I. Forcing, Basic Facts

§0. Introduction

In this chapter we start by introducing forcing and state the most important theorems on it (done in §1); we do not prove them as we want to put the stress on applying them. Then we give two basic proofs:
in §2, we show why CH (the continuum hypothesis) is consistent with ZFC, and in §3 why it is independent of ZFC. For this the $\aleph_1$-completeness and c.c.c.(=countable chain conditions) are used, both implying the forcing does not collapse $\aleph_1$ the later implying the forcing collapse no cardinal. In §4 we compute exactly $2^{\aleph_0}$ in the forcing from §3 (in §3 we prove just $V[G] \models$ "$2^{\aleph_0} \geq \lambda$"; we also explain what is a "Cohen real"). In §5 we explain canonical names.

Lastly in §6 we give more basic examples of forcing: random reals, forcing diamonds. The content of this chapter is classical, see on history e.g. [J]. (Except §7, 7.3 is A. Ostaszewski [Os] and 7.4 is from [Sh:98, §5], note that later Baumgartner has found a proof without collapsing and further works are:

P. Komjáth [Ko1], continuing the proof in [Sh:98] proved it consistent to have MA for countable partial orderings $+\neg$CH, and $\clubsuit$. Then S. Fuchino, S. Shelah and L. Soukup [FShS:544] proved the same, without collapsing $\aleph_1$ and M.Džamonja and S.Shelah [DjSh:604] prove that $\clubsuit$ is consistent with SH (no Souslin tree, hence $\neg$CH).)

§1. Introducing Forcing

1.1 Discussion. Our basic assumption is that the set theory ZFC is consistent. By Godel's completeness theorem it has a countable model. We make the following further assumptions about this model.

(a) The membership relation of the model is the real membership relation; and therefore the model is of the form $(V, \in)$.

(b) The universe V of the model is a transitive set, i.e., $x \in y \in V \to x \in V$.

Assumptions (a) and (b) are not essential but it is customary to assume them, and they simplify the presentation. So "V a model of ZFC", will mean "a countable model of ZFC satisfying (a) and (b)", and the letter V is used exclusively for such models.

Cohen's forcing method is a method of extending V to another model $V^\dagger$ of ZFC. It is obvious that whatever holds in the model $V^\dagger$ cannot be refuted by a proof from the axioms of ZFC, and therefore it is compatible with ZFC. If we show that a statement and its negation are both compatible with ZFC then we know that the statement is undecidable in ZFC.

Why do we look at extensions of V and not at submodels of V? After all, looking at subsets is easier since their members are already at hand: To answer this question we have to mention Godel's constructibility. The constructible sets are the sets which must be in a universe of set theory once the ordinals of that universe are there. Godel showed that the class L of the constructible sets is a model of ZFC and that one cannot prove in ZFC that there are any sets which are not constructible. Therefore, for all we know, V may contain only sets which are constructible and in this case every transitive subclass $V^\dagger$ of V which contains all ordinals of V and which is a model of ZFC must coincide with V, and therefore it gives us nothing new.

1.2 Discussion. Now we come to the concept of forcing. A forcing notion $P \in V$ is just a partially ordered set (not empty of course). Usually a partial order is required to satisfy $p \leq q \,\&\, q \leq p \Rightarrow p = q$, but we shall not (this is just a technicality), this is usually called pre-partial order or quasi order. It is also called a forcing notion. We normally assume that P as a minimal element denoted by $\emptyset_P$, i.e.

$$(\forall q \in P)(P \models \emptyset_P \leq q),$$

really from Chapter II on, we do not lose generality as by adding such a member we get an equivalent forcing notion, see §5. We want to add to V a subset G of P as follows.

(1) G is directed (i.e., every two members of G have an upper bound in G) and downward closed (i.e., if $x \leq y \in G$ then also $x \in G$).

Trivial examples of a set G which satisfies (1) is the empty set $\emptyset$ and $\{x : x \leq p\}$ for $p \in P$.

The following should be taken as a declaration of intent rather than an exactly formulated requirement.

(2) We want that $G \notin V$ and moreover G is "general" or "random" or "without any special property".

We aim at constructing a (transitive) set $V[G]$ which is a model of ZFC with the same ordinals as V, such that $V \subseteq V[G]$ and $G \in V[G]$, and which is minimal among the sets which satisfy these requirements.

So we can look at P as a set of approximations to G, each $p \in P$ giving some information on G, and $p \leq q$ means q gives more information; this view is helpful in constructing suitable forcing notions.

Where does the main problem in constructing such a set $V[G]$ lie? In the universe of set theory the ordinals of V are countable ordinals since V itself is countable. But an ordinal of V may be uncountable from the point of view of V (since V is a model of ZFC and the existence of uncountable ordinals is provable in ZFC). Since for each ordinal $\alpha \in V$ the information that α is countable is available outside V, G may contain i.e. code that information for each $\alpha \in V$. In this case every ordinal of V (and hence of $V[G]$) is countable in $V[G]$ and thus $V[G]$ cannot be a model of ZFC. How do we avoid this danger?

By choosing G to be "random" we make sure that it does not contain all that information.

While we choose a "random" G we do not aim for a random $V[G]$, but we want to construct a $V[G]$ with very definite properties. Therefore we can regard p as the assertion that $p \in G$ and as such p provides some information about G. All the members of G, taken together, give the complete information about G.

Now we come back to the second requirement on G and we want to replace the nebulous requirement above by a strict mathematical requirement.

1.3 Definition. (1) A subset $\mathcal{I}$ of P is said to be a *dense* subset of P if it satisfies

$$(\forall p \in P)(\exists q \in P)(p \leq q \,\&\, q \in \mathcal{I})$$

(2) Call $\mathcal{I} \subseteq P$ open (or upward closed) if for every $p, q \in P$

$$p \geq q \,\&\, q \in \mathcal{I} \Rightarrow p \in \mathcal{I}$$

1.4. Discussion. Since we want G to contain as many members of P as possible without contradicting the requirement that it be directed, we require:

$(2)'$ $G \cap \mathcal{I} \neq \emptyset$ for every dense open subset $\mathcal{I}$ of P which is in V.

1.4A Definition. A subset G of P which satisfies requirements (1) and $(2)'$ is called *generic* over V (we usually omit V), where this adjective means that G satisfies no special conditions in addition to those it has to satisfy.
The forcing theorem will assert that for a generic $G, V[G]$ is as we intended it to be.

Does $(2)'$ imply that $G \notin V$? Not without a further assumption, since if P consists of a single member p then $G = \{p\}$ satisfies (1) and $(2)'$ and $G \in V$. However if we assume that P has no trivial branch, in the sense that above every member of P there are two incompatible members, then indeed $G \notin V$ (incompatible means having no common upper bound). To prove this notice that if $G \in V$, then $P \setminus G$ is a dense open subset of P in V, remember that G is downward closed, and by $(2)'$ we would have $G \cap (P \setminus G) \neq \emptyset$, which is a contradiction.

1.5 The Forcing Theorem, Version A. (1) If G is a generic subset of P over V, then there is a transitive set $V[G]$ which is a model of ZFC, $V \subseteq V[G]$, $G \in V[G]$ and V and $V[G]$ have the same ordinals and we can allow V as a class of $V[G]$ (i.e. in the axioms guaranting (first order) definiable sets exists "$x \in V$" is allowed as a predicate).
(2) P has a generic subset G, moreover for every $p \in P$ there is a $G \subseteq P$ generic over V, $p \in G$. $\square_{1.5}$

1.6 Discussion. We shall not prove 1.5(1), but we shall prove 1.5(2). Since V is countable, P has at most $\aleph_0$ dense subsets in V; let us denote them with $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \ldots$ we shall construct by induction a sequence p_n. We take an arbitrary p_0. We choose p_{n+1} so that $p_n \leq p_{n+1} \in \mathcal{I}_n$; this is possible since $\mathcal{I}_n$ is dense. We take $G = \{q \in P : \exists n(q \leq p_n)\}$. It is easy to check that this G is generic.

Since we want to prove theorems about $V[G]$ we want to know what are the members of $V[G]$. We cannot have in V full knowledge on all the members of $V[G]$ since this would cause these sets to belong to V. So we have to agree that we do not know the set G, but, as we want as much knowledge on $V[G]$ as possible, we require that except for that we have in V full knowledge of all members of $V[G]$, more specifically V contains a prescription for building that member out of G. We shall call these prescriptions "names". We shall be guided in the construction of the names by the idea that $V[G]$ contains only those members that it has to.

Remember:

1.7 Definition. We define the rank of any $a \in V$:

$$\mathrm{rk}(a) \text{ is } \bigcup\{\mathrm{rk}(b) + 1 : b \in a\}$$

(note if $a = \emptyset, \mathrm{rk}(a) = 0$), the union of a set of ordinals is an ordinal, hence $\mathrm{rk}(a)$ is an ordinal if defined, and by the axiom of regularly $\mathrm{rk}(a)$ is defined for every a. So

1.8 Definition. We define what is a P-name (or name for P or a name in P) τ of rank $\leq \alpha$, and what is its interpretation $\tau[G]$. If P is clear we omit it.

This is done by induction on α. τ is a name of rank $\leq \alpha$ if it has the form $\tau = \{(p_i, \tau_i) : i < i_0\}$, $p_i \in P$ and each τ_i is a name of some rank $< \alpha$.

The interpretation $\tau[G]$ of τ is $\{\tau_i[G] : p_i \in G, i < i_0\}$

1.9 Definition.

(1) Let $\mathrm{rk}_n(\tau) = \alpha$ if τ is a name (for some P) of rank $\leq \alpha$ but not a name of rank $\leq \beta$ for any $\beta < \alpha$.

(2) For $a \in V$ and forcing notion P, $\dot{a}$ is a P-name defined by induction on $\mathrm{rk}(a)$;

$$\dot{a} = \{(p, \dot{b}) : p \in P, b \in a\}$$

(3) $\underset{\sim}{G} = \{(p, \dot{p}) : p \in P\}$ (when necessary we denote it by $(\underset{\sim}{G}_P)$.

(4) $\mathrm{rk}_r(\tau)$, the revised rank of a P-name τ is defined as follows: $\mathrm{rk}_r(\tau) = 0$ iff $\tau = \dot{a}$ for some $a \in V$

Otherwise

$$\mathrm{rk}_r(\tau) = \bigcup\{\mathrm{rk}_r(\sigma) + 1 : (p, \sigma) \in \tau \text{ for some } p\}$$

1.9A Remark. 1) Usually, we use $\underset{\sim}{\tau}, \underset{\sim}{f}, \underset{\sim}{a}$ etc. to denote P-names not necessarily of this form.

Eventually we lapse to denoting $\dot{a}$ (the P-name of a) by a, abusing our notation, in fact, no confusion arrives.

1.10 Claim. Given a forcing notion P, and $G \subseteq P$ generic over V, we have:

(1) $\mathrm{rk}_r(\tau) \leq \mathrm{rk}_n(\tau)$ and $\mathrm{rk}(\tau[G]) \leq \mathrm{rk}_n(\tau)$ for any P-name τ.

(2) for $a \in V$, $\dot{a}[G] = a$

(3) $\underset{\sim}{G}[G] = G$.

(4) $\mathrm{rk}_r(\tau), \mathrm{rk}_n(\tau)$ are well defined ordinals, for any P-name τ.

Proof. Trivial. $\square_{1.10}$

1.11 Discussion. Notice that while every name belongs to V, the values of the names are not necessarily in V since the definition of the interpretation of a name cannot be carried out in V. It turns out that these names are sufficient in the sense that the set of their values is a set $V[G]$ as required:

1.12 The Forcing Theorem (strengthened), Version B. In version A, in addition $V[G] = \{\tau[G] : \tau \in V$, and τ is a P-name $\}$. $\square_{1.12}$

We want to know which properties hold in $V[G]$. The properties we are interested in are the first order properties of $V[G]$, i.e., the properties given by formulas of the predicate calculus. We shall refer to the members of $V[G]$ by their names so we shall substitute the names in the formulas.

1.13 Definition. If $\tau_1 \ldots, \tau_n$ are names, for the forcing notion P, $\varphi(x_1, \ldots, x_n)$ a first-order formula of the language of set theory with an additional unary predicate for V, then we write $p \Vdash_P$ "$\varphi(\tau_1 \ldots, \tau_n)$" ($p$ forces $\varphi(\tau_1 \ldots, \tau_n)$ for the forcing P) *if* for every generic subset G of P which contains p we have:

$\varphi(\tau_1[G] \ldots, \tau_n[G])$ is satisfied (=is true) in $V[G]$,

in symbols $V[G] \models$ "$\varphi(\tau_1[G], \ldots, \tau_n[G])$".

1.14 The Forcing Theorem, Version C. If G is a generic subset of P *then* (in addition to the demands in versions A and B we have:) for every $\varphi(\tau_1 \ldots, \tau_n)$ as above there is a $p \in G$ such that $p \Vdash_P$ "$\neg\varphi(\tau_1 \ldots, \tau_n)$" or $p \Vdash_P$ "$\varphi(\tau_1 \ldots, \tau_n)$". Therefore $V[G] \models$ "$\varphi(\tau_1[G] \ldots, \tau_n[G])$" iff for some $p \in G$ $p \Vdash_P$ "$\varphi(\tau_1 \ldots, \tau_n)$". Moreover $\Vdash$ (as a relation) is definable in V. $\square_{1.14}$

This is finally the version we shall actually use, but we shall not prove this theorem either.

The *forcing relation* $\Vdash_P$ clearly depends on P. If we deal with a fixed P we can drop the subscript P. We refer to P as the *forcing notion.*

The rest of the section is devoted to technical lemmas which will help to use the forcing theorem.

1.15 Definition. For $p, q \in P$ we say that p and q are *compatible* if they have an upper bound. $\mathcal{I} \subseteq P$ is an *antichain* if every two members of $\mathcal{I}$ are incompatible. $\mathcal{I} \subseteq P$ is a *maximal antichain* if $\mathcal{I}$ is an antichain and there is no antichain $\mathcal{J} \subseteq P$ which properly includes $\mathcal{I}$. We say $\mathcal{I} \subseteq P$ is pre-dense (above $p \in P$) if for every $q \in P$ $(q \geq p)$ some $q^\dagger \in \mathcal{I}$ is compatible with q. We say $\mathcal{I} \subseteq P$ is dense above $p \in P$ if for every $q \in P$ such that $q \geq p$ there is r, $q \leq r \in \mathcal{I}$; we may omit "above p". We define "$\mathcal{I} \subseteq P$ is pre-dense above $p \in P$" similarly.

1.16 Lemma. Let G be a downward closed subset of P. Then: G is generic (over V) *iff* for every maximal antichain $\mathcal{I} \in V$ of P we have $|G \cap \mathcal{I}| = 1$.

Proof. Suppose G is generic. Since G is directed it cannot contain two incompatible members and hence $|G \cap \mathcal{I}| \leq 1$. Given $\mathcal{I} \in V$, a subset of P, let $\mathcal{J} = \{p \in P : (\exists q \in \mathcal{I}) p \geq q\} \in V$, i.e., $\mathcal{J}$ is the upward closure of $\mathcal{I}$. So $\mathcal{J}$ is obviously upward closed i.e. an open subset, we shall now show that if $\mathcal{I}$ is a maximal antichain of P, then $\mathcal{J}$ is dense. For any $r \in P$ clearly r is compatible with some member q of $\mathcal{I}$ (otherwise $\mathcal{I} \bigcup \{r\}$ would be an antichain properly including the maximal antichain $\mathcal{I}$), let $p \geq r, q$. Then, by the definition of $\mathcal{J}$, $p \in \mathcal{J}$ and we have proved the density of $\mathcal{J}$.
Since $\mathcal{J}$ is dense and open by Definition 1.4A we know $G \cap \mathcal{J} \neq \emptyset$, let $p \in G \cap \mathcal{J}$. Since $p \in \mathcal{J}$, there is a $q \in \mathcal{I}$ such that $q \leq p$, and since $p \in G$ and G is generic, $q \in G$ and so $q \in G \cap \mathcal{I}$, hence $|G \cap \mathcal{I}| \geq 1$. So (assuming $G \subseteq P$ is generic over V) we have proved: for every maximal antichain $\mathcal{I} \in V$ of P, $|G \cap \mathcal{I}| = 1$, thus proving the only if part of the lemma.

Now assume that for every maximal antichain $\mathcal{I} \in V$ we have $|G \cap \mathcal{I}| = 1$. First let $\mathcal{J} \in V$ be a dense subset of P and we shall prove $G \cap \mathcal{J} \neq \emptyset$. By Zorn's lemma there is an antichain $\mathcal{I} \subseteq \mathcal{J}$ which is maximal among the antichains in $\mathcal{J}$, i.e. the antichains of P which are subsets of $\mathcal{J}$. We claim that $\mathcal{I}$ is a maximal antichain. Let $r \in P$, we have to prove that r is compatible with some member of $\mathcal{I}$ (and hence $\mathcal{I}$ cannot be properly extended to an antichain). Since $\mathcal{J}$ is dense there is a $p \in \mathcal{J}$ such that $p \geq r$. Since $p \in \mathcal{J}$ and $\mathcal{I}$ is an antichain maximal in $\mathcal{J}$ necessarily p is compatible with some member q of $\mathcal{I}$, hence r is also compatible with q; so we have finished proving "$\mathcal{I}$ is a maximal antichain of P". So by our present assumption $|G \cap \mathcal{I}| = 1$ hence $G \cap \mathcal{J} \supseteq G \cap \mathcal{I} \neq \emptyset$.

