Chapter IX
Silver Machines

1. Silver Machines

Silver machines are a device for avoiding the use of the fine structure theory in proving results such as $\Box_\kappa$. The idea is as follows. In proving, say $\Box_\kappa$, as we did in Chapter IV, the main tool was the hierarchy of skolem functions $h_{\varrho_\alpha^n, A_\alpha^n}$. Of course, these functions, and the properties of them that we made use of, were obtained by our fine structure theory. But the fine structure theory itself was not used in the proof of $\Box_\kappa$. Any hierarchy of functions with similar properties would suffice. As we shall see, it is possible to construct such a functioinal hierarchy without using the fine structure theory. The idea is as follows.

We shall say that an ordinal α is *-definable from a class X of ordinals iff there is an $\mathscr{L}$-formula $\varphi(v_0, \dots, v_n)$ and elements $\beta_1, \dots, \beta_n, \gamma$ of X such that α is the unique ordinal for which

$$\models_{L_\gamma} \varphi(\mathring{\alpha}, \mathring{\beta}_1, \dots, \mathring{\beta}_n).$$

The idea behind the machine concept is this. Suppose we were to define a **-skolem function* for L as a function h such that $\mathrm{dom}(h) \subseteq \omega \times \mathrm{On}^{<\omega}$, $\mathrm{ran}(h) \subseteq \mathrm{On}$, and whenever α is *-definable from $X \subseteq \mathrm{On}$, then $\alpha \in h''(\omega \times X^{<\omega})$, where we use $X^{<\omega}$ to denote $\bigcup_{n<\omega} X^n$. In order to construct, say, a $\Box$-sequence, we might then go on to define a hierarchy of (set) functions convering to h, possessing some kind of condensation property. And to a point, this is the idea behind the definition of a Silver machine. But there are some differences. For instance, we shall not work with a single skolem function h but rather an infinite family of functions h_i, $i < \omega$. Although h_i will, in some sense, correspond to the function $h(i, -)$ of the above sketch, the index i will not be the Gödel number of a formula as was the case with the skolem functions of the fine structure theory, and for different indices i the functions h_i may be quite different in structure. (Hence there is no point in trying to combine them into one function.)

One remark concerning the use of the word "machine". This stems from the motivation which led Silver to develop the concept in the first place. "Silver hierarchy" would be a more suitable term for the structure we shall develop here (which is not quite the same as the original), but we shall, of course, stick to the established usage.

A structure

$$N = \langle X, <, (h_i)_{i<\omega} \rangle$$

is said to be *eligible* iff:

(i) $X \subseteq \mathrm{On}$;

(ii) $<$ is the usual ordering on X;

(iii) for each i, h_i is a partial function from $X^{k(i)}$ into X, for some integer $k(i)$.

If N is as above and λ is an ordinal, we set

$$N_\lambda = \langle X \cap \lambda, <, (h_i \cap \lambda^{k(i)+1})_{i<\omega} \rangle.$$

We sometimes write N_∞ instead of N.

If N, λ are as above and $A \subseteq X \cap \lambda$, $N_\lambda[A]$ denotes the closure of A under the functions of N_λ.

Let $N^j = \langle X^j, \langle, (h_i^j)_{i<\omega} \rangle$ be eligible structures of the same similarity type, for $j = 1, 2$. We write $N^1 \lhd N^2$ iff $X^1 \subseteq X^2$ and for all $i < \omega$ and all $x_1, \ldots, x_{k(i)} \in X^1$,

$$h_i^1(x_1, \ldots, x_{k(i)}) \simeq h_i^2(x_1, \ldots, x_{k(i)}).$$

A *machine* is an eligible structure of the form

$$M = \langle \mathrm{On}, <, (h_i)_{i<\omega} \rangle,$$

which satisfies the following three conditions:

I. *Condensation Principle.* If $N \lhd M_\lambda$, there is an α such that $N \cong M_\alpha$.

II. *Finiteness Principle.* For each λ there is a finite set $H \subseteq \lambda$ such that for any set $A \subseteq \lambda + 1$,

$$M_{\lambda+1}[A] \subseteq M_\lambda[(A \cap \lambda) \cup H] \cup \{\lambda\}.$$

III. *Skolem Property.* If α is *-definable from the set $X \subseteq \mathrm{On}$, then $\alpha \in M[X]$; moreover there is an ordinal $\lambda < [\sup(X) \cup \alpha]^+$, uniformly Σ_1 definable from $X \cup \{\alpha\}$, such that $\alpha \in M_\lambda[X]$.

Some explanatory comments are perhaps in order here. In the light of our introductory remarks, the inclusion of the Condensation Principle and of the Skolem Property in this definition should come as no surprise. But why the Finiteness Principle? This says that the hierarchy $(M_\lambda \mid \lambda \in \mathrm{On})$ grows very slowly, with only finitely many new ordinals being calculated at each stage. Hence events of set theoretic interest will occur only at limit levels of the hierarchy. This fact will be of considerable use to us, much as we used the fact that the structures L_α are only easily handled when α is a limit ordinal (as we saw in Chapter II).

There are several ways to construct a machine, but in essence the idea is that the machine should code the truth definition associated with *-definability. The following devices are introduced in order to facilitate our proof of the Condensation Property for the machine.

Suppose $N = \langle X, <, (h_i)_{i<\omega}\rangle$ is an eligible structure, where X is a set. Let $*$ denote $\sup(X)$, and set $X^* = X \cup \{*\}$. Define partial functions h_i^*, $i < \omega$, on X^* as follows:

(a) if $s \in X^{k(i)}$ and $h_i(s)$ is defined, set $h_i^*(s) = h_i(s)$;
(b) if $s \in X^{k(i)}$ and $h_i(s)$ is undefined, set $h_i^*(s) = *$;
(c) if $s \in (X^*)^{k(i)}$ contains $*$, set $h_i^*(s) = *$.

Let $N^* = \langle X^*, <, (h_i^*)_{i<\omega}, \{*\}\rangle$. Though essentially the same as N, N^* has the advantage (for us) that all of its functions are total, which is the reason for its introduction.

Suppose that S is a first-order language. The infinitary language $S^\#$ is obtained from S by allowing the formation of countably infinite conjunctions and disjunctions of quantifier free formulas. A *universal* sentence of $S^\#$ is a sentence of the form

$$\forall v_0 \ldots \forall v_n \varphi(v_0, \ldots, v_n)$$

where φ is quantifier free. A *universal theory* in $S^\#$ is a consistent set of universal sentences of $S^\#$.

Suppose that S is the first-order language of some eligible structure, and T is a theory in $S^\#$. We say that T is *α-categorical* if the structure $\langle \alpha, <\rangle$ has exactly one expansion to an S-structure satisfying T. (The definition of satisfaction for $S^\#$ is quite straightforward.)

The following lemma indicates how the above concepts can assist us in proving that our machine has the Condensation Property.

1.1 Lemma. *Let $M = \langle \mathrm{On}, <, (h_i)_{i<\omega}\rangle$ be an eligible structure. Let S be the language of the structures M_λ^*. Suppose there is a universal theory T in $S^\#$ such that:*

(i) *T is $(\alpha + 1)$-categorical for all α;*
(ii) *$M_\alpha^* \models T$ for all α.*

Then M has the Condensation Property.

Proof. Let $N \lhd M_\alpha$. Since $M_\alpha^* \models T$ and T is universal, we clearly have $N^* \models T$. The domain of N is a set of ordinals, so there is a unique ordinal $\bar{\alpha}$ and a unique isomorphism $\pi\colon N \cong \bar{N}$, where $\bar{N} = \langle \bar{\alpha}, <, (\bar{h}_i)_{i<\omega}\rangle$ is eligible. But $\bar{N}^* \models T$, so as T is $(\bar{\alpha}+1)$-categorical, $\bar{N}^* = M_{\bar{\alpha}}^*$. Hence $\bar{N} = M_{\bar{\alpha}}$, and we are done.

We are now ready to commence the construction of our machine. As a first step we define a certain well-ordering of $\mathrm{On}^{<\omega}$.

It is easily seen that the following rules do define a well-ordering of $\mathrm{On}^{<\omega}$. Let $s, t \in \mathrm{On}^{<\omega}$.

(i) If s is a proper subsequence of t, then any permutation of s precedes any permutation of t.
(ii) If s is a permutation of t, then s and t are ordered lexicographically.
(iii) If $s = (\alpha_1, \ldots, \alpha_n)$, $t = (\alpha_1, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_m)$ and $\beta_1, \ldots, \beta_m < \alpha_n$, then any permutation of t precedes any permutation of s. (In case $n = 1$ here, $\alpha_1, \ldots, \alpha_{n-1}$ is interpreted as the empty string.)

We denote by $<^*$ the well-ordering of $\mathrm{On}^{<\omega}$ so defined. For later use, we note that for $s, t \in \mathrm{On}^{<\omega}$, $\max(s) < \max(t)$ implies $s <^* t$. (Proving this should help the reader to understand the definition of $<^*$ more fully.)

For $s \in \mathrm{On}^{<\omega}$, we denote by $\hat{s}$ the ordinal corresponding to s in the ordering $<^*$, that is

$$\hat{s} = \mathrm{otp}(\langle\{t \mid t <^* s\}, <^*\rangle).$$

Define functions $P_n\colon \mathrm{On}^n \to \mathrm{On}$ by setting $P_n(s) = \hat{s}$. These are the "*pairing functions*". Note that $P_n((\alpha_1, \ldots, \alpha_n)) \geqslant \max(\alpha_1, \ldots, \alpha_n)$.

Define partial functions Q_i from On to On by

$$Q_i(\alpha) = \begin{cases} \alpha_i, & \text{if } \alpha = P_n((\alpha_1, \ldots, \alpha_n)) \quad \text{and} \quad i \leqslant n; \\ \text{otherwise undefined.} \end{cases}$$

These are the "*pairing inverses*".

Notice that

$$P = \langle \mathrm{On}, <, (P_n)_{n<\omega}, (Q_i)_{i<\omega} \rangle$$

is an eligible structure. Indeed, P satisfies two of the machine axioms, as we prove next.

1.2 Lemma. *P has the Finiteness Property.*

Proof. Let λ be given. For some n and some $\alpha_1, \ldots, \alpha_n \leqslant \lambda$,

$$\lambda = P_n((\alpha_1, \ldots, \alpha_n)).$$

Set

$$H = \{\alpha_1, \ldots, \alpha_n\} \cap \lambda.$$

Clearly, if $A \subseteq \lambda + 1$,

$$P_{\lambda+1}[A] \subseteq P_\lambda[(A \cap \lambda) \cup H] \cup \{\lambda\}. \quad \square$$

1.3 Lemma. *P has the Condensation Property.*

Proof. We use 1.1. Let S be the language of the structures P_λ^*. It is clear that, for fixed n, m, there is a first-order, quantifier free formula $\varphi_{n,m}(v_1, \ldots, v_n, v_{n+1}, \ldots, v_m)$ of S which says

$$\text{"}(v_1, \ldots, v_n) <^* (v_{n+1}, \ldots, v_m)\text{"}.$$

Let T_0 be the following universal $S^\#$ theory:

$$\begin{aligned} T_0 = {} & \{\forall x(x = * \vee x < *)\} \cup \{Q_n(*) = * \mid n \in \omega\} \\ & \cup \{\forall x_1 \ldots x_n(x_1 = * \vee \ldots \vee x_n = * \to P_n(\vec{x}) = *) \mid n \in \omega\} \\ & \cup \{\forall x_1 \ldots x_n y_1 \ldots y_m[\varphi_{n,m}(\vec{x}, \vec{y}) \to (P_n(\vec{x}) < P_m(\vec{y}) \vee P_m(\vec{y}) = *) \mid n, m \in \omega\} \\ & \cup \{\forall x \bigvee_{n<\omega} [x = P_n(Q_1(x), \ldots, Q_n(x))]\}. \end{aligned}$$

It is easily seen that T_0 is $(\alpha + 1)$-categorical for all α. And clearly, $P_\alpha^* \models T_0$ for all α. So by 1.1, P has the Condensation Property. $\square$

In order to incorporate the Skolem Property into our machine, we introduce a first-order language, Γ, appropriate for *-definability.

