpha_i, i < \alpha\} \subseteq P$ is such that $\alpha \leq \lambda$ and $\alpha_i \leq \lambda$ for $i < \alpha$, and such that for all $i < \alpha$ the set $\{p_{i,j} : j < \alpha_i\}$ is pre-dense in P. Then for all $p \in P$:

$$\{s \in \mathcal{S}_{\aleph_0}(\lambda) : (\exists q \in P)[q \geq p \text{ and } \{p_{i,j} : j < \alpha_i, j \in s\} \text{ is pre-dense above } q \text{ for any } i \in s, i < \alpha]\} \in \mathcal{D}_{\aleph_0}(\lambda).$$

2) Moreover, for any $\lambda \geq |P| + \aleph_1, Con_1(\lambda)$ is equivalent to $Con_2(\lambda)$; and if P is a complete Boolean algebra without 1 then $Con_1(\lambda)$ is equivalent to $Con_2(\lambda)$ for every uncountable λ.

Proof. We assume first $\neg Con_2(\lambda)$, i.e., there are $\alpha \leq \lambda$, $\alpha_i \leq \lambda$ for $i < \alpha$, $\{p_{i,j} : j < \alpha_i\}$ which is pre-dense in P and $p \in P$ such that the set
$T \stackrel{\text{def}}{=} \{s \in \mathcal{S}_{\aleph_0}(\lambda)$: for no $q \in P$ do we have: $q \geq p$ and $\{p_{i,j} : j < \alpha_i, j \in s\}$ is pre-dense above q for any $i \in s, i < \alpha\}$ is stationary.
Let $G \subseteq P$ be a generic subset of P such that $p \in G$. Now G meets every pre-dense set hence there is in $V[G]$ a function f on α such that $p_{i,f(i)} \in G$ for all $i < \alpha$. For the algebra (λ, f) we have $Sm((\lambda, f)) = Sm((\lambda, f))^{V[G]} \in \mathcal{D}_{\aleph_0}(\lambda)^{(V[G])}$. We shall show that $T \cap Sm((\lambda, f)) = \emptyset$ thus T which is stationary in V is no longer stationary in $V[G]$. Assume $s \in T \cap Sm((\lambda, f))$, then, as $T \in V$, clearly $s \in V$. Since $s \in Sm((\lambda, f))$ clearly s is closed under f hence $V[G] \models (\forall i \in s)$ $(i < \alpha \rightarrow (\exists j \in s)(j < \alpha_i \,\&\, p_{i,j} \in G))$, hence some $r \in G$ forces this statement. Since G is directed and $p \in G$ there is a $q \in G$ such that $q \geq p, r$ hence $q \Vdash_P$ "$(\forall i \in s)(i < \alpha \rightarrow (\exists j \in s)\ (j < \alpha_i \,\&\, p_{i,j} \in \underset{\sim}{G}_P))$". Therefore for every $i \in s$, such that $i < \alpha$ we know $\{p_{i,j} : j < \alpha_i \,\&\, j \in s\}$ is pre-dense above q (if this were not the case then for some $q^* \geq q$, q^* is incompatible with each member of $\{p_{i,j} : j < \alpha_i \,\&\, j \in s\}$ for some $i \in s$, $i < \alpha$, and for a generic G which contains q^* we cannot have $(\exists j \in s)(j < \alpha_i \,\&\, p_{i,j} \in G))$. Thus s satisfies exactly the condition of not belonging to T, which is a contradiction.

We have proved that $\neg Con_2(\lambda)$ implies $\neg Con_1(\lambda)$. We shall prove that $Con_2(\lambda)$ implies $Con_1(\lambda)$ for $\lambda \geq |P| + \aleph_1$, or for all $\lambda \geq \aleph_1$ if P is a complete

Boolean algebra without 1. This suffices to finish the proof of part (2) (of 1.10). It also suffices for part (1) i.e. for proving that $(\forall\lambda)Con_2(\lambda)$ implies $(\forall\lambda)Con_1(\lambda)$ since, as we shall now see if $\lambda > \mu \geq \aleph_1, Con_1(\lambda)$ implies $Con_1(\mu)$ (see 1.13, note that by 2.1 if $\lambda > \mu$, $Con_2(\lambda)$ implies $Con_2(\mu)$). For this purpose we shall prove the following lemmas (1.11, 1.12, 1.13 and then return to the proof of 1.10).

1.11 Lemma. For any sets D, E we denote by $D\bar{\cup}E$ the set $\{x \cup y : x \in D \,\&\, y \in E\}$. For all disjoint uncountable sets A, B we have: W is a closed unbounded (or stationary) subset of $\mathcal{S}_{\aleph_0}(A)$ iff $W\bar{\cup}\mathcal{S}_{\aleph_0}(B)$ is a closed unbounded (or stationary) subset of $\mathcal{S}_{\aleph_0}(A \cup B)$.

Proof. We can deal separately with the case any of the sets is empty, easily, so w.l.o.g. they are not empty. The proof that if W is closed unbounded in $\mathcal{S}_{\aleph_0}(A)$ then $W\bar{\cup}\mathcal{S}_{\aleph_0}(B)$ is closed unbounded in $\mathcal{S}_{\aleph_0}(A \cup B)$ is trivial. Now assume that W is stationary in $\mathcal{S}_{\aleph_0}(A)$, and suppose $W\bar{\cup}\mathcal{S}_{\aleph_0}(B)$ is not stationary in $\mathcal{S}_{\aleph_0}(A \cup B)$. Then there is a model $M = (A \cup B, F^n_\ell)_{n,\ell<\omega}$ such that $(W\bar{\cup}\mathcal{S}_{\aleph_0}(B)) \cap Sm(M) = \emptyset$. We can assume, without loss of generality, that the set of functions $\{F^n_\ell : n, \ell < \omega\}$ is closed under substitution. We define a function $\hat{F}^n_\ell$ for n-tuples of members of A as follows:

$$\hat{F}^n_\ell(a_0, \dots, a_{n-1}) = \begin{cases} F^n_\ell(a_0, \dots, a_{n-1}) & \text{if } F^n_\ell(a_0, \dots, a_{n-1}) \in A \\ \text{any member of } A & \text{otherwise} \end{cases}$$

Let $\hat{M} = (A, \hat{F}^n_\ell)_{n,\ell<\omega}$. We shall see that if $s \in Sm(\hat{M})$ then for some $t \in \mathcal{S}_{\aleph_0}(B)$, $s \cup t \in Sm(M)$. Let t be the subalgebra of M generated by s; we have to prove that $t \setminus s \subseteq B$. Let $b \in t \setminus s$, then since the set $\{F^n_\ell : n, \ell < \omega\}$ is closed under substitution, $b = F^n_\ell(a_0, \dots, a_{n-1})$ for some $n < \omega$, $\ell < \omega$ and $a_0, \dots, a_{n-1} \in s$. If $b \in A$ then by the definition of $\hat{F}^n_\ell$ we know that $\hat{F}^n_\ell(a_0, \dots, a_{n-1}) = b$, and since $s \in Sm(M)$, s is closed under $\hat{F}^n_\ell$, clearly $b \in s$, which cannot be the case since $b \in t \setminus s$. Therefore $b \notin A$, hence $b \in B$ and we have proved $t \setminus s \subseteq B$. We claim that $W \cap Sm(\hat{M}) = \emptyset$, contradicting our assumption that W is a stationary subset of $\mathcal{S}_{\aleph_0}(A)$. Suppose $s \in W \cap Sm(\hat{M})$, then, as we have shown, for some $t \in \mathcal{S}_{\aleph_0}(B)$ we have $s \cup t \in Sm(M)$. However $s \cup t \in W\bar{\cup}\mathcal{S}_{\aleph_0}(B)$ contradicting $(W\bar{\cup}\mathcal{S}_{\aleph_0}(B)) \cap Sm(M) = \emptyset$.

Thus we have proved the "only if" part. The "if" part can be proved similarly or by applying the "only if" part to $W_1 = \mathcal{S}_{\aleph_0}(A) \setminus W$. $\square_{1.11}$

1.12 Claim. (1) If f is a one-to-one function from A into B, then for $X \subseteq \mathcal{S}_{\aleph_0}(A)$: X is a stationary subset of $\mathcal{S}_{\aleph_0}(A)$ iff $\{a \in \mathcal{S}_{\aleph_0}(B) : f^{-1}(a) \in X\}$ is a stationary subset of $\mathcal{S}_{\aleph_0}(B)$.
(2) If $f : A \to B$ is one-to-one onto, *then* f induces a mapping from $\mathcal{P}(\mathcal{S}_{\aleph_0}(B))$ onto $\mathcal{P}(\mathcal{S}_{\aleph_0}(A))$ preserving Boolean operations and stationarity.
(3) If $V \subseteq V^\dagger$ are models of ZFC, $A, B \in V$, $V \models$ "$|A| = |B|$", then the stationarity of some $X \subseteq \mathcal{S}_{\aleph_0}(B)^V$ is destroyed in $V^\dagger$ iff the stationarity of some $X \subseteq \mathcal{S}_{\aleph_0}(A)^V$ is destroyed in $V^\dagger$ (where $X \in V$ of course)

Proof. Note that $\{a \subseteq \mathcal{S}_{\aleph_0}(A \cup B)$: if $y = f(x)$ then $x \in a \Leftrightarrow y \in a\} \in \mathcal{D}_{\aleph_0}(A \cup B)$.

The proof is left to the reader. $\square_{1.12}$

1.13 Claim. If $\lambda > \mu \geq \aleph_1$ then $Con_1(\lambda)$ implies $Con_1(\mu)$.

Proof. Let W be a stationary subset of $\mathcal{S}_{\aleph_0}(\mu)^V$. Then, as we have proved in 1.11, $\mathcal{S}_{\aleph_0}(\lambda \setminus \mu)^V\bar{\cup}W$ is a stationary subset of $\mathcal{S}_{\aleph_0}(\lambda)^V$. Since $Con_1(\lambda)$ holds $\mathcal{S}_{\aleph_0}(\lambda\setminus\mu)^V\bar{\cup}W$ is also a stationary subset of $\mathcal{S}_{\aleph_0}(\lambda)^{V[G]}$ in $V[G]$. We claim that W is a stationary subset of $\mathcal{S}_{\aleph_0}(\mu)^{V[G]}$ in $V[G]$. If this is not the case then there is a closed unbounded subset C of $\mathcal{S}_{\aleph_0}(\mu)^{V[G]}$ in $V[G]$ such that $C \cap W = \emptyset$. By Lemma 1.11 $\mathcal{S}_{\aleph_0}(\lambda \setminus \mu)^{V[G]}\bar{\cup}C$ is a closed unbounded subset of $\mathcal{S}_{\aleph_0}(\lambda)^{V[G]}$ in $V[G]$. Since $C \cap W = \emptyset$ we have $(\mathcal{S}_{\aleph_0}(\lambda \setminus \mu)^{V[G]}\bar{\cup}C) \cap (\mathcal{S}_{\aleph_0}(\lambda \setminus \mu)^V\bar{\cup}W) = \emptyset$ contradicting what we got that $\mathcal{S}_{\aleph_0}(\lambda \setminus \mu)^V\bar{\cup}W$ is a stationary subset of

$\mathcal{S}_{\aleph_0}(\lambda)^{V[G]}$ in $V[G]$. $\square_{1.13}$

Continuation of the Proof of 1.10. We return now to the proof that $Con_2(\lambda)$ implies $Con_1(\lambda)$ for any uncountable $\lambda \geq |P| + \aleph_1$ or for all $\lambda \geq \aleph_1$ if P is a complete Boolean algebra without 1. Let T be a stationary subset of $\mathcal{S}_{\aleph_0}(\lambda)$ in V. To prove that T is also a stationary subset of $\mathcal{S}_{\aleph_0}(\lambda)$ in $V[G]$ we have to prove that for every P-name $\underset{\sim}{M} = (\lambda, \underset{\sim}{F}^n_\ell)_{\ell,n<\omega}$ of an algebra, $\emptyset \Vdash_P$ "$T \cap Sm(\underset{\sim}{M}) \neq \emptyset$". Let $p \in P$, we shall prove that there is a $q \geq p$ such that $q \Vdash_P$ "$T \cap Sm(\underset{\sim}{M}) \neq \emptyset$". Let $h : {}^{\omega>}\lambda \to \lambda$ be a one-to-one function. We denote the restriction of h to n-tuples with h^n. Let h_ℓ, for $\ell < \omega$ be a function such that for $n > \ell$ we have $h_\ell(h^n(\beta_0, \ldots, \beta_{n-1})) = \beta_\ell$. For each $i < \lambda$ if $i = h(n, \ell, \beta_0, \ldots, \beta_{n-1})$ let $\mathcal{I}_i$ be a maximal antichain of P of conditions which force definite values for $\underset{\sim}{F}^n_\ell(\beta_0, \ldots, \beta_{n-1})$. If $|P| \leq \lambda$ then clearly $|\mathcal{I}_i| \leq \lambda$. If P is a Boolean algebra without 1 (and the order inherited from the Boolean algebra, not the inverse, so "x, y incompatible means $\models x \cup y = 1$), then for each $\beta < \lambda$ we can put in $\mathcal{I}_i$ the minimal condition (equivalently the lub in the Boolean algebra of all conditions) which force $\underset{\sim}{F}^n_\ell(\beta_0, \ldots, \beta_{n-1}) = \beta$, if there are such conditions and then $\mathcal{I}_i$ will be a maximal set of conditions which force definite values on $\underset{\sim}{F}^n_\ell(\beta_0, \ldots, \beta_{n-1})$ and $|\mathcal{I}_i| \leq \lambda$ and $\mathcal{I}_i$ is a maximal antichain. We define for $i < \lambda$ the ordinal α_i and the set $\{p_{i,j} : j < \alpha_i\}$ so that $\{p_{i,j} : j < \alpha_i\} = \mathcal{I}_i$. Let $\gamma(i, j)$ be such that $p_{i,j} \Vdash$ "$\underset{\sim}{F}^n_\ell(\beta_0, \ldots, \beta_{n-1}) = \gamma(i, j)$". Let $M^* = (\lambda, h^n, h_n, \gamma, n)_{n<\omega}$, then $Sm(M^*) \in \mathcal{D}_{\aleph_0}(\lambda)$. Let $W \stackrel{\text{def}}{=} \{s \in \mathcal{S}_{\aleph_0}(\lambda) : (\exists q \geq p)\ [\{p_{i,j} : j \in \alpha_i \cap s\}$ is pre-dense over q for all $i \in s]\}$; we know that $W \in \mathcal{D}_{\aleph_0}(\lambda)$ by the assumption $Con_2(\lambda)$. Since T is a stationary subset of $\mathcal{S}_{\aleph_0}(\lambda)$ there is an $s \in T \cap Sm(M^*) \cap W$. Since $s \in W$ let $q \geq p$ be such that $\{p_{i,j} : j \in \alpha_i \cap s\}$ is pre-dense above q for each $i \in s$. Assume $\beta_0, \ldots, \beta_{n-1} \in s$. By definition of M^* and since $s \in Sm(M^*)$, $\omega \subseteq s$. Thus $n, \ell \in s$ and since s is closed under h^{n+2} also $i = h^{n+2}(n, \ell, \beta_0, \ldots, \beta_{n-1}) \in s$. Since $\{p_{i,j} : j \in \alpha_i \cap s\}$ is pre-dense over q, every generic filter G which contains q contains some $p_{i,j}$ for $j \in \alpha_i \cap s$, and therefore $F^n_\ell(\beta_0, \ldots, \beta_{n-1}) = \gamma(i, j)$ in $V[G]$. Since $i, j \in s$ and since s is closed under the function γ we have $F^n_\ell(\beta_0, \ldots, \beta_{n-1}) \in s$. Thus we have in $V[G]$ that $\forall n \forall \ell (\forall \beta_0, \ldots, \beta_{n-1} \in s)\ [F^n_\ell(\beta_0, \ldots, \beta_{n-1}) \in s]$, hence $s \in Sm(\underset{\sim}{M})$. Since this holds for every G which contains q we have $q \Vdash$ "$s \in Sm(\underset{\sim}{M})$", i.e., $q \Vdash$ "$T \cap Sm(\underset{\sim}{M}) \neq \emptyset$" (since $s \in T$). So T is still stationary in V^P, as required.

$\square_{1.10}$

1.15 Observations. It can be seen, by means of $Con_2(\lambda)$ that in order that the forcing P be proper it suffices to require $Con_2(\lambda)$, or $Con_1(\lambda)$ for some $\lambda \geq 2^{|P|}$ (see Lemma 2.2). We can also replace, equivalently, $\alpha, \alpha_i \leq \lambda$ in $Con_2(\lambda)$ by $\alpha = \alpha_i = \lambda$, and we can replace the pre-dense sets by maximal antichains and $2^{|P|}$ by the number of the maximal antichains (see Lemma 2.2 in the next section). $\square_{1.15}$

1.16 Lemma. If P is a proper forcing *then* in $V[G]$ every countable set of ordinals is included in a countable set of ordinals of V (and hence $\aleph_1^V$ is uncountable in $V[G]$).

Proof. Let a be a countable set of ordinals in $V[G]$, then for some cardinal λ we have $a \in \mathcal{S}_{\aleph_0}(\lambda)^{V[G]}$; now the set $\{s \in \mathcal{S}_{\aleph_0}(\lambda)^{V[G]} : s \supseteq a\}$ is obviously a closed unbounded subset of $\mathcal{S}_{\aleph_0}(\lambda)^{V[G]}$ in $V[G]$. But $\mathcal{S}_{\aleph_0}(\lambda)^V$ is a stationary subset of $\mathcal{S}_{\aleph_0}(\lambda)^V$ in V, hence, since the forcing P is proper, it is also a stationary subset of $\mathcal{S}_{\aleph_0}(\lambda)^{V[G]}$ in $V[G]$. Thus $\mathcal{S}_{\aleph_0}(\lambda)^V \cap \{s \in \mathcal{S}_{\aleph_0}(\lambda)^{V[G]} : s \supseteq a\} \neq \emptyset$ and λ has a subset countable in V which includes a.

An alternative proof is that if $s \in \mathcal{S}_{\aleph_0}(\lambda)^{V[G]}$, then $W_1 = \{t : s \subseteq t \in \mathcal{S}_{\aleph_0}(\lambda)\} \in \mathcal{D}_{\aleph_0}(\lambda)$ in $V[G]$, but in V we have $W_0 = \mathcal{S}_{\aleph_0}(\lambda)^V \in \mathcal{D}_{\aleph_0}(\lambda)$, so in V W_0 is stationary, hence it is stationary in $V[G]$ hence $W_0 \cap W_1 \neq 0$ which is just what we need. $\square_{1.16}$

As a consequence, if α is an ordinal such that $\mathrm{cf}(\alpha) > \aleph_0$ in V, we have also $\mathrm{cf}(\alpha) > \aleph_0$ in $V[G]$.

§2. More on Properness

Discussion. It is worth noticing that one can use for a set of generators of $\mathcal{D}_{\aleph_0}(A)$ not only the set $\{Sm(M) : M$ is a model, the universe of M is A and M is an algebra with $\aleph_0$ operations $\}$ but also a somewhat wider set $\{Sm(M)$: M is a model, the universe of M is A and M is a partial algebra with $\aleph_0$ operations $\}$. This can be done since if M is a partial algebra, i.e., an algebra whose operations are not necessarily defined for all arguments, then $Sm(M)$ is also a closed unbounded subset of $\mathcal{S}_{\aleph_0}(A)$.

2.1 Claim. For $\mu < \lambda$, $Con_2(\lambda, P) \Rightarrow Con_2(\mu, P)$.

Proof. To see that let $\{p_{i,j} : j < \alpha_i, i < \alpha\}$ be as required by $Con_2(\mu)$, i.e., $\alpha \leq \mu$ and $\alpha_i \leq \mu$ for $i < \alpha$. Since $\mu < \lambda$ we can apply $Con_2(\lambda)$ and obtain $D \stackrel{\text{def}}{=} \{s \in \mathcal{S}_{\aleph_0}(\lambda) : \exists q[p \leq q \in P \,\&\, (\forall i \in s \cap \alpha)\ [\{p_{i,j} : j \in s \cap \alpha_i\}$ is pre-dense above $q]]\} \in \mathcal{D}_{\aleph_0}(\lambda)$. Since $\alpha_i \leq \mu$ for $i < \alpha$ and $\alpha \leq \mu$ we have $D = (D \cap \mathcal{S}_{\aleph_0}(\mu))\bar{\cup}\mathcal{S}_{\aleph_0}(\lambda \setminus \mu)$ (where $A\bar{\cup}B = \{x \cup y : x \in A \,\&\, y \in B\}$). By Lemma 1.11 for $T \subseteq \mathcal{S}_{\aleph_0}(A)$, if $T\bar{\cup}\mathcal{S}_{\aleph_0}(B) \in \mathcal{D}_{\aleph_0}(A \cup B)$ then $T \in \mathcal{D}_{\aleph_0}(A)$. Therefore $D \cap \mathcal{S}_{\aleph_0}(\mu) \in \mathcal{D}_{\aleph_0}(\mu)$ which establishes $Con_2(\mu)$ since $D \cap \mathcal{S}_{\aleph_0}(\mu)$ is exactly the set required for $Con_2(\mu)$. $\square_{2.1}$

2.2 Lemma. $Con_2(2^{|P|}) \Rightarrow (\forall \lambda \geq \aleph_0)Con_2(\lambda)$, and hence, since $\mu < \lambda$ and $Con_2(\lambda) \Rightarrow Con_2(\mu)$, therefore $(\exists \sigma \geq 2^{|P|})(Con_2(\sigma)) \Leftrightarrow (\forall \lambda)Con_2(\lambda)$.