Secondly to see that G is directed let $q, r \in G$ and let $\mathcal{J} = \{p \in P : p \geq q, r$ or p is incompatible with q or p is incompatible with $r\}$. Clearly $\mathcal{J} \in V$, to prove that $\mathcal{J}$ is dense let $s \in P$. If s is incompatible with q then $s \in \mathcal{J}$. Otherwise there is a $t \in P$ such that $s, q \leq t$. If t is incompatible with r then $t \in \mathcal{J}$, and we know that $t \geq s$. Otherwise there is a $w \in p$ such that $w \geq t, r$. Since $t \geq s, q$ we have $w \geq q, r$ and hence $w \in \mathcal{J}$. Since $w \geq t \geq s$ we know $\mathcal{J}$ is dense. By what we have shown above, $G \cap \mathcal{J} \neq \emptyset$. Let $p \in G \cap \mathcal{J}$. We shall see that p cannot be incompatible with q or with r, therefore, since $p \in \mathcal{J}$, $p \geq q, r$. We still have to prove that no two members of G, such as p and q, are incompatible. Suppose $p, q \in G$ and p and q are incompatible. We extend the antichain $\{p, q\}$, by Zorn's lemma to a maximal antichain $\mathcal{I} \in V$. We have $\mathcal{I} \cap G \supseteq \{p, q\}$, contradicting $|\mathcal{I} \cap G| = 1$.

As part of the assumption of 1.16 is "$G \subseteq P$ is downward closed", and we have proved G is directed, and $[\mathcal{J} \in V$ is a dense subset of $P \Rightarrow G \cap \mathcal{J} \neq \emptyset]$, we have proved that G is a generic subset of P over V (see Definition 1.4A). Hence we have finished proving also the if part of the lemma. $\square_{1.16}$

1.17 Lemma. If $\mathcal{J}$ is a pre-dense subset of P in V and G is a generic subset of P *then* $G \cap \mathcal{J} \neq \emptyset$.

Proof. Let $\mathcal{J}^\dagger = \{p \in P : (\exists q \in \mathcal{J}) p \geq q\}$. Let us prove that $\mathcal{J}^\dagger$ is a dense open subset of P. Now $\mathcal{J}^\dagger$ is obviously upward-closed. Let $r \in P$. Since $\mathcal{J}$ is pre-dense there is a $q \in \mathcal{J}$ such that q is compatible with r. Therefore, there is a $p \in P$ such that $p \geq q, r$. By the definition of $\mathcal{J}^\dagger$ we have $p \in \mathcal{J}^\dagger$. Thus we have proved that for every $r \in P$ there is a $p \in \mathcal{J}^\dagger$ such that $p \geq r$, and so $\mathcal{J}^\dagger$ is dense. Since $\mathcal{J} \in V$ and $\mathcal{J}^\dagger$ is constructed from $\mathcal{J}$ in V we have $\mathcal{J}^\dagger \in V$. Since G is generic over V we have $G \cap \mathcal{J}^\dagger \neq \emptyset$. Let $p \in G \cap \mathcal{J}^\dagger$. By the definition of $\mathcal{J}^\dagger$ there is a $q \in \mathcal{J}^\dagger$ such that $q \leq p$. Since G is downward closed we have $q \in G$ and hence $q \in G \cap \mathcal{J} \neq \emptyset$, which is what we had to prove. $\square_{1.17}$

1.18 Lemma. Let $q \in P$, and let $\mathcal{I}$ be a subset of P in V which is pre-dense above q. For every generic subset G of P if $q \in G$ then $G \cap \mathcal{I} \neq 0$.

Proof. Let $\mathcal{I}^\dagger = \mathcal{I} \bigcup \{p \in P : p \text{ is incompatible with } q\}$. Since $\mathcal{I} \in V$ also $\mathcal{I}^\dagger \in V$. Let us prove that $\mathcal{I}^\dagger$ is a pre-dense subset of P. Let $r \in P$. If r is incompatible with q then $r \in \mathcal{I}^\dagger$. If r is compatible with q then there is an $s \in P$ such that $s \geq r, q$. Since $\mathcal{I}$ is pre-dense above q, necessarily s is compatible with some member of $\mathcal{I}$, and hence r is compatible with the same member of $\mathcal{I}$ which neccessarily is also in $\mathcal{I}^\dagger$. Thus we have shown that $\mathcal{I}^\dagger$ is pre-dense. Let G be a generic subset of P such that $q \in G$. Since $\mathcal{I}^\dagger$ is pre-dense and $\mathcal{I}^\dagger \in V$ we have $G \cap \mathcal{I}^\dagger \neq \emptyset$. Let $t \in G \cap \mathcal{I}^\dagger$. Since $t, q \in G$, t is compatible with q, hence by the definition of $\mathcal{I}^\dagger$ we must have $t \in \mathcal{I}$ and thus $t \in G \cap \mathcal{I} \neq \emptyset$. $\square_{1.18}$

1.19 Lemma. Let $\mathcal{I} = \{p_i : i < i_0\}$ be an antichain in P and $\{\tau_i : i < i_0\}$ a corresponding indexed family of P-names (in V). *Then* there is a name τ such that: for every $i < i_0$ and for every generic G, if $p_i \in G$ then $\tau[G] = \tau_i[G]$ (and $\tau[G] = \emptyset$ if $G \cap \{p_i : i < i_0\} = \emptyset$). (We recall that a generic G contains at most one member of $\mathcal{I}$ and if $\mathcal{I}$ is a maximal antichain of P then G contains exactly one member of $\mathcal{I}$).

1.19A Remark. This means we can define a name by cases.

Proof. Suppose $\tau_i = \{\langle p_{i,j}, \tau_{i,j} \rangle : j < j_i\}$, (of course $j_i = 0$ is possible) and let $\tau = \{\langle r, \tau_{i,j} \rangle : j < j_i,\, i < i_0, r \geq p_{i,j} \text{ and } r \geq p_i\}$. $\square_{1.19}$

We note also:

1.20 Claim. Let G be a downward closed directed subset of P. The following are equivalent:

(a) G is generic.

(b) $G \cap \mathcal{I} \neq \emptyset$ for every dense open subset $\mathcal{I}$ of P.

(c) $G \cap \mathcal{I} \neq \emptyset$ for every dense subset $\mathcal{I}$ of P.

(d) $G \cap \mathcal{I} \neq \emptyset$ for every pre-dense subset $\mathcal{I}$ of P.

(e) $G \cap \mathcal{I} \neq \emptyset$ for every maximal antichain $\mathcal{I}$ of P.

1.20A Remark. Clearly for $\mathcal{I} \subseteq P$,

(1) $\mathcal{I}$ is dense open $\Rightarrow \mathcal{I}$ is dense $\Rightarrow \mathcal{I}$ is pre-dense,

(2) $\mathcal{I}$ is a maximal antichain of $P \Rightarrow \mathcal{I}$ is pre-dense.

Proof. By Remark 1.20A(1) clearly (d)⇒(c)⇒(b), by 1.20A(2) clearly (d)⇒(e) by 1.16 (e)⇒(a), trivially (a)⇒(b); by the closing up of subsets of P clearly (b)⇒(c)⇒(d); together we have finished. $\square_{1.20}$

§2. The Consistency of CH (The Continuum Hypothesis)

Usually the consistency of CH, i.e., of $2^{\aleph_0} = \aleph_1$, is proved by showing that it holds in L (the class of constructible sets) but we do not want to go in this way. So

2.1 Theorem. Model A. There is a model of ZFC in which $2^{\aleph_0} = \aleph_1$.

Proof. Let us first review the main points of the construction of the model. We start with a countable transitive model V of ZFC in which $2^{\aleph_0}$ is either $\aleph_1$ or greater. We shall extend it to a model $V[G]$ in which G will essentially be a counting of length $\aleph_1$ of all sets of natural numbers, i.e. a function g from ω_1 onto the family of sets of natural numbers. Each condition, i.e., each member of P, is an approximation of the generic object G and therefore will consist of partial information about the counting. Since every two members of G are compatible, the members of G, taken together, yield a counting of subsets of ω. The three things about which we have to worry in the proof are the following:

a) How do we know that every subset a of ω in V will occur in the counting given by G?

This will be answered by showing that the set of all partial countings in which a occurs is a dense subset of P, and hence G contains such a partial counting.

b) How will the new subsets of ω, i.e., the subset of ω which are in $V[G]$ but not in V, be counted when the members of P, being in V, can count only subsets of ω which are in V?

Here we shall make sure that $V[G]$ has no new subsets of ω.

c) Is $\aleph_1$ of V, which we have mapped on the set of all subsets of ω in $V[G]$ also the $\aleph_1$ of $V[G]$?

Here the answer is easily positive because V and $V[G]$ have the same sets of natural numbers.

In $V[G]$ we want to obtain a function g from $\aleph_1^{V[G]}$ (where $\aleph_1^{V[G]}$ denotes the ordinal which is the $\aleph_1$ of $V[G]$, i.e., the least ordinal α such that $V[G]$ does not contain a mapping of ω onto α) onto $\mathcal{P}(\omega)^{V[G]}$ (where $\mathcal{P}(\omega)$ is the power set of ω and the superscript $V[G]$ means that $\mathcal{P}(\omega)^{V[G]}$ is the power set of ω in $V[G]$). It will turn out that $\aleph_1^{V[G]} = \aleph_1^V$, and the only subsets of ω available in $V[G]$ are the members of $\mathcal{P}(\omega)^V$. By easy considerations for every countable ordinal α, the function $g{\restriction}\alpha$ has to belong to V (as it can be coded by a set of natural numbers). Therefore partial information about g is given by functions from countable ordinals into $\mathcal{P}(\omega)$ in V. Thus it is natural to define

$$P = \{f : f \text{ is a function from a countable ordinal into } \mathcal{P}(\omega)\}$$

where the definition is inside V: with $f \leq g$ *iff* $f \subseteq g$.
A member f of P is understood to "claim" that g is like f on the domain of f. When does a member $f^\dagger$ of P give us more information that f? When $f \subseteq f^\dagger$. Therefore we take the partial order $<$ on P to be proper inclusion.

Let G be a generic subset of P. By the definition of the concept of a generic object every two members of G are compatible, hence $\bigcup G = \bigcup\{f : f \in G\}$ is a function, we shall denote it with f_G. The domain of f_G is the union of the domains of the members of G and hence it is a union of ordinals $< \aleph_1^V$ and therefore the domain of f_G is an ordinal $\leq \aleph_1^V$. If A is a countable set in V then in V there is a function r from ω onto A. Since V is a transitive set, V contains already all the objects which are members of A in the universe and hence A has no new members in $V[G]$. Now $V[G]$ has the same set ω of natural numbers as V and therefore r maps ω on A also in $V[G]$ and A is countable in $V[G]$. In particular every ordinal which is countable in V is also countable in $V[G]$. Therefore, if $\aleph_1^V$ is uncountable in $V[G]$ then $\aleph_1^V$ is the least ordinal which is uncountable in $V[G]$, i.e., $\aleph_1^V = \aleph_1^{V[G]}$. If $\aleph_1^V$ is countable in $V[G]$ then $\aleph_1^V < \aleph_1^{V[G]}$, hence in either case $\aleph_1^V \leq \aleph_1^{V[G]}$. Therefore the domain of f_G is an ordinal $\leq \aleph_1^{V[G]}$ (since it is $\leq \aleph_1^V$). We have in fact deal with

2.2 Definition. 1) A first order formula $\varphi(x)$ (in the language of set theory) is upward absolute *if* when $V \subseteq V^\dagger$ are models of ZFC with the same ordinals (so by our conventions both are transitive sets) and $a \in V$, then: $V \models \varphi[a] \Rightarrow V^\dagger \models \varphi[a]$. Note that properties and function are interpreted by first order formulas.
2) We say φ is absolute if both φ and $\neg\varphi$ are upward absolute (i.e. we have "iff" above).

Obviously (or see [J]):

2.3 Lemma. 1) The following are upward absolute: "α an ordinal", "α not an ordinal" "α is (not) a natural number", "$\alpha = n$", "$\alpha \neq n$", "$\alpha = \omega$", "$\alpha \neq \omega$", "α is not a cardinal", "α is not regular", "A has cardinality $\leq \alpha$", "$A \subseteq B$", "$A \supseteq B$", "f is (not) a (one-to-one) function from A to (onto) B".
2) In fact any relation (function, property) defined by a $\sum_1$ - formula, i.e., by $(\exists y)\varphi(x_0 \dots, x_{n-1}, x_n, y)$ when φ has only bounded quantifiers i.e., of the form $(\forall z_1 \in z_2), (\exists z_1 \in z_2)$ is upward absolute. $\square_{2.3}$

We have now to prove the following facts.

2.4 Fact. Every $A \in \mathcal{P}(\omega)^V$ is in the range of f_G.

2.5 Fact. $\mathcal{P}(\omega)^{V[G]} = \mathcal{P}(\omega)^V$.

Once these facts are proved we know that the range of f_G is $\mathcal{P}(\omega)^{V[G]}$ and in $V[G]$ the function f_G maps an ordinal $\leq \aleph_1^{V[G]}$ onto $\mathcal{P}(\omega)$ hence in $V[G]$ the set $\mathcal{P}(\omega)$ has at most $\aleph_1$ subsets, which establishes $2^{\aleph_0} = \aleph_1$ in $V[G]$.

Remember that a subset $\mathcal{I}$ of P is said to be *pre-dense* in P if every member p of P is compatible with some member of $\mathcal{I}$. In particular $\mathcal{I}$ is pre-dense if $\mathcal{I}$ is a maximal antichain or if $\mathcal{I}$ is such that for every p in P there is a $q \in \mathcal{I}$ such that $q \geq p$.

Proof of Fact 2.4. Let $A \in \mathcal{P}(\omega)^V$; we want to prove that $A \in \mathrm{Rang}(f_G)$, i.e., for some $f \in G$ we have $A \in \mathrm{Rang}(f)$, or, in other words $G \cap \{f \in P : A \in \mathrm{Rang}(f)\} \neq \emptyset$. For this purpose it suffices to show that the set $\mathcal{I} = \{f \in P : A \in \mathrm{Rang}(f)\}$ is dense, since this set is obviously in V. Let $p \in P$ then $p : \alpha \to \mathcal{P}(\omega)$ in V, where α is a countable ordinal. Let f be the function with domain $\alpha + 1$, which is also a countable ordinal in V, into $\mathcal{P}(\omega)^V$ given by $f(\xi) = p(\xi)$ for $\xi < \alpha$ and $f(\alpha) = A$. Clearly $f \in P, f \geq p$ and $f \in \mathcal{I}$. Thus $\mathcal{I}$ is dense and Fact 2.4 is established. $\square_{2.4}$

Now Fact 2.5 will follow from 2.7, 2.8 below, thus completing the proof of Theorem 2.1.

2.6 Definition. 1) A forcing notion P is said to be $\aleph_1$-*complete*, or *countably complete* if every increasing (by $\leq$) sequence $\langle p_n : n < \omega \rangle$ of members of P, i.e., every sequence $\langle p_n : n < \omega \rangle$ such that $p_0 \leq p_1 \leq p_2 \leq \ldots$, has an upper bound p in P (i.e., $p \geq p_n$ for every $n < \omega$).
2) A partial order P is λ-complete if for any $\gamma < \lambda$ and increasing (by $\leq = \leq^P$) sequence $\langle p_i : i < \gamma \rangle$ of members of P, the sequence has an upper bound in P.

2.7 Lemma. Our present set P is countably complete in V.

Proof. Let $\langle p_n : n < \omega \rangle$ be a nondescending sequence of members of P in V. Since the p_n's are pairwise compatible $p = \bigcup_{n<\omega} p_n$ is a function. The domain of p is the union of the domains of the p_n's. Thus the domain of p is a countable union of countable ordinals, and is therefore a countable ordinal. The range of p is the union of the ranges of the p_n's and it consists therefore of members of $\mathcal{P}(\omega)^V$. Thus $p \in P$, and obviously $p \geq p_n$ for every $n < \omega$. $\square_{2.7}$

2.8 Theorem. 1) For every countably complete forcing notion P in V and every generic subset G of P, $V[G]$ contains no new ω-sequence of members of V, i.e., if $\langle a_n : n < \omega \rangle \in V[G]$ and $a_n \in V$ for $n < \omega$ then also $\langle a_n : n < \omega \rangle \in V$. In particular if $a \subseteq \omega$ and $a \in V[G]$ then also $a \in V$.
2) If the forcing notion P is λ-complete, $G \subseteq P$ is generic over V *then* $V[G]$ has no new (i.e.$\notin V$) bounded subsets of λ, not even new sequences of length $< \lambda$ of members of V.