The basic symbols of Γ are as follows:

variables: v_n $(n \in \omega)$;

connectives: $\wedge, \neg$;

predicates: $=, \in$;

constants: t_α^φ (for certain φ, α described below);

quantifiers: $\exists^\alpha$ $(\alpha \in \mathrm{On})$.

If φ is a formula of Γ, the *rank* of φ, $\varrho(\varphi)$, is the least α such that:

(i) if $\exists^\gamma$ occurs in φ, then $\gamma \leqslant \alpha$;
(ii) if t_γ^ψ occurs in φ, then $\gamma < \alpha$.

For each α and each Γ-formula φ of rank α, the language Γ has a constant t_α^φ, and this is the only occasion on which such a constant is defined.

The definitions of the language Γ and of the rank function ϱ thus proceed by means of a simultaneous recursion, which is easily seen to be well-defined.

The language Γ is interpreted in L as follows. The interpretation of t_α^φ is the set $\{x \in L_\alpha \mid \models_{L_\alpha} \varphi(\mathring{x})\}$, and the interpretation of $\exists^\alpha v_n$ is $(\exists x \in L_\alpha)$. Clearly, each member of L is denoted by a constant t_α^φ, and elements of $L_{\alpha+1}$ are just the interpretations of the constants t_α^φ as φ varies. For each α, L_α has a canonical name in Γ, namely $t_\alpha^{(v_0 = v_0)}$. This name is denoted by l_α. Similarly, α has a canonical name, $t_\alpha^{\mathrm{On}(v_0)}$, which we denote by o_α.

The formal definition of Γ in set theory is as follows.

$$v_n = \langle n + 6 \rangle;$$

$$t_\alpha^\varphi = \langle 0 \rangle ^\frown \varphi ^\frown \langle \omega + \alpha + 1 \rangle;$$

$$(x = y) = \langle 1 \rangle ^\frown x ^\frown y \qquad (x, y \text{ variables or constants});$$

$$(x \in y) = \langle 2 \rangle ^\frown x ^\frown y \qquad (x, y \text{ variables or constants});$$

$$(\varphi \wedge \psi) = \langle 3 \rangle ^\frown \varphi ^\frown \psi \qquad (\varphi, \psi \text{ formulas});$$

$$(\neg \varphi) = \langle 4 \rangle ^\frown \varphi \qquad (\varphi \text{ a formula});$$

$$(\exists^\alpha v_n \varphi) = \langle 5 \rangle ^\frown \langle \alpha \rangle ^\frown \langle n \rangle ^\frown \varphi \qquad (\varphi \text{ a formula}).$$

Thus each formula of Γ is a finite sequence of ordinals. Using the pairing functions P_n, we may now associate with each Γ-formula φ a unique single ordinal $\hat{\varphi}$. Similarly, each constant c of Γ is assigned an ordinal $\hat{c}$. If the ordinal α denotes a formula or a constant of Γ, we denote that formula/constant by $\ulcorner \alpha \urcorner$.

1.4 Lemma.

(i) *If* φ *is a subformula of* ψ, *then* $\hat{\varphi} < \hat{\psi}$.

(ii) *If* φ is $(\exists^\alpha v)\,\psi(v)$ *and* t *is* t^θ_γ, *where* $\gamma < \alpha$, *then* $\hat{\psi}(t) < \hat{\varphi}$.

(iii) *If* φ *is* $(t^{\theta_1}_{\alpha_1} = t^{\theta_2}_{\alpha_2})$ *or* $(t^{\theta_1}_{\alpha_1} \in t^{\theta_2}_{\alpha_2})$, *and if* $\varrho(\psi) \leqslant \max(\alpha_1, \alpha_2)$, *then* $\hat{\psi} < \hat{\varphi}$.

Proof. (i) If φ is a subformula of ψ, then as sequences of ordinals, φ is a subsequence of ψ, so $\varphi <^* \psi$. Hence $\hat{\varphi} < \hat{\psi}$.

(ii) This is a direct application of clause (iii) in the definition of $<^*$.

(iii) For definiteness, suppose $\alpha_1 \leqslant \alpha_2$. Thus $\max(\varphi) = \omega + \alpha_2 + 1$. Since $\varrho(\psi) \leqslant \alpha_2$, $\max(\psi) \leqslant \omega + \alpha_2$. Hence $\max(\psi) < \max(\varphi)$. So, as we remarked earlier, $\psi <^* \varphi$, giving $\hat{\psi} < \hat{\varphi}$. $\square$

Our machine will need to be able to handle the elementary syntax of Γ. Accordingly, we make the following definitions.

For $\nu \leqslant \omega.\omega$, let k_ν be the constant unary function with value ν. Let I be the unary function $I(\alpha) = \omega + \alpha$. Let J be the unary function $J(\alpha) = \alpha + 1$.

Let

$$N = \langle \mathrm{On}, <, (P_n)_{n<\omega}, (Q_i)_{i<\omega}, (k_\nu)_{\nu \leqslant \omega.\omega}, I, J \rangle.$$

The eligible structure N can clearly handle the besic syntax of the language Γ.

1.5 Lemma. *N has the Finiteness Property.*

Proof. Given λ, pick $n, \alpha_1, \ldots, \alpha_n$ so that $\lambda = P_n((\alpha_1, \ldots, \alpha_n))$ and let $H = \{\alpha_1, \ldots, \alpha_n\} \cap \lambda$. H is uniquely defined, and if $A \subseteq \lambda + 1$, then

$$N_{\lambda+1}[A] \subseteq N_\lambda[(A \cap \lambda) \cup H] \cup \{\lambda\}. \quad \square$$

1.6 Lemma. *N has the Condensation Property.*

Proof. If $\lambda \leqslant \omega.\omega$ and $X \lhd N_\lambda$, then $X = N_\lambda$ and there is nothing to prove. For the case $\lambda > \omega.\omega$, we use 1.1. Let S be the language of the structures N^*_λ. Let T_0 be as in the proof of 1.3. T_0 will take care of the P-part of N, so what we must do is extend T_0 to a universal $S^\#$ theory which uniquely characterises the functions k_ν, I, J. Let T_1 be the following universal $S^\#$ theory.

$$\begin{aligned}
T_1 = T_0 &\cup \{\forall x \forall y [k_\nu(x) = k_\nu(y) \mid \nu \leqslant \omega.\omega)\\
&\cup \{\forall x \forall y [[(x = *) \wedge (\bigwedge_{\nu \leqslant \omega.\omega} k_\nu(x) = *)] \vee [(x < *)\\
&\wedge \bigwedge_{\tau < \nu \leqslant \omega.\omega} (k_\tau(x) < k_\nu(x)) \wedge [\bigvee_{\nu \leqslant \omega.\omega} (k_\nu(x) = y) \vee (k_{\omega.\omega}(x) < y)]]]]\}\\
&\cup \{\forall x \forall y [[x = * \wedge I(x) = * \wedge J(x) = *] \vee [(x < *)\\
&\wedge [\bigvee_{\nu \leqslant \omega.\omega} ((x = k_\nu(x)) \wedge (I(x) = k_{\omega+\nu}(x)))\\
&\vee (x \geqslant k_{\omega.\omega}(x) \wedge I(x) = x)] \wedge [(x < J(x)) \wedge (y \leqslant x \vee J(x) \leqslant y)]]]\}.
\end{aligned}$$

Clearly, T_1 is $(\alpha + 1)$-categorical for all $\alpha \geqslant \omega . \omega$. Moreover, $N_\alpha^* \models T_1$ for $\alpha \geqslant \omega . \omega$. Hence by 1.3, if $X \lhd N_\alpha$, where $\alpha \geqslant \omega . \omega$, then $X \cong N_{\bar{\alpha}}$ for some unique $\bar{\alpha}$. $\square$

Now, L-truth for Γ-sentences is clearly definable. So we may define a function F from ordinals to $\{0, 1\}$ as follows:

$$F(\alpha) = \begin{cases} 1, & \text{if } \alpha = \hat{\varphi} \text{ where } \varphi \text{ is a true sentence of } \Gamma; \\ 0, & \text{if } \alpha = \hat{\varphi} \text{ where } \varphi \text{ is a false sentence of } \Gamma; \\ \text{otherwise undefined.} \end{cases}$$

We may now define functions G, H from ordinals to ordinals by:

$$G(\alpha) = \begin{cases} \hat{t}_\gamma^\psi, & \text{if } \alpha = (\exists^\delta v \varphi(v))^\wedge \text{ and } \exists^\delta v \varphi(v) \text{ is true and } \hat{t}_\gamma^\psi \text{ is least} \\ & \quad \text{such that } \gamma < \delta \text{ and } \varphi(t_\gamma^\psi) \text{ is true;} \\ \text{otherwise undefined.} \end{cases}$$

$$H(\alpha) = \begin{cases} \beta, & \text{if } \alpha = \varphi(v)^\wedge \text{ and } \beta \text{ is least such that } \varphi(o_\beta) \text{ is true;} \\ \text{otherwise undefined.} \end{cases}$$

Set

$$M = \langle \mathrm{On}, <, (P_n)_{n<\omega}, (Q_i)_{i<\omega}, (k_\nu)_{\nu \leqslant \omega . \omega}, I, J, F, G, H \rangle.$$

Clearly, M is an eligible structure. We show that M is a machine.

1.7 Lemma. *M has the Finiteness Property.*

Proof. The proof of 1.5 is still valid. $\square$

1.8 Lemma. *M has the Skolem Property.*

Proof. Let α be *-definable from $X = \{\beta_1, \ldots, \beta_n, \gamma)$, α being the unique ordinal such that $\models_{L_\gamma} \varphi(\mathring{\alpha}, \mathring{\beta}_1, \ldots, \mathring{\beta}_n)$ where φ is some $\mathscr{L}$-formula. Obtain the formula $\psi(v_0)$ of Γ from $\varphi(v_0, \ldots, v_n)$ by replacing v_i by o_{β_i}, for $i = 1, \ldots, n$, and each quantifier $\exists v$ by $\exists^\gamma v$. Clearly, if t is a constant of Γ, $\psi(t)$ will be a true sentence of Γ (in L) iff the interpretation of t in L is α.

Let $\delta = \psi(v_0)^\wedge$. Notice that δ is computable from $\beta_1, \ldots, \beta_n$, γ using the functions of M (in fact the functions of N). Let

$$\lambda = \sup\{\delta, \beta_1, \ldots, \beta_n, \gamma, \alpha\}.$$

Clearly, $\lambda < \sup(X \cup \{\alpha\})^+$, and λ is (uniformly) Σ_1 definable from $X \cup \{\alpha\}$. Then $\delta \in M_\lambda[X]$. But $H(\delta) = \alpha$. Hence $\alpha \in M_\lambda[X]$, as required. $\square$

1.9 Lemma. *M has the Condensation Property.*

Proof. If $\lambda \leqslant \omega . \omega$ and $X \lhd M_\lambda$, then by virtue of the functions k_ν, we have $X = M_\lambda$, so there is nothing more to prove. For the case $\lambda \geqslant \omega . \omega$ we use 1.1. Let S be the language of the structures M_λ^*. Let T_1 be the universal $S^\#$ theory defined in the proof of 1.6. As we saw in 1.6, T_1 will take care of the N-part of M. What we must do now is extend T_1 to a universal theory T in $S^\#$ which characterises uniquely the remaining functions of M.