Proof. It clearly suffices (see 2.1) to prove that for $\lambda > 2^{|P|}$ we have $Con_2(2^{|P|})$ $\Rightarrow Con_2(\lambda)$. Let p, $\langle p_{i,j} : j < \alpha_i, i < \alpha\rangle$ be as in $Con_2(\lambda)$. Let $\mathcal{I}_i$ denote the subset $\{p_{i,j} : j < \alpha_i\}$ of P. Let $\langle \mathcal{J}_i : i < 2^{|P|}\rangle$, be a listing possibly with repetitions of all pre-dense subsets of P. Let $\langle q_{i,j} : j < \beta_i\rangle$ be a listing of the members of $\mathcal{J}_i$, then we can have $\beta_i \leq |P|$. We define a partial function $F : \lambda \to 2^{|P|}$ by $F(i)$ = the first γ such that $\mathcal{J}_\gamma = \mathcal{I}_i$, for $i < \alpha$. We define also two partial functions G and H on $\lambda \times \lambda$, into λ by $G(i,j) =$ the γ such that $p_{i,j} = q_{F(i),\gamma}$, for $i < \alpha$, $j < \alpha_i$, and $H(i,j) =$ the least γ such that $p_{i,\gamma} = q_{F(i),j}$, for $i < \alpha$, $j < \beta_{F(i)}$. Since $Con_2(2^{|P|})$ holds the set $A \stackrel{\text{def}}{=} \{s \in \mathcal{S}_{\aleph_0}(2^{|P|}) : (\exists q \geq p)(\forall i \in s)(\{q_{i,j} : j \in s \cap \beta_i\}$ is pre-dense above $q)\}$ is in $\mathcal{D}_{\aleph_0}(2^{|P|})$. Therefore there is a partial algebra M with universe $2^{|P|}$ such that $A \supseteq Sm(M)$. Let N be the partial algebra whose universe is λ and whose partial operations are those of M together with F, G and H (which were defined above). We shall show that for every $s \in Sm(N)$ there is a $q \geq p$ such that for all $i \in s \cap \alpha$, $\{p_{i,j} : j \in s \cap \alpha_i\}$ is pre-dense above q. This will establish

$Con_2(\lambda)$ since the set which is required by $Con_2(\lambda)$ to be in $\mathcal{D}_{\aleph_0}(\lambda)$ has been shown to include $Sm(N)$ which is in $\mathcal{D}_{\aleph_0}(\lambda)$.

Let $s \in Sm(N)$; since N contains all the partial operations of M we have $s \cap 2^{|P|} \in Sm(M)$. Since $Sm(M) \subseteq A$ we have $s \cap 2^{|P|} \in A$; therefore there is a $q \geq p$ such that

$\otimes$ $(\forall i \in s \cap 2^{|P|})(\{q_{i,j} : j \in s \cap \beta_i\}$ is pre-dense above $q)$.

We shall show that for this q we have $(\forall i \in s \cap \alpha)(\{p_{i,j} : j \in s \cap \alpha_i\}$ is pre-dense above $q)$, which is all what is left to prove. Let $i \in s \cap \alpha$, since s is closed under F also $F(i) \in s \cap 2^{|P|}$ (since $\mathrm{Rang}(F) \subseteq 2^{|P|}$) hence, by $\otimes$, $\{q_{F(i),j} : j \in s \cap \beta_i\}$ is pre-dense above q. We shall see that $\{q_{F(i),j} : j \in s \cap \beta_i\} = \{p_{i,j} : j \in s \cap \alpha_i\}$ and this will establish that $\{p_{i,j} : j \in s \cap \alpha_i\}$ is pre-dense above q. For $j \in s \cap \beta_i$ we know $q_{F(i),j} = p_{i,H(i,j)}$ by the definition of H. Since $i, j \in s$ also $H(i,j) \in s$ and $H(i,j) < \alpha_i$ by the definition of H. Thus $q_{F(i),j} = p_{i,H(i,j)} \in \{p_{i,j} : j \in s \cap \alpha_i\}$. In the other direction, for $j \in s \cap \alpha_i$ $p_{i,j} = q_{F(i),G(i,j)} \in \{q_{F(i)} : j \in s \cap \beta_i\}$, since s is closed under G. $\square_{2.2}$

2.3 Theorem. Let $M = (|M|, \ldots,)$ be a model with countably many relations and functions, if M is uncountable then:

$$\{|N| \in \mathcal{S}_{\aleph_0}(|M|) : N \prec M\} \in \mathcal{D}_{\aleph_0}(|M|).$$

Proof. Let $M^\dagger$ be an algebra with universe M and with Skolem functions of M as operations (their choice is not unique, but is immaterial; we can e.g. expand M by a well ordering $<^*$ of its universe, and use all functions definable in $(M, <^*)$). Then, as is well known, $Sm(M^\dagger) \subseteq \{N \in \mathcal{S}_{\aleph_0}(|M|) : N \prec M\}$. Since $Sm(M^\dagger) \in \mathcal{D}_{\aleph_0}(|M|)$ also $\{|N| \in \mathcal{S}_{\aleph_0}(|M|) : N \prec M\} \in \mathcal{D}_{\aleph_0}(|M|)$. $\square_{2.3}$

2.4 Definition. For a cardinal λ we denote with $H(\lambda)$ the set of all sets whose transitive closure is of cardinality $< \lambda$. For a regular uncountable λ we know that $(H(\lambda), \in)$ is a model for all axioms of ZFC except maybe for the power set axiom. If not said otherwise we assume λ is like that, for simplicity.

2.5 Definition. Let N be an elementary substructure of $(H(\lambda), \in)$ and let $P \in N$ be a forcing notion. For $q \in P$ we say that q is (N, P)-generic, (or N-generic if it is clear which P we are dealing with), *if* for every subset $\mathcal{I}$ of P which is pre-dense and is in N the set $\mathcal{I} \cap N$ is pre-dense above q.

2.6 Lemma. A condition q is (N, P)-generic *iff* for every $\underset{\sim}{\tau}$ which is a name of an ordinal in the forcing notion P, if $\underset{\sim}{\tau} \in N$ then $q \Vdash$ "$\underset{\sim}{\tau} \in N$" (i.e., if the name is in N then q forces the value to be in N) *iff* for every P-name $\underset{\sim}{\tau} \in N$, $q \Vdash$ " if $\underset{\sim}{\tau} \in V$ then $\underset{\sim}{\tau} \in N$".

Proof. We prove only the first "iff", the second has the same proof. Assume that q is N-generic and let $\underset{\sim}{\tau} \in N$ be a name of an ordinal. Let $\mathcal{I} = \{r \in P : r \Vdash$ "$\underset{\sim}{\tau} = \alpha$", for some ordinal $\alpha\}$. $\mathcal{I}$ is obviously pre-dense. $\mathcal{I}$ is definable from P and $\underset{\sim}{\tau}$ in $(H(\lambda), \in)$, hence $\mathcal{I} \in N$. Since q is N-generic, $\mathcal{I} \cap N$ is pre-dense above q. Let f be the function on $\mathcal{I}$ defined by $f(r) =$ that α for which $r \Vdash$ "$\underset{\sim}{\tau} = \alpha$", then f is definable in $(H(\lambda), \in)$ from $\underset{\sim}{\tau}$, hence $f \in N$. Since $\mathcal{I} \cap N$ is pre-dense above q, $q \Vdash$ "$\underset{\sim}{G}_P \cap (\mathcal{I} \cap N) \neq \emptyset$", i.e., $q \Vdash$ "$(\exists r \in \mathcal{I} \cap N) r \in \underset{\sim}{G}_P$". Therefore if G is a subset of P generic over V and $q \in G$ then $\underset{\sim}{\tau}[G] = f(r)$ holds in $V[G]$, where $r \in (\mathcal{I} \cap N) \cap G$. Since $r \in N$, also $f(r) \in N$ (as $f \in N$) thus $\underset{\sim}{\tau}[G] \in N$ in $V[G]$. Therefore $q \Vdash$ "$\underset{\sim}{\tau} \in N$".

Now assume that for every P-name $\underset{\sim}{\tau}$ of an ordinal, if $\underset{\sim}{\tau} \in N$ then $q \Vdash$ "$\underset{\sim}{\tau} \in N$". Let $\mathcal{I} \in N$ be pre-dense in P. There is an $f \in H(\lambda)$ which maps $|\mathcal{I}|$ onto $\mathcal{I}$, hence there is such an f in N. We take $\underset{\sim}{\tau} = \mathrm{Min}\{i : f(i) \in \underset{\sim}{G}_P\}$. Since f, $P \in N$ and $\underset{\sim}{\tau}$ is definable from f and P in $(H(\lambda), \in)$, also $\underset{\sim}{\tau} \in N$, and $\underset{\sim}{\tau}$ is obviously a P-name of an ordinal. By our assumption $q \Vdash$ "$\underset{\sim}{\tau} \in N$", hence $q \Vdash$ "$(\exists i \in N)(f(i) \in \underset{\sim}{G}_P)$". Since f maps the members of $N \cap |\mathcal{I}|$ to members of $N \cap \mathcal{I}$, being in N, we have $q \Vdash$ "$(\exists r \in \mathcal{I} \cap N)(r \in \underset{\sim}{G}_P)$". Therefore $\mathcal{I} \cap N$ is pre-dense above q, which is what we had to prove. $\square_{2.6}$

2.7 Remark. For $\lambda \geq |P|$ it does not matter in $Con_2(\lambda)$ whether we require that for each $i < \alpha$ the set $\{p_{i,j} : j < \alpha_i\}$ is pre-dense or whether this set is pre-dense above p. Why? on the face of it the version where we require the set

$\{p_{i,j} : j < \alpha_i\}$ to be pre-dense is weaker since it makes a stronger assumption, but we now prove from it the stronger version. Suppose each $\{p_{i,j} : j < \alpha_i\}$ is pre-dense above p. Blow each such set by adding to it all members of P incompatible with p, to get the set $\{p_{i,j} : j < \beta_i\}$. Since $|P| \le \lambda$ we know $|\beta_i| \le \lambda$ so we can apply the weak version of $Con_2(\lambda)$ (β_i may be $> \lambda$ but since only the cardinality figures here it is O.K. as long as $\beta_i < \lambda^+$). We obtain a set A in $\mathcal{D}_{\aleph_0}(\lambda)$ such that for $s \in A$ we have a $q \ge p$ for which for each $i \in s \cap \alpha$ the set $\{p_{i,j} : j \in s \cap \beta_i\}$ is pre-dense above q. For $\alpha_i \le j < \beta_i$, $p_{i,j}$ is incompatible with p and hence also with q, therefore also the set $\{p_{i,j} : j \in s \cap \alpha_i\}$ is pre-dense above q which establishes the stronger version of $Con_2(\lambda)$. For P being a complete Boolean algebra with 1 omitted it suffice to add -p for $p \in P$, p not minimal, so $\lambda \ge \aleph_0$ suffice.

2.8 Theorem. (1) Let $\lambda > 2^{|P|}$, λ regular and assume $P \in H(\lambda)$ (this adds little since $H(\lambda)$ contains an isomorphic copy of P). P is a proper forcing notion iff for every countable elementary substructure N of $(H(\lambda), \in)$ satisfying P, $p \in N$ there is a condition q, $p \le q \in P$ such that q is (N, P)-generic.
(2) For $\lambda \ge 2^{|P|}$, $P \in H(\lambda)$, P is proper iff† $\{N : N \prec (H(\lambda), \in)$ is countable and there is an (N, P)-generic $q \ge p$ $\} \in \mathcal{D}_{\aleph_0}(H(\lambda))$ for every $p \in P$.

Proof. (1) We first prove that "if" part, i.e. suppose the condition of the theorem holds, and we shall prove $Con_2(2^{|P|})$ (suffice by 1.10 and 2.2). Let $\langle p_{i,j} : i < \alpha, j < \alpha_i\rangle$ be as in $Con_2(2^{|P|})$. Let $N \prec (H(\lambda), \in)$ be such that P, p, $\langle p_{i,j} : i < \alpha, j < \alpha_i\rangle \in N$. For $i \in N$, $\mathcal{I}_i \stackrel{\text{def}}{=} \{p_{i,j} : j < \alpha_i\} \in N$ since it is definable in $(H(\lambda), \in)$ from $\langle p_{i,j} : i < \alpha, j < \alpha_i\rangle$ and i. Let $q \ge p$ be (N, P)-generic; since $\{p_{i,j} : j < \alpha_i\} \in N$ and it is pre-dense we have that $\mathcal{I}_i \cap N = \{p_{i,j} : j \in N \cap \alpha_i\}$ is pre-dense above q. Therefore to establish $Con_2(2^{|P|})$ it suffices to prove that the set $A = \{N \cap 2^{|P|} : N \prec (H(\lambda) \in)$, and $p, P, \langle p_{i,j} : i < \alpha, j < \alpha_i\rangle \in N$, $|N| \le \aleph_0\}$ is in $\mathcal{D}_{\aleph_0}(2^{|P|})$. The set $A^\dagger = \{N \in \mathcal{S}_{\aleph_0}(H(\lambda)) : N \prec (H(\lambda), \in)$ and $p, P, \langle p_{i,j} : i < \alpha, j < \alpha_i\rangle \in N\}$

† We do not strictly distinguish between the set of N's and the set of their universes.

is in $\mathcal{D}_{\aleph_0}(H(\lambda))$ by Theorem 2.3. This implies that also $A \in \mathcal{D}_{\aleph_0}(2^{|P|})$, by the technique of using a model with operations closed under composition which we have already used several times, or more exactly by 1.12(1).

Now we prove the other direction of the theorem, so let $p \in P$, $\{p, P\} \subseteq N \prec (H(\lambda), \in)$, N countable and we shall find q as required. Assume that P is a proper forcing, i.e., $Con_2(2^{|P|})$. Since $\lambda > 2^{|P|}$ in $H(\lambda)$ there is a sequence $\langle p_{i,j} : i < \alpha, j < \alpha_i \rangle$ where $\alpha_i \leq |P|$, $\alpha \leq 2^{|P|}$ such that for $i < \alpha$ the set $\mathcal{I}_i \stackrel{\text{def}}{=} \{p_{i,j} : j < \alpha_i\}$ varies over all the pre-dense subset of P. By $Con_2(2^{|P|})$ there is a partial algebra with universe $2^{|P|}$ such that $Sm(M) \subseteq \{s \in \mathcal{S}_{\aleph_0}(2^{|P|}) : (\exists q \geq p)\ (\forall i \in s \cap \alpha)[\{p_{i,j} : j \in s \cap \alpha_i\}$ is pre-dense over $q]\}$; necessarily $M \in H(\lambda)$. Since there are such $\langle p_{i,j} : i < \alpha, j < \alpha_i \rangle$ and M in $H(\lambda)$ there are such also in N. N obviously contains all the natural numbers. Since M is given as a mapping of ω on all partial operations of M, all these operations belong to N and hence N is closed under them (if you prefer to see M as $(|M|, F)$, F a function with domain the vocabulary of M, which is countable, it works as well). Therefore $N \cap 2^{|P|} \in Sm(M)$ and therefore there is a $q \geq p$ such that $(\forall i \in N \cap \alpha)(\{p_{i,j} : j \in N \cap \alpha_i\}$ is pre-dense over $q)$. Let $\mathcal{I}$ be any subset of P in N which is pre-dense in P. Since in $(H(\lambda), \in)$ it is true that "for every pre-dense subset of P there is an $i < \alpha$ such that $\mathcal{I} = \mathcal{I}_i$ " (since this is the way we get $\langle p_{i,j} : i < \alpha, j < \alpha_i \rangle$ in $(H(\lambda), \in)$) and this sequence belongs to N; clearly this is true in N. Therefore $\mathcal{I} = \mathcal{I}_i$ for some $i \in N \cap \alpha$. For this i, if $j \in N \cap \alpha_i$ then also $p_{i,j} \in N$ (since $\langle p_{i,j} : j \in N \cap \alpha_i \rangle$ can be taken to be one-to-one, if $p_{i,j} \in N$ also $j \in N \cap \alpha_i$). Thus $\{p_{i,j} : j \in N \cap \alpha_i\} = \{p_{i,j} : j < \alpha_i\} \cap N = \mathcal{I}_i \cap N = \mathcal{I} \cap N$ and we know that this set is pre-dense above q (since $i \in N \cap \alpha$). Thus we have shown $q \geq p$ to be (N, P)-generic.

(2) Left to the reader. $\square_{2.8}$

2.9 Discussion. We shall now present another proof of the fact that if P satisfies c.c.c. then it is proper. We shall prove that if $\underset{\sim}{\tau}$ is a name of an ordinal, $N \prec (H(\lambda), \in)$ and $\underset{\sim}{\tau} \in N$ then $\emptyset \Vdash$ "$\underset{\sim}{\tau} \in N$". There is a maximal antichain $\mathcal{I}$ of P such that for each $p \in \mathcal{I}$, $p \Vdash$ "$\underset{\sim}{\tau} = \alpha$" for a unique α. Because of the c.c.c. $|\mathcal{I}| \leq \aleph_0$ so we can take $\mathcal{I} = \{p_i : i < \alpha\}$, $\alpha \leq \omega$. The sequence $\langle p_i : i < \alpha \rangle$ is in $H(\lambda)$ and its properties can be formulated in $(H(\lambda), \in)$. Therefore there is such a sequence in N. Since $\omega \subseteq N$ we have $p_i \in N$ for every $i < \omega$, and if α_i is the ordinal such that $p_i \Vdash$ "$\underset{\sim}{\tau} = \alpha_i$", then α_i is defined in $(H(\lambda), \in)$ from P, $\underset{\sim}{\tau}$ and p_i hence $\alpha_i \in N$. As $\{p_i : i < \alpha\}$ is a maximal antichain in P we conclude $\emptyset \Vdash$ "$\underset{\sim}{\tau} \in \{\alpha_i : i \in \alpha\}$", but $\{\alpha_i : i < \alpha\} \subseteq N$ (see above) so $\emptyset \Vdash$ "$\underset{\sim}{\tau} \in N$".

So for P satisfying the c.c.c., for λ, p, N as in 2.8(1), we have: p is (N, P)-generic.

2.10 Theorem. If the forcing notion P is $\aleph_1$-complete then it is proper.

Proof. Let λ be large enough, i.e., λ regular and $\lambda > 2^{|P|}$, let $p, P \in N \prec (H(\lambda), \in)$, $|N| = \aleph_0$. Let $\langle \mathcal{I}_i : i < \omega \rangle$ be a list of all pre-dense sets which are in N. We define the sequence $\langle p_n : n < \omega \rangle$ of members of $N \cap P$ by induction on n: $p_0 = p$ and $p_{n+1} \geq p_n, r_n$ for some $r_n \in \mathcal{I}_n \cap N$. There is such a $p_{n+1} \in N$ since "p_n is compatible with some members of $\mathcal{I}_n$"; $\mathcal{I}_n$ being pre-dense in N. By the $\aleph_1$- completeness of P there is a q such that $q \geq p_n$ for all $n < \omega$. Now q is (N, P)-generic since for every pre-dense subset $\mathcal{I}_n$ of P in N, $\mathcal{I}_n \cap N$ is pre-dense above q since $q \geq r_n$, $r_n \in \mathcal{I}_n \cap N$. (Remember a set Q is pre-dense above q if for every $p \geq q$ there is a member of Q which is compatible with p, but does not have to be $\geq q$). $\square_{2.10}$

Though the following is simple it has misled some.

2.11 Theorem. Let $P \in N \prec (H(\lambda), \in)$, and let G be a generic subset of P (over V). Let $N[G] = \{\underset{\sim}{\tau}[G] : \underset{\sim}{\tau}$ is a name $\& \underset{\sim}{\tau} \in N\}$. *Then* we have $N[G] = (N[G], \in) \prec (H^{(V[G])}(\lambda), \in)$ (and $N \subseteq N[G]$ of course).