Proof. 1) Let $\langle a_n : n < \omega \rangle \in V[G]$, then $\langle a_n : n < \omega \rangle$ has a name $\underset{\sim}{\tau}$. By the forcing theorem (i.e. 1.14) there is a $q \in G$ such that $q \Vdash$ "$\underset{\sim}{\tau}$ is an ω-sequence of members of V". We shall prove that the subset $\{p \in P : p \Vdash \text{“}\underset{\sim}{\tau} \in V\text{”}\}$ of P is pre-dense above q and, by 1.18, therefore G contains a p such that $p \Vdash$ "$\underset{\sim}{\tau} \in V$" and therefore $\underset{\sim}{\tau}[G] \in V$, i.e., $\langle a_n : n < \omega \rangle \in V$ is true in $V[G]$. Here we use the fact that $\{p \in P : p \Vdash \text{“}\underset{\sim}{\tau} \in V\text{”}\}$ is in V; this is the case since forcing is definable in V.

Let us prove now that $\mathcal{I} = \{p \in: p \Vdash \text{“}\underset{\sim}{\tau} \in V\text{”}\}$ is pre-dense above q. Let $q^\dagger \geq q$; it "knows" (i.e. forces) that every a_n is in V, since already q forces this statement, but even $q^\dagger$ does not necessarily "know" the identity of a_n. We shall see that we can extend $q^\dagger$ to a condition which "knows" a_0(=force a value), then to a condition which "knows" a_1, and so on, and as a consequence of the countably completeness of P, $q^\dagger$ can be finally extended to a condition p which "knows" all the a_n's. This will imply p "knows" that $\langle a_n : n < \omega \rangle$ is some particular member of V, and $p \Vdash$ "$\underset{\sim}{\tau} \in V$", which establishes the pre-density of $\mathcal{I}$ above q.

We define a sequence $\langle p_n : n < \omega \rangle$ of conditions and a sequence $\langle a_n : n < \omega \rangle$ of members of V as follows. Let us mention now that the forthcoming definition is carried out entirely within V and therefore the obtained sequences are members of V. We set $p_0 = q^\dagger$. For $n \geq 0$ we choose p_{n+1} and a_n so that $p_{n+1} \geq p_n$ and $p_{n+1} \Vdash$ "$\underset{\sim}{\tau}(n) = \dot{a}_n$", where $\underset{\sim}{\tau}(n) = \dot{a}_n$ is an abbreviation of "the n-th term of the sequence $\underset{\sim}{\tau}$ is a_n" and $\dot{a}_n$ is the P-name of a_n (see Definition 1.9(2) and 1.9A). Do such p_{n+1} and a_n exist? To prove their existence we go out of V, but this does not matter since once we know they exist the definition proceeds entirely within V. Let $G^\dagger$ be any generic subset which contains p_n.

In $V[G^\dagger]$ we have $\underset{\sim}{\tau}[G^\dagger]$ is an ω-sequence of members of V, since $q \leq p_n \in G^\dagger$ and $q \Vdash$ "τ is an ω-sequence of members of V". Let a_n be the n-th term of the sequence $\underset{\sim}{\tau}[G^\dagger]$, then $a_n \in V$ and $\underset{\sim}{\tau}[G^\dagger](n) = a_n$ is true in $V[G^\dagger]$. By the forcing theorem there is an $r \in G^\dagger$ such that $r \Vdash$ "$\underset{\sim}{\tau}(n) = \dot{a}_n$". Since $r, p_n \in G^\dagger$ they are compatible. Choose $p_{n+1} \geq r, p_n$ then also $p_{n+1} \Vdash$ "$\underset{\sim}{\tau}(n) = \dot{a}_n$", and p_{n+1} and a_n are as required. In order to choose a definite p_{n+1} in P we assume that we have some fixed well ordering of P in V and p_{n+1} is chosen to be the least member of P in that well-ordering for which there exists an a_n so that $p_{n+1} \geq p_n$ and $p_{n+1} \Vdash$ "$\underset{\sim}{\tau}(n) = \dot{a}_n$". Note that a_n is uniquely determined by p_{n+1} since if also for some $b \neq a_n$ we have $p_{n+1} \Vdash$ "$\underset{\sim}{\tau}(n) = \dot{b}$" then for every generic $G^\dagger$ which contains p_{n+1} we have that the n-th term of $\underset{\sim}{\tau}[G^\dagger]$ is both a_n and b, which is impossible.

Since P is countably complete there is a $p \in P$ such that $p \geq p_n$ for all $n < \omega$. We have, obviously $p \geq p_0 = q^\dagger$ and for every $n < \omega$ we know $p \geq p_{n+1}$ and hence $p \Vdash$ "$\underset{\sim}{\tau}(n) = \dot{a}_n$". Thus for every generic subset $G^\dagger$ of P which contains p we have: $\underset{\sim}{\tau}[G^\dagger]$ is an ω-sequence and $\underset{\sim}{\tau}[G^\dagger](n) = a_n$ for every $n < \omega$, hence $\underset{\sim}{\tau}[G^\dagger] = \langle a_n : n < \omega \rangle \in V$ (note $\langle a_n : n < \omega \rangle \in V$ since this sequence was defined in V). By the definition of the forcing relation we have $p \Vdash$ "$\underset{\sim}{\tau} \in V$", which is what we had to prove.

If $a \in V[G]$ and $a \subseteq \omega$ then let $\langle a_n : n < \omega \rangle$ be the characteristic function of a. Since each a_n is 0 or 1 we have $\langle a_n : n < \omega \rangle$ is a sequence of members of V and hence, by the present theorem $\langle a_n : n < \omega \rangle \in V$ and a can be easily obtained from $\langle a_n : n < \omega \rangle$ within V.

2) Left to the reader. $\square_{2.8,2.5,2.1}$

§3. On the Consistency of the Failure of CH

We first prove a technical lemma, and then prove that $2^{\aleph_0} = \aleph_\alpha$ is possible for almost any α.

3.1 The Existential Completeness Lemma. 1) If $p_0 \Vdash_P$ "$(\exists x)\varphi(x)$" *then* there is a name $\underset{\sim}{\tau}$ such that $p_0 \Vdash_P \varphi(\underset{\sim}{\tau})$, where $\varphi(x)$ is a formula which may mention names.

2) Moreover for every formula $\varphi(x)$ as above for some P-name $\underset{\sim}{\tau}$

$\Vdash_P$ "$(\exists x)[\varphi(x)] \to \varphi(\underset{\sim}{\tau})$" and $\Vdash_P$ "$\neg(\exists x)[\varphi(x)] \to \underset{\sim}{\tau} = \emptyset$".

Proof. 1) The idea of the proof is as follows. The condition p_0, "knows" that $(\exists x)\varphi(x)$ but this does not tell us directly that p_0 knows a particular name of a set x which satisfies $\varphi(x)$. However with more information than that in p_0 we know names of sets which satisfy $\varphi(x)$. What we have to do is to combine the various names to a single name which equals each of those names just when the name satisfies $\varphi(x)$.

Let

$$\mathcal{J} \stackrel{\text{def}}{=} \{q : q \Vdash_P \text{“}\neg(\exists x)\varphi(x)\text{” or for some name } \underset{\sim}{\tau} \text{ we have } q \Vdash_P \text{“}\varphi(\underset{\sim}{\tau})\text{”}\}.$$

$\mathcal{J}$ is defined in V, hence $\mathcal{J} \in V$. We shall now see that $\mathcal{J}$ is dense. Let $r \in P$, but r does not force $\neg(\exists x)\varphi(x)$. Then there is a generic $G \subseteq P$ such that $r \in G$, and $V[G] \models$ "$(\exists x)\varphi(x)$", by r's choice. Since every member of $V[G]$ is the value of some name we have $V[G] \models$ "$\varphi(\underset{\sim}{\tau}[G])$" for some P-name $\underset{\sim}{\tau}$. By the forcing theorem there is an $r^\dagger \in G$ such that $r^\dagger \Vdash_P$ "$\varphi(\underset{\sim}{\tau})$". Since G is directed and $r \in G$ without loss of generality $r^\dagger \geq r$ and by definition, $r^\dagger \in \mathcal{J}$. Now if $r \Vdash_P$ "$\neg(\exists x)\varphi(x)$", trivially $r \in \mathcal{J}$. Thus we have shown that $\mathcal{J}$ is dense. Let $\mathcal{I}$ be a maximal subset of $\mathcal{J}$ of pairwise incompatible members. We shall see that also $\mathcal{I}$ is pre-dense. Let $q \in P$ and suppose, in order to obtain a contradiction, that q is incompatible with every member of $\mathcal{I}$. By what we proved about $\mathcal{J}$ there is a $q^\dagger \in \mathcal{J}$ such that $q^\dagger \geq q$. Also $q^\dagger$ is, clearly, incompatible with every member of $\mathcal{I}$. Now $\mathcal{I} \bigcup \{q^\dagger\}$ is a subset of $\mathcal{J}$ of pairwise incompatible members which properly includes $\mathcal{I}$, which contradicts our choice of $\mathcal{I}$.

Let $\mathcal{I} = \{q_i : i < \alpha\}$, since $\mathcal{I} \subseteq \mathcal{J}$ there is for every $i < \alpha$ a name $\underset{\sim}{\tau}_i$ such that $q_i \Vdash_P$ "$\varphi(\underset{\sim}{\tau}_i)$" or $q_i \Vdash_P$ "$\neg(\exists x)\varphi(x)$". Let $\underset{\sim}{\tau}$ be the name

$$\underset{\sim}{\tau} = \begin{cases} \underset{\sim}{\tau}_i & \text{if } q_i \Vdash_P \text{"}\varphi(\underset{\sim}{\tau}_i)\text{"} \\ \emptyset & \text{otherwise} \end{cases}$$

which we have proved to exist (1.19) for pairwise incompatible q_i's. We claim that $p_0 \Vdash$ "$\varphi(\underset{\sim}{\tau})$". To prove that let G be a generic subset of P and $p_0 \in G$. Since $\mathcal{I}$ is pre-dense we have $G \cap \mathcal{I} \neq \emptyset$ and hence for some $i < \alpha$ we have $q_i \in G$. If $\underset{\sim}{\tau}_i$ is not defined $q_i \Vdash_P$ "$\neg(\exists x)\varphi(x)$", but then q_i, p_0 are incompatible, but both are in G, contradiction. So $q_i \Vdash$ "$\varphi(\underset{\sim}{\tau}_i)$" hence we have $V[G] \vDash \varphi(\underset{\sim}{\tau}_i[G])$. Also since $q_i \in G$ we have, by the definition of $\underset{\sim}{\tau}$, $\underset{\sim}{\tau}[G] = \underset{\sim}{\tau}_i[G]$, hence $V[G] \vDash$ "$\varphi(\underset{\sim}{\tau}[G])$", which establishes $p_0 \Vdash$ "$\varphi(\underset{\sim}{\tau})$".

2) The second part in the lemma was really proved too. $\square_{3.1}$

3.2 Theorem. Model B. There is a model in which the continuum hypothesis fails. Moreover, for every cardinal λ there is a forcing notion P such that $\Vdash_P$ "$2^{\aleph_0} \geq \lambda$ and every cardinal of V is a cardinal (of $V[G]$))".

Convention. We use the word "real" as meaning a subset of ω or its characteristic function.

Proof. We want to add to V λ real numbers (i.e., functions from ω into 2). Each condition gives us some information about them, so we have to make sure that the information contained in a single condition will not suffice to compute one of the reals since in this case this real will be already in V. Therefore we shall define the conditions so that each one will contain only a finite amount of information, and therefore each condition is clearly insufficient for computing any of the reals. We shall regard the λ reals, each of which is an ω-sequence of

0's and 1's, as written in a long sequence one after the other to form a sequence of length λ of 0's and 1's. The members of P will be finite approximations to this member of ${}^{\lambda}2$. Therefore we take $P = \{f : f$ is a finite function from λ into $\{0,1\}\}$ where by " a finite function from λ" we mean a function from a finite subset of λ. For the partial order on P we take proper inclusion, i.e., $f < g$ if $f \subset g$. This forcing is called "adding λ Cohen reals."

3.3 Lemma. $\Vdash_P$ "there are at least λ reals".

Proof of the Lemma. We shall prove the existence of the reals by giving them names. Since for every generic G there is a function g which satisfies $g = \bigcup_{f \in G} f$ we have $\Vdash_P$ "$(\exists x)(x = \bigcup_{f \in \underset{\sim}{G}_P} f)$" and by the first lemma in this section, 3.1 there is a name $\underset{\sim}{g}$ such that $\Vdash_P$ "$\underset{\sim}{g} = \bigcup_{f \in \underset{\sim}{G}_P} f$". Using again the same method we get, for every $i < \lambda$ a name $\underset{\sim}{a}_i$ defined by $\underset{\sim}{a}_i(n) = \underset{\sim}{g}(i+n)$. Now a_i is forced to be a name of a real number provided g is a name of a function on λ into $\{0,1\}$. We shall prove it in the next sublemma.

3.4 Sublemma. g is a function from λ into $\{0,1\}$ (i.e., this is forced).

Proof of the Sublemma. g is a function since G is a directed set of functions, hence its union g is a function. Also g is into $\{0,1\}$ since each member of G is into $\{0,1\}$. Next $\mathrm{Dom}(g) \subseteq \lambda$ since for every $f \in G$ we have $f \in P$ and hence $\mathrm{Dom}(f) \subseteq \lambda$; we still have to prove that $\mathrm{Dom}(g) = \lambda$. Let $i < \lambda$; it suffices to prove that the set $\mathcal{J}_i = \{f \in P : i \in \mathrm{Dom}(f)\}$, which is clearly is V, is pre-dense. Let $f \in P$, if $i \in \mathrm{Dom}(f)$ then $f \in \mathcal{J}_i$, otherwise let $f^{\dagger} = f \bigcup \{\langle i, 0\rangle\}$ then $f^{\dagger} \in P$ and $f^{\dagger} > f$ and $f^{\dagger} \in \mathcal{J}_i$. $\square_{3.4}$

We return to the Lemma 3.3. Note that since we have proved that $\Vdash_P$ "$\underset{\sim}{g}$ is a function from λ into $\{0,1\}$", it is enough to prove for each $i \neq j < \lambda$ that $\Vdash_P$ "$\underset{\sim}{a}_j \neq \underset{\sim}{a}_j$". For this it suffices to prove that for every $p \in P$ there is an $r \geq p \in P$ and an $n < \omega$ such that $r \Vdash_P$ "$\underset{\sim}{a}_i(n) \neq \underset{\sim}{a}_j(n)$", since this proves that the set of all $p \in P$ such that $p \Vdash_P$ "$\underset{\sim}{a}_i \neq \underset{\sim}{a}_j$" is pre-dense.

Let $p \in P$. Since $\mathrm{Dom}(p)$ is finite there is an n_0 such that for every $n \geq n_0$ $i+n \notin \mathrm{Dom}(p)$ and an n_1 such that for every $n \geq n_1$ we have $j+n \notin \mathrm{Dom}(p)$. We set $r = p\bigcup\{\langle i+k, 0\rangle, \langle j+k, 1\rangle$, where $k \geq n_0, n_1$. Clearly r is a function since $i+k, j+k \notin \mathrm{Dom}(p)$ and $i+k \neq j+k$ since $i \neq j$. Obviously $p < r$ and (r forces that) $\underset{\sim}{a}_i(k) = 0$, $\underset{\sim}{a}_j(k) = 1$ hence $\underset{\sim}{a}_i(k) \neq \underset{\sim}{a}_j(j)$. $\square_{3.3}$

Continuation of the Proof of 3.2. We started with λ, which is a cardinal of V and we proved that in $V[G]$ there are at least λ real numbers, but is λ in $V[G]$ the "same" cardinal as it was in V? As the matter stands now we do not even know whether the continuum hypothesis fails in $V[G]$ since even though λ may be a large cardinal in V it may be countable or $\aleph_1$ in $V[G]$. We shall now prove that all the cardinals of V are still cardinals in $V[G]$ so for example if $\lambda = \aleph_2^V$ then λ is still the third infinite cardinal in $V[G]$ and thus $\lambda = \aleph_2^{V[G]}$. We shall prove that the cardinals of V are not collapsed in $V[G]$ for forcing notions P which satisfy the countable chain condition and that P satisfies this condition. I.e. the cardinals of V are still cardinals in $V[G]$, and as V, $V[G]$ have the same ordinals and no non-cardinals of V are cardinals of $V[G]$ we have: V, $V[G]$ have the same cardinals; this is done in 3.6, 3.8 below. This is important general theorem which we shall use a lot. This will finish the proof of 3.2.

3.5 Definition.

(1) A forcing notion Q satisfies the countable chain condition (c.c.c.) if Q has no uncountable subset of pairwise incompatible members, i.e., if every uncountable subset of Q contains two compatible members.

(2) A forcing notion Q satisfies the λ-chain condition (λ-c.c.) if there are no λ pairwise incompatible members of P.

3.6 Lemma. If a forcing notion Q satisfies the c.c.c. then

(i) forcing with Q does not collapse cardinals and cofinalities, (i.e., $\Vdash_Q$ "every cardinal of V is a cardinal (of $V[G]$)), and the cofinality is preserved".

(ii) For every ordinal α and every Q-name $\underset{\sim}{\tau}$ there is, in V, a function F from α (to V) such that for every $\beta < \alpha$ we have $|F(\beta)| \leq \aleph_0$ and $\Vdash_Q$ "if $\underset{\sim}{\tau}$ is a function from α into V then $(\forall\beta < \alpha)[\underset{\sim}{\tau}(\beta) \in F(\beta)]$".

