Chapter III
ω_1-Trees in L

Tree theory forms a rich and interesting part of combinatorial set theory, having applications in other parts of set theory as well as in other areas of mathematics (in particular, in general topology). We study trees here because tree theory is greatly enhanced by the assumption $V = L$, and affords a good example of the application of the methods of constructibility theory. In this chapter we concentrate on ω_1-trees, and as we shall demonstrate, these arise out of some very basic questions in mathematics. Later chapters deal with generalisations to higher cardinals.

1. The Souslin Problem. ω_1-Trees. Aronszajn Trees

The Souslin Problem has its origin in a classical theorem of Cantor concerning the real line. In order to consider this theorem we need some definitions.

A *densely ordered set* is a linearly ordered set $\langle X, \leqslant \rangle$ such that whenever $x, y \in X$ and $x < y$, there is a $z \in X$ such that $x < z < y$.

An *interval* in a linearly ordered set $\langle X, \leqslant \rangle$ is a subset of X of the form

$$(x, y) = \{z \in X \mid x < z < y\}$$

for some $x, y \in X$, $x < y$. (We call this set the interval *determined by* x and y.)

An *ordered continuum* is a densely ordered set $\langle X, \leqslant \rangle$ such that whenever Y is a subset of an interval of X, there is a least $z \in X$ such that $(\forall y \in Y)(y \leqslant z)$ and a greatest $x \in X$ such that $(\forall y \in Y)(x \leqslant y)$. (We call z the *supremum* of Y, x the *infimum* of Y.)

A linearly ordered set is said to be *open* if it has no end-points.

A subset Y of a densely ordered set $\langle X, \leqslant \rangle$ is said to be *dense* in X if, whenever $x, z \in X$ are such that $x < z$, there is a $y \in Y$ such that $x < y < z$.

Cantor proved that, considered as a linearly ordered set, the real line ($\mathbb{R}$) is characterised, up to isomorphism, by being an open, ordered continuum having a countable dense subset (the rationals). In 1920, M. Souslin asked whether a natural weakening of these conditions still suffices to characterise $\mathbb{R}$.

Let us say that a linearly ordered set X has the *Souslin Property* if every set of pairwise disjoint, non-empty intervals of X is countable. (This condition is often referred to as the "countable chain condition".) Clearly, if a densely ordered set X has a countable dense subset Y, it must have the Souslin Property, since any non-empty interval of X must contain an element of Y. The question Souslin raised was this: Is it the case that $\mathbb{R}$ is characterised by being an open, ordered continuum having the Souslin Property? Although Souslin did not publish any indication that he thought a positive answer was likely, it has become common to refer to a positive answer as *The Souslin Hypothesis.*

We now know that the Souslin Problem cannot be solved in ZFC set theory, even if we assume GCH. We shall show that if we assume $V = L$, however, then the problem can be solved, with Souslin's Hypothesis being false.

We shall solve the Souslin Problem (assuming $V = L$) by first reformulating it in terms of trees. But before we do that, let us notice that the Souslin Hypothesis is equivalent (in ZFC) to the following assertion:

> *Every densely ordered set with the Souslin Property has a countable dense subset.*

(We shall denote this last assertion by *SH*.) The proof (of equivalence) in one direction is immediate. Assuming *SH*, if we are given an open, ordered continuum having the Souslin Property, then by *SH* it will have a countable dense subset, and so by Cantor's theorem it will be isomorphic to $\mathbb{R}$. For the proof in the other direction, suppose we are given a densely ordered set, X, with the Souslin Property. Let X' be obtained from X by introducing a copy of the rationals at each end (to obtain an *open* ordered set). Let X'' be the Dedekind completion of X'. It is easily seen that X'' is an open, ordered continuum with the Souslin Property. By the Souslin Hypothesis (as formulated by Souslin), X'' is isomorphic to $\mathbb{R}$. Hence X is isomorphic to a dense subset of an interval of $\mathbb{R}$. Thus X has a countable dense subset.

We shall prove that if $V = L$ then *SH* is false, by using $V = L$ to construct a densely ordered set having the Souslin Property but no countable dense subset. We achieve this by way of trees.