Proof. By repeating the Forcing theorems for N and $H(\lambda)$, Claim I 5.17 implies $N[G] \subseteq H(\lambda)^{V[G]} = H(\lambda)[G]$. Let $\varphi(x, y_1, \ldots, y_n)$ be a first order formula. We shall prove that if $(H(\lambda), \in)^{V[G]} \models (\exists x)\varphi(x, y_1, \ldots, y_n)$ for some $y_1, \ldots, y_n \in N[G]$ then there is an $x \in N[G]$ such that

$$(H(\lambda), \in)^{V[G]} \models \varphi(x, y_1, \ldots, y_n).$$

Then, by the Tarski-Vaught criterion we shall have $N[G] \prec (H(\lambda), \in)^{V[G]}$. Since $y_1, \dots, y_n \in N[G]$, let $\underset{\sim}{\tau}_1, \dots, \underset{\sim}{\tau}_n \in N$ be P-names such that $y_1 = \underset{\sim}{\tau}_1[G], \dots, y_n = \underset{\sim}{\tau}_n[G]$. So

$$V[G] \models \text{“}(H(\lambda), \in)^{V[G]} \models (\exists x)\varphi(x, \underset{\sim}{\tau}_1, \dots, \underset{\sim}{\tau}_n)\text{”}.$$

By the “existential completeness” of the forcing names (see I 3.1) there is a P-name $\underset{\sim}{\sigma}$ such that

$$\emptyset \Vdash \text{“}\underset{\sim}{\sigma} \in H(\lambda) \text{ and } (H(\lambda), \in)^{V[G]} \models (\exists x)\varphi(x, \underset{\sim}{\tau}_1, \dots, \underset{\sim}{\tau}_n) \rightarrow \varphi(\underset{\sim}{\sigma}, \underset{\sim}{\tau}_1, \dots, \underset{\sim}{\tau}_n)\text{”}.$$

By I 5.17 and I 5.13, there is a name $\underset{\sim}{\tau} \in H(\lambda)$ such that $\emptyset \Vdash \text{“}\underset{\sim}{\sigma} = \underset{\sim}{\tau}\text{”}$, therefore $\emptyset \Vdash \text{“}(H(\lambda), \in)^{V[G]} \models (\exists x)\ \varphi(x, \underset{\sim}{\tau}_1, \dots, \underset{\sim}{\tau}_2) \rightarrow \varphi(\underset{\sim}{\tau}, \underset{\sim}{\tau}_1, \dots, \underset{\sim}{\tau}_n)\text{”}$, where $\underset{\sim}{\tau} \in H(\lambda)$. Forcing statements relativized to $H^{V[G]}(\lambda)$ can be defined in $(H(\lambda), \in)$, hence $(H(\lambda), \in) \models (\exists \text{ a } P\text{-name } \underset{\sim}{\tau}) \left[\emptyset \Vdash \text{“}(H(\lambda), \in)^{V[G]} \models [(\exists x \varphi(x, \underset{\sim}{\tau}_1, \dots, \underset{\sim}{\tau}_n)) \rightarrow \varphi(\underset{\sim}{\tau}, \underset{\sim}{\tau}_1, \dots, \underset{\sim}{\tau}_n)]\text{”}\right]$. By the Tarski-Vaught criterion for $N \prec (H(\lambda), \in)$ there is such a name $\underset{\sim}{\tau} \in N$. Thus $V[G]$ satisfies:

$$(H(\lambda), \in) \models \text{“}(\exists x)\varphi(x, \underset{\sim}{\tau}_1[G], \dots, \underset{\sim}{\tau}_n[G]) \rightarrow \varphi(\underset{\sim}{\tau}[G], \tau_1[G], \dots, \underset{\sim}{\tau}_n[G])\text{”}.$$

We finish as $\underset{\sim}{\tau}_\ell[G] = y_\ell$ for $\ell = 1, \dots, n$ and $\underset{\sim}{\tau}[G] \in N[G]$. $\square_{2.11}$

2.11A Remark.

Do we have, in 2.11, also $(N[G], N, \in) \prec (H^{V[G]}(\lambda), H(\lambda), \in)$? This holds iff $N[G] \cap H(\lambda) = N$.

2.12 Theorem. Under the assumptions of the last theorem, the following three conditions are equivalent.

(a) $G \cap N$ is N-generic, i.e., for every $\mathcal{I} \in N$ which is pre-dense in P, $\mathcal{I} \cap N \cap G \neq \emptyset$.

(b) $N[G] \cap \mathrm{Ord} = N \cap \mathrm{Ord}$

(c) $N[G] \cap V = N \cap V$

(d) replace in (a) pre-dense by dense

(e) replace in (a) pre-dense by maximal antichain

Proof. (a) $\Rightarrow$ (c). Let $x \in N[G] \cap V$. We shall prove $x \in N$. Since $x \in N[G]$, $x = \underset{\sim}{\tau}[G]$ for some $\underset{\sim}{\tau} \in N$. Let $\mathcal{I} \stackrel{\text{def}}{=} \{p \in P : (\exists y)(p \Vdash \text{“}\underset{\sim}{\tau} = y \text{ (i.e } \dot{y})\text{”}) \vee p \Vdash \text{“}\tau \notin V\text{”}\} = \{p \in P : (\exists y \in H(\lambda))[p \Vdash \text{“}\tau = \dot{y}\text{”}) \vee p \Vdash \text{“}\tau \notin H(\lambda)\text{”}]\}$.

(Remember that by Claim I 5.17, if $\underset{\sim}{\tau} \in H(\lambda)$ and $p \Vdash \text{“}\underset{\sim}{\tau} = y\text{”}$, then $y \in H(\lambda)$.) $\mathcal{I}$ is obviously pre-dense in P. Since $\mathcal{I}$ is definable in $(H(\lambda), \in)$ from $\underset{\sim}{\tau}$, P and $\underset{\sim}{\tau}$, $P \in N$ also $\mathcal{I} \in N$. By (a) there is a $p \in \mathcal{I} \cap N \cap G$. Since $V[G] \models \text{“}\underset{\sim}{\tau}[G] \in V\text{”}$ we cannot have $p \Vdash \text{“}\underset{\sim}{\tau} \notin H(\lambda)\text{”}$, hence for some $y \in H(\lambda)$, $p \Vdash \text{“}\underset{\sim}{\tau} = y\text{”}$ and as $p, \underset{\sim}{\tau}$ are in N and y is definable from them, necessarily $y \in N$, hence $x = \underset{\sim}{\tau}[G] = y \in N \cap H(\lambda) \subseteq N \cap V$.

(c) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (a). Let $\mathcal{I} \in N$ be pre-dense in P. Let $\mathcal{J} = \{q \in P$: for some $p \in \mathcal{I}$ we have $p \leq q\}$. Let $\mathcal{I}^\dagger$ be a subset of $\mathcal{J}$, an antichain in P and maximal under those two conditions (for $\subseteq$). As $\mathcal{I}$ is pre-dense in P clearly $\mathcal{J}$ is dense and open hence $\mathcal{I}^\dagger$ is a maximal antichain of P. By the definition of $\mathcal{J}$, as $\mathcal{I}^\dagger \subseteq \mathcal{J}$ there is a function f from $\mathcal{I}^\dagger$ to $\mathcal{I}$ such that for every $p \in \mathcal{I}^\dagger$, $p \geq f(p)$ and $f(p) \in \mathcal{I}$. Since $\mathcal{I} \in N \prec (H(\lambda), \in)$, there is such an $\mathcal{I}^\dagger \in N$ and so w.l.o.g. ($\mathcal{I}^\dagger \in N$ and) $f \in N$, since in $H(\lambda)$ there is a sequence $\langle q_\beta : \beta < \alpha\rangle$ listing the members of $\mathcal{I}^\dagger$, there is such a sequence in N. Let $\underset{\sim}{\tau}$ be the canonical P-name such that for $\beta < \alpha$ we have: $\underset{\sim}{\tau}[G] = \beta$ if $q_\beta \in G$ (since $\mathcal{I}^\dagger$ is a maximal antichain of P there is one and only one such β). So $\underset{\sim}{\tau}$ is a P-name of an ordinal and $\underset{\sim}{\tau} \in N$. By (b) we have $\underset{\sim}{\tau}[G] \in N$. So $\gamma \stackrel{\text{def}}{=} \underset{\sim}{\tau}[G] \in N$. Since $\langle q_\beta : \beta < \alpha\rangle \in N$ also $q_\gamma \in N$. Since $\underset{\sim}{\tau}[G] = \gamma$, $q_\gamma \in G$ but $q_\gamma \in \mathcal{I}^\dagger$ so $f(q_\gamma) \in \mathcal{I}$, hence $f(q_\gamma) \in N$ (as q_γ, $f \in N$) and $f(q_\gamma) \in G$ (as $q_\gamma \geq f(q_\gamma)$ & $q_\gamma \in G$) hence clearly $f(q_\gamma) \in \mathcal{I} \cap N \cap G$ and so $\mathcal{I} \cap N \cap G \neq \emptyset$.

(e) $\Rightarrow$ (a) $\Rightarrow$ (d) Left to the reader. $\square_{2.12}$

2.13 Corollary. Assume P is a forcing notion and $P \in N \prec (H(\lambda), \in)$ and $q \in P$, then the following are equivalent:

(a) q in (N, P)-generic.

(b) $q \Vdash$ "$N[\underset{\sim}{G}_P] \cap \mathrm{Ord} = N \cap \mathrm{Ord}$".

(c) $q \Vdash$ "$N[\underset{\sim}{G}_P] \cap V = N \cap V$".

(d) for every maximal antichain $\mathcal{I}$ of P which belongs to N we have $q \Vdash$ "$N \cap \mathcal{I} \cap \underset{\sim}{G}_P \neq \emptyset$".

(e) for every dense open subset $\mathcal{I}$ of P which belongs to N we have: $q \Vdash$ "$N \cap \mathcal{I} \cap \underset{\sim}{G}_P \neq \emptyset$".

(f) $q \Vdash$ "$(N[\underset{\sim}{G}_P], N, \in) \prec (H(\lambda)^{V[\underset{\sim}{G}_P]}, H(\lambda), \in)$".

Proof. Each of the present (a) - (e) is equivalent to the statement that the corresponding condition in the last theorem holds for all generic subsets G of P which contain q. $\square_{2.13}$

§3. Preservation of Properness Under Countable Support Iteration

3.1 Definition. We call $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ (or $\langle \underset{\sim}{Q}_i : i < \alpha \rangle$) a system of countable support iterated forcing (or a CS iterated forcing system or a CS iteration) *if* the following holds (on canonical names see Definition I 5.12, Theorem I 5.13):

$P_i = \{f : \mathrm{Dom}(f)$ is a countable subset of i and
$(\forall j \in \mathrm{Dom}(f))[f(j)$ is a canonical P_j-name
and $\Vdash_{P_j}$ "$f(j) \in \underset{\sim}{Q}_j$"$)]\}$.

$\underset{\sim}{Q}_i$ is a P_i-name of a forcing notion.

The partial order $\leq_i$ on P_i is defined by

$$f \leq_i g \Leftrightarrow (\forall j \in \mathrm{Dom}(f))[g{\restriction}j \Vdash \text{“}f(j) \leq_{\underset{\sim}{Q}_j} g(j)\text{”}].$$

For every $j \notin \mathrm{Dom}(f)$ we take $f(j)$ to be a name $\emptyset_j = \emptyset_{Q_j}$ of the minimal member of Q_j. Let P_α be defined like P_i above. We say: the forcing notion defined by this system is the (partial order) P_α. We say $P_\alpha = \mathrm{Lim}_{<\aleph_1} \langle \underset{\sim}{Q}_j : j < \alpha \rangle$ or $P_\alpha = \mathrm{Lim}_{<\aleph_1} |\bar{Q}|$. We may omit the "$< \aleph_1$".

Instead "$f(i)$ is a canonical P_i-name" we can use other variants.

3.1A Fact. For Definition 3.1 the parallel of II 2.2A hold (only in part (7) $(\forall\alpha < \lambda)[|\alpha|^{\aleph_0} < \lambda]$ is needed), i.e. in Definition 3.1:

(1) If $i < j \leq \alpha$ then $P_i \subseteq P_j$ as sets and even as partial orders.

(2) If $i < j \leq \alpha$ and $p \in P_j$ then $p{\restriction}i \in P_i$; moreover $P_j \models$ "$p{\restriction}i \leq p$" and if $p{\restriction}i \leq q \in P_i$ then $r \stackrel{\text{def}}{=} q \cup p{\restriction}(j \setminus i)$ belong to P_j and is the least upper bound of q, p in P_j (actually a least upper bound).

(3) If $i < j \leq \alpha$ then $P_i \lessdot P_j$ and $q \in P_i, p \in P_j \Rightarrow P_j \models q \leq p \Leftrightarrow P_i \models q \leq p{\restriction}i$.

(4) If $j \leq \alpha$ is a limit ordinal of uncountable cofinality then $P_j = \bigcup\limits_{i<j} P_i$.

(5) The sequence $\langle \underset{\sim}{Q}_j : j < \alpha\rangle$ uniquely determines the sequence $\langle P_j, \underset{\sim}{Q}_j : j < \alpha\rangle$ and vice versa and similarly for $\langle P_i, \underset{\sim}{Q}_j : j < \alpha$, and $i \leq \alpha\rangle$.

(6) If $\underset{\sim}{Q}'_i$ is a P_i-name, such that $\Vdash$ "$\underset{\sim}{Q}'_i$ is a dense subset of $\underset{\sim}{Q}_i$" *then* $P'_i = \{f \in P_i$: for every $j \in \mathrm{Dom}(f)$ we have: $\Vdash_{P_i}$ "$f(j) \in \underset{\sim}{Q}'_i$"$\}$ is a dense subset of P_i. Moreover we can define and prove by induction on $i \leq \alpha$, $P''_i = \{f \in P_i$: for every $j \in \mathrm{Dom}(f)$ we have: $f(j)$ is a P''_i-canonical name of a member of $\underset{\sim}{Q}'_i\}$ is a dense subset of P_i and $\underset{\sim}{Q}''_i$ is a canonical P''_i-name satisfying $\Vdash_{P_i}$ "$\underset{\sim}{Q}''_i = \underset{\sim}{Q}'_i$" and $\langle P''_{j_0}, \underset{\sim}{Q}''_{j_1} : j_0 \leq i, j_1 < i\rangle$ is a FS iteration.

(6A) Assume Q'_i is a set of canonical P_i-names, such that for every P_i-name $\underset{\sim}{p}$ for some $\underset{\sim}{q} \in Q'_i$, $\Vdash_{P_i}$ "if $\underset{\sim}{p} \in Q_i$ then $\underset{\sim}{Q}_i \models \underset{\sim}{p} \leq \underset{\sim}{q}$". Then $P'_i = \{f \in P_i$: for every $j \in \mathrm{Dom}(f)$ we have: $f(i) \in Q'_i\}$ is a dense subset of P_i and $\langle P'_i, \underset{\sim}{Q}_i : i < \alpha\rangle$ satisfies (1) - (4) above.

(7) Moreover we can define and prove by induction on $i < \alpha$, F_i, P''_i such that

$$P''_i = \{f \in P'_i : \text{ for every } j \in \mathrm{Dom}(f), f(j) \in \mathrm{Rang}(F_j)\}$$

is a dense subset of P_i and F_i is a function with domain the P''_i-names of members of Q'_i, satisfying: $\underset{\sim}{p} \in Q'_i \Rightarrow F_i(\underset{\sim}{p})$ is a canonical P''_i-name forced to be equal to $\underset{\sim}{p}$. So letting $\underset{\sim}{Q}''_i = \{\underset{\sim}{p} : \underset{\sim}{p}$ a P''_i-name of a member of $Q_i\}$, we have: $\langle P''_i, \underset{\sim}{Q}''_i : i < \alpha\rangle$ is a countable support iteration, $P''_i \subseteq P_i$ is a dense subset.

(8) If $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha \rangle$ is a CS iteration, and for $i < \alpha$, $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i \in H(\lambda_i)$" and $\langle \lambda_i : i \leq \alpha \rangle$ is an increasing sequence of regulars satisfying $2^{\lambda_i} < \lambda_{i+1}$ and for limit $\delta \leq \alpha \Rightarrow (\sum_{i<\delta} \lambda_i)^{\aleph_0} < \lambda_\delta$ *then* $\bar{Q} \in H(\lambda_\alpha)$

3.1B The Definition by Induction Theorem. (one can construct $\underset{\sim}{Q}_i$'s by a given recursive recipe). If F is a function and α is an ordinal then there is a unique CS iterated forcing $\langle \underset{\sim}{Q}_j : j < \alpha_0 \rangle$ such that for all $j < \alpha_0$ $\underset{\sim}{Q}_j = F(\langle \underset{\sim}{Q}_i : i < j \rangle)$ and either $\alpha_0 = \alpha$ or else $F(\langle \underset{\sim}{Q}_i : i < \alpha_0 \rangle)$ is not suitable for $\underset{\sim}{Q}_{\alpha_0}$, i.e., it is not a name of a forcing notion in the forcing notion P_{α_0}.

Proof. This theorem is an obvious consequence of the standard definition-by-recursion theorem. $\square_{3.1B}$

3.2 Theorem. If $\langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ is a countable-support iterated forcing system and for each $i < \alpha$, $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i$ is proper" then P_α is proper.

Remark. The reader may look at the alternative proofs presented in the book: IX §2 (for one using alternative iteration) XII §1 (the one using games) and another proof later in this section.

Proof. In Theorem 2.8(1) we showed that P is a proper forcing iff for some $\lambda > 2^{|P|}$ every countable elementary substructure N of $(H(\lambda), \in)$ such that $P, p \in N$ has a q , $p \leq q \in P$ such that q is N-generic. As easily seen from the proof or by 2.8(2) it suffices to require this only for all such N which contain some fixed member y of $H(\lambda)$.

For our present proof we choose a regular cardinal λ which is large enough with respect to $|P_\alpha|$, and the definition of the iteration and we shall show that P_α is proper by showing that for every countable elementary substructure N of $(H(\lambda), \in)$ such that $\langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle \in N$, $P_\alpha \in N$ and for all $p \in P_\alpha \cap N$ there is a q, $p \leq q \in P_\alpha$ which is N-generic. We shall show, by induction on $j \leq \alpha$ such that $j \in N$, a somewhat stronger property:

$(*)$ For all $i < j$, $i \in N$ and for all $p \in N \cap P_j$, and $q \in P_i$ if q is (N, P_i)-generic and $q \geq p{\restriction}i$ then there is an $r \in P_j$ such that r is (N, P_j)-generic, $r \geq p$ and $r \geq q$ and $r{\restriction}i = q$ (we could add $\mathrm{Dom}(r) \cap [i, j) = N \cap [i, j)$).

For $j = 0$ the statement $(*)$ is vacuously true. Now we assume $(*)$ for j and prove it for $j+1$. Since $j+1 \in N$ also $j \in N$. Therefore, since $(*)$ holds for j we may assume, without loss of generality that $i = j$. Let G_j be a generic subset of P_j which contains q. By Theorem 2.11 we have

$$N[G_j] \prec (H(\lambda), \in)[G_j] = (H(\lambda), \in)^{V[G_j]},$$

since $P_j \in N$ (because j, $\langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle \in N$ and P_j is definable in $(H(\lambda), \in)$ from j and $\langle \underset{\sim}{Q}_i : i < \alpha \rangle$) and $\underset{\sim}{Q}_j \in N$ and hence $\underset{\sim}{Q}_j[G_j] \in N[G_j]$. Remember that $\underset{\sim}{Q}_j[G_j]$ is a proper forcing in $V[G_j]$. Since $p, j \in N$ also $p(j) = p_j \in N$ and $p_j[G_j] \in \underset{\sim}{Q}_j[G_j] \cap N[G_j]$; since λ is still sufficiently large and $\underset{\sim}{Q}_j[G_j]$ is proper there is an $r_j \in \underset{\sim}{Q}_j[G_j]$ such that $r_j \geq p_j[G_j]$ and r_j is $(N[G_j], \underset{\sim}{Q}_j[G_j])$-generic. Since the only requirement we had about the generic subset G_j of P_j was that is contains q, q forces the existence of an r_j as above. By the existential completeness lemma I 3.1 there is a name $\underset{\sim}{r}_j$ such that $q \Vdash_j$ "$\underset{\sim}{r}_j \in \underset{\sim}{Q}_j \,\&\, \underset{\sim}{r}_j \geq p_j \,\&\, \underset{\sim}{r}_j$ is $(N[\underset{\sim}{G}_j], \underset{\sim}{Q}_j)$-generic" and w.l.o.g. $\Vdash_{P_j}$ "$\underset{\sim}{r}_j \in \underset{\sim}{Q}_j$". We set now $r = q \cup \{\langle j, \underset{\sim}{r}_j \rangle\}$. Obviously $r \in P_{j+1}$ and $r \restriction j = q$. Also since $q \geq p \restriction j$ and $q \Vdash$ "$\underset{\sim}{r}_j \geq p_j$" we have $r \geq p$. We still have to prove that r is (N, P_{j+1})-generic.

By the corollary in 2.13 in order to prove that r is (N, P_{j+1})-generic it suffices to prove that for all generic subsets G of P_{j+1} which contain r, $N[G] \cap \mathrm{Ord} = N \cap \mathrm{Ord}$. Let G_j be the part of G up to j i.e. $G \cap P_j$. Since $r \in G$ clearly $q \in G_j$. Since q is (N, P_j)-generic we have $N[G_j] \cap \mathrm{Ord} = N \cap \mathrm{Ord}$. Let $G^* \subseteq \underset{\sim}{Q}_j[G_j]$, be the "$j$-th component" of G. Since $r \in G$ clearly $\underset{\sim}{r}_j[G] \in G^*$. Since $\underset{\sim}{r}_j[G_j]$ is $(N[G_j], \underset{\sim}{Q}_j[G_j])$-generic we have $N[G_j][G^*] \cap \mathrm{Ord} = N[G_j] \cap \mathrm{Ord}$, and using the equality above we get $N[G_j][G^*] \cap \mathrm{Ord} = N \cap \mathrm{Ord}$. We have to observe that $N[G] \subseteq N[G_j][G^*]$, then we have $N[G] \cap \mathrm{Ord} = N \cap \mathrm{Ord}$. For every P_{j+1}-name $\underset{\sim}{\tau}$ in N there is a name $\underset{\sim}{\tau}^* \in N$ as in Lemma II 1.5 ($\underset{\sim}{\tau}^*$ is definable from $\underset{\sim}{\tau}$ and P_{j+1}, and is hence in N). By Lemma II 1.5, $\underset{\sim}{\tau}[G] = \underset{\sim}{\tau}^*[G_j][G^*] \in N[G_j][G^*]$.

Now we come to deal with the case where j is a limit ordinal. Let $\langle \underset{\sim}{\tau}_n : n < \omega \rangle$ be a sequence of all P_j-names of ordinals which are in N. Note that $N \cap j$

is a countable set with no last element, so let $i = i_0 < i_1 < i_n < \cdots, (n < \omega)$, be a sequence cofinal in it.