Proof of Lemma 3.6. Proof of (ii): We define the function F on α by $F(\beta) = \{a \in V : (\exists q \in Q)(q \Vdash_Q$ "$\underset{\sim}{\tau}$ is a function from α and $\underset{\sim}{\tau}(\beta) = \dot{a}$")$\}$. We have to prove that the right hand side is a set, and not a proper class of V and moreover is countable. We shall assume it now and prove it later. The right hand side is the set of all possible values of $\underset{\sim}{\tau}(\beta)$ in all the $V[G]$'s in which $\underset{\sim}{\tau}[G]$ is a function from α into V. To see this suppose G is a generic set such that $\underset{\sim}{\tau}[G]$ is a function from α into V. Then for some $a \in V$ $V[G] \vDash$ "$\underset{\sim}{\tau}$ is a function from α and $\underset{\sim}{\tau}(\beta) = \dot{a}$". By the forcing theorem there is a $q \in Q$ such that $q \Vdash_Q$ "$\underset{\sim}{\tau}$ is a function from α and $\underset{\sim}{\tau}(\beta) = \alpha$". Hence $a \in F(\beta)$ and therefore $V[G] \vDash$ "$(\forall\beta < \alpha)[\underset{\sim}{\tau}(\beta) \in F(\beta)]$". Since this is the case for every generic G we have what is claimed in (ii).

Now we shall prove not only that the class $\{a \in V : (\exists q \in Q)(q \Vdash_Q \underset{\sim}{\tau}$ is a function on α and $\underset{\sim}{\tau}(\beta) = \dot{a}$")$\}$ is a set but even that it is a countable set, and thus $|F(\beta)| \leq \aleph_0$. Suppose $\{a_i : i < \omega_1\}$ is a subset of this class. For each such a_i there is a $q_i \in Q$ such that $q_i \Vdash_Q$ "$\underset{\sim}{\tau}$ is a function on α and $\underset{\sim}{\tau}(\beta) = a_i$". Since Q satisfies the c.c.c. there must be some $i \neq j$ such that q_i and q_j are compatible. Let $q \geq q_i, q_j$ and let $G \subseteq Q$ be a generic subset of Q which contains q. We have $q_i, q_j \in G$ and hence, since $q_i \Vdash_Q$ "$\underset{\sim}{\tau}(\beta) = a_i$" and $q_j \Vdash_Q$ "$\underset{\sim}{\tau}(\beta) = a_j$", we have $V[G] \vDash$ "$a_i = \underset{\sim}{\tau}(\beta) = a_j$", hence $a_i = a_j$. Thus we have shown that at least two of the a_i's must be equal therefore the class $\{a \in V : \exists q \in Q(q \Vdash_Q$ "$\underset{\sim}{\tau}$ is a function from α and $\underset{\sim}{\tau}(\beta) = a$")$\}$ must be countable. The proof that $F(\beta)$ is a set is similar.

Proof of (i). Let λ be an uncountable cardinal of V and suppose λ is not a cardinal in $V[G]$. Then there is an ordinal $\alpha < \lambda$ and a function $f \in V[G]$ which maps α onto λ. Let $\underset{\sim}{\tau}$ be a name for f in $V[G]$. By part (ii) of our lemma (already proved) there is a function F from α in V such that $f(\beta) \in F(\beta)$ and $|F(\beta)| \leq \aleph_0$ for every $\beta < \alpha$. We have therefore $\lambda = \mathrm{Rang}(f) \subseteq \bigcup_{\beta<\alpha} F(\alpha)$. But in V we have $|\bigcup_{\beta<\alpha} F(\beta)| \leq |\alpha|\aleph_0 < \lambda$, since λ is an uncountable cardinal

of V and $\alpha < \lambda$, and therefore also $|\alpha| < \lambda$. Thus we have proved that the uncountable cardinals are not collapsed in $V[G]$. Also $\aleph_0$ is not collapsed since the finite ordinals and ω of V are also finite ordinals and ω in $V[G]$.
The preservation of the cofinality is proved similarly. $\square_{3.6}$

Similarly one can prove for uncountable λ.

3.7 Claim. If Q satisfies the λ-c.c. then

(i) forcing by Q preserve cardinals and cofinalities which are $\geq \lambda$.

(ii) for every Q-name and ordinal α there is a function F with domain α, ($F \in V$) such that $\Vdash_Q$ " if $\underset{\sim}{\tau}$ is a function from α to V then $\underset{\sim}{\tau}(\beta) \in F(\beta)$ for every $\beta < \alpha$", and $|F(\beta)| < \lambda$ for $\beta < \alpha$.

3.8 Lemma. The forcing notion P which we use here for Model B satisfies the c.c.c.

3.8A Remark. Once we prove this lemma we know that all the cardinals in V here are cardinals also in $V[G]$ and therefore λ is a cardinal also in $V[G]$, and if λ is the α-th infinite cardinal $\aleph_\alpha$ in V it is also the α-th infinite cardinal in $V[G]$.

Proof. Suppose $\{f_i : i < \aleph_1\} \subseteq P$, in order to prove Lemma 3.8, it is enough to prove that two of the functions are compatible. By 3.10 below for some uncountable $A \subseteq \omega_1$ and finite w for every $i \neq j$ from A, we have $\mathrm{Dom}(f_i) \cap \mathrm{Dom}(f_j) = w$. The number of possible $f_i \restriction w$ is finite (i.e. $\leq 2^{|w|}$), so without loss of generality $f_i \restriction w = f^*$ for every $i \in A$. Now for any $i \neq j \in A$, we know $f_i \cup f_j \in P$ is a common upper bound of f_i, f_j. $\square_{3.8}$
We have promised the so called Δ-system lemma.

3.9 Definition. A family F of finite sets is called a Δ-system if there is a set w such that for any A, B in F we have $A \cap B = w$.
In this formulation our problem reduces to:

3.10 Lemma. Given an indexed family F of finite sets, $|F| = \aleph_1$ there is $F^\dagger \subseteq F; |F^\dagger| = \aleph_1$ such that $F^\dagger$ is a Δ-system.

Proof. The cardinality of each A in F is a member of ω but there are $\aleph_1$ elements in F, so by the pigeonhole principle some n is obtained uncountably many times. In similar cases in the future we will just say: w.l.o.g. $|A| = n$ for any A in F (since all we need is a family of $\aleph_1$ finite sets.)

Now we proceed by induction on n:

For $n = 1$: this is the pigeonhole principle for $\aleph_1$.

For $n > 1$ we distinguish two cases.

1) There is $a \in \bigcup\{A : A \in F\}$ such that there are uncountably many B in F such that $a \in B$, then you have an easy induction step (you take a "out" and put it back after using induction hypothesis).
2) If there is no such a then we build a sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ such that $\alpha \neq \beta \Rightarrow A_\alpha \cap A_\beta = \emptyset$ $(A_\alpha \in F)$; suppose A_β is defined for $\beta < \alpha$. Now $\bigcup_{\beta<\alpha} A_\beta$ is countable. The subfamily of F of members which contain an element of this union is clearly countable so there is $A \in F$, such that $A \cap \bigcup_{\beta<\alpha} A_\beta = \emptyset$, let $A_\alpha = A$ and we are done (in this case $w = \emptyset$). $\square_{3.10,3.2}$

§4. More on the Cardinality $2^{\aleph_0}$ and Cohen Reals

Now we want to find what is exactly the power of the continuum in $V[G]$ for the model from Theorem 3.2.

4.1 Theorem. Let $P = \{f : f \text{ is a finite function from } \lambda \text{ to } \{0,1\}\}$. We have already shown that $\Vdash_P$ "$2^{\aleph_0} \geq \lambda$", we shall show now that if $\lambda^{\aleph_0} = \lambda$ then $\Vdash_P$ "$2^{\aleph_0} = \lambda$".

Proof. The idea is to construct a family of $\lambda^{\aleph_0}$ canonical names for the real numbers and then prove that for every name $\underset{\sim}{\tau}$ there is a canonical name $\underset{\sim}{\tau}'$ of a real such that $\Vdash_P$ " if $\underset{\sim}{\tau}$ is a real then $\underset{\sim}{\tau} = \underset{\sim}{\tau}'$". Since a real is a function from ω into $\{0,1\}$, a real r is given by telling for each $n < \omega$ whether $r(n) = 0$ or $r(n) = 1$. For a P-name $\underset{\sim}{\tau}$ of a real the answer whether $\underset{\sim}{\tau}(n) = 0$ or $\underset{\sim}{\tau}(n) = 1$ depends on which condition is in G therefore we shall have a maximal antichain $\langle p_{n,i} : i \leq \alpha_n \rangle$ in P, where $\alpha_n \leq \omega$ (because of the c.c.c.), and a function f_n on α_n which tells us that if $p_{n,i} \in G$ then $\underset{\sim}{\tau}(n) = f_n(i)$. Since each generic $G \subseteq P$ contains exactly one of the $p_{n,i}$, for $i \leq \alpha_n$, this will give, for any given G, the value of $\underset{\sim}{\tau}(n)$ in a unique way. For any pair of sequences $\langle \langle p_{n,i} : i < \alpha_n \rangle : n < \omega \rangle$, $\langle f_n : n < \omega \rangle$ where for every $n < \omega$, f_n is a function from α_n into $\{0,1\}$, and $\langle p_{n,i} : i < \alpha_n \rangle$ a maximal antichain of P we can construct a P-name, which we shall denote with $\underset{\sim}{r}(\langle f_n : n < \omega \rangle, \langle \langle p_{n,i} : i < \alpha_n \rangle : n < \omega \rangle)$ such that $\underset{\sim}{r}$ is a name of a function from ω such that for every $n < \omega$ if $p_{n,i} \in G$ then $\underset{\sim}{r}(n) = f_n(i)$. There is no difficulty in obtaining such a name by the methods we described above (or by 3.1). A name of this form is called, in this proof, canonical.

Let us estimate now, in V, the number of the canonical names. For a fixed n, and a given α_n, there are $2^{|\alpha_n|} \leq 2^{\aleph_0}$ different suitable f_n's. Therefore there are $\leq \Pi_{n<\omega} 2^{\aleph_0} = 2^{\aleph_0}$ possibilities for $\langle f_n : n < \omega \rangle$. P is included in the set of all finite subsets of $\lambda \times 2$ and $|\lambda \times 2| = \lambda$ and so, obviously, also $|P| = \lambda$. The number of countable sequences from P is therefore $\lambda^{\aleph_0}$, thus the number of possibilities for $\langle p_{n,i} : i < \alpha_n \rangle$ is at most $\lambda^{\aleph_0}$, and for sequence $\langle \langle p_{n,i} : i < \alpha_n \rangle : n < \omega \rangle$ the number is at most $(\lambda^{\aleph_0})^{\aleph_0} = \lambda^{\aleph_0}$. For each sequence $\langle \langle p_{n,i} : i < \alpha_n \rangle : n < \omega \rangle$ we have at most $2^{\aleph_0}$ corresponding sequences $\langle f_n : n < \omega \rangle$, thus the total number of possibilities is at most $2^{\aleph_0} \cdot \lambda^{\aleph_0} = \lambda^{\aleph_0}$. Since $\lambda^{\aleph_0} = \lambda$ there are at most λ canonical names.

4.2 Lemma. For every P-name $\underset{\sim}{\tau}$ there is a canonical name $\underset{\sim}{r}$ (as defined in the beginning of the proof of 4.1 above) such that $\Vdash_P$ " if $\underset{\sim}{\tau}$ is a real then $\underset{\sim}{\tau} = \underset{\sim}{r}$".

Proof of the Lemma. For every n let $\mathcal{J}_n = \{p \in P : p \Vdash_P$ "$\underset{\sim}{\tau}$ is not a real " *or* $p \Vdash_P$ "$\underset{\sim}{\tau}$ is a real and $\underset{\sim}{\tau}(n) = \ell$" for some $\ell \in \{0,1\}\}$. For each n the set $\mathcal{J}_n$ is a dense subset of P, since every $q \in P$ can be extended to a $p \in \mathcal{J}_n$. (If

G is a generic subset of P which contains q then in $V[G]$ either $\underset{\sim}{\tau}[G]$ is not a real or $\underset{\sim}{\tau}[G]$ is a real in which case $\underset{\sim}{\tau}[G](n) = 0$ or $\underset{\sim}{\tau}[G](n) = 1$. By the forcing theorem some member p of G forces the statements mentioned above which hold in $V[G]$, and, without loss of generality, we can assume $p \geq q$. This p is in $\mathcal{J}_n$). Let $\mathcal{I}_n$ be a maximal antichain contained in $\mathcal{J}_n$. Since P satisfies the c.c.c. we have $|\mathcal{I}| \leq \aleph_0$ and we take $\alpha_n = |\mathcal{I}_n|$, $\mathcal{I}_n = \{p_{n,i} : i < \alpha_n\}$. We define f_n on α_n by:

$$f_n(i) = \begin{cases} 0 & \text{if } p_{n,i} \Vdash_P \text{“}\underset{\sim}{\tau}(n) = 0\text{”} \\ 1 & \text{otherwise} \end{cases}$$

Let $\underset{\sim}{\tau}^* = \underset{\sim}{\tau}(\langle f_n : n < \omega\rangle, \langle\langle p_{n,i} : i < \alpha_n\rangle : n < \omega\rangle)$, and we shall prove that for every generic $G \subseteq P$ we have $V[G] \models$ “ if $\underset{\sim}{\tau}$ is a real then $\underset{\sim}{\tau} = \underset{\sim}{\tau}^*[G]$”. Assume that $\underset{\sim}{\tau}[G]$ is indeed a real. Since τ^* is a name of a real, by its construction $\underset{\sim}{\tau}[G] = \underset{\sim}{\tau}^*[G]$ will be established once we prove that for every $n < \omega$ we have $\underset{\sim}{\tau}[G](n) = \underset{\sim}{\tau}^*[G](n)$. Since $\{p_{n,i} : i < \alpha_n\}$ is a maximal antichain in P (in V) G contains $p_{n,i}$ for a unique i. Since $p_{n,i} \in \mathcal{J}_n$ and since $p_{n,i}$ cannot force that $\underset{\sim}{\tau}$ is not a real (as $\underset{\sim}{\tau}[G]$ is a real in $V[G]$), we have $p_{n,i} \Vdash_P$ “$\underset{\sim}{\tau}(n) = \ell$”, for some $\ell \in \{0,1\}$. By the definition of $f_n(i)$ we have $f_n(i) = \ell$ for the same ℓ. By the definition of $\underset{\sim}{\tau}^*$ we have $\underset{\sim}{\tau}^*[G](n) = f_n(i)$ if $p_{n,i} \in G$. Since $p_{n,i} \Vdash_P$ “$\underset{\sim}{\tau}(n) = \ell$” we have $V[G] \models$ “$\underset{\sim}{\tau}[G](n) = \ell = f_n(i) = \underset{\sim}{\tau}[G](n)$” which is what we have to prove.

$\square_{4.2}$

Continuation of the proof of 4.1. To prove $\Vdash_P$ “$2^{\aleph_0} \leq \lambda$” we shall show that for every generic $G \subseteq P$ we have $V[G] \models$ “$2^{\aleph_0} \leq \lambda$”. Suppose $V[G] \models$ “$2^{\aleph_0} \geq \lambda^+$”, then in $V[G]$ there is a one-to-one function f from λ^+ into the set of reals. Let $\underset{\sim}{\sigma}$ be a name of this function. For every $i < \lambda^+$ there is a canonical name $\underset{\sim}{\tau}_i^*$ of a real such that $\Vdash_P$ “if $\underset{\sim}{\sigma}(i)$ is a real then $\underset{\sim}{\sigma}(i) = \underset{\sim}{\tau}_i^*$” (since we can either regard $\underset{\sim}{\sigma}(i)$ itself as a name and use the lemma above, or else use an earlier lemma, 3.1, which establishes the existence of a name $\underset{\sim}{\tau}$ such that $\Vdash_P$ “$\underset{\sim}{\sigma}(i) = \underset{\sim}{\tau}$” whenever $\Vdash_P$ “$(\exists x)(\underset{\sim}{\sigma}(i) = x)$” and then use the lemma to obtain $\Vdash_P$ “$\underset{\sim}{\tau} = \underset{\sim}{\tau}_i^*$”.) Since there are at most λ canonical names, and λ^+ of $V[G]$ is also $\geq \lambda^+$ of V there is a $j \neq i$ such that $\underset{\sim}{\tau}_j^* = \underset{\sim}{\tau}_i^*$. Now in $V[G]$ we have $f(i) = \underset{\sim}{\sigma}[G](i) = \underset{\sim}{\tau}_i^*[G] = \underset{\sim}{\tau}_j^*[G] = \underset{\sim}{\sigma}[G](j) = f(j)$, contradicting the fact that f is one-to-one. So we have proved 4.1. $\square_{4.1}$.