Notice first that the functions F, G have the following recursive definitions (by simultaneous recursion for F and G):

$$
\begin{aligned}
&[F(*) = *] \wedge \forall x[F(x) = 0 \vee F(x) = 1 \vee F(x) = *] \\
\wedge\ \forall \alpha[F(\alpha) = 1 \leftrightarrow\ &[\alpha = (t^\varphi_\nu = t^\psi_\nu)^\wedge \wedge F((\forall^\nu v)(\varphi(v) \leftrightarrow \psi(v))^\wedge) = 1] \\
&\vee [\alpha = (t^\varphi_\nu = t^\psi_\tau)^\wedge \wedge \nu < \tau \wedge F((\forall^\tau v)(\psi(v) \leftrightarrow v \in l_\nu \wedge \varphi(v))^\wedge) = 1] \\
&\vee [\alpha = (t^\psi_\tau = t^\varphi_\nu)^\wedge \wedge \nu < \tau \wedge F((\forall^\tau v)(\psi(v) \leftrightarrow v \in l_\nu \wedge \varphi(v))^\wedge) = 1] \\
&\vee [\alpha = (t^\varphi_\nu \in t^\psi_\nu)^\wedge \wedge F((\exists^\nu v)(\psi(v) \wedge (\forall^\nu w)(w \in v \leftrightarrow \varphi(w)))^\wedge) = 1] \\
&\vee [\alpha = (t^\varphi_\nu \in t^\psi_\tau)^\wedge \wedge \nu < \tau \wedge F((\exists^\tau v)(\psi(v) \\
&\qquad \wedge (\forall^\tau w)(w \in v \leftrightarrow w \in l_\nu \wedge \varphi(w)))^\wedge) = 1] \\
&\vee [\alpha = (t^\varphi_\tau \in t^\psi_\nu)^\wedge \wedge \nu < \tau \wedge F((\exists^\nu v)(\psi(v) \\
&\qquad \wedge (\forall^\tau w)(w \in v \leftrightarrow \varphi(w)))^\wedge) = 1] \\
&\vee [\alpha = (\varphi \wedge \psi)^\wedge \wedge F(\hat{\varphi}) = 1 \wedge F(\hat{\psi}) = 1] \\
&\vee [\alpha = (\neg \varphi)^\wedge \wedge F(\hat{\varphi}) = 0] \\
&\vee [\alpha = ((\exists^\nu v_n)\varphi)^\wedge \wedge G(\alpha) \neq *]] \\
\wedge\ \forall \alpha[F(\alpha) = 0 \leftrightarrow &\ldots\ldots\ldots];
\end{aligned}
$$

$$
\begin{aligned}
G(*) = * \wedge \forall\alpha\forall\beta[G(\alpha) = \beta \leftrightarrow \alpha = (\exists^\nu v \varphi(v))^\wedge \wedge \beta = (t^\psi_\tau)^\wedge \wedge \tau < \nu \\
\wedge F(\varphi(t^\psi_\tau)^\wedge) = 1 \wedge (\forall \gamma < \beta)(\gamma = t^\theta_\iota \wedge \iota < \nu \rightarrow F(\varphi(t^\theta_\iota)^\wedge) = 0].
\end{aligned}
$$

Using 1.4, it is easily seen that the above definitions are sound. The function H has the following definition:

$$
\begin{aligned}
[H(*) = *] \wedge \forall\alpha\forall\beta[H(\alpha) = \beta \leftrightarrow \alpha = \varphi(v)^\wedge \wedge F(\varphi(o_\beta)^\wedge) = 1 \\
\wedge (\forall \gamma < \beta)(F(\varphi(o_\gamma)^\wedge) = 0)].
\end{aligned}
$$

Roughly speaking, T will consist of T_1 together with the above definitions of F, G, H. That T will be $(\alpha + 1)$-categorical for all $\alpha \geqslant \omega . \omega$ and that M^*_α will be a model of T for all $\alpha \geqslant \omega . \omega$ is clear. What we need to check, though, is that it is possible to write the above definitions as universal sentences of $S^\#$.

The appearance of the constants 0, 1 causes no problems, since the functions k_0, k_1 yield these values for all $x \neq *$. And the functions of N also enable us to handle the passage from formulas to ordinals and back again. For the passage from formulas to ordinals this is clear. For the reverse passage, considering the definition of F as an example, we may commence the $F(\alpha) = 1$ clause thus:

$$F(\alpha) = 1 \leftrightarrow \bigvee_{n<\omega} [\alpha = P_n(Q_1(\alpha), \ldots, Q_n(\alpha)) \wedge \ldots].$$

For each α which denotes a formula there will be a unique n such that $\alpha = P_n(Q_1(\alpha), \ldots, Q_n(\alpha))$, and $(Q_1(\alpha), \ldots, Q_n(\alpha))$ will be $\ulcorner\alpha\urcorner$, so the relevant disjunct in the above will deal with $\ulcorner\alpha\urcorner$. Allied to this is the classification of a formula into its logical type. But there are only a finite number of types, and so we may form a disjunction over these. We leave it to the reader to check the fine details now, and declare the lemma proved. □

That completes the proof that M is a machine.

2. *The Combinatorial Principle* □

We use the machine constructed above in order to prove the combinatorial principle □ from $V = L$. More precisely, we prove the following theorem:

2.1 Theorem. *Assume $V = L$. Let A be a class of limit ordinals. Then there is a class $E \subseteq A$ such that:*

(i) *if $\kappa > \omega$ is regular and $A \cap \kappa$ is stationary in κ, then $E \cap \kappa$ is stationary in κ;*

(ii) *□(E) holds.* □

We recall that □(E) says that there is a sequence $(C_\alpha \mid \alpha \in S)$, where S is the class of all singular limit ordinals, such that: S

(i) C_α is a closed unbounded subset of α;

(ii) $\mathrm{otp}(C_\alpha) < \alpha$;

(iii) if $\bar{\alpha} < \alpha$ is a limit point of C_α, then $\bar{\alpha} \in S$, $\bar{\alpha} \notin E$, and $C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$.

Our proof of 2.1 using the machine M will be closely modelled upon the fine structure proof in VI.6. In one aspect the machine proof is better: it is uniform on α, avoiding the necessity of looking separately at different cases, which was a feature of the fine structure proof. With the machine, the analogue of the most difficult case in VI.6 works in all cases.

We assume $V = L$ from now on. M denotes the machine constructed in section 1. When we use the machine, a finite set of ordinals will often be referred to as a *parameter*. Since we may identify finite sets of ordinals with members of $\mathrm{On}^{<\omega}$ in a canonical manner, the well-ordering $<^*$ of $\mathrm{On}^{<\omega}$ gives us a well-ordering of all parameters.

Let α be a limit ordinal, $\beta \geqslant \alpha$. We say that α is *singular at* β if there is a parameter $p \subseteq \beta$ and a $\gamma < \alpha$ such that $M_\beta[\gamma \cup p] \cap \alpha$ is cofinal in α.

2.2 Lemma. *If $\alpha \in S$ there is a $\beta < \alpha^+$ such that α is singular at β.*

Proof. Let $\gamma = \mathrm{cf}(\alpha)$, and let f be the $<_L$-least map from γ cofinally into α. Let $\delta < \alpha^+$ be such that $f \in L_\delta$. Set $p = \{\alpha, \delta\}$.

For $\xi < \gamma$, $f(\xi)$ is the unique ordinal ζ such that $\vDash_{L_\delta}$ "ζ is the value at ξ of the $<_L$-least map from $\mathrm{cf}(\alpha)$ cofinally into α". So for each $\xi < \alpha$, $f(\xi)$ is *-definable from $\{\xi, \alpha, \delta\}$, and by the Skolem Property there is a $\beta(\xi) < [\max(f(\xi), \xi, \alpha, \delta\}]^+$ such that $f(\xi) \in M_{\beta(\xi)}[\{\alpha, \delta, \xi\}]$. Let $\beta = \sup_{\xi<\gamma}\beta(\xi)$. Since $\beta(\xi) < \alpha^+$ for all $\xi < \gamma$ and $\gamma < \alpha$, we have $\beta < \alpha^+$. Also, for each $\xi < \gamma$, $f(\xi) \in M_\beta[\{\xi, \alpha, \delta\}]$. Hence $\operatorname{ran}(f) \subseteq M_\beta[\gamma \cup p]$. Thus $M_\beta[\gamma \cup p] \cap \alpha$ is cofinal in α, and we are done. □

Let α be a limit ordinal, $\beta \geqslant \alpha$. Let us say that α is *semi-singular at* β iff there is a parameter $p \subseteq \beta$ such that whenever $p \subseteq X \lhd M_\beta$ and $X \cap \alpha$ is transitive, then $X \cap \alpha = \alpha$.

2.3 Lemma.

(i) *If α is singular at β, then α is semi-singular at β.*

(ii) *If* $\mathrm{cf}(\alpha) > \omega$ *and α is semi-singular at β (with parameter p), then α is singular at β (with parameter p).*

Proof. (i) Let $p \subseteq \beta$ be a parameter and let $\gamma < \alpha$ be such that $M_\beta[\gamma \cup p] \cap \alpha$ is cofinal in α. Set $p' = p \cup \{\gamma\}$. We show that α is semi-singular at β with parameter p'. Let $p' \subseteq X \lhd M_\beta$ be such that $X \cap \alpha$ is transitive. Since $\gamma \in X$, we have $\gamma \subseteq X$. So as $X \lhd M_\beta$, we have $M_\beta[\gamma \cup p] \subseteq X$. Hence $X \cap \alpha$ is confinal in α. Thus as $X \cap \alpha$ is transitive, we must have $X \cap \alpha = \alpha$.

(ii) Let α be semi-singular at β with parameter p. By recursion, define substructures $X_n \lhd M_\beta$ and ordinals $\alpha_n \leqslant \alpha$ is follows.

$$X_0 = M_\beta[p]; \qquad \alpha_0 = \sup(X_0 \cap \alpha);$$

$$X_{n+1} = M_\beta[\alpha_n \cup p]; \qquad \alpha_{n+1} = \sup(X_{n+1} \cap \alpha).$$

Set

$$X_\omega = \bigcup_{n<\omega} X_n, \qquad \alpha_\omega = \sup_{n<\omega}\alpha_n.$$

Clearly, $X_\omega \lhd M_\beta$ and $X_\omega \cap \alpha = \alpha_\omega$. Since $p \subseteq X_\omega$ therefore, we must have $X_\omega \cap \alpha = \alpha$, i.e. $\alpha_\omega = \alpha$. Since $\mathrm{cf}(\alpha) > \omega$, it follows that $\alpha_n = \alpha$ for some $n < \omega$. Let n be the least such. If $n = 0$, then $M_\beta[0 \cup p] \cap \alpha$ is cofinal in α, and if $n > 0$, then $\alpha_{n+1} < \alpha_n = \alpha$ and $M_\beta[\alpha_{n-1} \cup p] \cap \alpha$ is cofinal in α, so in either case α is singular at β (with parameter p). □

Let $\gamma < \alpha \leqslant \beta$. Let $p \subseteq \beta$ be a parameter. We shall say that (γ, p) *jumps below* α *in* M_β iff $M_\beta[\gamma \cup p] \cap \alpha \neq \gamma$.

2.4 Lemma. *Let $\alpha \in S$, $\beta \geqslant \alpha$, $p \subseteq \beta$ a parameter. The following are equivalent*:

(i) *α is semi-singular at β with parameter p;*

(ii) *for all $\gamma < \alpha$, (γ, p) jumps below α in M_β.*

Proof. (i) → (ii). Let $\gamma < \alpha$ and set $X = M_\beta[\gamma \cup p]$. Suppose $X \cap \alpha = \gamma$. Then since $p \subseteq X \lhd M_\beta$ and γ is transitive, we have $\gamma = X \cap \alpha = \alpha$, which is absurd. Hence $X \cap \alpha \neq \gamma$, proving (ii).

(ii) → (i). Let $p \subseteq X \lhd M_\beta$ be such that $X \cap \alpha$ is transitive. Set $\gamma = X \cap \alpha$. Suppose $\gamma < \alpha$. Then $M_\beta[\gamma \cup p] \cap \alpha \neq \gamma$. But $\gamma \cup p \subseteq X \lhd M_\beta$, so $M_\beta[\gamma \cup p] \subseteq X$, and we have $X \cap \alpha \neq \gamma$, a contradiction. Hence $\gamma = \alpha$, proving (i). □

The class E of 2.1 consists of all ordinals $\alpha \in A$ such that for some $\gamma \geqslant \alpha$ and some parameter $q \subseteq \gamma$: E

(i) α is not semi-singular at γ;

(ii) if $\bar{\alpha} \in A \cap \alpha$, then either $(\bar{\alpha}, q)$ jumps below α in M_γ or else $\bar{\alpha}$ is semi-singular at γ with a parameter in $M_\gamma[\bar{\alpha} \cup q]$.