For $p' \in P_j$ and $q' \in P_{i_n}$ such that $q' \geq p' \restriction i_n$ let $q' \bar{\cup} p'$ denote $q' \cup (p' \restriction (j \setminus i_n))$. Since $q' \geq p' \restriction i_n$, $q' \bar{\cup} p' \in P_j$ and $q' \bar{\cup} p' \geq p_1$. For $p_1, p_2 \in P_j$ we write $p_1 \approx p_2$ for $p_1 \leq p_2 \,\&\, p_2 \leq p_1$.

We define now two sequences $\langle q_n : n < \omega \rangle$ and $\langle p_n : n < \omega \rangle$ such that $q_0 = q, p_0 = p$ and for all $n < \omega$:

(1) $q_n \in P_{i_n}$ and q_n is (N, P_{i_n})-generic

(2) $q_{n+1} \restriction i_n = q_n$

(3) $p_n \in P_j$ and $\mathrm{Dom}(p_n) \subseteq N \cap j$

(4) $q_n \geq p_n \restriction i_n$

(5) $p_{n+1} \restriction i_n = p_n \restriction i_n$ and $q_n \bar{\cup} p_{n+1} \geq p_n$

(6) $q_n \Vdash_{P_{i_n}}$ "$(\exists s \in P_j \cap N)\ (\exists q^\dagger \geq q_n)(s \restriction i_n \leq q^\dagger \ \ \& \, q^\dagger \in \underset{\sim}{G}_{P_{i_n}}$
$\& \, q^\dagger \bar{\cup} s \cong q^\dagger \bar{\cup} p_n)$"

(7) $q_n \bar{\cup} p_{n+1} \Vdash_{P_j}$ "$\underset{\sim}{\tau}_n \in N$"

Let us assume that q_n and p_n are defined and that they satisfy (1) - (7). We shall now define p_{n+1} and q_{n+1}. Let G be a generic subset of P_{i_n} such that $q_n \in G$. We shall see that there are $q^\dagger \in G, s$ and s^* such that

(a) $q^\dagger \geq q_n$ and $s \in P_j \cap N$, $s \restriction i_n \leq q^\dagger$, $q^\dagger \bar{\cup} p_n \cong q^\dagger \bar{\cup} s$.

(b) $s^* \restriction i_n \leq q^\dagger$, $s^* \in P_j \cap N$, $s^* \geq s$ and s^* decides the value of τ_n.

By (6) there is an $s \in P_j \cap N$ and a $q^\dagger \geq q_n, s \restriction i_n \leq q^\dagger$ such that $q^\dagger \in G$ and $q^\dagger \bar{\cup} s \approx q^\dagger \bar{\cup} p_n$. The set $\mathcal{I}_0 = \{s^* : s^* \geq s$ and s^* decides the value of $\tau_n\}$ is obviously a pre-dense subset of P_j above s. This set belongs to N since it is definable from the parameters s and τ_n which are in N. Let $\mathcal{I}$ denote the set which consists of the restrictions of the members of $\mathcal{I}_0$ to i_n, since $\mathcal{I}_0 \in N$ and $s \in N \cap P_j$ clearly $\mathcal{I} \in N$ and $\mathcal{I}$ is a pre-dense subset of P_{i_n} above $s \restriction i_n$. Since q_n is (N, P_{i_n})-generic and $s \restriction i_n \leq q_n$ we have $\mathcal{I} \cap N$ is pre-dense above q_n. Therefore $\mathcal{I} \cap N \cap G \neq \emptyset$. Let $r \in \mathcal{I} \cap N \cap G$, then "$r$ is a restriction to i_n of a $s^* \in P_j$ such that $s^* \geq s$ and s^* decides $\underset{\sim}{\tau}_n$ and $s^* \restriction i_n \in G$" is true in $(H(\lambda)^{V[G]}, \in)$. By Tarski-Vaught's criterion there is such an s^* in $N[G]$, but $s^* \in P_j \subseteq V$ and by 2.13(c) (as $q_n \in G$, q_n is (N, P_{i_n})-generic) we know $N[G] \cap V = N$, together $s^* \in N$. Thus $r = s^* \restriction i_n \in G$, and we can take $q^\dagger$ to be $\geq s^* \restriction i_n$. Thus

$q^\dagger, s^*$ are as required by (a) and (b). Therefore $V[G] \models$ "$(\exists s^*)(\exists q^\dagger \in G)(s^*$ and $q^\dagger$ are as in (a) and (b), and s^* is the first such element in some fixed well ordering of $P_j)$". By the existential completeness lemma there is a P_{i_n}-name $\underset{\sim}{s}^*$ such that $q_n \Vdash$ "$(\exists q^\dagger \in \underset{\sim}{G})$ $[\underset{\sim}{s}^*, q^\dagger$ are as in (a) and (b) and $\underset{\sim}{s}^*$ is the least such]". Since each possible $\underset{\sim}{s}^*[G]$ is in N and it satisfies that $|\mathrm{Dom}(\underset{\sim}{s}^*)| \leq \aleph_0$ in $(H(\lambda), \in)$, it satisfies this also in N, hence $\mathrm{Dom}(s^*) \subseteq N$ (since an enumeration of $\mathrm{Dom}(s^*)$ is in N). We define p_{n+1} as follows. Let $p_{n+1}\restriction i_n = p_n\restriction i_n$. For $\gamma \in j \cap N \setminus i_n$ let $p_{n+1}(\gamma)$ be the P_γ-name of the member of $\underset{\sim}{Q}_\gamma$ determined by $\underset{\sim}{s}^*$ (i.e., if $\underset{\sim}{s}^*$ is a set of pairs of members of P_{i_n} and members of $P_j \cap N$ then $p_{n+1}(\gamma) = \{\langle r, t\rangle : (\exists r^\dagger \leq r)(\exists s)\ (\langle r^\dagger, s\rangle \in \underset{\sim}{s}^* \,\&\, (\bar{\exists} r'' \leq r)(\langle r'', t\rangle \in s(\gamma)))\})$.

Now let us define q_{n+1}. For each $s \in P_j \cap N$ such that $s\restriction i_n \leq q_n$ there is, by the induction hypothesis an $q_{n+1}(s) \in P_{i_{n+1}}$ such that $q_{n+1}(s)\restriction i_n = q_n$, $q_{n+1}(s) \geq s\restriction i_{n+1}$ and $q_{n+1}(s)$ is $(N, P_{i_{n+1}})$-generic. We define q_{n+1} as follows. The domain of q_{n+1} is the union of all the domains of the $q_{n+1}(s)$'s for $s \in N$ as above, and since N is countable the domain of q_{n+1} is countable. Let $q_{n+1}\restriction i_n = q_n$. For $i_n \leq \gamma < i_{n+1}$ such that $\gamma \in \mathrm{Dom}(q_{n+1})$ if $q_n \in G$, and G is a generic subset of P_{i_n}, then $V[G] \models (\exists u)(\exists s \in P_j \cap N)\ ([q_n \bar{\cup} s \approx q_n \bar{\cup} p_{n+1}] \,\&\, u = q_{n+1}(s))$. By the existential completeness lemma there is a P_{i_n}-name $\underset{\sim}{u}$ of a $P_{i_{n+1}}$-condition such that $q_n \Vdash_{P_{i_n}}$ "$(\exists s \in P_j \cap N)$ $(q_n \bar{\cup} s \approx q_n \bar{\cup} p_{n+1} \,\&\, \underset{\sim}{u} = q_{n+1}(s))$". Now $\underset{\sim}{u}$ determines canonically a P_γ-name of a Q_γ-condition: $\underset{\sim}{u}(\gamma)$, which is taken to be the value of $q_{n+1}(\gamma)$.

We shall not present here the proof that p_{n+1} and q_{n+1} thus defined satisfy (1) - (7).

Now we define $r = \cup_{n<\omega} q_n$. Clearly r belongs to P_j. We claim that for every $n, r \geq p_n$. To prove that we have to show that for every $\gamma \in \mathrm{Dom}(p_n)$, $r\restriction\gamma \Vdash$ "$p_n(\gamma) \leq r(\gamma)$". Since $\gamma \in \mathrm{Dom}(p_n)$ we have, by (3), $\gamma \in N$. Let k be minimal such that $\gamma < i_k$ then, by clause (4) we have $q_k \geq p_k\restriction i_k$, hence $q_k\restriction\gamma \Vdash$ "$p_k(\gamma) \leq q_k(\gamma)$". By the definition of r, $q_k\restriction\gamma = r\restriction\gamma, q_k(\gamma) = r(\gamma)$. Also, by (5) if $n \geq k$ then $p_k(\gamma) = p_n(\gamma)$, hence $r\restriction\gamma \Vdash_{P_\gamma}$ "$p_n(\gamma) \leq r(\gamma)$". So assume $n < k$ (hence $k > 0$); now for $\ell < k$ we have: $q_\ell \bar{\cup} p_{\ell+1}\restriction\gamma \Vdash_{P_\gamma}$ "$p_\ell(\gamma) \leq_{\underset{\sim}{Q}_\gamma} p_{\ell+1}(\gamma)$" (by clause (5)) but $q_k\restriction\gamma \geq q_\ell \bar{\cup} p_{\ell+1}\restriction\gamma$ for $\ell < k$ (by clauses (4) and (5)) hence $q_k\restriction\gamma \Vdash_{P_\gamma}$ "$p_0(\gamma) \leq p_1(\gamma) \leq \ldots \leq p_k(\gamma)$", hence $q_k\restriction\gamma \Vdash_{P_\gamma}$ "$p_n(\gamma) \leq p_k(\gamma)$", but we have proved above $q_k{\restriction}\gamma \Vdash_{P_\gamma}$ "$p_k(\gamma) \leq q_k(\gamma)$", hence $q_k{\restriction}\gamma \Vdash_{P_k}$ "$p_n(\gamma) \leq q_k(\gamma)$". However $r{\restriction}\gamma = q_k{\restriction}\gamma$ and $q_k(\gamma) = r(\gamma)$ (as $\gamma < i_k$) hence this means $r{\restriction}\gamma \Vdash_{P_\gamma}$ "$p_n(\gamma) \leq r(\gamma)$" as required. So we have really proved $P_j \models$ "$p_n \leq r$". Thus, by (7), $r \Vdash$ "$\underset{\sim}{\tau}_n \in N$" and therefore r is (N, P_j)-generic which finishes our proof. $\square_{3.2}$

3.3 Alternative proof of 3.2.

3.3A Advice to the reader. There are situations where it is enough to understand and believe the statement of a theorem (as opposed to its proof). For example, we took this attitude in Chapter 1 when we discussed the fundamental theorem of forcing.

However, this approach should not be used here. Not only is the preceding theorem basic for the theory to be developed in the rest of the book, it is (in the author's opinion) also essential for the reader to understand the proof, since variations and extensions of this proof will appear throughout the book.

To help the reader understand the proof of the Theorem 3.2 better we now give a reformulation of this proof which is due to Goldstern [Go]. This version emphasizes the fact that the conditions p_n are in N by constructing the whole sequence $(p_n : n < \omega)$ *before* constructing the generic conditions q_n.

N is an elementary submodel of some $H(\chi)$ for some large χ containing $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \varepsilon\rangle$.

3.3B Fact. If $\beta > \alpha$, $q \in P_\alpha$, $p \in P_\beta$, $q \geq^* p{\restriction}\alpha$, then $Q^+ \stackrel{\text{def}}{=} q \cup p{\restriction}[\alpha, \beta)$ is in P_β, and $q^+ \geq^* p$ (i.e., $q^+ \Vdash p \in G$).

3.3C "Existential Completeness Lemma". For any forcing P, and any condition $p \in P$, any formula $\varphi(x)$:

$$p \Vdash \exists\, x\, \varphi(x) \quad \textit{iff} \quad \text{there is a name } \underset{\sim}{\tau} \text{ such that } p \Vdash \varphi(\underset{\sim}{\tau}).$$

Proof. By I 3.1.

3.3D Preliminary Lemma. (This lemma does not require properness.) Assume $\alpha_1 \leq \alpha_2 \leq \beta$, $\underset{\sim}{p}_1$ is a P_{α_1}-name for a condition in P_β. Let $\mathcal{I}$ be a dense open set of P_β. Then $\emptyset_{P_{\alpha_2}} \Vdash_{P_{\alpha_2}} \exists p_2 \varphi(p_2)$, where $\varphi(p_2)$ is the conjunction of the following clauses:

(1) $p_2 \in P_\beta$, $p_2 \geq^* \underset{\sim}{p}_1$.

(2) $p_2 \in \mathcal{I}$.

(3) If $p_1 \restriction \alpha_2 \in G_{\alpha_2}$, then $p_2 \restriction \alpha_2 \in G_{\alpha_2}$.

3.3E Remark. By the existential completeness lemma there is an α_2-name $\underset{\sim}{p}_2$ for a condition in P_β such that $\Vdash_{P_{\alpha_2}} \varphi(\underset{\sim}{p}_2)$.

3.3F Remark. The P_{α_1}-name $\underset{\sim}{p}_1$ corresponds naturally to a P_{α_2}-name, which we also call $\underset{\sim}{p}_1$.

Proof. Assume not, then there exists a condition $r \in P_{\alpha_2}$ such that

$$r \Vdash \text{"there is no } p_2 \text{ satisfying (1)–(3)"}.$$

We may assume that r decides what $\underset{\sim}{p}_1$ is, (i.e. $r \Vdash \underset{\sim}{p}_1 = p_1$ for some $p_1 \in V$), and r also decides whether $p_1 \restriction \alpha_2 \in G_{\alpha_2}$.

Case 1. $r \Vdash p_1 \restriction \alpha_2 \notin G_{\alpha_2}$:

But then (3) is true for any p_2, so

$$r \Vdash \text{"there is no } p_2 \text{ satisfying (1)–(2)"}$$

which is a contradiction since $\mathcal{I}$ is a dense open.

Case 2. $r \Vdash p_1 \restriction \alpha_2 \in G_{\alpha_2}$, i.e. $r \geq^* p_1 \restriction \alpha_2$. Now let $r' = r \cup p_1 \restriction [\alpha_2, \beta) \geq^* p_1$, and find $r'' \in D$, $r'' \geq r'$. Then

$$r'' \restriction \alpha_2 \Vdash r'' \text{ satisfies (1)–(3)},$$

again a contradiction, because $r'' \restriction \alpha_2 \geq r$.

3.3G "Composition Fact". $q \in P_{\alpha+1}$ is $(P_{\alpha+1}, N)$-generic iff:

$q \restriction \alpha$ is (P_α, N)-generic, and $q \restriction \alpha \Vdash$ "$q(\alpha)$ is $(Q_\alpha, N[G_\alpha \cap N])$-generic."

Proof. See §2.

3.3H Induction Lemma. For all $\beta \in N \cap \varepsilon$, for all $\alpha \in N \cap \varepsilon$, all $\underset{\sim}{p} \in N$ assume $\underset{\sim}{p}$ is a P_α-name for a condition in P_β, and

(a) $q \in P_\alpha$

(b) q is (P_α, N)-generic.

(c) $q \Vdash_{P_\alpha}$ "$\underset{\sim}{p} \restriction \alpha \in G_\alpha \cap N$".

Then there is a condition q^+:

(a)$^+$ $q^+ \in P_\beta$, $q^+ \restriction \alpha = q$

(b)$^+$ q^+ is N-generic

(c)$^+$ $q^+ \Vdash_{P_\beta}$ "$\underset{\sim}{p} \in G_\beta \cap N$".

(Note that "q is N-generic" implies already "$q \Vdash \underset{\sim}{p} \in N$", so the main point of (c) is to say that $q \Vdash p \restriction \alpha \in G_\alpha$)

(For $\alpha = 0$ this shows that P_β is proper.)

Proof.

The proof is by induction on β.

Successor step.

Let $\beta = \beta' + 1$. Since we can first use the induction hypothesis on α, β'' to extend q to a condition $q' \in P_{\beta''}$ satisfying the appropriate version of (a)–(c), we may simplify the notation by assuming $\beta = \alpha + 1$.

Clearly $q \Vdash_{P_\alpha}$ "$N[G_\alpha] \prec H(\chi)^{V[G_\alpha]}$", $q \Vdash_{P_\alpha}$ "there is a $(Q_\alpha, N[G_\alpha])$-generic condition $\geq \underset{\sim}{p}(\alpha)$". By "existential completeness", there is a P_α-name $q^+(\alpha)$ for it. By the "composition fact", we are done.

Limit step.

Let $\beta \in N$ be a limit ordinal, $\beta \in N = \bigcup \alpha_n$, $\alpha_0 = \alpha$, $\alpha_n \in N$. Let $\langle \mathcal{I}_n : n < \omega \rangle$ enumerate all dense subsets of P_β that are in N.

First we will define a sequence $\langle \underset{\sim}{p}_n : n < \omega \rangle$, $\underset{\sim}{p}_n \in N$ such that *in* N the following will hold:

(0) $\underset{\sim}{p}_n$ is a P_{α_n}-name for a condition in P_β

(1) $\Vdash_{P_{\alpha_{n+1}}} \underset{\sim}{p}_{n+1} \geq^* \underset{\sim}{p}_n$

(2) $\Vdash_{P_{\alpha_{n+1}}} \underset{\sim}{p}_{n+1} \in \mathcal{I}_n$

(3) $\Vdash_{P_{\alpha_{n+1}}}$ "If $\underset{\sim}{p}_n \restriction \alpha_{n+1} \in G_{\alpha_{n+1}}$ then $\underset{\sim}{p}_{n+1} \restriction \alpha_{n+1} \in G_{\alpha_{n+1}}$".

For each n we thus get a name $\underset{\sim}{p}_n$ that is in N. For each n we can use the "preliminary lemma" (and Remark 3.3E before its proof) in N to obtain $\underset{\sim}{p}_{n+1}$.
Now we define a sequence $\langle q_n : n < \omega \rangle$, $q_n \in P_{\alpha_n}$, and q_n satisfies (a), (b), (c) (if we write q_n for q, p_n for p, and α_n for α).
$q_{n+1} = q_n^+$ can be obtained by the induction hypothesis, applied to α_n, α_{n+1}, and $\underset{\sim}{p}_n \restriction \alpha_{n+1}$. By (c)$^+$ we know

$$q_n^+ \Vdash_{P_{\alpha_n}} \text{"}(\underset{\sim}{p}_n[G_{\alpha_n}]) \restriction \alpha_{n+1} \in G_{\alpha_{n+1}}\text{"}.$$

Hence, by (3) and the genericity of q_{n+1} we have

$$q_{n+1} \Vdash_{P_{\alpha_{n+1}}} \text{"}(\underset{\sim}{p}_{n+1}[G_{\alpha_{n+1}}]) \restriction \alpha_{n+1} \in G_{\alpha_{n+1}} \cap N\text{"}.$$

Since $q_{n+1} \restriction \alpha_n = q_n$, $q = \lim q_n$ exists and is $\geq q_n$ for all n.

We have to show that $q \Vdash p \in G_\beta \cap N$ and that q is generic. Let G_β be a generic filter containing q. We will write p_n for $\underset{\sim}{p}_n[G_{\alpha_n}]$. (Note that $p_n \in N$, because q_n was N-generic and $q_n \in G_{\alpha_n}$.) Since $q_n \in G_\beta$, we have $p_n \restriction \alpha \in G_{\alpha_n} \cap N$ and $N \models p_n \geq^* p_{n-1} \geq^* \ldots \geq^* p_0$. Hence $p \restriction \alpha_n \in G_{\alpha_n} \cap N$ for all n, and therefore $p \in G_\beta \cap N$. Similarly, $p_n \in G_\beta$ for all n.
Consider a dense set $\mathcal{I}_n \subseteq P_\beta$. Since $q_{n+1} \Vdash \underset{\sim}{p}_{n+1} \in \mathcal{I}_n$, we have $p_{n+1} \in G_\beta \cap \mathcal{I}_n \cap N$. Hence q is generic. $\square_{3.3}$

More advice to the reader. It may also be helpful to look at the proof in Chapter XII, §1, which uses games. (Chapter XII, §1 can be read independently of chapters IV to XI.)

3.4 The General Associativity Theorem. Suppose $\langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ is a CS iterated forcing system each $\underset{\sim}{Q}_i$ proper then the parallel to II 2.4 holds.

Proof. Left to the reader. $\square_{3.4}$

3.5 Theorem. Suppose $\langle \underset{\sim}{Q}_j : j < \alpha \rangle$ is a $(< \kappa)$-support iterated forcing, $P_j = \mathrm{Lim}\langle \underset{\sim}{Q}_i : i < j \rangle$.

If $\Vdash_{P_j}$ "$\underset{\sim}{Q}_j \lessdot \underset{\sim}{Q}_j^\dagger, \underset{\sim}{Q}_j$ a dense subset of $\underset{\sim}{Q}_j^\dagger$" and $P_j^\dagger = \mathrm{Lim}\langle \underset{\sim}{Q}_i^\dagger : i < j \rangle$, *then* $P_j \lessdot P_j^\dagger$ is a dense subset of $P_j^\dagger$.

Remark. By Lemma I 5.1 (a), we can replace $\underset{\sim}{Q}_j$ by any equivalent $\underset{\sim}{Q}_j^\dagger$ (just use 3.5 a few times).