4.3 Definition. Cohen Generic Reals. Let us take $P = \{f : f$ is a finite function from ω into $\{0,1\}\}$, with $p \leq q$ being defined as $p \subseteq q$. This is called Cohen forcing. It is easily seen that if G is a generic subset of P then $\bigcup G$ is a real. Let us see now that we can reconstruct G from $\bigcup G$ by taking G to be the set of all finite subsets of $\bigcup G$. If $p \in G$ then p is obviously a finite subset of $\bigcup G$. If p is a finite subset of $\bigcup G$ then $p = \{\langle k_i, \ell_i \rangle : i < n\}$ for some $n < \omega$. Since $\langle k_i, \ell_i \rangle \in G$ there is a $p_i \in G$ such that $\langle k_i, \ell_i \rangle \in p_i$. Since G is directed there is a $q \in G$ such that $q \geq p_i$ for all $i < n$. Obviously $p = \{\langle k_i, \ell_i \rangle : i < n\} \leq q$, and since G is downward-closed we have also $p \in G$. Since G and $\bigcup G$ can be constructed from each other we can identify them and speak of a generic real $\bigcup G$. Given V, we shall call a real number g a Cohen real over V if for some $G \subseteq P$ which is generic over V we have $g = \bigcup G$.

4.3A Discussion. When we talk here about reals we talk about members of the cantor set ${}^{\omega}2$, given an open subset A of ${}^{\omega}2$ in V this A is also a set of reals in the (true) universe, but we are interested not in A itself in the universe but in the subset of ${}^{\omega}2$ in the universe that has there the same description that A has in V. For example, suppose A is $\{r \in {}^{\omega}2 : V \models r \supseteq p\}$ for some $p \in P$, then we are interested in the set $A^* = \{r \in {}^{\omega}2 : r \supseteq p\}$.

Obviously $A = A^* \cap V$ and A^* contains reals which are not in V (since, in the universe, $|A^*| = 2^{\aleph_0}$ while $|V| = \aleph_0$) and hence not in A. The analogous situation for the real line $\mathbb{R}$ is when we look at a rational interval $(a,b)^V = \{x \in \mathbb{R}^V : a < x < b\}$, and the corresponding set in the universe is $(a,b) = \{x \in \mathbb{R} : a < x < b\}$. Obviously $(a,b)^V = (a,b) \cap V$ and (a,b) contains reals which are not in V (since $|(a,b)| = 2^{\aleph_0}$ and $|V| = \aleph_0$) and hence not in $(a,b)^V$. Let us denote with B_p the basic open set $\{r \in {}^{\omega}2 : r \supseteq p\}$ in the Cantor space, and with B_p^V the corresponding set $\{r \in {}^{\omega}2 : V \models r \supseteq p\}$ in V. Obviously $B_p^V = B_p \cap V$. Given an open set A^V in V we define $A^* = \bigcup\{B_p : p \in P, B_p^V \subseteq A^V\}$. A^* is obviously an open subset of ${}^{\omega}2$ and $A^V = A^* \cap V$. For a closed set

C^V we take $A = {}^\omega 2 \cap V \setminus C^V$ and $C^* = {}^\omega 2 \setminus A^*$, we can write it also as $C^* = \{r \in {}^\omega 2 : (\forall n < \omega)\ [B^V_{r\restriction n} \cap C^V \neq \emptyset]\}$. One can easily see that if A^V is a clopen (i.e. closed and open) set then A^* is the same set whether we regard A^V as an open set or as a closed set.

We shall use A^V only when $A^V \in V$; in fact for every Borel set $A \in V$ there is a unique Borel set A^* in the universe such that they have a common definition (in V) and then $A^* \cap V = A$.

The close connection between Cohen forcing and Cohen (generic) real is a quite universal phenomena for the forcing in use.

4.4 Theorem. A real number r is a Cohen real (or Cohen generic real) over V *iff* r belongs to no C^* where C^V is a closed nowhere dense subset of ${}^\omega 2$ in V, or in other words, if r belongs to every set A^* for every set $A \in V$ which is a dense open set of ${}^\omega 2$ in V.

Proof. Let r be a Cohen real and let A^V be a dense open set in V. We have $r = \bigcup G$ for some generic subset G of P where P is from 4.3. Let $\mathcal{I} = \{p \in P : B^V_p \subseteq A^V\}$. Since A^V is a dense open set, clearly $\mathcal{I}$ is a dense subset of P, and hence $G \cap \mathcal{I} \neq \emptyset$. Let $p \in G \cap \mathcal{I}$. Since $p \in G$ clearly $r = \bigcup G \supseteq p$, i.e., $r \in B_p$. Since $p \in \mathcal{I}$ we have $B^V_p \subseteq A^V$ hence $B_p \subseteq A^*$ and $r \in A^*$.

Now assume that for every dense open set $A^V, r \in A^*$. Let G be the set of all finite subsets of r, then $r = \bigcup G$. We shall prove that G is a generic subset of P over V and hence r is a Cohen real (over V). Let $\mathcal{I}$ be a dense subset of P. Then $A^V \stackrel{\text{def}}{=} \bigcup\{B^V_p : p \in \mathcal{I}\}$ is a dense open subset of $({}^\omega 2)^V$, and therefore $r \in A^*$, hence $r \in B^V_q$ for some $q \in P$ such that $B^V_q \subseteq A^V$. Since B^V_q is a closed set in the compact space ${}^\omega 2 \cap V$ and $\{B^V_p : p \in \mathcal{I}\}$ is an open cover of A^V and hence of B^V_q there is a finite subset $\{B^V_{p_i} : 1 \leq i \leq n\}$, with $\{p_i : 1 \leq i \leq n\} \subseteq \mathcal{I}$ which covers B^V_q. Let m be such that m includes the domains of all p_i, $1 \leq i \leq n$, and q. Let $r^\dagger$ be such that $r^\dagger(j) = r(j)$ for $j < m$ and $r^\dagger(j) = 0$ for $j > m$. Since $r \in B_q$ also $r^\dagger \in B^V_q$ and since $r^\dagger \in V$ also $r^\dagger \in B^V_q$. Therefore for some $1 \leq i \leq n$, $r^\dagger \in B^V_{p_i}$ and $r^\dagger \restriction \mathrm{Dom}(p_i) = p_i$. But $r^\dagger$

coincides with r on m which includes $\mathrm{Dom}(p_i)$, hence $r{\restriction}\mathrm{Dom}(p_i) = p_i$. Lastly $r{\restriction}\mathrm{Dom}(p_i) \in G$ since it is a finite subset of r, hence $p_i \in G$. Since $p_i \in \mathcal{I}$ we have $G \cap \mathcal{I} \neq 0$ and G is generic. $\square_{4.4}$

4.5 Corollary. The set of all Cohen reals over a model V is a comeager subset of ${}^{\omega}2$.

Proof. The reals which are not Cohen reals are exactly the reals which do not belong to some set A^* where A^V is a dense open subset of ${}^{\omega}2 \cap V$ in V. Let $\mathcal{I} = \{p \in P : B_p^V \subseteq A^V\}$. Since A^V is a dense open subset of ${}^{\omega}2 \cap V$ in V, clearly $\mathcal{I}$ is a dense subset of P and $A^* = \bigcup\{B_p : p \in \mathcal{I}\}$ is a dense open subset of ${}^{\omega}2$. The set of the reals which are not Cohen reals is $\bigcup\{{}^{\omega}2 \setminus A^* : A^V \in V, A^V$ is dense open in $({}^{\omega}2)^V\}$. This is the union of $\aleph_0$ nowhere dense sets ($2^{\omega} \setminus A^*$ is nowhere dense since A^* is dense open, the union is countable since V is) and is thus a meager set. Therefore the set of all Cohen reals is comeager. $\square_{4.5}$

Remark. Does the Cohen forcing collapse cardinals? No, since P is countable and hence, obviously it satisfies the c.c.c. See Lemma 3.6 (i).

§5. Equivalence of Forcings Notions, and Canonical Names

We deal with forcing with a subset of P.

5.1 Lemma. Let $(P, \leq)$ be a forcing notion in V and let Q be a dense subset of P.

(a) If G is a generic subset of P over V *then* $H \stackrel{\mathrm{def}}{=} G \cap Q$ is a generic subset of Q over V and $G = \{p \in P : (\exists q \in H) p \leq q\}$.

(b) If H is a generic subset of Q over V *then* $G \stackrel{\mathrm{def}}{=} \{p \in P : (\exists q \in H) p \leq q\}$ is a generic subset of P over V and $H = G \cap Q$.

(c) The assertion (a) and (b) above establish a one-one correspondence between the generic subsets of P and those of Q. If G and H are corresponding generic sets then $V[G] = V[H]$ and therefore the same statements are forced by the forcing notions P and Q (they are equivalent - see Definition 5.2 below).

(d) Any Q-name is a P-name, and if $\underset{\sim}{\tau}_1, \dots, \underset{\sim}{\tau}_n$ are Q-names, $\varphi(x_1, \dots, x_n)$ a first order formula then $\Vdash_P \varphi(\underset{\sim}{\tau}_1, \dots, \underset{\sim}{\tau}_n)$ iff $\Vdash_Q \varphi(\underset{\sim}{\tau}_1, \dots, \underset{\sim}{\tau}_n)$.

(e) For any P-name $\underset{\sim}{\tau}$ there is a Q-name $\underset{\sim}{\sigma}$ such that $\Vdash_P$ "$\underset{\sim}{\tau} = \underset{\sim}{\sigma}$".

Proof of the Lemma. (a) It is obvious that $G \cap Q$ is downward closed in Q. Let $p, p^\dagger \in G \cap Q$. Since G is directed there is a $p'' \in G$ such that $p'' \geq p, p^\dagger$. Now Q is dense, and hence $\{q \in Q : q \geq p''\}$ is dense above p''. Since $p'' \in G$ clearly $G \cap \{q \in Q : q \geq p''\} \neq \emptyset$; let q be such that $p'' \leq q \in G \cap Q$, so $q \geq p'' \geq p, p^\dagger$. Thus we have seen that $G \cap Q$ is directed. Let $\mathcal{I} \in V$ be dense in Q, then, as easily seen, $\mathcal{I}$ is also dense in P. Therefore we have $\mathcal{I} \cap (Q \cap G) = (\mathcal{I} \cap Q) \cap G = \mathcal{I} \cap G \neq \emptyset$. We can conclude by 1.20 that $G \cap Q$ is a generic subset of Q (over V).

Obviously $\{p \in P : (\exists q \in G \cap Q) p \leq q\} \subseteq G$. In the other direction, if $r \in G$ then, as we have seen for p'' above, there is a $q \in G \cap Q$ such that $q \geq p$. Therefore $r \in \{p \in P : (\exists q \in G \cap Q) p \leq q\}$ and so

$$\{p \in P : (\exists q \in G \cap Q) p \leq q\} = G.$$

(b) G is obviously downward closed and directed. Let $\mathcal{I}$ be a dense upward closed subset of P, then $\mathcal{I} \cap Q$ is obviously a dense subset of Q. Then $\emptyset \neq H \cap (\mathcal{I} \cap Q) = H \cap \mathcal{I} \subseteq G \cap \mathcal{I}$. Lastly $H = G \cap Q$ is obvious by the definition of G as H is downward closed.

(c) Since we have $G \in V[H]$, because G can be easily computed from H, we can evaluate all the P-names in $V[H]$ and we get therefore $V[G] \subseteq V[H]$. Similarly $H \in V[G]$ implies $V[H] \subseteq V[G]$, hence $V[H] = V[G]$.

(d),(e) are left to the reader. $\square_{5.1}$

In fact in 5.1 we have proved that P, Q are equivalent where

5.2 Definition. The forcing notions P, Q are equivalent if there are $\underset{\sim}{\tau}, \underset{\sim}{\sigma}$, a P-name and a Q-name respectively such that:

(1) $\Vdash_P$ "$\underset{\sim}{\tau}$ is a generic (over V) subset of Q".

(2) $\Vdash_Q$ "$\underset{\sim}{\sigma}$ is a generic subset of P"

(3) for $G \subseteq P$ generic, $G = \underset{\sim}{\sigma}[\underset{\sim}{\tau}[G]]$

(4) for $G \subseteq Q$ generic, $G = \underset{\sim}{\tau}[\underset{\sim}{\sigma}[G]]$

Clearly equivalence of forcing notion is an equivalence relation.

5.3 Definition. (1) A function f from P into Q is called a complete embedding if: for any maximal antichain $\mathcal{I} \subseteq P$, $f(\mathcal{I}) = \{f(p) : p \in \mathcal{I}\}$ is a maximal antichain of Q and $f \restriction \mathcal{I}$ is one to one of course and $P \models$ "$p \leq q$" $\Rightarrow Q \models$ "$f(p) \leq f(q)$".

(2) We write $P \lessdot Q$ if $P \subseteq Q$ (which mean: $p \in P \Rightarrow p \in Q$, for p, $q \in P$ we have $P \models$ "$p \leq q$" $\Leftrightarrow Q \models$ "$p \leq q$"), and the identity mapping is a complete embedding of P into Q.

5.4 Lemma. 1) If f is a complete embedding of P into Q, *then* there is a Q-name $\underset{\sim}{\sigma}$, $\Vdash_Q$ "$\underset{\sim}{\sigma}$ is a generic subset of P". If in addition the range of f is a dense subset of Q, then P, Q are equivalent.

2) If $P \lessdot Q$, and p_1, $p_2 \in P$ then: p_1, p_2 are compatible in P iff p_1, p_2 are compatible in Q.

3) Assume $P \subseteq Q$. Then $P \lessdot Q$ iff every pre-dense $\mathcal{I} \subseteq P$ is pre-dense in Q too.

Proof. Easy. $\square_{1.5}$

5.5 Definition. (1) For a forcing notion P, and $p, q \in P$, we say $p \approx q$ (p, q are equivalent) if any $r \in P$ is compatible with p iff it is compatible with q. Clearly $\approx$ is an equivalence relation on P.

(2) We define $P/\approx$ as follows: the members are $p/\approx$ (for $p \in P$) and we define a partial order: $(p/\approx) \leq (q/\approx)$ *iff* there are $p^\dagger \in p/\approx$ and $q^\dagger \in q/\approx$ such that every $r \in P$ compatible with $q^\dagger$ is compatible with $p^\dagger$.

5.6 Claim. (1) In Definition 5.5 we have:

(a) $(p/\approx) \leq (q/\approx)$ *iff* $q \Vdash_P$ "$p \in \underset{\sim}{G}_P$".

(b) $\approx$ is an equivalence relation.

(c) in 5.5 (2), "there are $p^\dagger \in (p/\approx), q^\dagger \in (q/\approx)$" can be replaced by "for every $p^\dagger \in (p/\approx)$ and $q^\dagger \in (q/\approx)$"

(d) $(p/\approx) = (q/\approx)$ iff $(p/\approx) \leq (q/\approx) \,\&\, (q/\approx) \leq (p/\approx)$.

(e) $P/\approx$ is a partial order.

(2) The function $p \to p/\approx$ is a complete embedding of P into $P/\approx$ with a dense range ; hence $P, P/\approx$ are equivalent. The function preserves "$p \leq q$", "p, q compatible", "p, q incompatible" (though not necessarily "$\neg p \leq q$").

Proof. 1)(a) If the right side fails, let $G \subseteq P$ be generic over V such that $q \in G$ but $p \notin G$. Choose a maximal antichain $\mathcal{I}$ such that $p \in \mathcal{I} \in V$, then for some r we have $G \cap \mathcal{I} = \{r\}$ and q, r are compatible (because they are in G) while p, r are not compatible (because $r \neq p$ as one is in G the other not and $\{r, p\} \subseteq \mathcal{I}$). By clause (c) of 5.6(1) this implies the failure of the left side, so we have proved the only if direction. The other direction in the "iff" is proved similarly.

(b), (c) Easy.

(d), (e) Left to the reader.

2) Easy. $\square_{5.6}$

It is interesting to note that in each equivalence class of equivalent forcing we can choose canonically a representative (unique up to isomorphism), which is essentially a complete Boolean algebra (without the one).

5.7 Definition. For any forcing notion P let:

(1) A set $A \subseteq P$ is called open-regular if it is open and for every $p \in P \setminus A$ there is $q \geq p$ incompatible with A, i.e., $(\forall r \in P)(r \geq q \to r \notin A)$.

(2) $RO(P)$ is the partial order whose set of elements is the nonempty open regular subsets of P, and $A \leq B$ (for $A, B \in RO(P)$), if $B \subseteq A$, equivalently if every $p \in P$ incompatible with A (i.e. incompatible with every $q \in A$) is incompatible with B.

(3) For $p \in P$ let $ro(p) = \{q \in P : \text{there is no } r \geq q \text{ incompatible with } p\}$.

5.8 Theorem. (1) The mapping $p \mapsto ro(p)$ is a complete embedding of P into $RO(P)$;

(2) $RO(P)/\approx$, if we add to it a maximal element 1, becomes a complete Boolean algebra;

(3) If B is a complete Boolean algebra, and $P = B \setminus \{1_B\}$, (with the usual ordering of a Boolean algebra) *then* $RO(P)/\approx$ is isomorphic to P, moreover, we can use the isomorphism $p \mapsto ro(p)$.

Proof. Well known. $\square_{5.8}$

5.9 Theorem. The forcing notions P, Q are equivalent *iff*

$RO(P)/\approx$ and $RO(Q)/\approx$ are isomorphic.