2.5 Lemma. *Let $\kappa > \omega$ be a regular cardinal, and assume that $A \cap \kappa$ is stationary in κ. Then $E \cap \kappa$ is stationary in κ.*

Proof. Let C be a club subset of κ. We show that $C \cap E \neq \emptyset$. Let $f: \kappa \to \kappa$ be defined by C

$$f(\alpha) = \text{the least element of } C \text{ greater than } \alpha.$$ f

Then $f \in L_{\kappa^+}$ so for some ordinal $\theta < \kappa^+$, f is the θ-th element of L in the well-ordering $<_L$. Let $\varrho < \kappa^+$ be such that $\varrho > \theta$ and (say) L_ϱ is a model of ZF^-. By absoluteness, θ ϱ

$$f = [\text{the } \theta\text{-th element of } L \text{ in the ordering } <_L]^{L_\varrho}.$$

Let α be, if possible, the least ordinal in $A \cap \kappa$ such that α

(i) $M_{\kappa^+}[\alpha \cup \{\theta, \varrho\}] \cap \kappa = \alpha$;

(ii) if $p \subseteq M_{\kappa^+}[\alpha \cup \{\theta, \varrho\}]$ is a parameter, then α is not semi-singular at κ^+ with parameter p.

We show that α is well-defined here. Define a chain

$$X_0 \lhd X_1 \lhd \ldots \lhd X_\nu \lhd \ldots \lhd M_{\kappa^+} \ (\nu < \kappa)$$

by recursion, as follows. Let $X_0 \lhd M_{\kappa^+}$ be such that $\theta, \varrho \in X_0$ and $\alpha_0 = X_0 \cap \kappa \in \kappa$. If $X_\nu \lhd M_{\kappa^+}$ is defined and $\alpha_\nu = X_\nu \cap \kappa \in \kappa$, let $X_{\nu+1} \lhd M_{\kappa^+}$ be such that $\alpha_\nu \in \alpha_{\nu+1} = X_{\nu+1} \cap \kappa \in \kappa$. If $\lim(\nu)$ and X_η is defined for all $\eta < \nu$ and such that $\alpha_\eta = X_\eta \cap \kappa \in \kappa$ for all $\eta < \nu$, let $X_\nu = \bigcup_{\eta<\nu} X_\eta$, $\alpha_\nu = \sup_{\eta<\nu} \alpha_\eta$. Since $\kappa > \omega$ is regular, this definition causes no difficulty. Since $\{\alpha_\nu \mid \nu < \kappa\}$ is club in κ, we can find a $\nu < \kappa$ such that $\lim(\nu)$ and $\alpha_\nu \in A \cap \kappa$. X_ν, α_ν

Since $\alpha_\eta \cup \{\theta, \varrho\} \subseteq X_\eta \lhd X_\nu \lhd M_{\kappa^+}$ for all $\eta < \nu$, we have

$$M_{\kappa^+}[\alpha_\nu \cup \{\theta, \varrho\}] \subseteq X_\nu.$$

But $X_\nu \cap \kappa = \alpha_\nu$. Thus

$$M_{\kappa^+}[\alpha_\nu \cup \{\theta, \varrho\}] \cap \kappa = \alpha_\nu.$$

Thus α_ν satisfies condition (i) above.

Now suppose that $p \subseteq M_{\kappa^+}[\alpha_\nu \cup \{\theta, \varrho\}]$ is a parameter. For some $\eta < \nu$, $p \subseteq M_{\kappa^+}[\alpha_\eta \cup \{\theta, \varrho\}]$. Thus $p \subseteq X_\eta \lhd M_{\kappa^+}$, and

$$X_\eta \cap \alpha_\nu = X_\eta \cap \kappa \cap \alpha_\nu = \alpha_\eta < \alpha_\nu.$$

Thus α_ν is not semi-singular at κ^+ with parameter p. This shows that α_ν satisfies condition (ii) above.

It follows that α is well-defined, and indeed that $\alpha \leqslant \alpha_\nu$.

Now let

π, γ
$$\pi: M_\gamma \cong M_{\kappa^+}[\alpha \cup \{\theta, \varrho\}].$$

Then

$$\pi \restriction \alpha = \mathrm{id} \restriction \alpha \quad \text{and} \quad \pi(\alpha) = \kappa.$$

$\bar{\theta}, \bar{\varrho}, q$ Let $\bar{\theta} = \pi^{-1}(\theta)$, $\bar{\varrho} = \pi^{-1}(\varrho)$, and set $q = \{\bar{\theta}, \bar{\varrho}\}$. We show that γ, q satisfy the definition of E for α. Notice that $\gamma = M_\gamma[\alpha \cup q]$.

Suppose that α were semi-singular at γ. Then for some parameter $p \subseteq \gamma$, α will be semi-singular at γ with parameter p. Let $\delta < \alpha$. Then by 2.4, $M_\gamma[\delta \cup p] \cap \alpha \neq \delta$. Applying π, and using the fact that $\pi \restriction \alpha = \mathrm{id} \restriction \alpha$, we have $M_{\kappa^+}[\delta \cup \pi(p)] \cap \alpha \neq \delta$. Thus, whenever $\delta < \alpha$, $(\delta, \pi(p))$ jumps below α in M_{κ^+}. So by 2.4, α is semi-singular at κ^+ with parameter $\pi(p)$. But $\pi(p) \subseteq \pi''\gamma = M_{\kappa^+}[\alpha \cup \{\theta, \varrho\}]$, so this contradicts the choice of α. Hence α is not semi-singular at γ.

Now let $\bar{\alpha} \in A \cap \alpha$ be such that $(\bar{\alpha}, q)$ does not jump below α in M_γ. Thus

$$M_\gamma[\bar{\alpha} \cup q] \cap \alpha = \bar{\alpha}.$$

Applying π and using the fact that $\pi \restriction \alpha = \mathrm{id} \restriction \alpha$, we get

$$M_{\kappa^+}[\bar{\alpha} \cup \{\theta, \varrho\}] \cap \alpha = \bar{\alpha}.$$

Using property (i) of α we get

$$M_{\kappa^+}[\bar{\alpha} \cup \{\theta, \varrho\}] \cap \kappa = \bar{\alpha}.$$

So by the minimality of α there is a parameter $p \subseteq M_{\kappa^+}[\bar{\alpha} \cup \{\theta, \varrho\}]$ such that $\bar{\alpha}$ is semi-singular at κ^+ with parameter p. Let $\delta < \bar{\alpha}$. By 2.4,

$$M_{\kappa^+}[\delta \cup p] \cap \bar{\alpha} \neq \delta.$$

Applying π^{-1},

$$M_\gamma[\delta \cup \pi^{-1}(p)] \cap \bar{\alpha} \neq \delta.$$

So as $\delta < \bar{\alpha}$ was arbitrary, 2.4 tells us that $\bar{\alpha}$ is semi-singular at γ with parameter $\pi^{-1}(p)$. Since $\pi^{-1}(p) \subseteq M_\gamma[\bar{\alpha} \cup q]$, this completes the proof that $\alpha \in E$.

We obtain the contradiction which proves the lemma by showing that $\alpha \in C$. Let $\nu < \alpha$. Then $f(\alpha)$ is definable from ν, θ in L_ϱ. Hence $f(\nu)$ is *-definable from $\{\nu, \theta, \varrho\}$. So by the Skolem Property for M, $f(\nu) \in M_{\kappa^+}[\alpha \cup \{\theta, \varrho\}] \cap \kappa = \alpha$. Hence $f''\alpha \subseteq \alpha$. Thus by definition of f, α is a limit point of C. Hence $\alpha \in C$, and we are done. $\square$

As a first step towards the construction of a $\square(E)$-sequence, we construct a sequence $(C_\alpha \mid \alpha \in S)$ such that:

(i) C_α is a club subset of α;

(ii) if $\bar{\alpha}$ is a limit point of C_α, then $\bar{\alpha} \in S$ and $C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$.

Let $\alpha \in S$. By 2.2 and 2.3, we may define $\beta(\alpha)$ as the least ordinal β such that α is semi-singular at β. Let $p(\alpha)$ be the $<^*$-least parameter p such that α is semi-singular at $\beta(\alpha)$ with parameter p. $\beta(\alpha)$ $p(\alpha)$

2.6 Lemma. *$\beta(\alpha)$ is a limit ordinal.*

Proof. Let $\beta = \beta(\alpha), p = p(\alpha)$. Suppose that $\beta = \lambda + 1$. By the Finiteness Property for M there is a finite set $H \subseteq \lambda$ such that for any set $A \subseteq \beta$.

$$(*) \qquad M_\beta[A] \subseteq M_\lambda[(A \cap \lambda) \cup H] \cup \{\lambda\}.$$

Set $q = p \cup H$. We show that α is semi-singular at λ with parameter q, thereby contradicting the definition of $\beta(\alpha)$ (which is greater than λ), and hence proving the lemma.

Let $q \subseteq X \lhd M_\lambda$, $X \cap \alpha$ transitive. Set $Y = M_\beta[X]$. Since $\lambda < \beta$, we have

$$M_\lambda[X] \subseteq M_\beta[X] = Y.$$

But by $(*)$,

$$Y = M_\beta[X] \subseteq M_\lambda[X] \cup \{\lambda\}.$$

Hence either $Y = M_\lambda[X]$ or else $Y = M_\lambda[X] \cup \{\lambda\}$. In either case we have

$$Y \cap \alpha = M_\lambda[X] \cap \alpha.$$

But $X \lhd M_\lambda$, so $M_\lambda[X] = X$, and we therefore have $Y \cap \alpha = X \cap \alpha$, so $Y \cap \alpha$ is transitive. But $p \subseteq Y \lhd M_\beta$, so this means that $Y \cap \alpha = \alpha$. Hence $X \cap \alpha = \alpha$, as required. $\square$

2.7 Lemma. *Let $\alpha \in S$, and set $\beta = \beta(\alpha), p = p(\alpha)$. For every $\gamma < \alpha$ there is a $\delta < \beta$ such that (γ, p) jumps below α in M_δ.*

Proof. Let $\gamma < \alpha$. By 2.4, (γ, p) jumps below α in M_β; i.e. $M_\beta[\gamma \cup p] \cap \alpha \neq \gamma$. So for some ordinal $\xi, \gamma < \xi < \alpha$, we have $\xi \in M_\beta[\gamma \cup p]$. Let $\xi_1, \ldots, \xi_n$ be a finite sequence of ordinals such that $\xi_n = \xi$ and for each i, either $\xi_i \in \gamma \cup p$ or else ξ_i is obtained from $\xi_1, \ldots, \xi_{i-1}$ by an application of an M-function. By 2.6, β is a limit ordinal, so we can find a $\delta < \beta$ such that $\delta > \max(p), \ \xi_1, \ldots, \xi_n$. Clearly, $\xi \in M_\delta[\xi_1, \ldots, \xi_n]$, so $M_\delta[\gamma \cup p] \cap \alpha \neq \gamma$, as required. $\square$

2.8 Lemma. *Let $\alpha \in S$, and set $\beta = \beta(\alpha)$, $p = p(\alpha)$. Then $\beta = M_\beta[\alpha \cup p]$.*

Proof. Let $X = M_\beta[\alpha \cup p]$. Since $X \lhd M_\beta$, the Condensation Property for M gives us a unique π and a unique λ such that $\pi\colon X \cong M_\lambda$. Clearly, $\pi \restriction \alpha = \mathrm{id} \restriction \alpha$. Since $X = X[\alpha \cup p]$, we have, applying π and setting $q = \pi''p, \lambda = M_\lambda[\alpha \cup q]$. But

$\lambda \leqslant \beta$ and by an easy isomorphism argument, α is semi-singular at λ, so $\lambda = \beta$. Again, the same easy isomorphism argument shows that α is semi-singular at λ with parameter q, so as $q \leq^* p$ we have $q = p$. Thus $\beta = M_\beta[\alpha \cup p]$, as stated. $\square$

2.9 Lemma. *Let $\alpha \in S$, $\mathrm{cf}(\alpha) > \omega$, and set $\beta = \beta(\alpha)$, $p = p(\alpha)$. Then for some $\theta < \alpha$, $M_\beta[\theta \cup p]$ is cofinal in β.*

Proof. By 2.3(ii) there is a $\theta < \alpha$ such that $M_\beta[\theta \cup p] \cap \alpha$ is cofinal in α. We show that $M_\beta[\theta \cup p]$ is cofinal in β. Suppose not, and pick $\delta < \beta$ such that $M_\beta[\theta \cup p] \subseteq \delta$.