Proof. Left to the reader. $\square_{3.5}$

§4. Martin's Axiom Revisited

Why is c.c.c. forcing so popular? I think the main reason is that such forcing notions preserve cardinalities and cofinalities, so why shall we not be interested in the property "P does not collapse cardinals " instead "P satisfies the c.c.c. ". In particular Magidor and Stavi had wondered on the role of the c.c.c. mainly in MA and asked:

"Is it consistent that for any forcing notion P of power $\aleph_1$ not collapsing cardinals (i.e., $\aleph_1$) and dense $\mathcal{I}_i \subseteq P$ (for $i < \aleph_1$) there is a directed $G \subseteq P$, such that $G \cap \mathcal{I}_i \neq \emptyset$ for $i < \aleph_1$?"

In particular Baumgartner, Harrington and Kleinberg [BHK] proved that if $S \subseteq \omega_1$ is stationary co-stationary, and CH holds, then there is a forcing notion $P_S = \{C : C$ a countable closed subset of $S\}$ with the order $C_1 \leq C_2$ iff $C_1 = C_2 \cap (\mathrm{Sup} C_1 + 1)$ which does not change cardinalities and cofinalities and which collapse S (i.e., collapse its stationarity, i.e., $\Vdash_{P_S}$ "$S \subseteq \omega_1$ is not stationary ".

So why not include such forcing in MA? Because we can find pairwise disjoint stationary sets $S_n \subseteq \omega_1, \omega_1 = \cup_{n<\omega} S_n$. If we make each S_n in turn not stationary, ω_1 must be collapsed. More exactly, if we try to iterate the forcings P_{S_n}, after ω steps $\aleph_1$ collapses, no matter how the limit is taken. It does not matter if we look at the desired version of MA, in some V and let $\mathcal{I}^n_\alpha = \{C \in P_{S_n} : \sup(C) \geq \alpha\}$. Thus if $G \cap \mathcal{I}^n_\alpha \neq \emptyset$ for $n < \omega, \alpha < \omega_1$, then in V, each S_n is not stationary.

You can still argue that CH is the cause of the problem but we shall prove in Theorem 4.4 that even $2^{\aleph_0} > \aleph_1, S \subseteq \omega_1$ stationary co-stationary there is a forcing notion P of power $\aleph_1$, not changing cardinalities and cofinalities but still collapsing S.

So it is natural to change the question to "P of power $\aleph_1$, not collapsing stationary subsets of $\aleph_1$", and we shall answer it positively, assuming there is a model V of ZFC with a strongly inaccessible cardinal.

The natural scheme is to iterate (by CS iteration) proper forcing of power $\aleph_1$, in an iteration of length ω_2. However to prove the consistency of almost anything by iterating proper forcing we usually have to prove the κ-chain condition is satisfied, where κ will be the new $\aleph_2$ and the length of the iteration. We have a problem even if $|P| = \aleph_1, \Vdash_P$ "$|\underset{\sim}{Q}| = \aleph_1$", $P * \underset{\sim}{Q}$ have a large power because of the many names. We can overcome this either by using κ strongly inaccessible, or showing that the set of names which are essentially hereditarily countable is dense.

Another problem is that "not destroying stationary subsets of ω_1" is not the same as "proper". However we shall prove that if P is not proper, then $\Vdash_{\mathrm{Levy}(\aleph_1, 2^{|P|})}$ "P destroys a stationary subset of ω_1". So instead of "honestly" dealing with a candidate P i.e., a forcing notion which does not destroy stationary subsets of ω_1, but is not proper we cheat and make it to destroy a stationary subset of ω_1.

4.1 Theorem. Suppose $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \kappa \rangle$ is a CS iteration $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i$ is a proper forcing notion which has power $< \kappa$", κ is regular and $(\forall \mu < \kappa)\mu^{\aleph_0} < \kappa$.

Then $P_\kappa = \lim \bar{Q}$ satisfies the κ-c.c., and each $P_i (i < \kappa)$ even has a dense subset of power $< \kappa$. Hence for $i < \kappa, \Vdash_{P_i}$ "$2^{\aleph_0} < \kappa$".

Proof. Easily (twice use 3.4), w.l.o.g. the set of elements of $\underset{\sim}{Q}_i$ is a cardinal $\underset{\sim}{\mu}_i < \kappa$; (i.e., $\underset{\sim}{\mu}_i$ is a P_i-name of a cardinal $< \kappa$).

4.1A Definition. For a forcing notion P and P-name $\underset{\sim}{Q}$ of a forcing notion with set of elements $\underset{\sim}{\mu}$ (a P-name of cardinal) with minimal element $\emptyset_{\underset{\sim}{Q}}$ (can demand it to be 0) we define a hereditary countable P-name of a member of $\underset{\sim}{Q}$: it is the closure of the set of ordinals $< \underset{\sim}{\mu}$ (see $(*)$ below) by the two operations (a) and (b) (see below):

$(*)$ the names $\dot{\alpha}$ for $\alpha < \underset{\sim}{\mu}$ or more exactly for an ordinal α the P-name $\underset{\sim}{\tau}_\alpha$ is such that $\underset{\sim}{\tau}_\alpha[G] = \alpha$ if $\alpha < \underset{\sim}{\mu}[G_P]$ and $\underset{\sim}{\tau}_\alpha[G] = \emptyset_{\underset{\sim}{Q}}[G_P]$ if $\alpha \geq \underset{\sim}{\mu}[G_P]$. Of course we can restrict to $\underset{\sim}{\tau}_\alpha$ such that $\nVdash_P \alpha \geq \underset{\sim}{\mu}$. Also if $\underset{\sim}{\mu} = \mu$ we can use just $\dot{\alpha}$, $\alpha < \mu$

(a) if $\underset{\sim}{\tau}_n (n < \omega)$ are such names, and $p_n \in P$ (for $n < \omega$) *then* let $\underset{\sim}{\tau}$ be the $\underset{\sim}{\tau}_n$ for the least n satisfying $p_n \in \underset{\sim}{G}_P$, and $\emptyset_{\underset{\sim}{Q}}$ if there is no such n.

(b) if $\underset{\sim}{\tau}_{n,m} (n < \omega, m < \omega)$ are such names, let $\underset{\sim}{\tau}$ be the least ordinal $\alpha < \underset{\sim}{\mu}$ such that for every n, $\{\underset{\sim}{\tau}_{n,m} : m < \omega\}$ is pre-dense over τ (in Q); and $\emptyset_{\underset{\sim}{Q}}$ if there is no such α. (Remember: the members of $\underset{\sim}{Q}$ are ordinals $< \underset{\sim}{\mu}$).

We shall prove by induction on $\xi \leq \kappa$ that P_ξ satisfies the κ-chain condition.

Suppose this holds for every $\zeta < \xi$, so for $\zeta < \xi$ by Claim I 3.7 clause (ii) we have $< \kappa$ possible values for $\underset{\sim}{\mu}_\zeta$, each is $< \kappa$ so

$$\mu_\zeta \stackrel{\text{def}}{=} \sup\{\mu : \nVdash_{P_\zeta} \text{“}\mu \neq \underset{\sim}{\mu}_\zeta\text{”}\}$$

is $< \kappa$ (as κ is regular). So for $\zeta < \xi$ w.l.o.g. $\underset{\sim}{\mu}_\zeta = \mu_\zeta$ (as we can add to $\underset{\sim}{Q}_\zeta$ the ordinals i, $\underset{\sim}{\mu}_\zeta \leq i < \mu_\zeta$ such that i is $\approx$ to the minimal element $\emptyset_{Q_i}$). Let us define by induction on ζ, $P^\dagger_\zeta = \{f : f$ a function with domain a countable subset of ζ, $f(i)$ is a hereditarily countable $P_i^\dagger$-name of an ordinal $< \mu_i\}$. Let $P_\xi^\dagger \subseteq P_\xi$ inherit its order. We now can prove by induction on $\zeta \leq \xi$, that $P_\zeta^\dagger$ is a dense subset of P_ζ, using the proof that properness is preserved by CS iteration. It is clear that $|P_\xi^\dagger| \leq |\xi|^{\aleph_0}$, so for $\xi < \kappa$, P_ξ has a dense subset of power $< \kappa$. So we finish.

For $\xi = \kappa$, if $p_i \in P_\kappa$ for $i < \kappa$, clearly $S \stackrel{\text{def}}{=} \{i < \kappa : \mathrm{cf}(i) = \aleph_1\}$ is stationary, $f(i) = \mathrm{Sup}[i \cap \mathrm{Dom}(p_i)] < i$ is a pressing down function, hence by Fodor Lemma on some stationary $S_1 \subseteq S, h$ has a constant value γ. There is a closed unbounded $C \subseteq \kappa$ such that: if $\beta \in C, \alpha < \beta$, then $\mathrm{Dom}(p_\alpha) \subseteq \beta$. So $S_1 \cap C$ is still stationary, hence has power κ and for $\alpha, \beta \in C \cap S_1, p_\alpha, p_\beta$ are compatible iff $p_\alpha{\restriction}\gamma, p_\beta{\restriction}\gamma$ are compatible (in P_γ or P_κ, does not matter). But we have proved that P_γ satisfies the κ-chain condition, so we finish. $\square_{4.1}$

We have proved

4.1B Claim. For $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha\rangle$, a *CS* iteration of proper forcing, such that for each $i < \alpha$ it is forced that $\underset{\sim}{Q}_i$ is with set of elements $\subseteq$ Ord, we have, for $i < \alpha$:

1) $P_i' = \{f \in P_i$: for $j \in \mathrm{Dom}(f)$, $f(j)$ is a hereditarily countable P_j'-name$\}$ is a dense subset of P_i, and $i < j \leq \alpha$ and $f \in P_j' \Rightarrow f{\restriction}i \in P_i'$
2) If $\underset{\sim}{f}$ is a $\underset{\sim}{P}_\alpha$-name of a function from ω to Ord and $\mathrm{cf}(\alpha) > \aleph_0$ *then* for a dense open set of $q \in P_\alpha$, for some $\beta < \alpha$ and P_β-name $\underset{\sim}{g}$ of a function from ω to Ord, $q \Vdash_{P_\alpha}$ "$\underset{\sim}{f} = \underset{\sim}{g}$".

4.2 Theorem. Suppose P is not proper, then there is an $\aleph_1$-complete forcing notion Q, in fact $Q = \mathrm{Levy}(\aleph_1, 2^{|P|})$ will do, such that $|Q| \leq 2^{|P|}$ and $\Vdash_Q$ "P collapses some stationary $S \subseteq \omega_1$".

Proof. As P is not proper, there is a stationary $S \subseteq \mathcal{S}_{\aleph_0}(\mu)$ which P destroys, $\aleph_0 < \mu \leq 2^{|P|}$. So there are P-names $\underset{\sim}{F}_n^\ell$ of n-place functions from μ to μ, such that $\Vdash_P$ "$Sm((\mu, \underset{\sim}{F}_0^\ell, \ldots)) \cap S = \emptyset$". Let $Q = \mathrm{Levy}(\aleph_1, \mu) = \{f : f$ a function from some $\alpha < \omega_1$ into $\mu\}$.

Fact A. $\Vdash_Q$ "S is a stationary subset of $\mathcal{S}_{\aleph_0}(\mu)$".

This is because Q is $\aleph_1$ -complete hence, by 2.10, proper.

Fact B. The statement $\Vdash_P$ "$S \subseteq \mathcal{S}_{\aleph_0}(\mu)$ is not stationary " is absolute, i.e., if it holds in V it holds in V^Q.

We just have to check that the P-names $\underset{\sim}{F}{}^\ell_n$ continue to satisfy the suitable requirement (and Q adds no new member to P and no new member to S).

Fact C. $\Vdash_Q$ " the ordinal μ has power $\aleph_1$".

This is trivial.

Fact D. If forcing by P destroys a stationary subset of $\mathcal{S}_{\aleph_0}(A)(A = \mu$ in our case), A of power $\aleph_1$, *then* forcing by P destroys some stationary subset of ω_1. (follows from Lemma 1.5 and 1.12(3)). $\square_{4.2}$

4.3 Theorem. Suppose ZFC has a model with a strongly inaccessible cardinal κ. Then ZFC has a model in which $2^{\aleph_0} = \aleph_2$ and

$(*)$ If P is a forcing notion of power $\aleph_1$ not destroying stationary subsets of ω_1, and $\mathcal{I}_i \subseteq P$ is dense for $i < \omega_1$ *then* there is a directed $G \subseteq P$ satisfying $G \cap \mathcal{I}_i \neq \emptyset$ for $i < \omega_1$.

Proof. Notice that *if* $V \models$ "$\kappa > \aleph_0 \,\&\, \kappa^{<\kappa} = \kappa \,\&\, |P| \leq \kappa \,\&\, (\exists \lambda \leq \kappa)[P$ has the λ-c.c.]" *then* $V^P \models$ "$\kappa^{<\kappa} = \kappa \,\&\, \kappa > \aleph_0$".

This is proved exactly as the parallel fact in Theorem II 3.4. Now let $\{S_\alpha : \alpha < \kappa\}$ be a partition of κ to κ sets such that $\beta \in S_\alpha \Rightarrow \beta \geq \alpha$, and $|S_\alpha| = \kappa$. Define by induction on $i < \kappa$ a CS iterated forcing system $\langle P_i, \underset{\sim}{Q}_i : i < \kappa \rangle$. Let $\langle \underset{\sim}{\leq}_\xi : \xi \in S_\alpha \rangle$ be a list of the canonical P_α-names of partial orders on ω_1. The induction hypothesis for $i < \kappa$ is:

(1) Q_j is proper for $j < i$.

(2) the density of P_i is $< \kappa$ (i.e. it has a dense subset of cardinality $< \kappa$).

Assuming $\langle P_j, \underset{\sim}{Q}_j : j < i\rangle$ is already defined, let

$$\underset{\sim}{Q}_i = \begin{cases} \langle \omega_1, \underset{\sim}{\leq}_i \rangle & \text{if } \Vdash_{P_i} \text{“}(\omega_1, \underset{\sim}{\leq}_i) \text{ is proper”} \\ \mathrm{Levy}(\aleph_1, 2^{\aleph_1}) & \text{otherwise} \end{cases}$$

where $\mathrm{Levy}(\aleph_1, 2^{\aleph_1})$ means “$2^{\aleph_1}$ and the Levy collapse are interpreted in V^{P_i}”, so are P_i-names. Clearly P_i is proper (by Theorem 3.2 and remembering that $\mathrm{Levy}(\aleph_1, 2^{\aleph_1})$ is $\aleph_1$-complete hence by Theorem 2.10 proper). We still have to check that density$(P_i) < \kappa$ but it is easy, note that we use Theorem 4.1. Finally also P_κ is proper by 3.2 and (again by Theorem 4.1) satisfy the κ-c.c. which makes it possible to prove $V^{P_\kappa} \models (*)$ exactly as in the proof of Theorem II 3.4, but using 4.2 above. $\square_{4.3}$

Note that in view of Theorem 4.1 we have a parallel of MA for proper forcing without assuming an inaccessible. We now return to a promise.

4.4 Theorem. Suppose $S \subseteq \omega_1$ is stationary co-stationary (i.e., also $\omega_1 \setminus S$ is stationary too). *Then* there is a forcing notion P_S which shoots a closed unbounded $C \subseteq S$ (i.e., add such a set) without collapsing cardinals (or changing cofinalities).

Remark. So we cannot answer Magidor, Stavi's question positively in the original version.

Remark. Assuming CH this was done by Baumgartner, Harrington, Kleinberg [BHK]. Without CH, Abraham [A] and Baumgartner [B3] introduce forcing notions which add a new closed unbounded subset of ω_1 (for different purposes). We can adapt each for proving 4.4., and will use a forcing similar to Abraham's.

Proof. Let $P = \mathrm{Levy}(\aleph_0, < \aleph_1)$. So P is essentially adding $\aleph_1$ Cohen reals. If $G_P \subseteq P$ is (directed and) generic over V, then G_P is also generic over L (the constructible universe) as $P \in L$ and also over $L[S]$. By Theorem I 6.7 the forcing P satisfies the countable chain condition and $\aleph_1^{L[G_P]} = \aleph_1^V = \aleph_1^{V[G_P]}$ and $V, V[G_P]$ have the same cardinals and cofinalities. Let

$Q = \{C : C$ a closed bounded subset of S which belongs to $L[S, G_P]\}$

$C_1 \leq C_2$ iff $C_1 = C_2 \cap (\mathrm{Max}(C_1) + 1)$

Clearly Q is a forcing notion of power $\aleph_1$, so it cannot collapse cardinals or regularity of cardinals except possibly $\aleph_1$ (all finite subsets of S belong to Q). So we shall prove that $P * \underset{\sim}{Q}$ does not collapse $\aleph_1$. So let (in V) $N \prec (H(\lambda), \in)$, N countable, $(p, \underset{\sim}{q}) \in P * \underset{\sim}{Q} \in N, (p, \underset{\sim}{q}) \in N$, $\delta \overset{\text{def}}{=} N \cap \omega_1 \in S$ and suppose $G_P \subseteq P$ is generic over V and $p \in G_P$. Note that as S is a stationary subset of ω_1, there is such N (in V). So it is enough to find $(q', \underset{\sim}{p}') \geq (q, \underset{\sim}{p})$ which is $(N, P * \underset{\sim}{Q})$-generic. As P satisfies the countable chain condition, p is (N, P)-generic (by 2.9), hence $N[G_P] \cap V = N$. Clearly $Q[G_P] \in L[S, G_P] \subseteq V[G_P]$, now $N[G_P]$ does not necessarily belongs to $L[S, G_P]$, but $N[G_P] \cap L[S, G_P]$ is $N[G_\gamma] \cap L_\delta[S, G_P] = L_\delta[S, G_P] \in L[S, G_P]$ and is a countable set in $L[S, G_P]$. In $L[S, G_P]$ we have an enumeration of $\underset{\sim}{Q}[G_P] \cap N[G_P]$ (of length ω), say $\langle q_n : n < \omega \rangle$ (but not of the set of dense subsets); in fact we have it even in $L[S, G_P \restriction (\delta + 1)]$ (and as we use $\mathrm{Levy}(\aleph_0, < \aleph_1)$ not $\mathrm{Levy}(\aleph_0, < \kappa)$ even in $L[S, G_P \restriction \delta])$. Now in $L[S, G_P]$ there is a Cohen generic real over $V[G_P \restriction (\delta + 1)]$ say $r^* \in {}^\omega\omega$ and we use it to construct a sequence $\bar{C} = \langle C_n : n < \omega \rangle$ such that $C_n \in \underset{\sim}{Q}[G_P]$ in $L[S, G_P]$ i.e. $\bar{C} \in L[S, G_P]$, e.g. we choose C_n by induction on n; we let $C_0 = \underset{\sim}{q}[G_P]$ and we let C_{n+1} be $q_{m(n)}$ where $m(n)$ is the first natural number m such that: $m \geq r^*(n)$ and $\underset{\sim}{Q}[G_P] \models$ "$C_n \leq q_m$". Let $G \overset{\text{def}}{=} \{q : q \in \underset{\sim}{Q}[G_P]$, and $q \in N[G_P]$ and for some n, $\underset{\sim}{Q}[G_P] \models q \leq C_n\}$. Clearly $G \subseteq N[G_P] \cap \underset{\sim}{Q}[G_P]$ is generic over $V[G_P \restriction (\delta+1)]$. So $q^\dagger = \bigcup_n C_n \cup \{\delta\} \in \underset{\sim}{Q}[G_P]$ is $(N[G_P], \underset{\sim}{Q}[G_P])$-generic. Going to names we finish. $\square_{4.4}$

Remark. Also N in $V[G_P]$ is O.K. as it still belongs to some $V[G_P \restriction \alpha]$ for some $\alpha < \omega_1$.

§5. On Aronszajn Trees

5.1 Definition. 1) A cardinal κ is said to have the *tree* property if every tree of height κ in which every level has $< \kappa$ members has a branch of length κ. A tree which is a counterexample to the tree property of κ is called a *κ-Aronszajn tree.* By the König infinity lemma $\aleph_0$ has the tree property.

2) A κ-Aronszajn tree in which every antichain is of cardinality $< \kappa$ is called a *κ-Souslin tree.* An $\aleph_1$-Aronszajn tree, and an $\aleph_1$-Souslin tree are called an Aronszajn and a Souslin tree, respectively. A λ^+-Aronszajn tree is said to be *special* if it is the union of λ antichains, if $\lambda = \aleph_0$ we may omit it. A special Aronszajn tree cannot be Souslin, since in a Souslin tree every antichain is countable, hence the tree, being uncountable, cannot be the union of $\aleph_0$ antichains. A λ-wide Aronszajn tree is a tree with ω_1 levels, λ nodes and no ω_1-branch.

5.1A Remark. It is easy to show that an Aronszajn tree T is special iff there is a function $f : T \to \mathbb{Q}$ which is order preserving. A λ^+-tree which is special is a λ^+-Aronszajn tree. The following was proved by Aronszajn, [Ku35] (and 5.3 is a well known generalization).