Proof. The proof is easy, using:

5.10 Claim. Suppose $A \in RO(P)$ and $\mathcal{I}$ a maximal antichain such that for every $p \in \mathcal{I}$ either $p \in A$ or p is incompatible with A. *Then* A, $\bigcup_{p \in A \cap \mathcal{I}} ro(p)$ are equal (in $RO(P)$, union as in a complete Boolean algebra). $\square_{5.10,5.9}$

5.11 Definition. For any $a \in V^\dagger$ and $V \subseteq V^\dagger$ we define $\mathrm{rk}_V(a)$: $\mathrm{rk}_V(a) = 0$ iff $a \in V$, and $\mathrm{rk}_V(a) = \bigcup\{\mathrm{rk}_V(b) + 1 : b \in a\}$ otherwise.

5.12 Definition. We define when a P-name $\underset{\sim}{\tau}$ is canonical by induction on its rank α: if $\underset{\sim}{\tau} = \{(p_i, \underset{\sim}{\tau}_i) : i < i_0\}$, then τ is canonical if:

(1) If $\Vdash_P$ "$\mathrm{rk}(\underset{\sim}{\tau}) \leq \beta$", then $\beta \geq \alpha$, and if $\Vdash_P$ "$\mathrm{rk}_V(\underset{\sim}{\tau}) \leq \beta$" then $\beta \geq 1 + \mathrm{rk}_r(\underset{\sim}{\tau})$

(2) if $\Vdash_P$ "$\underset{\sim}{\tau}$ has power $< \lambda$", λ is a regular cardinal and P satisfies the λ-c.c. *then* $i_0 < \lambda$"

(3) each τ_i is canonical, moreover: if $p_i \Vdash$ "rk$(\underset{\sim}{\tau}_i) \leq \beta$" then rk$_n(\underset{\sim}{\tau}_i) \leq \beta$, and if $p_i \Vdash$ "rk$_V(\underset{\sim}{\tau}_i) \leq \beta$" then $\beta \geq 1 + \text{rk}_r(\underset{\sim}{\tau}_i)$ and if $\Vdash_P$ "$\underset{\sim}{\tau}_i$ has power $< \lambda$", λ a regular cardinal and $P\restriction\{p : p \geq p_i\}$ satisfies the λ-c.c. then $|\underset{\sim}{\tau}_i| < \lambda$.

5.13 Theorem. For every P-name $\underset{\sim}{\tau}$ there is a canonical P-name $\underset{\sim}{\sigma}$ such that $\Vdash_P$ "$\underset{\sim}{\tau} = \underset{\sim}{\sigma}$".

Proof. We prove by induction on the rank of $\underset{\sim}{\tau}$, that

(*) if $r \in P$, $r \Vdash$ "rk$_V(\underset{\sim}{\tau}[\underset{\sim}{G}]) \leq \alpha$ and rk$_V(\underset{\sim}{\tau}[\underset{\sim}{G}]) \leq \beta$, and $\underset{\sim}{\tau}[\underset{\sim}{G}]$ has power $< \lambda$", and λ is a regular cardinal, and $P\restriction\{r' : r \leq r' \in P\}$ satisfies the λ-c.c., *then* we can find a canonical P-name $\underset{\sim}{\sigma}$ such that: $r \Vdash$ "$\underset{\sim}{\tau} = \underset{\sim}{\sigma}$" and rk$_n(\underset{\sim}{\sigma}) \leq \alpha$ and rk$_r(\underset{\sim}{\sigma}) \leq 1 + \beta$, and if $\underset{\sim}{\sigma} = \{(q_i, \underset{\sim}{\sigma}_i) : i < i_0\}$ then $i_0 < \lambda$.

If rk$_n(\underset{\sim}{\tau}) = 0$, let $\underset{\sim}{\sigma} = \underset{\sim}{\tau}$ and there are no problems. If not, but $\Vdash_P$ "$\underset{\sim}{\tau} \in V$" let $\mathcal{I}$ be a maximal antichain of P above r such that for every $p \in \mathcal{I}$ for some $a_p \in V$, $p \Vdash_P$ "$\underset{\sim}{\tau} = \underset{\sim}{a}_p$". We then let $\sigma = \{(p, \dot{b}) : p \in \mathcal{I}, b \in a_p\}$. So assuming neither occurs, we can find a maximal antichain $\mathcal{I}$ of P above r, so that for each $p \in \mathcal{I}$, for some ordinals α_p and β_p and λ_p we have $p \Vdash$ "rk$(\underset{\sim}{\tau}[\underset{\sim}{G}]) = \alpha_p$ and rk$_V(\underset{\sim}{\tau}[\underset{\sim}{G}]) = \beta_p$, and the power of $\underset{\sim}{\tau}[\underset{\sim}{G}]$ is λ_p, (which is a cardinal in $V[\underset{\sim}{G}]$)". Let $\underset{\sim}{f}_p$ be a P-name such that $p \Vdash$ "$\underset{\sim}{f}_p$ a function from λ_p onto $\underset{\sim}{\tau}$", for each $p \in \mathcal{I}$ (use Lemma 3.1). Let $\underset{\sim}{\tau} = \{(p_i, \underset{\sim}{\tau}_i) : i < i_0\}$, and let $\mathcal{J}_\gamma$ for $\gamma < \sup\{\lambda_p : p \in \mathcal{I}\}$) be a maximal antichain of P above r, so that each $q \in \mathcal{J}_\gamma$ is above some member p of $\mathcal{I}$, and if $\gamma < \lambda_p$, for some $i = i(q, \gamma)$, $q \geq p_i$, $q \Vdash$ "$\underset{\sim}{f}_p(\gamma) = \underset{\sim}{\tau}_i$" and $q \Vdash_P$ "rk$(\underset{\sim}{\tau}_i[\underset{\sim}{G}]) = \alpha_q^{i,\gamma}$, rk$_V(\underset{\sim}{\tau}[\underset{\sim}{G}]) = \beta_q^{i,\gamma}$ and the power of $\underset{\sim}{\tau}_i[G]$ is $\lambda_q^{i,\gamma}$ (which is a cardinal in $V[\underset{\sim}{G}]$)".

Now apply (*) by the induction hypothesis for $\underset{\sim}{\tau}_{i(q,\gamma)}, q$ for any $q \in \mathcal{J}_\gamma$ and get canonical $\underset{\sim}{\sigma}_{\gamma,q}$. Let

$$\underset{\sim}{\sigma} = \{(q, \underset{\sim}{\sigma}_{\gamma,q}) : \gamma < \sup\{\lambda_p : p \in \mathcal{I}\} \text{ and } q \in \mathcal{J}_\gamma\}$$

As $P\restriction\{r' : r \le r' \in P\}$ satisfies the λ-c.c., $|\mathcal{I}| < \lambda$, and as $r \Vdash$ "$\tau[\underset{\sim}{G}]$ has power $< \lambda$", each λ_p ($p \in \mathcal{I}$) is $< \lambda$. As λ is regular, $\bigcup_{p\in\mathcal{I}} \lambda_p$ is $< \lambda$, so we can finish. $\square_{5.13}$

5.14 Theorem. If P_1, P_2 are equivalent and disjoint, then there is Q, $P_\ell \lessdot Q$, P_ℓ a dense subset of Q, (for $\ell = 1, 2$). $\square_{5.14}$

5.15 Claim. The λ-c.c. property is preserved by equivalence. $\square_{5.15}$

5.16 Definition. (1) $Tc(x)$ (the transitive closure of x) is defined by induction on the rank of x,

$$Tc(x) = \{x\} \cup \bigcup_{y\in x} Tc(y)$$

and it is the minimal transitive set to which x belongs
(2) $H(\lambda) = \{x : |Tc(x)| < \lambda\}$

5.17 Claim. Let λ be an uncountable regular cardinal.

(1) If $\underset{\sim}{\tau}$ is a canonical P-name, $P \in H(\lambda)$ and $V \models \mathrm{cf}(\lambda) > |P|$ *then:*

$$\underset{\sim}{\tau} \in H(\lambda) \textit{ iff } \Vdash_P \text{“}\underset{\sim}{\tau} \in H(\lambda)^{V[\underset{\sim}{G}]}\text{”}$$

(2) $(H(\lambda), \in)$ satisfies all axioms of ZFC except possibly the power set axiom.

Proof. (1) If $\underset{\sim}{\tau} \in H(\lambda)$ we can prove by induction on $\mathrm{rk}_n(\underset{\sim}{\tau})$ that $\underset{\sim}{\tau}[G] \in H(\lambda)^{V[G]}$ hence obviously $\Vdash_P$ "$\underset{\sim}{\tau} \in H(\lambda)^{V[\underset{\sim}{G}]}$". Now P satisfies the $|P|^+$-chain condition and $H(\lambda) = \bigcup\{H(\mu^+) : \mu < \lambda\}$ so w.l.o.g. λ is a successor cardinal; clearly $|P| < \lambda$ (by the definition of $H(\lambda)$). By Claim 3.7, λ is a cardinal also in $V[G]$ so Definition 5.12(1) gives the other direction.
(2) Is obvious. Furthermore if λ is strongly inaccessible $(H(\lambda), \in)$ is a model of all ZFC. $\square_{5.17}$

5.18 Claim. 1) If P is a forcing notion satisfying the c.c.c. *then* the number of canonical P-names of an ordinal $< \mu$ is $\leq (|P| + \mu + \aleph_0)^{\aleph_0}$.
2) In (1) the number of canonical P-names of a function from μ to λ, is $\leq (|P| + \mu + \lambda + \aleph_0)^{\mu}$.
3) If P satisfies the κ-c.c., κ regular, the numbers in (1) and (2) should be $(|P| + \mu + \aleph_0)^{<\kappa}$, $(|P| + \mu + \lambda + \aleph_0)^{<(\kappa+\mu^+)}$ respectively.

Proof. Included, essentially, in the proof of 4.1. $\square_{5.18}$

§6. Random Reals, Collapsing Cardinals and Diamonds

Random Reals.

6.1 Definition. Let $\mathcal{C}$ be the set of all closed subsets of the real line, $\mathcal{B}$ the set of all Borel subsets of the real line and $\mathcal{M}$ the set of all (Lebesgue) measurable subsets of the real line. For $X \in \{\mathcal{C}, \mathcal{B}, \mathcal{M}\}$ we take $P_X = \{A \in X : A$ is of positive Lebesgue measure $\}$, and $p \leq q$ if $q \subseteq p$. We use here the real line, but we could have used also the Cantor space ${}^{\omega}2$ as the real line. For $A \in \mathcal{M}$, let $\mathrm{Leb}(A)$ be the Lebesgue measure of A.

6.1A Discussion. Every measurable set of positive measure includes a closed set of positive measure, and thus $P_{\mathcal{C}}$ is dense in $P_{\mathcal{M}}$. Since $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{M}$, the three notion of forcing $P_{\mathcal{M}}$, $P_{\mathcal{B}}$, and $P_{\mathcal{C}}$ are interchangeable (in fact equivalent see 5.1(c) and Definition 5.2). We shall work mostly with $P_{\mathcal{C}}$ and when we do not care with which of these notions we work we shall write just P.

First we shall prove that $P_{\mathcal{M}}$ satisfies the c.c.c. Let p_i, $i < \aleph_1$ be pairwise incompatible conditions. For each $i < \aleph_1$ we know $\mathrm{Leb}(p_i) > 0$, where Leb denotes Lebesgue measure. Therefore there is a positive integer n_i such that $\mathrm{Leb}(p_i \cap (-n_i, n_i)) \geq \frac{1}{n_i}$. Since there are $\aleph_1$ i's and $\aleph_0$ positive integers there is an uncountable subset S of $\aleph_1$ and an $n < \omega$ such that $n_i = n$ for all $i \in S$. Since for $i, j < \aleph_1$, $i \neq j$, p_i and p_j are incompatible we have $\mathrm{Leb}(p_i \cap p_j) = 0$. Let $T \subseteq S$ contain $2n^2 + 1$ members, then $\mathrm{Leb}((-n, n)) \geq \mathrm{Leb}(\bigcup_{i \in T} p_i \cap$

$(-n,n)) = \sum_{i\in T} \mathrm{Leb}(p_i \cap (-n,n)) \geq \sum_{i\in T} \frac{1}{n} = (2n^2+1)\frac{1}{n} = 2n + \frac{1}{n}$, which is a contradiction.

For every positive real ε in V the set $\{[n\varepsilon, (n+1)\varepsilon]$: n is an integer $\}$ is obviously a maximal antichain, therefore G contains a closed interval of length ε. For $p \in P_{\mathcal{C}}$ let $p^{V[G]}$ denote the closed set with the same description in $V[G]$ as p has in V. If $p_i \in G \subseteq P_{\mathcal{C}}$ for $i = 1, \ldots, n$ we have $\mathrm{Leb}(\bigcap_{i=1}^n p_i^V) \neq \emptyset$. Since $p_i^{V[G]} \supseteq p_i^V$ we have $\bigcap_{i=1}^n p_i^{V[G]} \neq \emptyset$. Let us consider the set $\{p^{V[G]} : p^V \in G \subseteq P_{\mathcal{C}}\}$. The intersection of every finite subset of it is nonvoid and its members are closed sets, some of them bounded, hence $\cap_{p\in G} p^{V[G]} \neq \emptyset$ by compactness. Moreover, since for every positive integer m the set $\{p^{V[G]} : p^V \in G\}$ contains an interval of length $\frac{1}{m}$ (since if the length of p^V is $\frac{1}{m}$ so is the length of $p^{V[G]}$) necessarily the set $\cap_{p\in G} p^{V[G]}$ consists of a single real g. A real obtained in this way is called a *random real* (over V). We shall see that we can reconstruct G from g. Now let us mention that since $P_{\mathcal{C}}$ is dense in $P_{\mathcal{B}}$, we have, for every $p \in P_{\mathcal{B}} \cap G$ a condition $q \geq p$ such that $q \in P_{\mathcal{C}} \cap G$, i.e., $q \subseteq p$. Since the generic real g belongs to $q^{V[G]}$, and since $q \subseteq p$ implies $q^{V[G]} \subseteq p^{V[G]}$ also $g \in p^{V[G]}$. Thus for every Borel set p^V in G we have $g \in p^{V[G]}$.

To reconstruct G from g it suffices to tell which closed sets belong to G, since if we deal with $P_{\mathcal{B}}$ or $P_{\mathcal{M}}$, G will consist of all members, of $P_{\mathcal{B}}$ or $P_{\mathcal{M}}$ which are $\leq$ these closed sets. Now we shall see that in $V[G_{P_{\mathcal{C}}}]$, $G = \{A^V \in P_{\mathcal{C}} : g \in A^{V[G]}\}$. We have already seen that if $A^V \in G$ then $g \in A^{V[G]}$. Now we assume that $g \in A^{V[G]}$ and prove that $A^V \in G$. Assume $A^V \notin G$ and extend $\{A^V\}$ to an antichain S of $P_{\mathcal{C}}$ which is maximal among the antichains which consist of pairwise disjoint sets. We shall see that S is a maximal antichain of $P_{\mathcal{C}}$. If this is not the case there is an $E \in P_{\mathcal{C}}$ such that E is incompatible with every member of S, i.e., $\mathrm{Leb}(E \cap B) = 0$ for every $B \in S$. By the c.c.c. S is countable, hence $\mathrm{Leb}(E \setminus \bigcup S) = \mathrm{Leb}(E) \setminus \mathrm{Leb}(E \cap \bigcup S)$, but $\mathrm{Leb}(E \cap \bigcup S) \leq \sum_{B\in S} 0 = 0$ hence $\mathrm{Leb}(E \setminus S) = \mathrm{Leb}(E) > 0$.
Now $E \setminus \bigcup S$ is a Borel set of positive measure, hence it includes a closed set W of positive measure. For every $B \in S$ we know that $W \cap B \subseteq W \cap \bigcup S = \emptyset$, contradicting the maximality of S. Now that we have proved that S is a maximal antichain we know that G contains some member B^V of S, which

is different from A^V (since $A^V \notin G$) and therefore $B^V \cap A^V = \emptyset$. Since $B^V \in G$ we have $g \in B^{V[G]}$, hence $g \in B^{V[G]} \cap A^{V[G]}$. Let $[n, n+1]^{V[G]}$ be an interval which contains g, then $B^{V[G]} \cap A^{V[G]} \cap [n, n+1]^{V[G]} \neq \emptyset$.

On the other hand, since $(B^V \cap [n, n+1]^V) \cap [A^V \cap [n, n+1]^V) = \emptyset$ the distance between $B^V \cap [n, n+1]^V$ and $A^V \cap [n, n+1]^V$ is some $d > 0$. Let e be a rational number $< d$. There is a finite number of intervals of length e which separate $V_B^V \cap [n, n+1]^V$ from $A^V \cap [n, n+1]^V$. The "same" intervals separate also $B^{V[G]} \cap [n, n+1]^{V[G]}$ from $A^{V[G]} \cap [n, n+1]^{V[G]}$ contradicting $B^{V[G]} \cap A^{V[G]} \cap [n, n+1]^{V[G]} \neq \emptyset$.

6.2 Theorem. A real number r is random over V *iff* for every Borel set $A^V \in V$ such that $\mathrm{Leb}(A^V) = 0$ we have $r \notin A^*$ *iff* this holds for every G_σ set A^V (where A^* is the Borel set in the universe with the same description as A.)