Let $\gamma < \alpha$. By 2.4, $M_\beta[\gamma \cup p] \cap \alpha \neq \gamma$. If $\gamma \leqslant \theta$, we have $M_\beta[\gamma \cup p] \subseteq M_\delta[\gamma \cup p]$, so $M_\delta[\gamma \cup p] \cap \alpha \neq \gamma$. And if $\gamma > \theta$, then $M_\beta[\theta \cup p] \subseteq M_\delta[\theta \cup p] \subseteq M_\delta[\gamma \cup p]$, so as $M_\beta[\theta \cup p] \cap \alpha$ is cofinal in α, $M_\delta[\gamma \cup p] \cap \alpha \neq \gamma$. In either case, therefore, (γ, p) jumps below α in M_δ. Since $\gamma < \alpha$ was arbitrary, 2.4 tells us that α is semi-singular at δ, contrary to $\delta < \beta$. $\square$

We are now able to define C_α, $\alpha \in S$ to satisfy conditions (i) and (ii) specified above.

β, p Fix $\alpha \in S$, and set $\beta = \beta(\alpha)$, $p = p(\alpha)$. We define increasing, continuous se-
$\delta(\nu), \alpha_\nu$ quences of ordinals, $(\delta(\nu) \mid \nu \leqslant \lambda)$, $(\alpha_\nu \mid \nu \leqslant \lambda)$, for some limit ordinal $\lambda \leqslant \alpha$, by recursion, as follows.

$$\delta(0) = \alpha_0 = 0;$$

$$\delta(\nu + 1) = \text{the least } \delta \leqslant \beta \text{ such that } \alpha_\nu \cup p \subseteq \delta \text{ and } (\alpha_\nu, p) \text{ jumps below } \alpha \text{ in } M_\delta;$$

$$\alpha_{\nu+1} = \text{the least } \gamma \leqslant \alpha \text{ such that } (\gamma, p) \text{ does not jump below } \alpha \text{ in } M_{\delta(\nu+1)};$$

$$\delta(\eta) = \sup\nolimits_{\nu<\eta} \delta(\nu), \qquad \text{if } \lim(\eta);$$

$$\alpha_\eta = \sup\nolimits_{\nu<\eta} \alpha_\nu, \qquad \text{if } \lim(\eta).$$

λ The definition breaks down when an ordinal λ is reached for which $\delta(\lambda) \geqslant \beta$ or $\alpha_\lambda \geqslant \alpha$.

Note that by continuity, for limit η, (α_η, p) does not jump below α in $M_{\delta(\eta)}$.

We show that $(\delta(\nu) \mid \nu \leqslant \lambda)$ is increasing. Suppose $\delta(\nu + 1) \leqslant \delta(\nu)$. Since (α_ν, p) jumps below α in $M_{\delta(\nu+1)}$, it follows that (α_ν, p) jumps below α in $M_{\delta(\nu)}$. This contradicts the properties of α_ν. Hence $\delta(\nu) < \delta(\nu + 1)$.

Next we show that for limit η, α_η is the least $\gamma \leqslant \alpha$ such that (γ, p) does not jump below α in $M_{\delta(\eta)}$, just as is the case at successor stages. We prove this by induction on η. Suppose $\gamma < \alpha_\eta$ were such that (γ, p) does not jump below α in $M_{\delta(\eta)}$. Pick $\nu < \eta$ such that $\alpha_\nu > \gamma$. Then as $\delta(\nu) < \delta(\eta)$, $M_{\delta(\nu)}[\gamma \cup p] \cap \alpha = \gamma$. By definition if ν is a successor ordinal, and by induction hypothesis if ν is a limit ordinal, this implies that $\alpha_\nu \leqslant \gamma$, contrary to the choice of ν. This proves the result.

We now show that $(\alpha_\nu \mid \nu < \lambda)$ is increasing. Well, we clearly cannot have $\alpha_{\nu+1} = \alpha_\nu$. But if $\alpha_{\nu+1} < \alpha_\nu$, then by the properties of α_ν, $(\alpha_{\nu+1}, p)$ must jump below α in $M_{\delta(\nu)}$, and hence also in $M_{\delta(\nu+1)}$, contrary to the known properties of $\alpha_{\nu+1}$. Hence $\alpha_\nu < \alpha_{\nu+1}$.

Now, if $\delta(\nu) < \beta$ and $\alpha_\nu < \alpha$, then by 2.7, $\delta(\nu + 1) < \beta$, so by 2.4, $\alpha_{\nu+1} < \alpha$. Hence $\lim(\lambda)$. Suppose $\delta(\lambda) < \beta$. Then by 2.4, $\alpha_\lambda < \alpha$, which contradicts the choice of λ. Thus $\delta(\lambda) = \beta$. It follows that $\alpha_\lambda = \alpha$. For if $\alpha_\lambda < \alpha$, then by 2.4, (α_λ, p) jumps below α in $M_{\delta(\lambda)}$, contrary to the properties of α_λ, $\delta(\lambda)$.

We set

$$C_\alpha = \{\alpha_\eta \mid \eta < \lambda\},$$ C_α

a club subset of α. We shall show that if $\bar{\alpha}$ is a limit point of C_α, then $\bar{\alpha} \in S$ and $C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$. But before that, we note for later use that as $(\delta(\nu) \mid \nu < \lambda)$ is strictly increasing and cofinal in β and $(\alpha_\nu \mid \nu < \lambda)$ is strictly increasing and cofinal in α, we have:

2.10 Lemma. $\mathrm{cf}(\beta(\alpha)) = \mathrm{cf}(\alpha)$. □

For $\eta < \lambda$, now, set

$$Y_\eta = M_{\delta(\eta)}[\alpha_\eta \cup p].$$ Y_η

Since $Y_\eta \lhd M_{\delta(\eta)}$, the Condensation Property gives an isomorphism

$$\pi_\eta \colon M_{\psi(\eta)} \cong Y_\eta.$$ π_η, $\psi(\eta)$

Let $\pi_\eta^{-1}(p) = p_\eta$. Notice that $\pi_\eta \restriction \alpha_\eta = \mathrm{id} \restriction \alpha_\eta$. p_η

2.11 Lemma. *Let* $\eta < \lambda$, $\lim(\eta)$. *Then* $\alpha_\eta \in S$ *and* $\beta(\alpha_\eta) = \psi(\eta)$, $p(\alpha_\eta) = p_\eta$.

Proof. We show first that α_η is semi-singular at $\psi(\eta)$ with parameter p_η. By 2.4 it suffices to show that for all $\gamma < \alpha_\eta$, (γ, p_η) jumps below α_η in $M_{\psi(\eta)}$.

Let $\gamma < \alpha_\eta$. By the properties of α_η,

$$M_{\delta(\eta)}[\alpha_\eta \cup p] \cap \alpha = \alpha_\eta,$$
$$M_{\delta(\eta)}[\gamma \cup p] \cap \alpha \neq \gamma.$$

Combining these two facts gives

$$M_{\delta(\eta)}[\gamma \cup p] \cap \alpha_\eta \neq \gamma.$$

But $\alpha_\eta \cup p \subseteq Y_\eta \lhd M_{\delta(\eta)}$. So we get

$$Y_\eta[\gamma \cup p] \cap \alpha_\eta \neq \gamma.$$

Applying π_η^{-1} gives

$$M_{\psi(\eta)}[\gamma \cup p_\eta] \cap \alpha_\eta \neq \gamma,$$

as required.

Since α_η is semi-singular at $\psi(\eta)$, we must have $\alpha_\eta \in S$, of course, so the first part of the lemma is proved.

Suppose that $\beta(\alpha_\eta) \neq \psi(\eta)$. Then by the above, $\beta(\alpha_\eta) < \psi(\eta)$. Since $\delta(\eta) = \sup_{v<\eta} \delta(v)$ and $Y_\eta = \bigcup_{v<\eta} Y_v$, we can pick $v < \eta$ such that $\delta(v) > \pi_\eta(\beta(\alpha_\eta))$ and $\pi_\eta(p(\alpha_\eta)) \subseteq Y_v$.

Now, (α_v, p) does not jump below α in $M_{\delta(v)}$, so

$$M_{\delta(v)}[\alpha_v \cup p] \cap \alpha_\eta = \alpha_v.$$

But $\pi_\eta(p(\alpha_\eta)) \subseteq Y_v = M_{\delta(v)}[\alpha_v \cup p]$, so it follows that

$$M_{\delta(v)}[\alpha_v \cup \pi_\eta(p(\alpha_\eta))] \cap \alpha_\eta = \alpha_v.$$

Thus as $\delta(v) > \pi_\eta(\beta(\alpha_\eta))$,

$$M_{\pi_\eta(\beta(\alpha_\eta))}[\alpha_v \cup \pi_\eta(p(\alpha_\eta))] \cap \alpha_\eta = \alpha_v.$$

But clearly,

$$\alpha_\eta \cup \pi_\eta(p(\alpha_\eta)) \subseteq \pi_\eta{}''M_{\beta(\alpha_\eta)} \lhd M_{\pi_\eta(\beta(\alpha_\eta))}.$$

Hence

$$(\pi_\eta{}''M_{\beta(\alpha_\eta)})\,[\alpha_v \cup \pi_\eta(p(\alpha_\eta))] \cap \alpha_\eta = \alpha_v.$$

Applying π_η^{-1},

$$M_{\beta(\eta)}[\alpha_v \cup p(\alpha_\eta)] \cap \alpha_\eta = \alpha_v.$$

But $\alpha_v < \alpha_\eta$. So by 2.4, α_η is not semi-singular at $\beta(\alpha_\eta)$ with parameter $p(\alpha_\eta)$. This is absurd, of course. Hence $\beta(\alpha_\eta) = \psi(\eta)$. It follows at once that $p(\alpha_\eta) \leqslant^* p_\eta$.

Suppose that $p(\alpha_\eta) <^* p_\eta$. Then $\pi_\eta(p(\alpha_\eta)) <^* p$. So by definition of p, α is not semi-singular at β with parameter $\pi_\eta(p(\alpha_\eta))$. So by 2.4, there is a $\gamma < \alpha$ such that

$$M_\beta[\gamma \cup \pi_\eta(p(\alpha_\eta))] \cap \alpha = \gamma.$$

Suppose first that $\gamma < \alpha_\eta$. By the above, we get

$$M_{\delta(\eta)}[\gamma \cup \pi_\eta(p(\alpha_\eta))] \cap \alpha_\eta = \gamma.$$

So as $\alpha_\eta \cup \pi_\eta(p(\alpha_\eta)) \subseteq Y_\eta \lhd M_{\delta(\eta)}$,

$$Y_\eta[\gamma \cup \pi_\eta(p(\alpha_\eta))] \cap \alpha_\eta = \gamma.$$

Applying π_η^{-1},

$$M_{\psi(\eta)}[\gamma \cup p(\alpha_\eta)] \cap \alpha_\eta = \gamma.$$

But $\psi(\eta) = \beta(\alpha_\eta)$, so by 2.4 we have a contradiction.