5.2 Theorem. There is a special Aronszajn tree T.

Proof. The members of the α-th level T_α of T will be increasing bounded sequences of rational numbers (of length α) with $\trianglelefteq$ as the tree relation. When we come to define T_α we assume that for all $\beta < \gamma < \alpha$ and for all $x \in T_\beta$ and every rational $q > \mathrm{Sup}(\mathrm{Rang}(x))$ there is a $y \in T_\gamma$ such that $x \lhd y$ and $\mathrm{Sup}(\mathrm{Rang}(y)) = q$, and $|T_\beta| = \aleph_0$. If $\alpha = 0$ take $T_\alpha = \{<>\}$. If $\alpha = \gamma + 1$ take $T_\alpha = \{z\hat{\ } < q >: z \in T_\gamma \,\&\, q \in \mathbb{Q} \,\&\, q > \mathrm{Sup}(\mathrm{Rang}(z))\}$, where $\mathbb{Q}$ is the set of all rational numbers. Obviously $|T_\alpha| = |T_\beta| \cdot \aleph_0 = \aleph_0$. The induction hypothesis holds also for $\beta < \alpha$, as easily seen. If α is a limit ordinal then for every $x \in \cup_{\beta<\alpha} T_\beta$ and every $q > \mathrm{Sup}(\mathrm{Rang}(x))$ we shall construct a sequence y of length α which extends x such that $\mathrm{Sup}(\mathrm{Rang}(y)) = q$. Let $\langle \beta_n : n < \omega \rangle$ be a (strictly) increasing sequence such that $x \in T_{\beta_0}$ and $\mathrm{Sup}\{\beta_n : n < \omega\} = \alpha$. Let $\langle q_n : n < \omega \rangle$ be an increasing sequence of rationals such that $q_0 \geq \mathrm{Sup}(\mathrm{Rang}(x))$ and $\mathrm{Sup}_{n<\omega} q_n = q$. We define now a member $x_n \in T_{\beta_n}$ such that $\mathrm{Sup}(\mathrm{Rang}(x_n)) = q_n$ as follows: $x_0 = x$. Assume $x_n \in T_{\beta_n}$ is defined; $\text{Sup}(\text{Rang}(x_n)) = q_n < q_{n+1}$ then by the induction hypothesis there is an $x_{n+1} \in T_{\beta_{n+1}}$ such that $x_n \lhd x_{n+1}$ and $\text{Sup}(\text{Rang}(x_{n+1})) = q_{n+1}$. Take $y = \bigcup_{n<\omega} x_n$, then the length of y is $\cup_{n<\omega}\beta_n = \alpha$ and $\text{Sup}(\text{Rang}(y)) = \text{Sup}(\text{Rang}(x_n)) = \text{Sup}_{n<\omega} q_n = q$. As y was chosen for x and q we let $y = y_{x,q}$. Lastly let $T_\alpha = \{y_{x,q}\colon x \in \bigcup_{\beta<\alpha} T_\alpha$ and $\text{Sup}(\text{Rang}(x)) < q \in \mathbb{Q}\}$. Since we introduced one such y for each $x \in \cup_{\beta<\alpha} T_\beta$ and $q > \text{Sup}(\text{Rang}(x))$ and there are only $\aleph_0$ such pairs clearly $|T_\alpha| \le \aleph_0$.

T has no branch of length ω_1 since if S is such a branch then $\cup S$ is an increasing sequence of rationals of length ω_1, which is impossible.

By our construction of T, for every $x \in T$ we know $\text{Sup}(\text{Rang}(x))$ is a rational number. Therefore $T = \bigcup_{q\in\mathbb{Q}}\{x \in T : \text{Sup}(\text{Rang}(x)) = q\}$, and each set $\{x \in T : \text{Sup}(\text{Rang}(x)) = q\}$ is clearly an antichain. Thus the tree T is a special Aronszajn tree. $\square_{5.2}$

When we want to construct a κ^+-Aronszajn tree we use, instead of the rationals, the set $\mathbb{Q}_\kappa$ of all sequences of ordinals $< \kappa$ of length κ which are eventually 0, ordered lexicographically. We can proceed as in the construction of the $\aleph_1$-Aronszajn tree, but when we construct T_α, for a limit ordinal α such that $\text{cf}(\alpha) < \kappa$, we have to put in T_α all the increasing sequences y of members of $\mathbb{Q}_\kappa$ of length α such that $y{\restriction}\beta \in T_\beta$ for every $\beta < \alpha$. Otherwise we have no assurance that we can carry out the construction of T_α for a limit ordinal α such that $\text{cf}(\alpha) = \kappa$. In order to be sure that $|T_\alpha| \le \kappa$, for every $\alpha < \kappa^+$ we need that $\kappa^{<\kappa} = \sum_{\mu<\kappa} \kappa^\mu = \kappa$, since this will enable us to prove that if for a limit ordinal α with $\text{cf}(\alpha) < \kappa$ we construct T_α as mentioned above we still have $|T_\alpha| \le \kappa$.
So we have presented a proof of the well known:

5.3 Theorem. If $\kappa = \kappa^{<\kappa}$ then there is a κ^+-Aronszajn tree.

If the continuum hypothesis holds then $\aleph_1^{\aleph_0} = \aleph_1$ and therefore there is an $\aleph_2$-Aronszajn tree. Therefore, if we look for a model with no $\aleph_2$-Aronszajn tree, the continuum hypothesis should fail to hold in such a model. There is a theorem which says that in such a model $\aleph_2$ is a weakly compact cardinal in L, hence the consistency of the inexistence of $\aleph_2$-Aronszajn trees is at least as strong as the consistency of the existence of a weakly compact cardinal; we shall see that these two consistency assumptions are equivalent. Mitchell had proved this theorem, and Baumgartner gave a simpler proof by proper forcing.

The following theorem is due to Baumgartner, Malitz and Reinhart [BMR].

5.4 Theorem. For every tree T of height ω_1 with no branch of length ω_1 (no restrictions on its cardinality) there is a c.c.c. forcing notion P such that in the generic extension of V by P the tree T is special. If $|T| < 2^{\aleph_0}$ *then* by Martin's axiom it follows that T is special.

As a consequence, if we assume Martin's axiom and $2^{\aleph_0} > \aleph_1$, then all Aronszajn trees are special and hence there are no Souslin trees.

Proof. Let P be the set of all finite functions p from T into ω such that if $p(x) = p(y)$ then x and y are incomparable. For every $x \in T$ the set $\mathcal{I}_x$ of all members of P whose domain contains x is obviously dense in P, hence if G is a generic subset of $P, F = \cup G$ is defined on all of T and if $F(x) = F(y)$ then x and y are incomparable. If we have Martin's axiom and $|T| < 2^{\aleph_0}$ then there are $< 2^{\aleph_0}$ dense sets $\mathcal{I}_x$ and the directed set G can be taken to intersect all of them, and $F = \cup G$ is as above, i.e., it specializes T since $T = \cup_n \{x : F(x) = n\}$.

We still have to prove that P satisfies the c.c.c. Suppose there is an uncountable subset W of P whose members are pairwise incompatible. Without loss of generality we can assume that all members of W have the same cardinality, that their domains form a Δ-system with the heart s and that for all $p \in W, p\restriction s$ is the same function. Denote $W = \{p_\alpha : \alpha < \omega_1\}$ and let $\mathrm{Dom}(p_\alpha) \setminus s = \{x_{\alpha,1}, \dots, x_{\alpha,n}\}$. Let $\alpha, \beta < \omega_1$, p_α and p_β are incompatible, hence $p_\alpha \cup p_\beta \notin P$. Since p_α and p_β coincide on s and the rest of their domains are disjoint we must have for some $1 \leq k, \ell \leq n$, $p_\alpha(x_{\alpha,k}) = p_\beta(x_{\beta,\ell})$ while $x_{\alpha,k}$ and $x_{\beta,\ell}$ are comparable. Let $Y_{\alpha,k,\ell} = \{\beta < \omega_1 : \beta \neq \alpha, p_\alpha(x_{\alpha,k}) = p_\beta(x_{\beta,\ell})$ and $x_{\alpha,k}$ and $x_{\beta,\ell}$ are comparable $\}$. As we saw $\cup_{1\leq k,\ell\leq n} Y_{\alpha,k,\ell} = \omega_1 \setminus \{\alpha\}$. Let E be a uniform ultrafilter on ω_1, then for every α there are k and ℓ such that $Y_{\alpha,k,\ell} \in E$, let $k(\alpha)$ and $\ell(\alpha)$ be such. Therefore for an uncountable subset A of $\omega_1, k(\alpha) = k$ and $\ell(\alpha) = \ell$ for $\alpha \in A$. Let $\alpha, \beta \in A$ then $Y_{\alpha,k,\ell}$,

$Y_{\beta,k,\ell} \in E$ hence $Y_{\alpha,k,\ell} \cap Y_{\beta,k,\ell} \in E$ and therefore $|Y_{\alpha,k,\ell} \cap Y_{\beta,k,\ell}| = \aleph_1$. Let $\gamma \in Y_{\alpha,k,\ell} \cap Y_{\beta,k,\ell}$ then $x_{\alpha,k}$ and $x_{\beta,k}$ are comparable with $x_{\gamma,\ell}$. Now $x_{\gamma,\ell}$'s with different $\gamma \in Y_{\alpha,k,\ell} \cap Y_{\beta,k,\ell}$ are different, and since there are only countably many members of T below $x_{\alpha,k}$ or below $x_{\beta,\ell}$ (in T's sense) there must be some $\gamma \in Y_{\alpha,k,\ell} \cap Y_{\beta,k,\ell}$ such that $x_{\gamma,\ell}$ is greater than both $x_{\alpha,k}$ and $x_{\beta,k}$ (in T's sense) and since T is a tree, $x_{\alpha,k}$ is comparable with $x_{\beta,k}$. This holds for all $\alpha, \beta \in A$ hence T has a linearly ordered subset of cardinality $\aleph_1$: $\{x_{\alpha,k} : \alpha \in A\}$, and therefore a branch of length ω_1, contradicting our assumption. $\square_{5.4}$

§6. Maybe There Is No $\aleph_2$-Aronszajn Tree

Toward this we mention (see for history, 6.2 below):

6.1 Lemma. 1) Assume $V \models$ "$2^{\aleph_0} > \aleph_1 \,\&\, T$ is an $\aleph_2$-Aronszajn tree." Let P be an $\aleph_1$-complete forcing notion. Then $V[P] \models$ "T has no cofinal branches". ($\aleph_2$ may become of cardinality $\aleph_1$ in $V[P]$ so it does not have to stay an $\aleph_2$-Aronszajn tree.)

2) Assume:

(a) T is a tree with δ^* levels such that $\mathrm{cf}(\delta^*) > \aleph_0$

(b) for no limit $\delta < \delta^*$ of cofinality $\aleph_0$ can we find pairwise distinct $x_\eta \in T_\delta$ for $\eta \in {}^\omega 2$ such that: $[\alpha < \delta \Rightarrow \{x_\eta \restriction \alpha : \eta \in {}^\omega 2\}$ is finite$]$ ($x_\eta \restriction \alpha$ is the unique $y <_T x_\eta$, $y \in T_\alpha$)

(c) P is an $\aleph_1$-complete.

Then forcing by P add no new δ^*-branch to T.

Proof. 1) Assume that $p_0 \Vdash$ "$\underset{\sim}{B}$ is a cofinal branch in T". We shall define in V two functions $F : {}^{\omega>}2 \to T\restriction\alpha$ for some $\alpha < \omega_2$ and $S : {}^{\omega>}2 \to P$ such that:

(i) $F(<>) =$ the root of T, $S(<>) = p_0$

(ii) for all $x \in 2^{<\omega}$ we have $S(x) \Vdash$ "$F(x) \in \underset{\sim}{B}$".

(iii) $x \lhd y \Rightarrow S(x) <_P S(y)$, $F(x) <_T F(y)$, and

(iv) $F(x\char`^ < 0 >)$ and $F(x\char`^ < 1 >)$ are incomparable in T.

$F(\eta)$ and $S(\eta)$ are defined by induction on the length of η. Assume $S(\eta)$, $F(\eta)$ are defined we shall define $S(\eta\hat{}\langle\ell\rangle)$, $F(\eta\hat{}\langle\ell\rangle)$ for $\ell = 0, 1$. Since $S(\eta) \geq_P p_0, S(\eta)$ has, for every $\beta < \omega_2$, an extension which forces some member of T_β (i.e., the set of vertices of height β in the tree) to be in $\underset{\sim}{B}$. If $\{t : F(\eta) <_T t$ and there is $p \geq_P S(\eta)$ such that $p \Vdash$ "$t \in \underset{\sim}{B}$"$\}$ was a set of pairwise comparable members of T then they would be a branch of T in V, contradicting our hypothesis. Therefore there are two incomparable t's in this set, take one to be $F(\eta\hat{} < 0 >)$ and the other to be $F(\eta\hat{} < 1 >)$ and choose $S(\eta\hat{} < 0 >)$ and $S(\eta\hat{} < 1 >)$ as conditions $\geq S(\eta)$ such that $S(\eta\hat{} < \ell >) \Vdash$ "$F(\eta\hat{} < \ell >) \in \underset{\sim}{B}$" for $\ell \in \{0, 1\}$. Since the range of F is countable it is included in some $T{\restriction}\alpha$ for some $\alpha < \omega_2$. Since P is $\aleph_1$ -complete, for every $\eta \in 2^\omega, P$ contains a condition p_η which is an upper bound of $\{S(\eta{\restriction}n) : n < \omega\}$. Since $p_\eta \geq p_0$ there is a $q_\eta \geq p_\eta$ and a $t_\eta \in T_\alpha$ such that $q_\eta \Vdash$ "$t_\eta \in \underset{\sim}{B}$". Let $\nu \neq \eta$, and $\nu, \eta \in 2^\omega$. Let n be the least such that $\nu{\restriction}n \neq \eta{\restriction}n$, then by requirement (iv) above we have that $F(\nu{\restriction}n)$ and $F(\eta{\restriction}n)$ are incomparable in T. Now $P \models$ "$q_\eta \geq p_\eta \geq S(\eta{\restriction}n)$", hence also $q_\eta \Vdash$ "$F(\eta{\restriction}n) \in \underset{\sim}{B}$". Since q_η forces that $\underset{\sim}{B}$ is a branch of T and that $t_\eta, F(\eta{\restriction}n) \in \underset{\sim}{B}$ clearly t_η and $F(\eta{\restriction}n)$ are comparable in T. Since the height of $F(\eta{\restriction}n)$ is $< \alpha$ and the height of t_η is α we have $F(\eta{\restriction}n) <_T t_\eta$. Similarly also $F(\nu{\restriction}n) <_T t_\nu$ and since $F(\eta{\restriction}n)$ and $F(\nu{\restriction}n)$ are $<_T$-incomparable also t_η and t_ν are $<_T$-incomparable and hence different. Thus T_α contains $2^{\aleph_0} > \aleph_1$ different members t_η, contradicting the assumption that T is an $\aleph_2$-Aronszajn tree.

2) Similar proof. $\square_{6.1}$

6.2 Theorem. If ZFC is consistent with the existence of a weakly compact cardinal then ZFC is consistent with $2^{\aleph_0} = \aleph_2$ and the non-existence of $\aleph_2$-Aronszajn trees.

Remark. By what was mentioned in the last section we have "iff" in this theorem. Mitchell had proved the theorem and Baumgartner [B3] gave a simpler proof by proper forcing.

Proof. Let κ be a weakly compact cardinal. We shall use a system $\langle P_i, \underset{\sim}{Q}_i : i < \kappa\rangle$ of iterated forcing with countable support. Q_i will be the composition of two forcing notions $Q_{i,0}$ and $Q_{i,1}$. Now $Q_{i,0}$ will be the forcing notion of countable functions from ω_1 into ω_2 (in $V[P_i]$) which collapses $\aleph_2$ i.e. Levy$(\aleph_1, \aleph_2)$. Now $Q_{i,0}$ is obviously $\aleph_1$-complete. $V[P_i][Q_{i,0}]$ contains wide trees of cardinality $\aleph_1$ and ω_1 levels with no ω_1-branches (e.g., $\{\langle\alpha,\beta\rangle : \beta < \alpha < \omega_1\}$ with $\langle\alpha,\beta\rangle <_T \langle\alpha^\dagger,\beta^\dagger\rangle$ iff $\alpha = \alpha^\dagger \,\&\, \beta < \beta^\dagger$). Let W be the disjoint union of all such trees (up to isomorphisms), as a single tree with at most $2^{\aleph_1}$ roots (we take all the trees to be $\omega_1 \times \{i\}$ with some partial orderings). The tree have ω_1 levels, $\leq 2^{\aleph_1}$ nodes and no ω_1-branch. By Theorem 5.4 there is a c.c.c. forcing $Q_{i,1}$ which makes this tree special, and hence makes every $\aleph_1$-wide tree of cardinality $\aleph_1$ special, provided it has no ω_1- branch.

Note that in V^{P_i}, $Q_{i,0}$ has cardinality $\leq 2^{\aleph_1}$, and $Q_{i,1}$ in $V^{P_i * \underset{\sim}{Q}_{i,0}}$ has cardinality $\leq 2^{\aleph_1}$. Let us notice that these descriptions of $Q_{i,0}$ in $V[P_i]$ and $Q_{i,1}$ in $V[P_i][Q_{i,0}] = V[P_i * Q_{i,0}]$ really yield corresponding names $\underset{\sim}{Q}_{i,0}$ and $\underset{\sim}{Q}_{i,1}$ by the Lemma of the existential completeness which we proved.

Since $Q_{i,0}$ is $\aleph_1$-complete over $V[P_i]$ and since $Q_{i,1}$ satisfies the c.c.c. over $V[P_i][Q_{i,0}]$, both are proper, hence $Q_{i,0} * Q_{i,1}$ is proper and therefore also each $P_i, i \leq \kappa$, is proper. Thus $\aleph_1$ is not collapsed even in $V[P_\kappa]$ (by 3.2). Let λ be an inaccessible cardinal, $\lambda \leq \kappa$. Our construction of P_i and Q_i are such that for $i < \lambda$ we have $|Q_i| < \lambda$, hence each P_i has a dense subset of cardinality $< \lambda$. Therefore, as we proved in 4.1 also P_λ satisfies the λ-c.c. and therefore λ is not collapsed in $V[P_\lambda]$ and thus $V[P_\lambda] \vDash$ "$\lambda \geq \aleph_2$" (since $\aleph_1$ too is not collapsed). Before finishing we prove two lemmas.

6.3 Lemma. $V[P_\lambda] \vDash$ " there are at least λ real numbers " for every $\lambda \leq \kappa$.

Proof. It suffices to prove that for every $i < \lambda$ there is a real in $V[P_{i+1}] \setminus V[P_i][Q_{i,0}]$. We shall see that a forcing notion such as $Q_{i,1}$ introduces a Cohen real over the previous universe. Let us simplify the notation by writing $V^\dagger$ for $V[P_i][Q_{i,0}]$, and Q for $Q_{i,1}$ and T for the tree in $V^\dagger$ which Q makes special. Let $\langle a_j : j < \omega\rangle$ be an ascending sequence in T in $V^\dagger$ (i.e., $j < k < \omega \rightarrow$

$a_j <_T a_k$). Now Q introduces a function $\underset{\sim}{F}$ on T into ω such that if $a, b \in T$ and $\underset{\sim}{F}(a) = \underset{\sim}{F}(b)$ then a and b are $<_T$-incomparable. For $j < \omega$ let $\underset{\sim}{t}_j = 0$ if $\underset{\sim}{F}(a_j)$ is even and $\underset{\sim}{t}_j = 1$ if $\underset{\sim}{F}(a_j)$ is odd. We shall see that $\underset{\sim}{t} = \langle \underset{\sim}{t}_j : j < \omega \rangle$ is a Cohen real over $V^\dagger$, i.e., for every dense subset $\mathcal{I}$ of ${}^{\omega>}2$ in $V^\dagger, t\restriction n \in \mathcal{I}$ for some $n < \omega$. For $p \in Q$ let $p^* = \{\langle j, s\rangle : j < \omega$ and $a_j \in \mathrm{Dom}(p)$ and $([s = 0 \,\&\, p(a_j)$ is even$]$ or $[s = 1 \,\&\, p(a_j)$ is odd$]\,)\}$. Let $Q_\mathcal{I} = \{p \in Q : p^* \in \mathcal{I}\}$; we shall see that $Q_\mathcal{I}$ is a dense subset of Q in $V^\dagger$. Clearly $Q_\mathcal{I} \in V^\dagger$ since it is defined in $V^\dagger$. For $q \in Q$ let $r \geq q^*, r \in \mathcal{I}$; there is such an r since $\mathcal{I}$ is dense in ${}^{\omega>}2$. Let n be a strict upper bound of the range of q. Let $p = q \cup \{\langle a_j, 2n + 2j + r(j)\rangle : j \in \mathrm{Dom}(r) \,\&\, a_j \notin \mathrm{Dom}(q)\}$. Obviously $p \in Q$, and $p^* = r \in \mathcal{I}$ hence $p \in Q_\mathcal{I}$. Since $p \geq q$ we know $Q_\mathcal{I}$ is dense. Let G be the generic subset of Q then there is a $p \in G \cap Q_\mathcal{I}$ such that $p^* \in \mathcal{I}$, and for some $n < \omega, p^* \in {}^n2$ (as $p^* \in Q = {}^{\omega>}2$). Since $p \subseteq \underset{\sim}{F}[G]$ we have $\underset{\sim}{t}[G]\restriction n = p^*$, hence $\underset{\sim}{t}[G]\restriction n \in \mathcal{I}$, which establishes that t is a Cohen real. $\square_{6.3}$