Proof. Assume that r is random over V, $r \in \cap\{B^{V[G]} : B^V \in G \subseteq P_{\mathcal{B}}\}$. Let A^V be a Borel set with $\mathrm{Leb}(A^V) = 0$. As easily seen the complement $\mathbb{R}^V \setminus A^V$, where $\mathbb{R}^V$ is the set of all reals in V, is such that $\{\mathbb{R}^V \setminus A^V\}$ is a maximal antichain in $P_{\mathcal{B}}$ and hence $\mathbb{R}^V \setminus A^V \in G$. Therefore $r \in \mathbb{R}^{V[G]} \setminus A^{V[G]} \subseteq \mathbb{R} \setminus A^*$, i.e., $r \notin A^*$.

Now assume that $r \notin A^*$ for every G_σ - set A^V (i.e. countable intersection of open sets) such that $\mathrm{Leb}(A^V) = 0$. Define $G \subseteq P_{\mathcal{C}}$ by $B^V \in G$ iff $r \in B^*$. If $A^V \supseteq B^V \in G$ then also $A^* \supseteq B^*$ and $r \in B^V$, hence $r \in A^*$ and $A^V \in G$. If $A^V, B^V \in G$ then $r \in A^* \cap B^*$. If $\mathrm{Leb}(A^V \cap B^V) = 0$ then since $A^V \cap B^V$ is closed and therefore is a G_σ-set, we would get $r \notin A^* \cap B^*$. Therefore $\mathrm{Leb}(A^V \cap B^V) > 0$, and since $r \in A^* \cap B^*$, we get $A^V \cap B^V \in G$. Finally, let S be a maximal antichain in $P_{\mathcal{C}}$, we have to prove $G \cap S \neq \emptyset$ in order to prove that G is generic. Let $E^V = \mathbb{R}^V \setminus \bigcup\{A : A \in S\}$, so E is obviously a G_σ - set. Since S is maximal $\mathrm{Leb}(E^V) = 0$, therefore $r \notin E^*$. Since $E^V = \mathbb{R}^V \setminus \bigcup\{A : A \in S\}$, so we have $E^* = \mathbb{R} \setminus \bigcup_{B \in S} B^*$. Since $r \notin E^*$ we have $r \in B^*$ for some $B \in S$, and therefore $B \in G$ and $G \cap S \neq 0$. $\square_{6.2}$

The Levy Collapse.

6.3 Definition. 1) $\mathrm{Levy}(\aleph_0, \lambda) = \{f : f$ is a finite function from ω into $\lambda\}$, where λ is an uncountable cardinal. For the partial order on $\mathrm{Levy}(\aleph_0, \lambda)$ we choose inclusion.
2) $\mathrm{Levy}(\kappa, \lambda) = \{f : f$ is a partial function from κ to λ such that $|\mathrm{Dom}(f)| < \kappa\}$ ordered by inclusion.

6.4 Discussion. In $V[G]$, where G is a generic subset of $\mathrm{Levy}(\aleph_0, \lambda)$ it is easily seen that $g = \bigcup G$ is a function on ω onto λ. Therefore $|\lambda|^{V[G]} = \aleph_0$.

Since in $V[G]$ there are no cardinals between λ and λ^+ all the ordinals $< \lambda^+$ are countable in $V[G]$ so $(\lambda^+)^V = \aleph_1^{V[G]}$ provided λ^+ is a cardinal in $V[G]$, thus we must check what is the fate of the chain condition for $\mathrm{Levy}(\aleph_0, \lambda)$. $\mathrm{Levy}(\aleph_0, \lambda)$ has λ pairwise incompatible members, for example $\{\langle 0, \alpha\rangle : \alpha < \lambda\}$. However, it is easy to see that $|\mathrm{Levy}(\aleph_0, \lambda)| = \lambda$ and hence $\mathrm{Levy}(\aleph_0, \lambda)$ satisfies the λ^+-chain condition and the cardinal λ^+ is not collapsed by forcing with $\mathrm{Levy}(\aleph_0, \lambda)$.

We define

6.5 Definition. 1) $\mathrm{Levy}(\aleph_0, <\lambda) = \{f : f$ is a finite function from $\lambda \times \omega$ into λ such that $f(0, n) = 0$ and for $\alpha \neq 0$ we have $f(\alpha, n) < \alpha\}$. The partial order on $\mathrm{Levy}(\aleph_0, <\lambda)$ is inclusion.
2) $\mathrm{Levy}(\kappa, <\lambda) = \{f : f$ is a partial function from $\lambda \times \kappa$ into λ such that $|\mathrm{Dom}(f)| < \kappa$ and $f(0, \alpha) = 0$ and $\alpha > 0 \Rightarrow f(\alpha, i) < \alpha\}$ ordered by inclusion.

6.6 Discussion. Let G be a generic subset of $\mathrm{Levy}(\aleph_0, <\lambda)$ over V, and let $f_G = \bigcup G$. Obviously for every $0 < \alpha < \lambda$ the function $f_G(\alpha, -)$ is a mapping of ω onto α, and hence α is countable in $V[G]$. What about λ, is it, too, countable in $V[G]$ or else is it $\aleph_1^{V[G]}$? If λ is singular in V it stays singular also in $V[G]$, hence it cannot be $\aleph_1^{V[G]}$ and it is countable.

6.7 Theorem. If λ is regular then $\mathrm{Levy}(\aleph_0, <\lambda)$ satisfies the λ-chain condition, and hence $\lambda = \aleph_1^{V[G]}$ in $V[G]$.

We shall prove the following more general version.

6.8 Theorem. Let λ be a regular uncountable cardinal, $|W| = \lambda$, let $\langle A_x : x \in W\rangle$ be such that $|A_x| < \lambda$ for $x \in W$ and let $P = \{f : f$ is a finite function on W such that $f(x) \in A_x$ for each $x \in \mathrm{Dom}(f)\}$ be a forcing notion such that: if $f, g \in P$ agree on $\mathrm{Dom}(f) \cap \mathrm{Dom}(g)$ then f, g are compatible. *Then* P satisfies the λ-chain condition.

Proof. Without loss of generality we can assume that $W = \lambda$ and $A_x \subseteq \lambda$ for each $x \in W$. Let $\langle f_\alpha : \alpha < \lambda\rangle$ be a sequence of members of P. For $\alpha < \lambda$ we define $h(\alpha) = \mathrm{Max}[\{0\} \bigcup(\mathrm{Dom}(f_\alpha) \cap \alpha)]$. Since f_α is a finite function we have $h(\alpha) < \alpha$ for $\alpha > 0$. Thus h is a regressive function on $\lambda \setminus \{0\}$ and therefore it has a fixed value, γ_0 on a stationary subset S of λ by Fodor's Lemma. Let $P^\dagger$ be the set of all members of P whose domain is included in $\gamma_0 + 1$. For every finite subset u of $\gamma_0 + 1$

$$|\{f \in P^\dagger : \mathrm{Dom}(f) = u\} = |\{f \in P : \mathrm{Dom}(f) = u\}| =$$

$$= \Pi_{\alpha \in u} |A_\alpha| \leq \aleph_0 + \mathrm{Max}_{\alpha \in u} |A_\alpha| < \lambda.$$

The number of finite subsets u of $\gamma_0 + 1$ is $\leq |\gamma_0| + \aleph_0 < \lambda$ hence, since λ is regular $|P^\dagger| = \sum_{u \subseteq \gamma_0+1,\ u \text{ is finite}} |\{f \in P : \mathrm{Dom}(f) = u\}| < \lambda$. For each $g \in P^\dagger$ let $S_g = \{\alpha \in S : f_\alpha \restriction (\gamma_0 + 1) = g\}$. Clearly $\bigcup_{g \in P^\dagger} S_g = S$ since for every $\alpha \in S$ we have $\alpha \in S_{f_\alpha \restriction (\gamma_0+1)}$. Since $|P^\dagger| < \lambda$ one of the S_g's say S_{g_0} must be stationary (since the union of $< \lambda$ nonstationary sets is nonstationary). Let $C = \{\delta < \lambda : \delta$ a limit ordinal satisfying $(\forall \alpha < \delta)[\mathrm{Dom}(f_\alpha) \subseteq \delta]\}$, clearly C is a closed unbounded subset of λ, hence $S' = S_{g_0} \cap C$ is a stationary subset of λ. Let $\alpha \in S_{g_0}$ and let $\beta \in S_{g_0}$ be such that $\alpha < \beta \in S$ hence β is a strict upper bound of $\mathrm{Dom}(f_\alpha)$, we shall see that f_α and f_β are compatible. Since $\beta \in S_{g_0} \subseteq S$, clearly $\mathrm{Max}(\mathrm{Dom}(f_\beta) \cap \beta) = \gamma_0$ and since $\mathrm{Dom}(f_\alpha) \subseteq \beta$ we have $\mathrm{Dom}(f_\alpha) \cap \mathrm{Dom}(f_\beta) \subseteq \gamma_0 + 1$. Since $\alpha, \beta \in S_{g_0}$ we have $f_\alpha \restriction (\gamma_0 + 1) = f_\beta \restriction (\gamma_0 + 1) = g_0$, hence f_α and f_β are compatible. $\square_{6.8,6.7}$

On Diamonds.

6.9 Theorem. If λ is regular uncountable cardinal, $S \subseteq \lambda$ stationary, then for some forcing notion P:

(1) $\Vdash_P$ "$\diamondsuit_S$ holds " (see below)

(2) forcing with P, preserve the cardinals $\mu \leq \lambda$ (in fact it is λ-complete and hence add no new α-sequences of ordinals for $\alpha < \lambda$.

(3) $|P| \leq \sum_{\mu<\lambda} 2^\mu$, so if $\lambda^{<\lambda} = \lambda$ forcing with P does not change cardinalities and cofinalities,

where: "$\diamondsuit_S$ holds " means: there is a sequence $\langle A_\alpha : \alpha \in S\rangle$, $A_\alpha \subseteq \alpha$ such that for any $A \subseteq \lambda$ the set $\{\alpha \in S : A \cap \alpha = A_\alpha\}$ is stationary.

Proof. Now the idea is as usual to construct a generic object, by approximations inside V.

So we define: $P = \{\langle A_i : i \in S \cap \alpha\rangle : \alpha < \lambda$ and $A_i \subseteq i$ for every $i \in S \cap \alpha\}$, ordered by: being an intial segment.

Part (3) is easy by the definition of P.

Part (2) holds as P is obviously λ-complete (take union) and the inequality stated in (3) is straigthforward - [the fact that no new α-sequences are added follows from regularity of λ]. Now what we are left with is:

$(*)$ if G is P-generic then $V[G] \models$ "$\diamondsuit_S$ holds"

Let A be included in λ, and let C be closed and unbounded subset of λ, both in $V[G]$, then C, A have names $\underset{\sim}{C}, \underset{\sim}{A}$ repectively and for some f in G, $f \Vdash$ "$\underset{\sim}{C}$ is a club of λ and $\underset{\sim}{A}$ is a subset of λ". Let $f \leq p \in P$. All we have to do is find $q > p$ such that $q = \langle A_i : i \in S \cap \alpha\rangle$, $\alpha < \lambda$, $A_i \subseteq i$ and $q \Vdash$ "$i \in \underset{\sim}{C} \,\&\, \underset{\sim}{A} \cap i = A_i$" for some $i \in S \cap \alpha$; this will prove that the set of such q's is dense above f hence that G contains one of them so $V[G] \models$ "$\{\alpha \in S : A \cap \alpha = A_\alpha$ and $\alpha \in \underset{\sim}{C}\} \neq \emptyset$" and as this holds for any club $C \in V[G]$ clearly $V[G] \models$ "$\{\alpha \in S : A \cap \alpha = A_\alpha\}$ is stationary" i.e., $V[G] \models$ "$\diamondsuit_S$ holds as exemplified by $\underset{\sim}{\bar{A}}$" where $\underset{\sim}{\bar{A}} = \bigcup\{f : f \in G\}$.

So let us find q, we define by induction on $\zeta < \lambda$, $\alpha_\zeta < \lambda$, $p_\zeta = \langle A_i : i \in S \cap \alpha_\zeta\rangle$, B_ζ and β_ζ such that

1) for $\xi < \zeta$, $\alpha_\xi < \alpha_\zeta$, $p_\xi < p_\zeta$, and $p_0 = p$

2) $p_{\zeta+1} \Vdash$ "$\beta_\zeta \in \underset{\sim}{C}$"

3) $p_{\zeta+1} \Vdash$ "$\underset{\sim}{A} \cap \alpha_\zeta = B_\zeta$" for some $B_\zeta \in V$ such that $B_\zeta \subseteq \alpha_\zeta$.

4) for limit ζ, we have $p_\zeta = \bigcup_{\xi<\zeta} p_\xi = \langle A_i : i \in S \cap \alpha_\zeta \rangle$.

5) $\alpha_\zeta < \lambda$ is (strictly) increasing continuous and β_ζ is strictly increasing continous.

6) $\alpha_\zeta < \beta_{\zeta+1} < \alpha_{\zeta+1}$.

The definition is easy, remember for ζ successor $f \Vdash$ "$\underset{\sim}{C}$ is an unbounded subset of λ", so there is $\beta \in \underset{\sim}{C}$, $\beta > \alpha_\zeta$. For ζ limit remember $f \Vdash$ "$\underset{\sim}{C}$ is closed ".

For (3) remember that P is λ-complete hence does not add new bounded subset of λ.

Note that for ζ limit $\alpha_\zeta = \beta_\zeta$ (by clauses 5),6)) and that $\xi < \zeta \Rightarrow B_\xi = B_\zeta \cap \alpha_\xi$. In the end $\{\beta_\zeta : \zeta < \lambda\}$ is a club in V, but S is a stationary subset of λ (in V) hence for some limit ζ, $\beta_\zeta \in S$, but then $p_\zeta \Vdash$ "for $\varepsilon < \zeta$, $\underset{\sim}{A} \cap \alpha_\varepsilon = B_\varepsilon$ and $\beta_\varepsilon \in \underset{\sim}{C}$ hence $\underset{\sim}{A} \cap \beta_\zeta = B_\zeta = \bigcup_{\xi<\zeta} B_\xi$ and $\beta_\zeta \in \underset{\sim}{C}$". Let $q = \langle A'_i : i \in S \cap (\beta_\zeta + 1) \rangle$, where A'_i is: A_i if $i < \beta_\zeta$ and B_ζ if $i = \beta_\zeta$. Easily $p \leq q$, and q is as required.

Note that we also have proved that S remains stationary. $\square_{6.9}$

§7. ♣ Does Not Imply ◇

Note: ♣ is a weak version of the diamond.

7.1 Definition. For a regular uncountable cardinal λ and stationary $S \subseteq \lambda$ set of limit ordinals let us state the combinatorial principle ♣:

♣(S) = "there exists a witness i.e. a sequence $\langle A_\alpha : \alpha \in S \rangle$ such that for every $\alpha \in S$ we have $A_\alpha \subseteq \alpha$ and $\sup A_\alpha = \alpha$ and for every unbounded subset X of λ there exists an $\alpha \in S$ such that $A_\alpha \subseteq X$".

When (S) is omitted, it is $(\aleph_1)$.

7.2 Observation. This form of stating ♣ implies the apparantly stonger form of ♣:

"$\langle A_\alpha : \alpha \in S\rangle$ is as above, and for every unbounded X in λ there are stationarily many points $\alpha \in S$ such that $A_\alpha \subseteq X$".