A *tree* is a partially ordered set $\mathbf{T} = \langle T, \leqslant_T \rangle$ such that for every $x \in T$, the set

$$\hat{x} = \{ y \in T \mid y <_T x \}$$

is well-ordered by $\leqslant_T$.

The order-type of the set $\hat{x}$ under $<_T$ is called the *height* of x in $\mathbf{T}$, denoted by $ht_{\mathbf{T}}(x)$.

If α is an ordinal, the α-th *level* of $\mathbf{T}$ is the set

$$T_\alpha = \{ x \in T \mid ht_{\mathbf{T}}(x) = \alpha \}.$$

We often write $T \upharpoonright \alpha$ to denote the set $\bigcup_{\beta < \alpha} T_\beta$, and $\mathbf{T} \upharpoonright \alpha$ for the restriction of the structure $\mathbf{T}$ to this set.

Sometimes we blur the distinction between a tree and its underlying set, writing T instead of $\mathbf{T}$, etc.

In a tree $\mathbf{T}$, if we are at any point x, there is only one path "downwards", namely $\hat{x}$, though there may be several (or none) paths "upwards" from x. It is customary to represent trees pictorally as in Figure 1, using vertical connecting lines to denote the ordering $<_T$ in the upward direction, drawing the levels of the tree on a horizontal line.

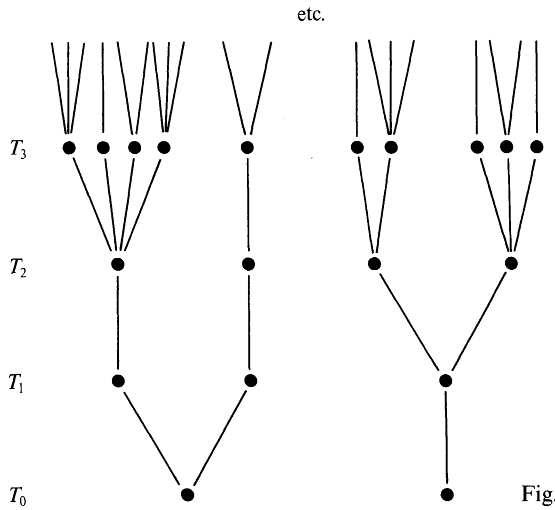

Fig. 1

Let $\mathbf{T}$ be a tree. A linearly ordered subset b of T with the property that whenever $x \in b$, then $y <_T x$ implies $y \in b$, is called a *branch* of $\mathbf{T}$. If α is the order-type of b under $<_T$, we say that b is an *α-branch*. A branch is *maximal* if it is not properly contained in any other branch of $\mathbf{T}$. By the Axiom of Choice, every branch can be extended to a maximal branch. Every set $\hat{x}$ is a branch of $\mathbf{T}$. If x has no *successors* in $\mathbf{T}$ (i.e. there are no points $y \in T$ such that $x <_T y$), then $\hat{x} \cup \{x\}$ is a maximal branch of $\mathbf{T}$.

An *antichain* of $\mathbf{T}$ is a subset of T, no two elements of which are comparable under the ordering $<_T$. An antichain is *maximal* if it is not properly contained in any other antichain of $\mathbf{T}$ (or, equivalently, iff every point of $\mathbf{T}$ is comparable with some member of the antichain under $<_T$). By the Axiom of Choice, every antichain of $\mathbf{T}$ can be extended to a maximal antichain. If $T_\alpha \neq \emptyset$, then T_α is a maximal antichain of $\mathbf{T}$.

Let θ be an ordinal, λ a cardinal. A tree $\mathbf{T}$ is said to be a *(θ, λ)-tree* iff:

(i) $(\forall \alpha < \theta)(T_\alpha \neq \emptyset)$;
(ii) $T_\theta = \emptyset$;
(iii) $(\forall \alpha < \theta)(|T_\alpha| < \lambda)$.

In words, a (θ, λ)-tree is one of "height" θ and "width" less than λ. (We demand $|T_\alpha| < \lambda$ rather than $|T_\alpha| \leqslant \lambda$ in (iii) to allow for the case where λ is a limit cardinal.)

A tree $\mathbf{T}$ is said to have *unique limits* if, whenever α is a limit ordinal and $x, y \in T_\alpha$, if $\hat{x} = \hat{y}$ then $x = y$.