6.4 Lemma. For every inaccessible $\lambda \leq \kappa, V[P_\lambda] \models$ "$\lambda = \aleph_2$".

Proof. Let G be a generic subset of P_λ. We saw already that $V[G] \models$ "$\lambda \geq \aleph_2$". Suppose now that $V[G] \models$ "$\aleph_2 = \mu$", where $\mu < \lambda$. Let $F \in V[G]$ be a function on $\mu \times \omega_1$ such that for all $0 < \alpha < \mu$ we have $\{F(\alpha, \beta) : \beta < \omega_1\} = \alpha$, i.e., $F(\alpha, -)$ is a mapping of ω_1 on α. Let $\underset{\sim}{F}$ be a name of F and let $p_0 \in G$ force that F is as we described. For each $\alpha < \mu$ and $\beta < \omega_1$ let $\mathcal{I}_{\alpha,\beta}$ be a maximal antichain of members of P_λ which are $\geq p_0$ and which give definite values to $\underset{\sim}{F}(\alpha, \beta)$. Since as we saw, P_λ satisfies the λ-c.c. condition, clearly $|\mathcal{I}_{\alpha,\beta}| < \lambda$. Let $\mathcal{I} = \cup_{\alpha<\mu, \beta<\omega_1} \mathcal{I}_{\alpha,\beta}$. Since λ is regular $|\mathcal{I}| < \lambda$. Since each member of $\mathcal{I} \cup \{p_0\}$ is a countable function on λ there is a $\gamma < \lambda$ such that $\mathcal{I} \cup \{p_0\} \subseteq P_\gamma$. Let $G_\gamma = G \cap P_\gamma$, then clearly $F \in V[G_\gamma]$ (since we define $F(\alpha, \beta)$ in $V[G_\gamma]$ to be that γ for which there is a $q \in \mathcal{I}_{\alpha,\beta} \cap G_\gamma$ such that $q \Vdash$ "$\underset{\sim}{F}(\alpha, \beta) = \gamma$"). Then $V[G_\gamma] \models \mu \leq \aleph_2$, but since $V[G] \models$ "$\mu = \aleph_2$" we have $V[G_\gamma] \models$ "$\mu = \aleph_2$". But since we force above $V[G_\gamma]$ with $Q_{\gamma,0}[G_\gamma]$ which collapses the $\aleph_2$ of $V[G_\gamma]$, we have $V[G_\gamma][G_{\gamma,0}] \models$ "$\mu < \aleph_2$" hence $V[G] \models$ "$\mu < \aleph_2$", which is a contradiction. $\square_{6.4}$

Continuation of the Proof of Theorem 6.2. Now let us go on with the proof of the theorem. Assume that there is a $p_0 \in P_\kappa$ such that $p_0 \Vdash_{P_\kappa}$ " there is an $\aleph_2$-Aronszajn tree," i.e., $p_0 \Vdash_{P_\kappa}$ " there is a κ-Aronszajn tree T on κ and a function F on κ such that for $\alpha < \kappa, F(\alpha)$ is the rank of α in T, (since by the lemma $V[P_\kappa] \models$ "$\aleph_2 = \kappa$"), i.e. $p_0 \Vdash_{P_\kappa}$ " there is a transitive relation T on κ such that for all $\alpha, \beta, \gamma \in \kappa$ we have: $[\alpha T\gamma \,\&\, \beta T \gamma \Rightarrow \alpha T \gamma]$ and there is a function F from κ into κ such that for all $\alpha, \beta < \kappa$: if $\alpha T \beta$ then $F(\alpha) < F(\beta)$, for all $\alpha \in \kappa$ and $\gamma < F(\alpha)$ there is a $\beta T \alpha$ such that $F(\beta) = \gamma$, and for all $\gamma \in \kappa$ there is a $\beta \in \kappa$ such that for all $\alpha \in \kappa$ if $F(\alpha) = \gamma$ then $\alpha \leq \beta$, and for all $B \subseteq \kappa$ there are $\alpha, \beta \in B$ such that $\alpha \neq \beta \wedge \neg \alpha T \beta \wedge \neg \beta T \alpha$, or else there is a $\beta < \kappa$ such that $B \subseteq \beta$". This implies, by the existential completeness that there are canonical names of $\underset{\sim}{T}$ and $\underset{\sim}{F}$ of relations on κ such that:

$(*)$ $p_0 \Vdash$ "$\underset{\sim}{T}$ is a transitive relation on κ and $(\forall \alpha, \beta, \gamma < \kappa)(\alpha \underset{\sim}{T} \gamma \,\&\, \beta \underset{\sim}{T} \gamma \rightarrow \alpha \underset{\sim}{T} \gamma)$ and $(\forall \alpha, \beta, \gamma < \kappa)$ $[\alpha = \beta \vee \alpha \underset{\sim}{T} \beta \vee \beta \underset{\sim}{T} \alpha]$ and $(\exists \underset{\sim}{F} : \kappa \rightarrow \kappa)(\forall \alpha, \beta < \kappa)$ $[(\alpha \underset{\sim}{T} \beta \rightarrow F(\alpha) < F(\beta)) \,\&\, (\beta < \underset{\sim}{F}(\alpha) \rightarrow (\exists \gamma < \kappa)(\gamma \underset{\sim}{T} \alpha \wedge \underset{\sim}{F}(\gamma) = \beta))]$ and
$(\forall \gamma < \kappa)(\exists \beta < \kappa)(\forall \alpha < \kappa)(\underset{\sim}{F}(\alpha) = \gamma \rightarrow \alpha < \beta)$"

and for every canonical name $\underset{\sim}{B}$ of a subset of κ

$(**)$ $p_0 \Vdash_\kappa$ "$(\exists \alpha, \beta \in \underset{\sim}{B})(\alpha \neq \beta \wedge \neg \alpha \underset{\sim}{T} \beta \wedge \neg \beta \underset{\sim}{T} \alpha) \vee (\exists \beta \in \kappa)(\underset{\sim}{B} \subseteq \beta)$".

Now a name $\underset{\sim}{X}$ of a subset of $\kappa \times \kappa$ (like $\underset{\sim}{T}$) or κ (like $\underset{\sim}{B}$) or even a subset of $H(\kappa)^V$, we can assume the name is canonical (see Definition I 5.12 and Theorem I 5.13). So $\underset{\sim}{X}$ is a subset of $\{(p, x) : p \in P_\kappa \text{ and } x \in H(\kappa)\}$, so a subset of $H(\kappa)$ and even assume that for each $x \in H(\kappa)$ the set $\mathcal{I}_{\underset{\sim}{X},x} \overset{\text{def}}{=} \{p : (p, x) \in \underset{\sim}{X}\}$ is an antichain of P_κ. But P_κ satisfies the κ.c.c. hence $x \in H(\kappa) \Rightarrow |\mathcal{I}_{\underset{\sim}{X},x}| < \kappa$, hence $E = \{\mu < \kappa : \mu \text{ strong limit singular, and } [j < \mu \Rightarrow \bar{Q} \restriction j \in H(\mu)] \text{ and } x \in H(\mu) \Rightarrow \mathcal{I}_{\underset{\sim}{X},x} \in H(\mu)\}$ is a club of κ. So for $\mu \in E$, $\underset{\sim}{X} \cap H(\mu)$ is a P_μ-name.

Consider now the structure $(H(\kappa), \in, \underset{\sim}{T}, \underset{\sim}{F})$. The statement $(*)$ is a first order statement about this structure, and that $(**)$ holds for every $\underset{\sim}{B}$ as mentioned is a Π^1_1 statement about this structure i.e. a statement of the form: for every subset X of the model some first model sentence holds. We now use one of the equivalent forms of the definition of weakly compact (can be read from the proof, or see e.g. [J]). Since κ is weakly compact and therefore Π^1_1-

indescribable there is an inaccessible cardinal $\lambda < \kappa$, from E such that $(H(\lambda), \in, \underset{\sim}{T} \cap \lambda \times \lambda, \underset{\sim}{F} \cap \lambda \times \lambda)$ is an elementary substructure of $(H(\kappa), \in, \underset{\sim}{T}, \underset{\sim}{F})$ and satisfies the Π^1_1 statement mentioned above. P_κ and $\Vdash_{P_\kappa}$ are definable in $(H(\kappa), \in, \underset{\sim}{T}, \underset{\sim}{F})$ and the same definitions give P_λ and $\Vdash_{P_\lambda}$ in $(H(\lambda), \in, \underset{\sim}{T} \cap \lambda \times \lambda, \underset{\sim}{F} \cap \lambda \times \lambda)$. Therefore $(\underset{\sim}{T} \cap \lambda \times \lambda)[G_\lambda]$ is a λ-Aronszajn tree in $V[G_\lambda]$ (where G is the generic subset of P_κ over V and $G_\lambda = G \cap P_\lambda$). We claim that $(\underset{\sim}{T} \cap \lambda \times \lambda)[G_\lambda]$ is the part of the tree $\underset{\sim}{T}[G]$ up to level λ. Let $\langle \alpha, \beta \rangle \in (\underset{\sim}{T} \cap \lambda \times \lambda)[G_\lambda]$ then $\alpha, \beta < \lambda$ and for some $p \in G_\lambda$, $p \Vdash_{P_\lambda} \langle \underset{\sim}{\alpha}, \underset{\sim}{\beta} \rangle \in (T \cap \lambda \times \lambda)$, hence $p \Vdash_{P_\kappa}$ "$\langle \alpha, \beta \rangle \in \underset{\sim}{T}$" (by the relation between the two above mentioned structures), hence, since $p \in G$, $\langle \alpha, \beta \rangle \in T[G]$, shows that $(\underset{\sim}{T} \cap \lambda \times \lambda)[G_\lambda]$ is included in the part of $\underset{\sim}{T}[G]$ up to level λ. The proof of the equality of these two trees will be completed once we show that in $\underset{\sim}{T}[G]$ all the ordinals in the levels below λ are $< \lambda$. Let $\mu < \lambda$ then, since $(*)$ holds for the structure $(H(\lambda), \ldots)$ there is a $p \in G_\lambda$ and an ordinal $\beta < \lambda$ such that $p \Vdash_{P_\lambda} (\forall \alpha \in \lambda)\ (\underset{\sim}{F} \cap \lambda \times \lambda)(\alpha) = \mu \to \alpha < \beta)$, therefore $p \Vdash_{P_\kappa} (\forall \alpha \in \kappa)\ (\underset{\sim}{F}(\alpha) = \mu \to \alpha < \beta)$, and since $p \in G$ we have in $V[G]$ that all the ordinals in the level μ are $< \beta < \lambda$.

Thus in $V[P_\lambda]$ we know $\underset{\sim}{T} \cap (\lambda \times \lambda)$ is a λ-Aronszajn tree, i.e., an $\aleph_2$-Aronszajn tree. Now we saw that in $V[P_\lambda]$ there are at least λ real numbers, i.e., $2^{\aleph_0} \geq \aleph_2$ in $V[P_\lambda]$ and we know $Q_{\lambda,0}$ is an $\aleph_1$-complete forcing notion in $V[P_\lambda]$, therefore, by Lemma 6.1 $T \cap (\lambda \times \lambda)$ still has no cofinal branch in $V[P_\lambda][Q_{\lambda,0}]$. Since in $V[P_\lambda][Q_{\lambda,0}]$ we have $|\lambda| = \aleph_1$ there is a subset a of λ of order-type ω_1. Let b be the set of all the ordinals in $T \cap \lambda$ whose level is in a, so $T \cap b$ is an $\aleph_1$-tree with no cofinal branch in $V[P_\lambda][Q_{\lambda,0}]$. Now $Q_{\lambda,1}$ makes this tree special (dealing with a tree isomorphic to it). Thus the tree $T{\restriction}\{t : F(t) \in a\}$ is a special $\aleph_1$-tree in $V[P_{\lambda+1}]$ and therefore it stays so also in $V[P_\kappa]$ since the function which makes it special is in $V[P_{\lambda+1}]$ and hence also in $V[P_\kappa]$. Thus in $V[P_\kappa]$ we have $T{\restriction}\lambda$ is a tree which has no cofinal branch and $\{t \in T : F(t) \in a\}$ is (in $V^{P_{i+1}}$) an $\aleph_1$-wide Aronszajn tree, but this is a contradiction since $T \cap (\lambda \times \lambda)$ is the part up to level λ of the κ-tree T and as such it must have branches of length λ. $\square_{6.2}$

We used a weakly compact κ to obtain a generic extension in which there are no $\aleph_2$-Aronszajn trees. If all we want is to obtain an extension of V in which there are no special $\aleph_2$-Aronszajn trees (i.e., trees which are the union of $\aleph_1$ antichains) then it suffices to use a Mahlo cardinal κ (see [B3] on this).

§7. Closed Unbounded Subsets of ω_1 Can Run Away from Many Sets

Baumgartner [B3] has proved the consistency of the following with $ZFC+2^{\aleph_0} = \aleph_2$: if $A_i \subseteq \omega_1$, for $i < \omega_1$, is infinite countable then there is a closed unbounded $C \subseteq \omega_1$ such that $A_i \not\subseteq C$ for every $i < \omega_1$. We prove a somewhat stronger assertion.

7.1 Theorem. $ZFC + 2^{\aleph_0} = \aleph_2$ is consistent with:

(*) if $A_i \subseteq \omega_1$ has no last element and is nonempty and has order type $< \mathrm{Sup} A_i$, for $i < \omega_1$, *then* there is a closed unbounded subset C of ω_1, such that $C \cap A_i$ is bounded in A_i for every i.

Remark. Abraham improved Baumgartner's result to:

$ZFC + 2^{\aleph_0} =$ anything +

(**) there are $\aleph_2$ closed unbounded subsets of $\aleph_1$, the intersection of any $\aleph_1$ of them is finite.

Galvin proved previously that CH implies (*) fail. Our proof is similar to [Sh80 §4].

Proof. We start with a model V satisfying CH, and use CS iterated forcing of length ω_2, such that in the intermediate stages CH still holds. So by 4.1, it suffices to prove.

7.2 Lemma. Suppose V satisfies CH. There is a proper forcing notion of power $\aleph_1$ which adds a closed unbounded subset of C of ω_1 and $\mathrm{Sup}[C \cap A] < \mathrm{Sup} A$ for any infinite $A \subseteq \omega_1$ with no last element and order type $< \mathrm{Sup}(A)$, which belongs to V.

Discussion. A plausible forcing to exemplify 7.2 is:

$$Q = \{C : C \text{ a closed countable subset of } \omega_1 \text{ so that if } A \in V \text{ is a set of ordinals } < \omega_1,\ \delta = \mathrm{Sup}(A) \text{ is a limit ordinal, } A \text{ has order type } < \delta, \text{ then } \mathrm{Sup}[A \cap C] < \delta\}$$

(so if $\delta > \sup(C)$ this holds trivially), with the order

$$C_1 < C_2 \text{ iff } C_2 \cap [(\mathrm{Max}C_1) + 1] = C_1.$$

So the elements of Q are approximations to the required C. It is clear that a generic subset of Q gives a C as required, provided that $\aleph_1$ is not collapsed; hence the main point is to prove the properness of Q. Unfortunately it seems Q is not proper, in fact has no infinite members. However if we want to add a $C \in Q$ which is (N, Q)-generic for some $N \prec (H(\lambda), \in)$ a c.c.c., forcing is enough. So we could first force with some P, $|P| = \aleph_2, P$ satisfies the c.c.c. such that

$$\Vdash_P \text{“}2^{\aleph_0} = \aleph_2 \text{ and MA holds”}.$$

Then we define $\underset{\sim}{Q}$ in $V[G_P]$ for $G_P \subseteq P$ generic over V; similar to the definition above but members of the forcing notion are from $V[G_P]$ whereas the A for which we demand $\sup(C) \cap A < \sup(A)$ are from V. So

$Q = \{C \in V[G]$: C a closed countable subset of ω_1, and if $A \in V$ is infinite with no last element and (order type A) $<$ Sup$A \leq$ Max$C$, then $\mathrm{Sup}(C \cap A) < \mathrm{Sup}A\}$.

Now Q is proper, and adds a C as required, so $P * \underset{\sim}{Q}$ adds a C as required. Unfortunately $P * \underset{\sim}{Q}$ also collapses $\aleph_2$, so if we are willing to use some strongly inaccessible $\kappa > \aleph_0$, there is no problem. Otherwise, we use a restricted version of MA, which is consistent with $2^{\aleph_0} = \aleph_1$, so 7.3, 7.4, 7.5 below prove Lemma 7.2 hence Theorem 7.1. Throughout we use the order $<$ on Q defined above.

7.3 Claim. Suppose

(a) V satisfies CH, P is a forcing notion of power $\aleph_1$ satisfying the c.c.c. and $G \subseteq P$ is generic over V.

(b) $\delta < \omega_1$ is a limit ordinal, $(\forall \alpha, \beta < \delta)[\alpha + \beta < \delta]$, $R \in V[G]$ is a countable family of closed bounded subsets of δ, ordered by $C_1 \leq C_2$ iff $C_1 = C_2 \cap (\mathrm{Max} C_1 + 1)$

(c) Define (in $V[G]$):
$Q = Q_R \stackrel{\text{def}}{=} \{(C, \{(A_i, \alpha_i) : i < n\}) : C \in R, n < \omega$, for each $i < n, A_i \in V$ is a subset of δ of order type $< \delta$, and $C \cap A_i \subseteq \alpha_i$ and α_i is an ordinal $< \delta\}$,
the order is
$(C^1, \{(A^1_i, \alpha^1_i) : i < n^1\}) \leq (C^2, \{(A^2_i, \alpha^2_i) : i < n^2\})$ *iff* $C^1 < C^2, n^1 \leq n^2$ and for every $i < n^1$ for some $j < n^2$ we have $(A^1_i, \alpha^1_i) = (A^2_j, \alpha^2_j)$ and $C^2 \setminus C^1$ is disjoint to A^1_i.

Then

1) Q_R satisfies the c.c.c.

2) if for every $C \in R$, $\{\beta < \delta : C \cup \{\beta\} \in R\}$ has order type δ then $\Vdash_Q$ "$\bigcup_{C \in \underset{\sim}{G}_Q} C$ is an unbounded subset of δ".

Proof. 1) Trivial, as any two conditions with the same first coordinate are compatible, and there are only countably many possibilities for the first coordinate.

(2) Trivial, because if $A_1, \ldots, A_n$ are subsets of δ of order type $< \delta$, their union has order type $< \delta$ by Dushnik, Miller [DM]. So for every $p = (C, \{(A_\ell, \alpha_\ell) : \ell < n\}) \in Q$ the set

$$B_p = \{\beta < \alpha : p \leq (C \cup \{\beta\}, \{(A_\ell, \alpha_\ell) : \ell < n\}) \in Q \text{ and } \beta > \max(C)\}$$

has order type δ because

$$B_p = \{\beta : C \cup \{\beta\} \in R\} \setminus (\bigcup_{\ell < n} A_\ell \cup (\max C + 1))$$

and the first set has order type δ (by an assumption) whereas the second has order type $< \delta$ (by the previous sentence as $\mathrm{otp}(A_\ell) < \delta$ by the definition of Q, and $\mathrm{otp}([0, \max C + 1)) < \delta$ by the assumption on R). Now $B_p \subseteq \delta$ being of order type δ, necessarily is unbounded in δ and we are done. $\square_{7.3}$

7.4 Claim. Suppose V satisfies CH. There is a forcing notion P of power $\aleph_1$ satisfying the c.c.c., such that the following statement is forced:

(*) Suppose $\delta < \omega_1$ is limit, $(\forall\alpha, \beta < \delta)[\alpha + \beta < \delta]$ (equivalently δ is an ordinal power of ω) and R is a countable family of closed bounded subsets of δ such that $(\forall C \in R)(\forall\beta)$ $(\mathrm{Max}C < \beta < \delta \to C \cup \{\beta\} \in R)$ and for $n < \omega$ we have: $\mathcal{I}_n$ is a dense open subset of R such that $\mathcal{I}_n^\dagger = \{(C, \emptyset) : C \in \mathcal{I}_n\} \subseteq Q_R$ is pre-dense in Q_R (where Q_R is from 7.3 clause (c)). *Then* there are $C_n \in R$, such that $C_n < C_{n+1}$, $\mathrm{Sup}_n \mathrm{Max}C_n = \delta, C_{n+1} \in \mathcal{I}_n$ and for every $A \in V, A \subseteq \delta$ of order type $< \delta$, $\mathrm{Sup}[A \cap (\cup_n C_n)] < \delta$; moreover we can choose $C_0 \in R$ arbitrarily.

Proof. We can use an FS iteration $\langle P_i, \underset{\sim}{Q}_i : i < \omega_1\rangle$ of forcing notions satisfying the c.c.c., such that for every possible R and δ (which are in V or appear in V^{P_i} for some $i < \omega$) for uncountably many $j < \omega_1$ in V^{P_j} we have $Q_j = Q_R$. $\square_{7.4}$

7.5 Claim. Suppose V satisfies CH, P is as in 7.4, and $\underset{\sim}{Q} = \{C : C \in V^P$ a closed bounded subset of ω_1, such that for every infinite countable $A \in V$, $A \subseteq \omega_1$ with no last element, and (order type of A) $< \mathrm{sup}A$, if $\mathrm{Sup}A \leq \mathrm{Max}C$ then $\mathrm{Sup}(A \cap C) < \mathrm{Sup}A\}$ ordered by: $C_1 \leq C_2$ iff $C_1 = C_2 \cap (\ \mathrm{max}C + 1)$. *Then* $\underset{\sim}{Q}$ is proper (i.e., $\Vdash_P$ "$\underset{\sim}{Q}$ is proper ") and has cardinality $\leq \aleph_1$.

Proof. Let $G \subseteq P$ be generic over V, λ be regular big enough and let in $V[G]$

$S \stackrel{\mathrm{def}}{=} \{N : N \prec (H(\lambda), \in), P \in N,\ H(\aleph_2)^V \in N,\ G \in N$, and there is a sequence $\langle N_i : i < \delta\rangle$ such that $N_i \prec N$, $\langle N_j : j \leq i\rangle \in N_{i+1}$, $N = \cup_{i<\delta} N_i$ and $\delta = N \cap \omega_1$ (automatically $\underset{\sim}{Q}[G] \in N)\}$

It is easy to check the following facts, and by 2.8 they imply Q is proper, so we finish the proof of 7.2, hence 7.1.