Proof. Assume the first form holds, let X be unbounded and let C be an arbitrary club in λ. We must show that there is a point $\alpha \in C \cap S$ such that $A_\alpha \subseteq X$. Define by induction an increasing sequence $\langle \beta_i : i < \lambda\rangle$ of ordinals in X as follows: γ_i is the first member of C greater than all $\langle \beta_i : j < i\rangle$, and β_i is the first member of X greater than γ_i. As both C and X are unbounded, this is well defined. Now $X' \stackrel{\text{def}}{=} \langle \beta_i : i < \lambda\rangle$ is unbounded, therefore there exists an $\alpha \in S$ such that $A_\alpha \subseteq X'$. But the demand $\sup A_\alpha = \alpha$ implies that α is also a limit of members of C - as between two consecutive members of X' there is a member of C. By closeness of C, $\alpha \in C$. $\square_{7.2}$

Next we prove

7.3 Fact. if $CH + \clubsuit_S$ hold, then also $\diamondsuit_S$ holds, where S is a stationary subset of $\aleph_1$.

Proof. suppose that $\langle A_\alpha : \alpha \in S\rangle$ is a witness that $\clubsuit_S$ holds. Using CH let $\langle B_i : i < \aleph_1\rangle$ be a list in which every bounded subset of $\aleph_1$ appears $\aleph_1$ times, and such that $\sup(B_i) \leq i$. To get such a list start with a function g from $\aleph_1$ onto $\mathcal{S}_{\leq\aleph_0}(\aleph_1) \times \aleph_1$ (this is where CH comes in, where $\mathcal{S}_{\leq\aleph_1}(A)$ is the family of countable subsets of A). Define $i : \aleph_1 \to \mathcal{S}_{\leq\aleph_0}(\aleph_1)$ by cases as follows: suppose $g(i) = (B_\alpha, \beta)$. Set B_i to be B_α if $\sup(B_\alpha) \leq i$, and $\emptyset$ otherwise. So it is easy to check that i is a listing of all bounded subsets of $\aleph_1$ which satisfies our requirements. Now define a sequence $\langle D_\alpha : \alpha \in S\rangle$ as follows: $x \in D_\alpha$ iff there is an $i \in A_\alpha$ such that $x \in B_i$. Let us verify now that this sequence demonstrates $\diamondsuit_S$. Let X be a subset of $\aleph_1$. If X is bounded, let X' be the set of indices of X in our list, namely all i such that $B_i = X$. From our assumption that each bounded set appears $\aleph_1$ times in the list, X' is unbounded, and so there are stationarily many points $\alpha \in S$ such that $A_\alpha \subseteq X'$. This implies that D_α is X. Now suppose X is unbounded, and define by induction a function

$j : \aleph_1 \to \aleph_1$ as follows: $j(\alpha)$ = minimal j greater than all the ordinals in $\{j(\beta) : \beta < \alpha\}$ such that B_i is the bounded set $X \cap \sup\{j(\beta) : \beta < \alpha\}$. Now as j is strictly increasing, for every α we have $j(\alpha) \geq \alpha$ (the first counterexample yields an immediate contradiction). So there is a club C whose members are closed under j. Set X' as the range of j. This is clearly an unbounded set by the monotonicity of j. So there are stationarily many $\delta \in S$ for which $A_\delta \subseteq X'$. Hence there are stationary many $\delta \in S \cap C$ for which $A_\delta \subseteq X'$. For each such δ, let us prove that D_δ is exactly $X \cap \delta$: As A_δ contains only members of X', which are in particular indices of bounded subsets of the form $X \cap \alpha$, it is clear that D_δ - which is the union of these sets - is contained in $X \cap \delta$. To see equality, fix an arbitrary $\beta < \delta$. It suffice to show that in the union D_δ appears an initial segment of the form $X \cap \alpha$ with $\alpha \geq \beta$. As $\delta \in C$, clearly δ is closed under the function j so $j(\beta) < \delta$, and since A_δ is cofinal in δ, there is an ordinal γ such that $\delta > \gamma > j(\beta)$ and $\gamma \in A_\delta$. But $A_\delta \subseteq X'$ hence $\gamma = j(\alpha)$ for some α, now by the monotonicity of j, $\alpha > \beta$. So by definition of j, $B_{j(\alpha)}$ is an initial segment of X which is obtained by cutting X somewhere higher than $j(\beta)$, but the latter is greater or equal to β. So D_δ includs $X \cap \beta$ for arbitrarily large $\beta < \delta$, hence D_δ include $X \cap \delta$; hence equality follows. $\square_{7.3}$

7.4 Theorem. ♣ does not imply ◇.

Discussion.

It is clear that the principle ◇ implies ♣, and under CH, we have seen that ♣ implies ◇. It was asked whether ♣ → ◇. As we shall now see, the answer is negative. We shall build a model of ZFC in which ♣ holds, but CH fails. As trivially ◇ → CH, this necessarily implies that ◇ also fails.

Proof. Out intention is to begin with a ground model satisfying GCH (or just $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$; for example a model of $V = L$) which has a ◇-sequence on $\aleph_2$. Using the ◇ we will define a ♣-sequence on $\aleph_2$ which is immune to $\aleph_1$-complete forcing, i.e. is still a ♣-sequence in the universe after forcing with an $\aleph_1$-complete forcing notion. The next step is adding $\aleph_3$ subsets to $\aleph_1$ using an $\aleph_1$-complete forcing notion. The last stage will be collapsing $\aleph_1$, thus making of the cardinal that previously was $\aleph_2$ our present $\aleph_1$. Although this will not be done by an $\aleph_2$-complete forcing, our $\clubsuit$-sequence on $\aleph_2$, which has has now dropped to $\aleph_1$, will remain a $\clubsuit$-sequence, while there will be $\aleph_2$ subsets of $\aleph_0$ (which were formerly $\aleph_3$ subsets of $\aleph_1$). So we change our universe three times. We will present each stage of our plan in detail:

Stage A: defining a $\clubsuit$-sequence on $\aleph_2$ using $\diamondsuit(\aleph_2)$

We first note here that a $\diamondsuit$-sequence can guess not only ordinary subsets of $\aleph_2$, but also elementary substructures of any structure over $\aleph_2$. Suppose a structure M over $\aleph_2$ (in a countable language) is given. For each relation symbol or a function symbol R fix a subset of A_R $\aleph_2$ of size $\aleph_2$ and a 1-1 function $F_R : \aleph_2^{lg(R)} \to A_R$ such that two different A_R's are disjoint where we let $\ell g(R) = n$ if R is an n-place relation symbol, and $\ell g(R) = n+1$ if R is an n-place function symbol. Thus the set $F_R(R^M)$ codes the relation R^M. Let A be the union of all $F_R(R^M)$. So A codes the structure M. If $\langle A_i : i \in S\rangle$ is a $\diamondsuit$-sequence, then A is guessed stationarily often, namely for stationarily many $i \in S$ we have $A \cap i = A_i$. Clearly, the set $C = \{i < \aleph_2$: for all R we have $F_R(R^{M\restriction i}) \subseteq i\}$ is closed unbounded in $\aleph_2$. Moreover, the set of α such that $M\restriction\alpha$ is an elementary substructure of M is a club by the Skolem-Löwenhein theorem and the continuity of elementary chains. So the intersection of the two clubs with our stationary set is the stationary set S^M of all α such that $M\restriction\alpha$ is an elementary substructure of M and for all R we have $F_R(R^{M\restriction\alpha}) \subseteq \alpha$ and $A_\alpha = \bigcup_R F_R(R^{M\restriction\alpha})$ (that is, $M\restriction\alpha$ contains its own coding and A_α equals this coding). In short we say that for every α in S^M, A_α guess the elementary substructure $M\restriction\alpha$.

Let S be the subset of $\aleph_2$ of all ordinals having cofinality $\aleph_0$. We wish to make use of $\diamondsuit_S$. Why does this principle hold in our ground model? Because, e.g. we could have forced it easily by a preliminary forcing which appeares in 6.9.

Now we come to defining the $\clubsuit$-sequence. Let us choose coding for a language with two relation signs, $<^*$ and a two place relation $R(\ ,\ )$, and let $\langle M_\alpha = (\alpha, <^*_\alpha, R_\alpha) : \alpha \in S\rangle$ be a diamond sequence for such models. So for every structure M with universe $\aleph_2$ for this language, for stationarily many $\delta \in S$ the substructures $M \restriction \delta$ of it are guessed by our $\diamondsuit$-sequence i.e. $M \restriction \delta = M_\delta$. Restrict attention now only to those places in which the guessed substructure satisfies the following sentences (the set of such $\delta \in S$ will be called S'):

(i) $<^*$ is a partial order.

(ii) if $\beta <^* \gamma$ then $R(\beta, x) \to R(\gamma, x)$

(iii) $(\forall \alpha)(\forall \beta)(\exists \gamma > \alpha)(\exists \xi > \beta)\ [R(\gamma, \xi)]$

Note that there are stationarily many such places (by guessing one such structure on $\aleph_2$, for example). In each such place δ we define now a subset of δ of order type ω which we call D_δ : let δ be fixed, and let $\langle \beta_n^\delta : n < \omega \rangle$ be a cofinal increasing sequence in δ; we define by induction on $n < \omega$ a sequence $\langle \gamma_n^\delta, \xi_n^\delta : n < \omega \rangle$ such that

1. $\gamma_n^\delta <_\delta^* \gamma_{n+1}^\delta$
2. $\xi_n^\delta > \beta_n^\delta$
3. $R_\delta(\gamma_n^\delta, \xi_n^\delta)$

So let γ_0^δ, $\xi_0^\delta < \delta$ be such that $R(\gamma_0^\delta, \xi_0^\delta)$ & $\beta_0^\delta < \xi_0^\delta$, exists by clause (iii) above, and the induction step i.e. choosing γ_{n+1}^δ, ξ_{n+1}^δ such that $\gamma_n^\delta <_\delta^* \gamma_{n+1}^\delta$ & $\beta_{n+1}^\delta < \xi_{n+1}^\delta$ & $R(\gamma_{n+1}^\delta, \xi_{n+1}^\delta)$ is handled using (iii). Set $D_\delta = \{\xi_n^\delta : n < \omega\}$.

Our ♣-sequence will be the sequence $\langle D_\delta : \delta \in S' \rangle$.

Claim B. $\langle D_\delta : \delta \in S' \rangle$ is a ♣-sequence.

Proof. Suppose X is unbounded in $\aleph_2$. Define a structure $(\aleph_2, <^*, R)$ with $<^*$ be the natural order on $\aleph_2$, and $R(y, x)$ iff $x \in X$. So for stationarily many δ in S' the model is guessed by A_δ, namely $\langle \delta, A_\delta \rangle$ is an elementary substructure of our structure. So each $\xi_n \in X$ - which implies that $D_\delta \subseteq X$.

Stage C: adding $\aleph_3$ subsets to $\aleph_1$.

We will force now with $P_1 = \{$countable functions from $\aleph_3$ to $2\}$, ordered by inclusion. The advantage of our ♣-sequence to other ♣-sequences is that it is preserved under the forcing notions we are about to apply, and this is what we are about to check now. We must check first, however, the weaker condition, that this forcing preserves stationarity of stationary subsets of $\aleph_2$ namely that a subset S of $\aleph_2$ which intersects every "old" club ("old" meaning "in the universe before the forcing") intersects also every "new" club. This follows from the following couple of claims:

Claim D. P_1 satisfies the $\aleph_2$-.c.c.

Claim E. Every P that satisfies the $\aleph_2$-.c.c preserves the stationarity of subsets of $\aleph_2$.

Proof of Claim D. let χ be any regular cardinal which is large enough to have the power set of P_1 in $H(\chi)$. Suppose that $\mathcal{I}$ is a maximal antichain of P_1 whose cardinality is greater than $\aleph_1$. Now define an increasing sequence of elementary submodels of $(H(\chi), \in)$ of length $\aleph_1$ called $\langle N_i : i < \aleph_1 \rangle$, as follows:

(i) P_1, $\mathcal{I}$ are members of N_0, and N_0 is countable, has cardinality $\aleph_1$.

(ii) every countable subset of N_i is an element of N_{i+1}.

(iii) $j < i \Rightarrow N_j \prec N_i$.

There is no problem to carry out the construction, because at each stage in the construction the cardinality of the model at hand is at most $\aleph_1$, therefore it may have at most $\aleph_1$ countable subsets (we have CH in the ground model). So close these $\aleph_1$ elements together with everything you already have in an elementary submodel of cardinality $\aleph_1$ using the Skolem-Löwenheim method. At limits take unions, for example. Let us denote the union of the increasing chain as N. So N is an elementary submodel of $(H(\chi), \in)$ of cardinality $\aleph_1$. Furthermore, every countable set of elements of N is a subset of one of the models along the construction (say N_i) by the regularity of $\aleph_1$, therefore it is an element of N_{i+1}, and therefore of N. In short we say that "N is closed under countable subsets".

We recall that P_1, $\mathcal{I} \in N$. As $\mathcal{I}$ is greater in cardinality than N, there must be an element p of $\mathcal{I}$ which is not a member of N. (Remember that p is a countable function from $\aleph_3$ to 2.) Look at p', which we define as $p \cap N$. From the countability of p, p' is clearly countable, and as N is closed under countable sets and p' is certainly a countable subset of N, we have $p' \in N$. Now $(H(\chi), \in)$ satisfies the sentence "there is a member of $\mathcal{I}$ which extends p'" (it is p). As N is an elementary submodel of it, and contains as members all the constants mentioned in this sentence - therefore there must be an element $q \in \mathcal{I} \cap N$ which according to N extends p'. We lack only one more detail to derive a contradiction: the domain of q is countable, therefore there is an enumeration of it *in* N. This implies that $\mathrm{Dom}(q)$ is contained in N. So inspect the union $q \cup p$. Clearly q extends the part of p which is in N, and contradicts nothing that is outside of N, as we just saw. Therefore p and q are compatible members of $\mathcal{I}$.

Proof of Claim E. Let S be a stationary subset of $\aleph_2$ in V, and suppose $\underset{\sim}{C}$ is a P_1-name for a club of $\aleph_2$ with the condition p in the generic set forcing $p \Vdash$ "$\underset{\sim}{C}$ is a club in $\aleph_2$". Attach to each ordinal $\alpha < \aleph_2$ a maximal antichain $\mathcal{I}_\alpha$ of extentions of p such that for each $q \in \mathcal{I}_\alpha$ there is a $\beta(q)$ such that $q \Vdash$ "$\beta(q)$ is the minimal member of $\underset{\sim}{C}$ above α". Define as B_α the set of all ordinals γ for which some member of $\mathcal{I}_\alpha$ forces that γ is the first member of $\underset{\sim}{C}$ above α. Clearly B_α has cardinality $\leq |\mathcal{I}_\alpha|$ which is $< \aleph_2$ hence B_α is bounded in $\aleph_2$. Using our standard argument, we see that $C^* := \{\delta < \aleph_2 :$ for all $\alpha < \delta$ the set B_α is included in $\delta\}$ is a club of $\aleph_2$. But this club, being defined in V, is an old club! Therefore it meets S, let us say in δ. As for each $\mathcal{I}_\alpha$, $\alpha < \delta$ the generic set must choose a condition from $\mathcal{I}_\alpha$, and B_α is contained in δ, δ is a limit of the realization of $\underset{\sim}{C}$, therefore in it.

Stage F. So we know now that our S' from the definition of the ♣-sequence is still a stationary subset of $\aleph_2$ after forcing with P_1. But is the sequence still a ♣-sequence? We verify this now.

Let $\underset{\sim}{X}$ be a P_1-name, $p \in P_1$ and $p \Vdash_{P_1}$ "$\underset{\sim}{X} \subseteq \aleph_2$ is unbounded and $\underset{\sim}{C}$ is a club of $\aleph_2$". We show that the set of conditions which force "there exists $\delta \in S \cap \underset{\sim}{C}$ such that $D_\delta \subseteq \underset{\sim}{X}$" is dense above p. Fix an elementary submodel

N of $(H(\chi), \in)$, $|N| = \aleph_2$, N closed under subsets of size $\leq \aleph_1$, with p, P_1, $\underset{\sim}{X}$ and $\underset{\sim}{C}$ members of N. Enumerate in the sequence $\langle p_i : i < \aleph_2 \rangle$ the member of $P_1 \cap M$ which are $\geq p$ and define a structure M with the same language as above with universe $\aleph_2$ with: $i <^* j$ iff $P_1 \models p_i \leq p_j$, and $R(i, x)$ iff as $p_i \Vdash$ "$x \in \underset{\sim}{X}$". From the definition of our diamond sequence $\langle M_\alpha : \alpha \in S' \rangle$, stationarily many elementary substructures of this structure are guessed by it i.e. $S'' = \{\alpha \in S' : M_\alpha = N \restriction \alpha\}$ is stationary. Let α be one such coordinate so $D_\alpha = \{\xi_n^\alpha : n < \omega\}$ where $\xi_n^\alpha < \xi_{n+1}^\alpha < \alpha = \bigcup\limits_{n<\omega} \xi_n^\alpha$, $N \models \gamma_n^\alpha <_\alpha^* \gamma_{n+1}^\alpha$, and $N \models R(\gamma_n^\alpha, \xi_n^\alpha)$. Then we have at hand now an increasing sequence in P_1 of conditions $\langle q_n^\alpha : n < \omega \rangle$ such that $q_n \Vdash$ "$\xi_n^\alpha \in \underset{\sim}{X}$", just let $q_n^\alpha = p_{\gamma_n^\alpha}$. From elementaricity, this is really an increasing sequence of conditions, and let us bound it by q, which existed by $\aleph_1$-completeness. So $q \Vdash$ "$D_\alpha \subseteq \underset{\sim}{X}$" and q is certainly an extention of p.

So we have added $\aleph_3$ subsets to $\aleph_1$ forcing with P_1 without destroying out $\clubsuit$-sequence nor collapsing cardinals.

Stage G: collapsing $\aleph_1$.

We collapse now $\aleph_1, P_2$ being the finite functions from $\aleph_0$ to $\aleph_1$. This is a forcing notion of size $\aleph_1$ and therefore leaves the cardinals above $\aleph_1$ unchanged. To show that our sequence, which was formerly on $\aleph_2$ is now on $\aleph_1$ still a $\clubsuit$-sequence, it will suffice to prove the following.

Fact H. In the world after the collapse every new unbounded set of (the new) $\aleph_1$ contains an old unbounded set of (the old) $\aleph_2$.

Proof. Suppose p is a condition and $p \Vdash$ "$\underset{\sim}{X}$ is an unbounded subset of $\aleph_2$". So for every $i < \aleph_2$ there is an ordinal $\tau(i)$ and an extention $q(i)$ of p such that $q(i) \Vdash$ "$\aleph_2 > \tau(i) > i$ and $\tau(i) \in \underset{\sim}{X}$". By the pigeon hole principle there are $\aleph_2$ coordintes i with a fixed $q(*) = q(i)$. Set $X \stackrel{\text{def}}{=} \{\tau(i) : q(*) = q(i)\}$ - then surely X is unbounded, so $q(*)$ is an extention of p which forces that X is an unbounded subset of $\underset{\sim}{X}$.

From the lemma we easily deduce both the preservation of stationary sets (in a new club there is an old unbounded set, the closure of which is an old club; alternatively use Claim E) and the preservation of our ♣-sequence, for instead of guessing a new set, it is enough to guess an old subset. $\square_{7.4}$