Now suppose that $\gamma \geqslant \alpha_\eta$. By 2.8 we have

$$p_\eta \subseteq \psi(\eta) = \beta(\alpha_\eta) = M_{\beta(\alpha_\eta)}[\alpha_\eta \cup p(\alpha_\eta)].$$

Applying π_η,

$$p \subseteq M_{\delta(\eta)}[\alpha_\eta \cup \pi_\eta(p(\alpha_\eta))].$$

Hence

$$p \subseteq M_\beta[\alpha_\eta \cup \pi_\eta(p(\alpha_\eta))].$$

So as $\alpha_\eta \leqslant \gamma$,

$$p \subseteq M_\beta[\gamma \cup \pi_\eta(p(\alpha_\eta))].$$

Thus

$$M_\beta[\gamma \cup p] \subseteq M_\beta[\gamma \cup \pi_\eta(p(\alpha_\eta))].$$

So by choice of γ,

$$M_\beta[\gamma \cup p] \cap \alpha = \gamma.$$

This contradicts 2.4. Hence we must have $p(\alpha_\eta) = p_\eta$, and the proof is complete. □

2.12 Lemma. *Let $\eta < \lambda$, $\lim(\eta)$. Set $\bar{\alpha} = \alpha_\eta$, $\bar{\beta} = \beta(\alpha_\eta)$, $\bar{p} = p(\alpha_\eta)$, and define $\bar{\lambda}$,* $\eta, \bar{\alpha}, \bar{\beta}, \bar{p}, \bar{\lambda}$
$(\bar{\delta}(\nu) \mid \nu < \bar{\lambda})$, $(\bar{\alpha}_\nu \mid \nu < \bar{\lambda})$, $(\bar{Y}_\nu \mid \nu < \bar{\lambda})$ from $\bar{\alpha}$ just as λ, $(\delta(\nu) \mid \nu < \lambda)$, $(\alpha_\nu \mid \nu < \lambda)$, $\bar{\delta}(\nu), \bar{\alpha}_\nu, \bar{Y}_\nu$
$(Y_\nu \mid \nu < \lambda)$ were defined from α above. Let $\pi = \pi_\eta$. Then for all $\nu < \bar{\lambda}$: π

(i) $\bar{\alpha}_\nu = \alpha_\nu$;
(ii) $\pi(\bar{\delta}(\nu + 1)) = \delta(\nu + 1)$;
(iii) $\pi'' M_{\bar{\delta}(\nu)} = M_{\delta(\nu)} \cap Y_\eta$;
(iv) $\pi'' \bar{Y}_\nu = Y_\nu$.

Proof. We first of all prove (i)–(iii) by a simultaneous induction on ν.

By 2.11 we have:

$$\pi : M_{\bar{\beta}} \cong Y_\eta = M_{\delta(\eta)}[\bar{\alpha} \cup p], \qquad \pi(\bar{p}) = p, \qquad \pi \restriction \bar{\alpha} = \mathrm{id} \restriction \bar{\alpha}.$$

Since $\bar{\alpha}_0 = \alpha_0 = 0$ and $\bar{\delta}(0) = \delta(0) = 0$, the first step in the induction is trivial. Limit stages are immediate by continuity. So assume now that the result holds at $\nu < \bar{\lambda}$. Set $\bar{\delta} = \bar{\delta}(\nu + 1)$, $\delta = \pi(\bar{\delta})$. We prove that $\delta = \delta(\nu + 1)$, $\bar{\alpha}_{\nu+1} = \alpha_{\nu+1}$. Our proof of the first of these equalities will also yield $\pi'' M_{\bar{\delta}} = M_\delta \cap Y_\eta$.

Note that by definition of δ.

$$(\pi \restriction \bar{\delta}) : M_{\bar{\delta}} \lhd M_\delta.$$

Applying π to $\bar{\alpha}_\nu \cup \bar{p} \subseteq \bar{\delta}$ gives $\alpha_\nu \cup p \subseteq \delta$. Also, we have

$$M_{\bar{\delta}}[\bar{\alpha}_\nu \cup \bar{p}] \cap \bar{\alpha} \neq \bar{\alpha}_\nu; \qquad M_{\bar{\delta}}[\bar{\alpha}_{\nu+1} \cup \bar{p}] \cap \bar{\alpha} = \bar{\alpha}_{\nu+1},$$

so as $\bar{\alpha}_{\nu+1} < \bar{\alpha} = \alpha_\eta$, we conclude that

$$M_{\bar{\delta}}[\bar{\alpha}_\nu \cup \bar{p}] \cap \alpha_\eta \neq \bar{\alpha}_\nu.$$

Applying $\pi \upharpoonright \bar{\delta}$ gives

$$M_\delta[\alpha_\nu \cup p] \cap \alpha_\eta \neq \alpha_\nu.$$

Thus $\delta(\nu + 1) \leqslant \delta$. We show that $\delta \leqslant \delta(\nu + 1)$ as well. We have

$$M_{\delta(\nu+1)}[\alpha_\nu \cup p] \cap \alpha \neq \alpha_\nu; \qquad M_{\delta(\nu+1)}[\alpha_{\nu+1} \cup p] \cap \alpha = \alpha_{\nu+1},$$

so combining these two results gives

$$M_{\delta(\nu+1)}[\alpha_\nu \cup p] \cap \alpha_{\nu+1} \neq \alpha_\nu.$$

Thus for some $\xi \in \alpha_{\nu+1}$, $\xi > \alpha_\nu$, we have $\xi \in M_{\delta(\nu+1)}[\alpha_\nu \cup p]$. Hence we can find a finite sequence $\xi_1, \ldots, \xi_n$ of ordinals in $\delta(\nu + 1)$ such that $\xi_n = \xi$ and for all $i = 1, \ldots, n$, either $\xi_i \in \alpha_\nu \cup p$ or else ξ_i is the value of some M function at some members of $\{\xi_1, \ldots, \xi_{i-1}\}$. Now,

$$\xi_1, \ldots, \xi_n \in M_{\delta(\nu+1)}[\alpha_\nu \cup p] \subseteq M_{\delta(\eta)}[\alpha_\eta \cup p] = Y_\eta,$$

so we can define $\bar{\xi}_i = \pi^{-1}(\xi_i)$ for $i = 1, \ldots, n$. Since $\alpha_\nu < \xi_n = \xi < \alpha_{\nu+1} < \alpha_\eta$, we have $\alpha_\nu < \bar{\xi}_n = \xi < \alpha_{\nu+1} < \alpha_\eta$. And for each i, either $\bar{\xi}_i \in \alpha_\nu \cup p$ or else $\bar{\xi}_i$ is the value of some M-function at members of $\{\bar{\xi}_1, \ldots, \bar{\xi}_{i-1}\}$. So, if we set $\bar{\varrho} = \max(\bar{\xi}_1, \ldots, \bar{\xi}_n)$, we have $\bar{\xi} \in M_{\bar{\varrho}+1}[\alpha_\nu \cup \bar{p}]$. Hence

$$M_{\bar{\varrho}+1}[\alpha_\nu \cup \bar{p}] \cap \bar{\alpha} \neq \alpha_\nu.$$

Thus by choice of $\bar{\delta}$, $\bar{\delta} \leqslant \bar{\varrho} + 1$. Now set $\varrho = \pi(\bar{\varrho})$. Since $\bar{\varrho} = \max(\bar{\xi}_1, \ldots, \bar{\xi}_n)$, we have $\varrho = \max(\xi_1, \ldots, \xi_n) < \delta(\nu + 1)$. Also,

$$\varrho \in M_{\delta(\nu+1)}[\alpha_\nu \cup p] \subseteq M_{\delta(\eta)}[\alpha_\eta \cup p] = Y_\eta.$$

But the function $J(\gamma) = \gamma + 1$ is an M-function, and $\delta(\eta)$ is a limit ordinal, so it follows that

$$\varrho + 1 \in M_{\delta(\eta)}[\alpha_\eta \cup p] = Y_\eta.$$

Hence $\pi(\bar{\varrho} + 1) = \varrho + 1$. Since $\bar{\delta} \leqslant \bar{\varrho} + 1$, applying π gives $\delta \leqslant \varrho + 1 \leqslant \delta(\nu + 1)$, as required.

We now have $\pi(\bar{\delta}(\nu + 1)) = \delta(\nu + 1)$. It follows at once that

$$\pi'' M_{\bar{\delta}(\nu+1)} = M_{\delta(\nu+1)} \cap Y_\eta.$$

We prove that $\bar{\alpha}_{\nu+1} = \alpha_{\nu+1}$.

By definition,

(i) $M_{\bar{\delta}}[\alpha_{\nu+1} \cup p] \cap \bar{\alpha} = \alpha_{\nu+1}$,

(ii) $\gamma < \alpha_{\nu+1} \rightarrow M_{\bar{\delta}}[\gamma \cup p] \cap \alpha_{\nu+1} \neq \gamma$.

Since $(\pi \restriction \bar{\delta})\colon M_{\bar{\delta}} \lhd M_\delta$ and $\alpha_{\nu+1} \cup p \subseteq M_\delta \cap Y_\eta = \pi'' M_{\bar{\delta}}$, (i) and (ii) give

(i)′ $M_{\bar{\delta}}[\alpha_{\nu+1} \cup \bar{p}] \cap \bar{\alpha} = \alpha_{\nu+1}$,
(ii)′ $\gamma < \alpha_{\nu+1} \to M_{\bar{\delta}}[\gamma \cup \bar{p}] \cap \alpha_{\nu+1} \neq \gamma$.

Hence $\bar{\alpha}_{\nu+1} = \alpha_{\nu+1}$.

That completes the proof of (i)–(iii). We are left with (iv).

Using (iii) we have

$$\pi'' \bar{Y}_\nu = \pi''(M_{\bar{\delta}(\nu)}[\bar{\alpha}_\nu \cup \bar{p}]) = (M_{\delta(\nu)} \cap Y_\eta)\,[\alpha_\nu \cup p].$$

But

$$M_{\delta(\nu)}[\alpha_\nu \cup p] = Y_\nu \subseteq Y_\eta,$$

so we have

$$(M_{\delta(\nu)} \cap Y_\eta)\,[\alpha_\nu \cup p] = M_{\delta(\nu)}[\alpha_\nu \cup p] = Y_\nu.$$

Thus $\pi'' \bar{Y}_\nu = Y_\nu$, proving (iv). □

2.13 Corollary. *Let $\alpha \in S$. If $\bar{\alpha}$ is a limit point of C_α, then $\bar{\alpha} \in S$ and $C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$.*

Proof. Using the above notation, $C_\alpha = \{\alpha_\nu \mid \nu < \lambda\}$ and for some limit ordinal η, $\bar{\alpha} = \alpha_\eta$. By 2.12, $C_{\bar{\alpha}} = \{\alpha_\nu \mid \nu < \bar{\lambda}\}$. But $(\alpha_\nu \mid \nu < \lambda)$ is strictly increasing and $\sup_{\nu < \bar{\lambda}} \alpha_\nu = \bar{\alpha} = \sup_{\nu < \eta} \alpha_\nu$, so $\bar{\lambda} = \eta$ and $C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$. □

Our next step in obtaining $\square(E)$ is to thin down the sets C_α to sets C'_α such that:

(i) C'_α is a closed subset of α;
(ii) if $\mathrm{cf}(\alpha) > \omega$, then C'_α is unbounded in α;
(iii) if $\bar{\alpha}$ is a limit point of C'_α, then $\bar{\alpha} \in S$ and $C'_{\bar{\alpha}} = \bar{\alpha} \cap C'_\alpha$;
(iv) $\mathrm{otp}(C'_\alpha) < \alpha$.

It will then be a fairly easy matter to turn $(C'_\alpha \mid \alpha \in S)$ into a $\square(E)$-sequence.