Fact A. $S \in \mathcal{D}_{\aleph_0}((H(\lambda))$ (in $V[G]$, remember we do not distinguish strictly between N and its set of elements).

Fact B. In $V[G]$: if $N \in S, \delta = N \cap \omega_1$, $R \overset{\text{def}}{=} \underset{\sim}{Q}[G] \cap N$, and $\mathcal{I} \in N$ is a dense open subset of $\underset{\sim}{Q}[G]$, *then*

(a) $\{(C, \{(A_i, \alpha_i) : i < n\}) \in Q_R : C \in \mathcal{I} \cap R\}$ is a dense subset of Q_R and $\{(C, \emptyset) : C \in \mathcal{I} \cap R\}$ is a pre-dense subset of Q_R.

(b) For every $C \in \underset{\sim}{Q}[G] \cap N$ there is C^*, $C \leq C^* \in \underset{\sim}{Q}[G]$ and C^* is $(\underset{\sim}{Q}[G], N)$-generic.

Proof. Let $\langle N_i : i < \delta \rangle$ be as in the definition of S. Let $p_0 = (C_0, \{(A_\ell, \alpha_\ell) : \ell < n\}) \in Q_R$ and let $\mathcal{I} \in N$ be a dense open subset of $\underset{\sim}{Q}[G]$, so for some $i < \delta$ we have $\underset{\sim}{\mathcal{I}}$, $p \in N_i$. Let δ_j be $N_j \cap \omega_1$. Let $A'_\ell = \{j < \delta : A_i \cap [\delta_j, \delta_{j+1}) \neq \emptyset\}$, so $\operatorname{otp}(A'_\ell) \leq \operatorname{otp}(A_\ell) < \delta$. hence as in the proof of 7.3 there is $j \in (i, \delta)$ which does not belong to A'_ℓ for $\ell < \omega$. Now $p_1 = (C_0 \cup \{\delta_j\}, \{(A_\ell, \alpha_\ell) : \ell < n\})$ belongs to R and is $\geq p$. There is $C_2 \in \underset{\sim}{Q}[G] \cap \underset{\sim}{\mathcal{I}}[G]$ such that $C \cup \{\delta_j\} \leq C_2$ (in $\underset{\sim}{Q}[G]$), hence there is such C_2 in N_{j+1} (as relevant parameters belong to it). Now $p_2 = (C_2, \{(A_\ell, \alpha_\ell) : \ell < n\})$ belongs to $\underset{\sim}{Q}[G] \cap N$ and is $\geq p_1 \geq p_0$. This proves clause (a).

Let $\langle \mathcal{I}^0_n : n < \omega \rangle$ list the dense open subsets of $\underset{\sim}{Q}[G]$ which belong to N and $C \in \underset{\sim}{Q}[G] \cap N$, and let $\mathcal{I}_n = \mathcal{I}^0_n \cap N$, and let $R = \underset{\sim}{Q}[G] \cap N$ with the inherited order. Trivially $C \in R \;\&\; \max(C) < \beta < N \cap \omega_1 \Rightarrow C \leq_R C \cup \{\beta\}$, and the other assumption in $(*)$ of 7.4 holds by the previous paragraph, so there is $\langle C_n : n < \omega \rangle$ as guaranteed there (with $C \leq C_0$).

Now $C^* = \bigcup_{n<\omega} C_n \cup \{N \cap \omega_1\}$ belongs to $\underset{\sim}{Q}[G]$, $C_n \leq C^*$ and $C_{n+1} \in \mathcal{I}_n \subseteq \mathcal{I}^0_n$, so as $\mathcal{I}_n$ was open, $C^* \in \mathcal{I}_n$. So we have proved clause (b) too. $\square_{7.5,7.2,7.1}$

§8. The Consistency of SH + CH + There Are No Kurepa Trees

8.1 Definition. For any regular κ, a κ-Kurepa tree is a κ-tree such that the number of its κ-branches is $> \kappa$. Let the κ-Kurepa Hypothesis (in short κ–KH) be the statement "there exists κ-Kurepa tree". We may write "KH" instead of ω_1-KH. (Be careful: KH says "there *are* Kurepa trees", but SH says "there are *no* Souslin trees"!)

Solovay proved that Kurepa trees exist if $V = L$, more generally Jensen [Jn] proved the existence of κ-Kurepa's trees follows from Jensen's $\diamondsuit^+$, which holds in L for every regular uncountable κ which is not "too large". But $\neg$KH is consistent with of ZFC + GCH, which was first shown by Silver in [Si67], starting from a strongly inaccessible κ. The method of his proof is as follows: collapse every λ, $\omega_1 < \lambda < \kappa$ using Levy's collapse $\mathrm{Levy}(\aleph_1, < \kappa) = \{p : |p| \leq \aleph_1$ & p is a function with $\mathrm{Dom}(p) \subseteq \kappa \times \omega_1 \wedge \forall \langle \alpha, \xi \rangle \in \mathrm{Dom}(p)(p(\alpha, \xi) \in \alpha)\}$. Now $\mathrm{Levy}(\aleph_1, < \kappa)$ can be viewed as an iteration of length κ, and satisfied the κ-c.c. on the one hand, and $\aleph_1$-completeness on the other hand. Therefore $\aleph_1$ does not get collapsed, as well as any cardinal $\aleph_\alpha \geq \kappa$. Suppose now that $T \in V^P$ is an ω_1-tree. So it has appeared already at an earlier stage along the iteration, say $T \in V^{P'}$, where V^P is obtained from $V^{P'}$ by an $\aleph_1$-complete forcing. In $V^{P'}$ the tree T has at most $2^{\aleph_1}$ branches, and this is less than κ. Note that by 6.1(2) the tree T can have no new ω_1-branches in V^P. So T is not a Kurepa tree in V^P.

Devlin in [De1] and [De2] has shown, starting from a strongly inaccessible, the consistency of GCH + SH + $\neg$KH. For a proof by iteration see Baumgartner [B3].

8.2 Remark. In both proofs the inaccessible cardinal is necessary, for $\neg$KH implies that $\aleph_2$ is an inaccessible cardinal of L.

The main point in Silver's proof, is the fact that $\aleph_1$-complete forcing notions do not add new branches to ω_1-trees. In this section we prove that the property of not adding branches is preserved under CS iterations and use this to give another proof of CON(SH + $\neg$KH) from the consistency of "$(\exists \kappa)$ κ inaccessible". This serve as a prelude and motivation to Chapter V (and even more Chapter VI), which deals with preservation of such properties. In chapter V we will show that moreover the iteration we construct here does not add reals, so (since we start from a model of CH) we will get a model of "CH + SH + $\neg$KH".

8.3 Definition. A forcing notion Q is *good* for an ω_1-tree T (so the α-th level of T is T_α etc.), if for any countable elementary submodel $N \prec H(\chi, \in)$, for χ large enough, with $T, Q \in N$, and every condition $p \in N \cap Q$, there exists an $(N, Q)-$generic condition $q \geq p$ such that if $\underset{\sim}{\tau} \in N$ is a name, $q \Vdash$ "either $\underset{\sim}{\tau}[\underset{\sim}{G}_P]$ is an old branch of T or $\underset{\sim}{\tau}[\underset{\sim}{G}_P] \cap T_{<\delta_N}$ is not a branch of $T_{<\delta_N}$ with a bound $x \in T_{\delta_N}$", where δ_N denotes $N \cap \omega_1 = sup(N \cap \omega_1)$.

8.4 Fact. Q is good for an ω_1-tree T iff Q is proper and Q does not add a new branch to T.

Proof. $\Rightarrow$: Suppose Q is good for T. The properness of Q follows trivially. Let $p \Vdash_Q$ "$\underset{\sim}{\tau}$ is a new branch of T", and we shall derive a contradiction; let $\{T, p, Q\} \in N \prec (H(\chi), \in)$, χ large enough and N countable. So let $q \geq p$ be as in the definition of *good*.

If $\underset{\sim}{\tau}[G]$ is an old branch — we are done. If not, $\underset{\sim}{\tau}[G] \cap T_{<\delta_N} \neq B_x = \{y : y <_T x\}$ for all $x \in T_{\delta_N}$. But this implies that $\underset{\sim}{\tau}[G]$ being linearly ordered by $<_T$ has no member of level $\geq \delta_N$, so it cannot be a ω_1-branch of T.

Conversely, suppose that Q is proper and does not add a new ω_1-branch to T. Let $\underset{\sim}{\tau}, p \in N$ be as in the definition, and pick $q \geq p$ which is (N, Q)-generic, and a generic subset G of P over V with $q \in G$. So $\underset{\sim}{\tau}[G] \in N[G] \prec H(\chi, \in)[G]$, and $\underset{\sim}{\tau}[G]$ is either an old ω_1-branch, or is not an ω_1-branch at all. In the first case we are done. Now if $\underset{\sim}{\tau}[G]$ is not an ω_1-branch, then either $(\exists \alpha)\underset{\sim}{\tau}[G] \cap T_\alpha = \emptyset$ or $\exists x, y \in \underset{\sim}{\tau}[G]$ such that x, y are not comparable in T. By elementaricity of $N[G]$, such an α or such x, y exist also in $N[G]$. So q forces what is required by the definition. $\square_{8.4}$

8.5 Theorem. If T is an ω_1-tree and $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ is a countable support iteration such that for all i the forcing notion $\underset{\sim}{Q}_i$ (is forced to be) good for T, *then* also $P_\alpha = \text{Lim}(\bar{Q})$ is good for T.

Proof. We break here the proof into two parts. The first part is nothing more than another proof of the preservation of properness under countable support iteration. It is meant to help those readers who find the proof in III 3.2 hard to follow. In the second part we show how to extend the first part in order to get a full proof of the theorem.

Let $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha \rangle$ be a countable support iteration such that $\Vdash_{P_j}$ "$\underset{\sim}{Q}_j$ is proper" for all $j < \alpha$. We fix some regular χ which is large enough for what we need. As in III 3.2 we prove by induction on $j \leq \alpha$ the condition $(*)_j$, which is stronger than the properness of P_j:

$(*)_j$ P_j is good for T, and

(a) forcing with P_j add no ω_1-branches to T and:

(b) *for all* $i < j$ and countable $N \prec (H(\chi), \in)$ such that $i, j, \underset{\sim}{Q} \in N$ and $p \in P_j \cap N$ and an (N, P_i)- generic $q \in P_i$ which satisfies $q \geq p{\restriction}i$ *there is* r such that:

(i) $r \in P_j$

(ii) $r{\restriction}i = q$

(iii) r is (N, P_j)-generic

(iv) $p \leq r$

(v) $\mathrm{Dom}(q) \cap [i, j) = N \cap [i, j)$.

The proof is split to cases. Note that though in the statement $(*)_j$(b) we say "for $i < j$" it holds for $i = j$ too.

Case 1. $j = 0$

Trivial.

Case 2. j a successor ordinal

Let $j = j_1 + 1$, now (a) of $(*)_j$ holds as (a) of $(*)_{j_1}$ holds and $\underset{\sim}{Q}_i$ is good for T, so we shall deal with (b) of $(*)_j$. So by the induction hypothesis applied to j_1 and i (see remark above) w.l.o.g. $i = j_1$ and continue as in the proof of III 3.2.

Case 3. j a limit ordinal

We first look at a case of clause (b) of $(*)_j$ and/or of clause (b) of $(*)_j$ (by 8.4 this suffice) so i, N, p, q are given as there.

As N is countable, we can pick a sequence of ordinals of order type ω, $\langle i_n : n < \omega \rangle$ which is cofinal in $N \cap j$ and such that $i_0 = i$ and $i_n \in N$ for all n. Let $\langle \mathcal{I}_n : n < \omega \rangle$, enumerate all the dense sets of P_j in N. Let $\langle (\underset{\sim}{\tau}_n, x_n) : n < \omega \rangle$

enumerate the pairs $(\underset{\sim}{\tau}, x)$ where $\underset{\sim}{\tau}$ is a P_j-name from N and $x \in T_{\delta_N}$ where $\delta_N \stackrel{\text{def}}{=} N \cap \omega_1$.

We define by induction on n a condition q_n and a P_{i_n}-name of conditions $\underset{\sim}{p}_n$ such that:

(a) $q_n \in P_{i_n}$ is (N, P_{i_n})-generic.

(b) $q_0 = q$

(c) $q_{n+1} \restriction i_n = q_n$, $\mathrm{Dom}(q_n) = \mathrm{Dom}(q_0) \cup ([i_0, i_n) \cap N)$

(d) $\underset{\sim}{p}_0 = p$ and $\underset{\sim}{p}_n$ is a P_{i_n}-name of a member of P_j

(e) $q_n \Vdash_{P_{i_n}}$ "$\underset{\sim}{p}_n \in P_j \cap \mathcal{I}_n \cap N$"

(f) $q_n \Vdash_{P_{i_n}}$ "$\underset{\sim}{p} \restriction i_n \in \underset{\sim}{G}_{P_{i_n}}$"

(g) $q_n \Vdash_{P_{i_n}}$ "$\underset{\sim}{p}_n \leq \underset{\sim}{p}_{n+1}$"

(h) if $G_j \subseteq P_j$ is generic over V and $q_n \in G_j \cap P_{i_n}$ and $\underset{\sim}{p}_n[G_j \cap P_{i_n}] \in G_{P_j}$ then either $(\exists \alpha < \delta_N)(\underset{\sim}{\tau}_{i_n}[G_j] \cap T_\alpha \nsubseteq \{t : t <_T x_n\}$ or $T \cap \underset{\sim}{\tau}_{i_n}[G_j]$ is an ω_1-branch of T from V.

Let us carry the induction. For $n = 0$ there is no problem.

Suppose now that we have defined $q_n, \underset{\sim}{p}_n$ and let us define $\underset{\sim}{p}_{n+1}, q_{n+1}$. Pick a generic subset $G_{P_{i_n}}$ of P_{i_n} such that $q_n \in G_{i_n}$. So by clause (e) we have $\underset{\sim}{p}_n[G_{i_n}] \in P_j \cap N$, let $p_n^* \stackrel{\text{def}}{=} \underset{\sim}{p}_n[G_{i_n}]$. Define, in V, the set $\mathcal{J}_n = \{u \in P_{i_n} : (\exists r)[p_n^* \leq r \;\&\; r \in \mathcal{I}_n \;\&\; u = r \restriction i_n]\} \in V$. As $\underset{\sim}{p}_n[G_{i_n}]$ belongs to N, so does $\mathcal{J}_n$. Clearly, $\mathcal{J}_n$ is dense above $\underset{\sim}{p}_n[G_{i_n}] \restriction i_n$. Define $\hat{\mathcal{J}}_n = \mathcal{J}_n \cup \{u : u$ is incompatible with $p_n^* \restriction i_n\} \in V$. So $\hat{\mathcal{J}}_n \in N$ is a dense subset of P_{i_n}. By the genericity of q_n, the set $\hat{\mathcal{J}}_n \cap N$ is predense above q_n, and as a consequence there exists a condition $u_0 \in \hat{\mathcal{J}}_n \cap N \cap G_{i_n}$. As by clause (f) we have $p_n^* \restriction i_n = \underset{\sim}{p}_n[G_{i_n}] \restriction i_n \in G_{i_n}$, clearly u_0 cannot be incompatible with it, but $u_0 \in \hat{\mathcal{J}}_n$ so by the definition of $\hat{\mathcal{J}}_n$ necessarily $u_0 \in \mathcal{J}_n$. There is, therefore, a condition $r_0 \in \mathcal{I}_n$ such that $u_0 = r_0 \restriction i_n$. By elementaricity of N, we can assume that $r_0 \in N$.

In $V[G_{i_n}]$, for any $p \in P_j/G_{i_n} = \{p \in P_j : p{\restriction}i_n \in G_{i_n}\}$ let $B^n_p = B^n_p[G_{i_n}] = \{t \in T : p \not\Vdash_{P_j/G_{i_n}}$ "$t \notin \underset{\sim}{T}_n$"$\}$, equivalently $\{t \in T :$ for some p satisfying $p \le p' \in P_j/G_{i_n}$ we have $p' \Vdash$ "$t \in \underset{\sim}{T}_n$"$\}$. We now choose p^0_{n+1}, α_n such that:

(i) $p^0_{n+1} \in P_j/G_{i_n}$

(ii) $r_0 \le_{P_j} p^0_{n+1} \in N[G_{i_n}]$

(iii) one of the following occurs

(a) $B^n_p[G_{i_n}] \cap T_{\alpha_n} \not\subseteq \{t : t <_T x_n\}$

(b) $p^0_{n+1} \Vdash_{P_j/G_{i_n}}$ "$T \cap \underset{\sim}{T}_n$ is an ω_1-branch of T".

Why is this possible? If for some r, $r_0 \le r \in P_j/G_{i_n}$ and $B^n_r[G_{i_n}]$ is disjoint from some T_α then there are such r, $\alpha_n \in N[G_{i_n}]$ and $p^0_{n+1} = r$ is as required. If for some $\alpha < \omega_1$, $B^n_{r_0}[G_{i_n}] \cap T_\alpha$ has at least two members, then there is such $\alpha_n < \omega_1$, and so there is $t_n \in B^n_{r_0}[G_{i_n}] \cap T_{\alpha_n} \subseteq N$ such that $\neg(t_n <_T x_n)$. By the definition of $B^n_{r_0}[G_{i_n}]$ there is p^0_{n+1} satisfying $r_0 \le p^0_{n+1} \in P_j/G_{i_n}$ such that $p^0_{n+1} \Vdash_{P_j}$ "$t_n \in \underset{\sim}{T}_n$", and p^0_{n+1}, α_n, t_n are as required. Again by elementaricity w.l.o.g. $p^0_{n+1} \in N[G_{i_n}]$ so $p^0_{n+1} \in N$ (as $N[G_{i_n}] \cap V = N$).

Define now $\underset{\sim}{p}_{n+1}$, a P_{i_n}-name by cases. Let $\underset{\sim}{p}_{n+1}[G_{i_n}]$ be $\underset{\sim}{p}_n[G_{i_n}]$ if q_n is not in the generic set G_{i_n}, and equals p^0_{n+1} as described above otherwise. For the definition of q_{n+1} we utilize the induction hypotheses $(*)_{i_n}$. We have just given a prescription, i.e. a name, for $\underset{\sim}{p}_{n+1}$. We can choose a maximal antichain $\mathcal{J} = \{u^n_\zeta : \zeta < \zeta_n(*)\}$ of P_{i_n} of conditions which decide this name, namely $u^n_\zeta \Vdash$ "$\underset{\sim}{p}_{n+1} = p^{n+1}_\zeta$" for some p^{n+1}_ζ, and $u^n_\zeta \ge q_{i_n}$ or u_ζ is incompatible with q_{i_n}.

For each $\zeta < \zeta(*)$ we can apply the induction hypothesis $(*)_{i_{n+1}}$ holds so apply $(*)_{i_{n+1}}$ clause (b) with i_n, i_{n+1}, N, u_ζ, p^{n+1}_ζ here standing for i, j, N, q, p there and get $q^{n+1}_\zeta \in P_{i_n+1}$ as guaranteed there. Define q_{n+1} as follows: $\mathrm{Dom}(q_{n+1}) = (\mathrm{Dom}(q_n)) \cup (N \cap [i_n, i_{n+1}))$, for $\gamma \in \mathrm{Dom}(q_{n+1})$: if $\gamma \in \mathrm{Dom}(q_n)$ then $q_{n+1}(\gamma) = q_n(\gamma)$; if $\gamma \in N \cap [i_n, i_{n+1})$ then $q_{n+1}(\gamma)$ is a P_γ-name: if $\zeta < \zeta_n(*)$ and $u^n_\zeta \in \underset{\sim}{G}_{P_\gamma}$ then it is $p^{n+1}_\zeta(\gamma)$. Check that q_{n+1} is as required.

So we have succeed to carry the induction. Let $q = \bigcup_{n<\omega} q_n$, clearly $q \in P_j$ and $q{\restriction}i_n = q_n$. As in the proof of III 3.2 we can show that:

$(*)$ if $G_j \subseteq P_j$ is generic over V and $q \in G_j$ then $\underset{\sim}{p}_n[G_j \cap P_{i_n}] \in G_j$.

So q is (N, P_j)-generic and as

$$q_n \Vdash_{P_{i_n}} \text{“} p{\restriction}i_{n+1} = p_0{\restriction}i_{n+1} \leq_{P_{i_n}} p_1{\restriction}i_{n+1} \leq \cdots \leq \underset{\sim}{p}_{i_{n+1}}{\restriction}i_{n+1} \leq q_{n+1}\text{”}$$

clearly q is above p. This show that q is as required in clause (b) of $(*)_j$. But by the choice of p^0_{n+1} (and the list $\langle(\underset{\sim}{\tau}_n, x_n) : n < \omega\rangle$) necessarily $q \Vdash$ “for every $\tau \in N[\underset{\sim}{G}_j]$, if τ is not an old ω_1-branch of T then $\tau \cap N[\underset{\sim}{G}_j]$ is not of the form $\{t : t <_T x\}$, for $x \in T_{\delta_N}$”. So q is as required in clause (b) of $(*)_j$ and in Definition 8.3. $\square_{8.5}$

8.6 Theorem. If CON(ZFC + κ is inaccessible) then CON(ZFC + GCH +SH + $\neg$KH).

Proof. Described in 8.1, using 8.5. For CH we need to use the results from chapter V, sections 6 and 7.