Let $\alpha \in S$, and set $\beta = \beta(\alpha)$, $p = p(\alpha)$. Define $\lambda, (\delta(\nu) \mid \nu < \lambda)$, $(\alpha_\nu \mid \nu < \lambda)$, β, p $(Y_\eta \mid \eta < \lambda)$, $(\pi_\eta \mid \eta < \lambda)$, $(\psi(\eta) \mid \eta < \lambda)$, $(p_\eta \mid \eta < \lambda)$ as before.

2.14 Lemma. *Let $\eta_1 < \eta_2 < \lambda$ be limit ordinals. Then $\sup Y_{\eta_1} < \sup Y_{\eta_2}$.*

Proof. Since $\alpha_{\eta_1} < \alpha_{\eta_2}$, 2.4 gives

$$M_{\beta(\alpha_{\eta_2})}[\alpha_{\eta_1} \cup p(\alpha_{\eta_2})] \cap \alpha_{\eta_2} \neq \alpha_{\eta_1}.$$

Applying $\pi_{\eta_2}\colon M_{\beta(\alpha_{\eta_2})} \cong Y_{\eta_2}$ and using 2.11,

$$Y_{\eta_2}[\alpha_{\eta_1} \cup p] \cap \alpha_{\eta_2} \neq \alpha_{\eta_1}.$$

Hence

$$M_{\sup Y_{\eta_2}}[\alpha_{\eta_1} \cup p] \cap \alpha_{\eta_2} \neq \alpha_{\eta_1}.$$

But

$$M_{\delta(\eta_1)}[\alpha_{\eta_1} \cup p] \cap \alpha = \alpha_{\eta_1}.$$

So as $\sup Y_{\eta_1} \leqslant \delta(\eta_1)$, we have

$$M_{\sup Y_{\eta_1}}[\alpha_{\eta_1} \cup p] \cap \alpha_{\eta_2} = \alpha_{\eta_1}.$$

Thus $\sup Y_{\eta_1} < \sup Y_{\eta_2}$. □

γ In defining C'_α there are two cases to consider. Let γ be the least ordinal such that $M_\beta[\gamma \cup p]$ is cofinal in β.

Case 1. γ is a limit ordinal.
Set

$$C'_\alpha = \{\alpha_\eta \mid \lim(\eta) \wedge (\exists \xi \leqslant \alpha_\eta)\,[\sup Y_\eta = \sup M_\beta[\xi \cup p]]\}.$$

2.15 Lemma. *C'_α is closed in α.*

Proof. By the continuity of the sequence $(Y_\eta \mid \eta < \lambda)$. □

2.16 Lemma. *Let* $\mathrm{cf}(\alpha) > \omega$. *Then C'_α is unbounded in α.*

Proof. Let $H = \{\sup Y_\eta \mid \lim(\eta)\}$, $K = \{\sup M_\beta[\delta \cup p] \mid \delta < \gamma\}$. Clearly, H and K are club in β. (By 2.10, $\mathrm{cf}(\beta) = \mathrm{cf}(\alpha) > \omega$.) Hence $H \cap K$ is club in β. So we can pick arbitrarily large limit ordinals $\eta < \lambda$ so that $\sup Y_\eta \in K$. For any such η, $\sup Y_\eta = \sup M_\beta[\delta \cup p]$ for some $\delta < \gamma$. But

$$\sup M_\beta[\alpha_\eta \cup p] \geqslant \sup M_{\delta(\eta)}[\alpha_\eta \cup p] \geqslant \sup Y_\eta.$$

Hence we can find such a $\delta \leqslant \alpha_\eta$. Then $\alpha_\eta \in C'_\alpha$. □

2.17 Lemma. $\mathrm{otp}(C'_\alpha) < \alpha$.

Proof. Define $\theta: C'_\alpha \to \mathrm{On}$ by letting $\theta(\alpha_\eta)$ be the least $\xi \leqslant \alpha_\eta$ such that $\sup Y_\eta = \sup M_\beta[\xi \cup p]$. By 2.14, θ is order-preserving. But by definition of γ, $\mathrm{ran}(\theta) \subseteq \gamma$. Hence $\mathrm{otp}(C'_\alpha) \leqslant \gamma$. But by 2.9, $\gamma < \alpha$. □

2.18 Lemma. *Let $\bar\alpha$ be a limit point of C'_α. Then $\bar\alpha \in S$, $\bar\alpha$ falls under Case 1, and $C'_{\bar\alpha} = \bar\alpha \cap C'_\alpha$.*

Proof. Since $C'_\alpha \subseteq C_\alpha$, we know at once that $\bar\alpha \in S$. Let $\bar\gamma$ be least such that $M_{\bar\beta}[\bar\gamma \cup \bar p]$ is cofinal in $\bar\beta$. We must show that $\lim(\bar\gamma)$ and that $C'_{\bar\alpha} = \bar\alpha \cap C'_\alpha$.

Let $\bar\alpha = \alpha_\varrho$. Then ϱ is a limit of limit ordinals $\eta < \varrho$ for which $\alpha_\eta \in C_\alpha$. For each such η there is a least $\xi_\eta \leqslant \alpha_\eta$ such that $\sup Y_\eta = \sup M_\beta[\xi_\eta \cup p]$. Since the sequence $(Y_\eta \mid \eta < \lambda)$ is continuous, taking the supremum over all such η gives

$$\sup Y_\varrho = \sup M_\beta[\xi \cup p],$$

where $\xi = \sup_\eta \xi_\eta$. We show that $\bar\gamma = \xi$. Since $\lim(\xi)$, this proves $\lim(\bar\gamma)$.

Let $\tau = \sup Y_\varrho$. Then since $\tau = \sup M_\beta[\xi \cup p]$, we have

$$M_\beta[\xi \cup p] = M_\tau[\xi \cup p].$$

But $\tau \leqslant \delta(\varrho) < \beta$. Hence

$$M_\beta[\xi \cup p] = M_{\delta(\varrho)}[\xi \cup p].$$

So

$$\tau = \sup M_{\delta(\varrho)}[\xi \cup p].$$

But

$$\xi \cup p \subseteq Y_\varrho \lhd M_{\delta(\varrho)}.$$

So,

$$\tau = \sup Y_\varrho[\xi \cup p].$$

Thus

$$\sup Y_\varrho = \sup Y_\varrho[\xi \cup p].$$

Applying π_ϱ^{-1},

$$\bar{\beta} = \sup M_{\bar{\beta}}[\xi \cup \bar{p}].$$

Hence $\bar{\gamma} \leqslant \xi$. Suppose that $\bar{\gamma} < \xi$. Then for some η, $\bar{\gamma} < \xi_\eta$. Now,

$$\sup M_{\bar{\beta}}[\bar{\gamma} \cup \bar{p}] = \bar{\beta}.$$

So

$$\sup M_{\bar{\beta}}[\xi_\eta \cup \bar{p}] = \bar{\beta}.$$

Applying π_ϱ,

$$\sup Y_\varrho[\xi_\eta \cup p] = \sup Y_\varrho.$$

But

$$Y_\varrho[\xi_\eta \cup p] = M_{\delta(\varrho)}[\xi_\eta \cup p].$$

Hence

$$\sup M_{\delta(\varrho)}[\xi_\eta \cup p] = \sup Y_\varrho.$$

Thus

$$\sup M_\beta[\xi_\eta \cup p] \geqslant \sup Y_\varrho.$$

But

$$\sup M_\beta[\xi_\eta \cup p] = \sup Y_\eta < \sup Y_\varrho,$$

so we have a contradiction. Hence $\bar{\gamma} = \xi$.

Since $\bar{\alpha}$ is a limit point of C_α, we know that $C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$. So

$$C_{\bar{\alpha}} = \{\alpha_\eta \mid \eta < \varrho\}.$$

Hence,

$$C'_{\bar{\alpha}} = \{\alpha_\eta \mid \lim(\eta) \wedge \eta < \varrho \wedge (\exists \xi \leqslant \alpha_\eta)\,[\sup \bar{Y}_\eta = \sup M_{\bar{\beta}}[\xi \cup \bar{p}]]\}.$$

Using 2.12(iv) we get at once,

$$C'_{\bar{\alpha}} = \pi_\varrho{}''C'_{\bar{\alpha}} = \bar{\alpha} \cap C'_\alpha. \quad \square$$

That completes the construction and study of C'_α in Case 1.

Case 2. Otherwise.

We commence by defining a descending sequence of ordinals $\eta_1, \ldots, \eta_n$ for some n. First let η be least ($< \alpha$) such that $\sup M_\beta[\eta \cup p] = \beta$. Since we are not in Case 1, η is a successor ordinal. Set $\eta_1 = \eta - 1$. Thus

$$\varphi_1 = \sup M_\beta[\eta_1 \cup p] < \beta.$$

Now suppose that $\eta_1, \ldots, \eta_{i-1}$ are defined, where $i > 1$. Let η be least ($< \alpha$) such that $\sup M_\beta[\eta \cup p \cup \{\eta_1, \ldots, \eta_{i-1}\}] = \beta$. If $\lim(\eta)$, then $n = i - 1$ and the definition stops. Otherwise set $\eta_i = \eta - 1$. Then

$$\varphi_i = \sup M_\beta[\eta_i \cup p \cup \{\eta_1, \ldots, \eta_{i-1}\}] < \beta.$$

Since $\eta_1 > \eta_2 > \eta_3 > \ldots$, the definition stops after finitely many steps. We set

$$q = q(\alpha) = \{\eta_1, \ldots, \eta_n\}, \qquad \varphi = \varphi(\alpha) = \max(\varphi_1, \ldots, \varphi_n).$$

Set

$$C'_\alpha = \{\alpha_\eta \mid \lim(\eta) \wedge q \subseteq \alpha_2 \wedge \sup Y_\eta > \varphi$$
$$\wedge (\exists \xi \leqslant \alpha_\eta)[\sup Y_\eta = \sup M_\beta[\xi \cup p \cup q]]\}.$$

Since we shall have no further need to refer to the γ of Case 1, we now define γ to be the least ordinal such that $M_\beta[\gamma \cup p \cup q]$ is cofinal in β. By definition of q, we have $\lim(\gamma)$.

2.19 Lemma.

(i) *C'_α is closed in α;*

(ii) *if* $\mathrm{cf}(\alpha) > \omega$, *then C'_α is unbounded in α;*

(iii) $\mathrm{otp}(C'_\alpha) < \alpha$.

Proof. Just replace p by $p \cup q$ in the proof of 2.15, 2.16, and 2.17, (also, γ has a new meaning now of course.) □

2.20 Lemma. *Let $\bar{\alpha}$ be a limit point of C'_α, say $\bar{\alpha} = \alpha_\varrho$. Then:*

(i) *$\bar{\alpha}$ falls under Case 2;*

(ii) $\pi_\varrho(q(\bar{\alpha})) = q$ *(i.e. $q(\bar{\alpha}) = q$);*

(iii) $\sup[\pi_\varrho{}''\varphi(\bar{\alpha})] = \varphi(\alpha)$;

(iv) $C'_{\bar{\alpha}} = \bar{\alpha} \cap C'_\alpha$.

Proof. We prove (i)–(iii); (iv) then follows easily, much as in 2.18. In fact (iii) itself follows from (i) and (ii) as we now prove. It suffices to show that for each $i = 1, \ldots, n$,

$$\sup[\pi_\varrho{}''\sup M_{\bar{\beta}}[\bar{\eta}_i \cup \bar{p} \cup \{\bar{\eta}_1, \ldots, \bar{\eta}_{i-1}\}]] = \sup M_\beta[\eta_i \cup p \cup \{\eta_1, \ldots, \eta_{i-1}\}],$$

i.e.

$$\sup \pi_\varrho{}'' M_{\bar{\beta}}[\bar{\eta}_i \cup \bar{p} \cup \{\bar{\eta}_1, \ldots, \bar{\eta}_{i-1}\}] = \sup M_\beta[\eta_i \cup p \cup \{\eta_1, \ldots, \eta_{i-1}\}].$$

In fact we prove that

$$\pi_\varrho{}''M_{\bar\beta}[\bar\eta_i \cup \bar p \cup \{\bar\eta_1, \ldots, \bar\eta_{i-1}\}] = M_\beta[\eta_i \cup p \cup \{\eta_1, \ldots, \eta_{i-1}\}].$$

We have

$$\begin{aligned}\pi_\varrho{}''M_{\bar\beta}[\bar\eta_i \cup \bar p \cup \{\bar\eta_1, \ldots, \bar\eta_{i-1}\}] &= Y_\varrho[\eta_i \cup p \cup \{\eta_1, \ldots, \eta_{i-1}\}] \\ &= M_{\delta(\varrho)}[\eta_i \cup p \cup \{\eta_1, \ldots, \eta_{i-1}\}],\end{aligned}$$

by definition of Y_ϱ. But $\delta(\varrho) \geqslant \sup Y_\varrho > \varphi$, so by definition of φ,

$$\delta(\varrho) > \sup M_\beta[\eta_i \cup p \cup \{\eta_1, \ldots, \eta_{i-1}\}].$$

Hence

$$M_\beta[\eta_i \cup p \cup \{\eta_1, \ldots, \eta_{i-1}\}] = M_{\delta(\varrho)}[\eta_i \cup p \cup \{\eta_1, \ldots, \eta_{i-1}\}].$$

That proves (iii), assuming (i) and (ii). We must therefore prove (i) and (ii) to be done. In fact the proof of (i) is contained in the proof of (ii) so we simply concentrate on (ii). We prove by induction on i that for each $i = 1, \ldots, n$, $\bar\eta_i$ is defined and $\bar\eta_i = \eta_i$, and that if η is least such that $M_{\bar\beta}[\eta \cup p \cup \{\bar\eta_1, \ldots, \bar\eta_n\}]$ is cofinal in $\bar\beta$, then $\lim(\eta)$.

Suppose we have proved that for all $j = 1, \ldots, i-1$, $\bar\eta_j$ is defined and $\bar\eta_j = \eta_j$. Since $\sup Y_\varrho > \varphi_i$, we have

$$\begin{aligned}\sup Y_\varrho &> \sup M_\beta[\eta_i \cup p \cup \{\eta_1, \ldots, \eta_{i-1}\}] \\ &\geqslant \sup M_{\delta(\varrho)}[\eta_i \cup p \cup \{\eta_1, \ldots, \eta_{i-1}\}] \\ &= \sup Y_\varrho[\eta_i \cup p \cup \{\eta_1, \ldots, \eta_{i-1}\}].\end{aligned}$$

Applying π_ϱ^{-1}, we get

$$\bar\beta > M_{\bar\beta}[\eta_i \cup \bar p \cup \{\bar\eta_1, \ldots, \bar\eta_{i-1}\}].$$

If we can show that $\bar\beta = \sup M_{\bar\beta}[(\eta_i + 1) \cup \bar p \cup \{\bar\eta_1, \ldots, \bar\eta_{i-1}\}]$, then by definition we shall have $\bar\eta_i = \eta_i$.

Since $\bar\alpha = \alpha_\varrho \in C'_\alpha$, there is a $\xi \leqslant \bar\alpha$ such that

$$\sup Y_\varrho = \sup M_\beta[\xi \cup p \cup q].$$

Hence as $\delta(\varrho) \geqslant \sup Y_\varrho$,

$$M_\beta[\xi \cup p \cup q] = M_{\delta(\varrho)}[\xi \cup p \cup q].$$

But $\xi \cup p \cup q \subseteq Y_\varrho \lhd M_{\delta(\varrho)}$, so

$$M_{\delta(\varrho)}[\xi \cup p \cup q] = Y_\varrho[\xi \cup p \cup q].$$

Thus

$$\sup Y_\varrho = \sup Y_\varrho[\xi \cup p \cup q].$$

Applying π_ϱ^{-1},

$$\bar\beta = \sup M_{\bar\beta}[\xi \cup \bar p \cup q].$$

Now,

$$\sup M_\beta[\xi \cup p \cup q] = \sup Y_\varrho \leqslant \delta(\varrho) < \beta,$$

and

$$\sup M_\beta[(\eta_i + 1) \cup p \cup q] = \beta,$$

so we must have $\eta_i + 1 > \xi$. Thus

$$\bar\beta = \sup M_{\bar\beta}[(\eta_i + 1) \cup \bar p \cup q].$$

But $\{\eta_i, \eta_{i+1}, \ldots, \eta_n\} \subseteq \eta_i + 1$. Hence

$$\begin{aligned}\bar\beta &= \sup M_{\bar\beta}[(\eta_i + 1) \cup \bar p \cup \{\eta_1, \ldots, \eta_{i-1}\}]\\ &= \sup M_{\bar\beta}[(\eta_i + 1) \cup \bar p \cup \{\bar\eta_1, \ldots, \bar\eta_{i-1}\}],\end{aligned}$$

as required.

For each limit ordinal $\eta < \varrho$ now, let $\xi_\eta < \alpha_\eta$ be least such that

$$\sup Y_\eta = \sup M_\beta[\xi_\eta \cup p \cup q].$$

By 2.14, if $\xi = \sup_{\eta<\varrho} \xi_\eta$, we have $\lim(\xi)$,

$$\sup Y_\varrho = \sup(\bigcup_{\eta<\varrho} Y_\eta) = \sup M_\beta[\xi \cup p \cup q],$$

and for each $\eta < \varrho$, $\sup M_\beta[\xi_\eta \cup p \cup q] < \sup Y_\varrho$. Since $\sup Y_\varrho \leqslant \delta(\varrho)$, it follows that ξ is the least ordinal such that $\sup M_{\delta(\varrho)}[\xi \cup p \cup q] = \sup Y_\varrho$, and hence that ξ is the least ordinal such that $\sup Y_\varrho[\xi \cup p \cup q] = \sup Y_\varrho$. Applying π_ϱ^{-1}, we see that ξ is the least ordinal such that $\sup M_{\bar\beta}[\xi \cup \bar p \cup q] = \bar\beta$. Since $\lim(\xi)$, this means that the definition of $\bar q$ stopped at stage $n + 1$, so $\bar q = q$. The proof is complete. $\square$

To complete the proof of 2.1 now, we use the sequence $(C'_\alpha \mid \alpha \in S)$ to build a $\square(E)$-sequence. The following lemma sums up what we know about the sets C'_α.

2.21 Lemma.

(i) *C'_α is a closed subset of α;*

(ii) *if $\mathrm{cf}(\alpha) > \omega$, then C'_α is unbounded in α;*

(iii) $\mathrm{otp}(C'_\alpha) < \alpha$;

(iv) *if $\bar\alpha$ is a limit point of C'_α, then $\bar\alpha \in S$ and $C'_{\bar\alpha}$.* $\square$

The following lemma will enable us to avoid the class E on limit points of the final $\square(E)$-sequence.

2.22 Lemma. *If $\alpha \in E$, then $C'_\alpha \cap A$ is bounded in α.*

Proof. Let $\beta = \beta(\alpha)$, $p = p(\alpha)$, and adopt the notation used in the definition of C_α and C'_α. Since $\alpha \in E$ there is a $\gamma \geqslant \alpha$ and a parameter $q \subseteq \gamma$ such that:

(a) α is not semi-singular at γ;

(b) if $\bar{\alpha} \in A \cap \alpha$, then either $(\bar{\alpha}, q)$ jumps below α in M_γ or else $\bar{\alpha}$ is semi-singular at γ with a parameter in $M_\gamma[\bar{\alpha} \cup q]$.

By (a), $\gamma < \beta$. Since $\sup_{\eta<\lambda} \delta(\eta) = \beta$ and $\sup_{\eta<\lambda} \alpha_\eta = \alpha$ and $\beta = M_\beta[\alpha \cup p]$, we can find an ordinal $\eta_0 < \lambda$ such that

$$q \cup \{\gamma\} \subseteq M_{\delta(\eta_0)}[\alpha_{\eta_0} \cup p].$$

Suppose that $C'_\alpha \cap A$ were unbounded in α. Then we could find a limit ordinal $\eta < \lambda$ such that $\eta \geqslant \eta_0$ and $\alpha_\eta \in A$. By the definition of α_η,

$$M_{\delta(\eta)}[\alpha_\eta \cup p] \cap \alpha = \alpha_\eta.$$

So as $q \subseteq M_{\delta(\eta)}[\alpha_\eta \cup p]$,

$$M_{\delta(\eta)}[\alpha_\eta \cup q] \cap \alpha = \alpha_\eta.$$

So as $\gamma < \delta(\eta)$,

$$M_\gamma[\alpha_\eta \cup q] \cap \alpha = \alpha_\eta.$$

Thus by (b) above (with $\bar{\alpha} = \alpha_\eta$), α_η must be semi-singular at γ with some parameter in $M_\gamma[\alpha_\eta \cup q]$. Consider the isomorphism

$$\pi_\eta\colon M_{\psi(\eta)} \cong M_{\delta(\eta)}[\alpha_\eta \cup p].$$

Let $\bar{\gamma} = \pi_\eta^{-1}(\gamma)$, $\bar{q} = \pi_\eta^{-1}(q)$. Using 2.4, we see easily that (since $\pi_\eta^{-1} \upharpoonright \alpha_\eta = \mathrm{id} \upharpoonright \alpha_\eta$) α_η is semi-singular at $\bar{\gamma}$ with a parameter in $M_{\bar{\gamma}}[\alpha_\eta \cup \bar{q}]$. But by 2.11, $\bar{\gamma} < \xi(\eta) = \beta(\alpha_\eta)$, so this is impossible. Hence C'_α must be bounded in α. □

2.23 Corollary. *If $\alpha \in S$ and $\bar{\alpha} < \alpha$ is a limit point of $C'_\alpha \cap A$, then $\bar{\alpha} \notin E$.*

Proof. Let $\bar{\alpha} < \alpha$ be a limit point of $C'_\alpha \cap A$. By 2.21 (iv), $C'_{\bar{\alpha}} = \bar{\alpha} \cap C'_\alpha$. But $\bar{\alpha}$ is a limit point of $C'_\alpha \cap A$ and hence of $C'_{\bar{\alpha}} \cap A$, so $\sup C'_{\bar{\alpha}} \cap A = \bar{\alpha}$. So by 2.22 $\bar{\alpha} \notin E$. □

Now define sets C''_α by;

$$C''_\alpha = \begin{cases} C'_\alpha - \sup(C'_\alpha \cap A), & \text{if } \sup(C'_\alpha \cap A) < \alpha, \\ \text{the closure of } (C'_\alpha \cap A), & \text{if } \sup(C'_\alpha \cap A) = \alpha. \end{cases}$$

Clearly, the sets C''_α have the following properties:

(i) C''_α is a closed subset of α;
(ii) if $\mathrm{cf}(\alpha) > \omega$, then C''_α is unbounded in α;
(iii) $\mathrm{otp}(C''_\alpha) < \alpha$;
(iv) if $\bar{\alpha} \in C''_\alpha$, then $\bar{\alpha} \in S$, $\bar{\alpha} \notin E$, and $C''_{\bar{\alpha}} = \bar{\alpha} \cap C''_\alpha$.

Define sets D_α, $\alpha \in S$, by recursion, thus:

$$D_\alpha = \begin{cases} \bigcup \{D_\gamma \mid \gamma \in C''_\alpha\}, & \text{if } \sup(C''_\alpha) = \alpha; \\ \bigcup \{D_\gamma \mid \gamma \in C''_\alpha\} \cup \{\alpha_n \mid n < \omega\}, & \text{if } \sup(C''_\alpha) < \alpha, \end{cases}$$

where $(\alpha_n \mid n < \omega)$ is any sequence cofinal in α with $\alpha_0 = \sup(C''_\alpha)$.

As in IV.5.1, it is easily seen that $(D_\alpha \mid \alpha \in S)$ is a $\square(E)$-sequence. The proof of 2.1 is complete.

Exercises

1. Using the argument from Chapter V.5 as a model, obtain a machine proof of the Covering Lemma.

2. Obtain machine proofs of the results in Chapter VII concerning trees and large cardinals in L.