A (θ, λ)-tree $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ has unique limits and each of the following conditions is satisfied:

(i) $|T_0| = 1$;
(ii) if $\alpha, \alpha + 1 < \theta$ and $x \in T_\alpha$, then there are distinct $y_1, y_2 \in T_{\alpha+1}$ such that $x <_T y_1$ and $x <_T y_2$;
(iii) if $\alpha < \beta < \theta$ and $x \in T_\alpha$, there is a $y \in T_\beta$ such that $x <_T y$.

Let κ be an infinite cardinal. A *κ-tree* is a normal (κ, κ)-tree.

It is trivial to show that every ω-tree has an ω-branch. (By recursion, pick $x_n \in T_n$ so that $x_n <_T x_{n+1}$.) And it is tempting to imagine that this simple result generalises to ω_1-trees. However, as was first demonstrated by N. Aronszajn, there are ω_1-trees having no ω_1-branch. Such trees are now known as *Aronszajn trees.*

1.1 Theorem. *There is an Aronszajn tree (i.e. an ω_1-tree with no ω_1-branch).*

Proof. By recursion on the levels, we construct an ω_1-tree $\mathbf{T}$. The elements of T_α will be strictly increasing α-sequences of rational numbers, and the tree ordering will be $x <_T y$ iff x is an initial segment of y (i.e. iff $x \subset y$). Notice that if b were an ω_1-branch of such a tree, $\bigcup b$ would be a strictly increasing ω_1-sequence of rationals, which is impossible. Hence our tree certainly can have no ω_1-branches, and our problem is simply to construct the tree. In order to do this, we ensure that at each stage in the construction, $\mathbf{T} \upharpoonright \alpha$ satisfies the following condition:

> $P(\alpha)$: $\mathbf{T} \upharpoonright \alpha$ is a normal (α, ω_1)-tree, and for every $\beta < \gamma < \alpha$ and every $x \in T_\beta$ and every rational $q > \sup(x)$, there is a $y \in T_\gamma$ such that $x \subset y$ and $q > \sup(y)$,

where $\sup(x)$ here denotes the supremum (in the reals) of the range of values of the rational sequence x.

To commence the construction, we set

$$T_0 = \{\emptyset\}.$$

If $\mathbf{T} \upharpoonright (\alpha + 1)$ is defined and satisfies $P(\alpha + 1)$, we define

$$T_{\alpha+1} = \{x ^\frown \langle q \rangle \mid x \in T_\alpha \wedge q \in \mathbb{Q} \wedge q > \sup(x)\}.$$

Clearly, $\mathbf{T} \upharpoonright (\alpha + 2)$ then satisfies $P(\alpha + 2)$.

Finally, suppose α is a limit ordinal and $\mathbf{T} \upharpoonright \alpha$ has been defined and satisfies $P(\alpha)$. (Notice that if $P(\beta)$ is valid for $\mathbf{T} \upharpoonright \beta$ for all $\beta < \alpha$ then $P(\alpha)$ is automatically valid for $\mathbf{T} \upharpoonright \alpha$.) The construction of T_α depends upon the following claim.

Claim. For each $x \in T \upharpoonright \alpha$ and each rational $q > \sup(x)$, there is an α-branch b of $\mathbf{T} \upharpoonright \alpha$ such that $x \in b$ and $\sup(\bigcup b) \leqslant q$.

To prove the claim, given x, q as above, pick a strictly increasing ω-sequence $(\alpha_n \mid n < \omega)$ of ordinals, cofinal in α, so that $x \in T \upharpoonright \alpha_0$. Since $P(\alpha)$ is valid, we can inductively pick elements $y_n \in T_{\alpha_n}$ so that $x \subset y_0 \subset y_1 \subset y_2 \subset \ldots$ and $\sup(y_n) < q$. Set

$$b = \{y \in T \upharpoonright \alpha \mid (\exists n < \omega)(y \subset y_n)\}.$$

Celarly, b is an α-branch of $\mathbf{T} \upharpoonright \alpha$ which contains x and is such that $\sup(\bigcup b) \leqslant q$, proving the claim.

Using the claim, we construct T_α as follows. For each $x \in T \upharpoonright \alpha$ and each rational $q > \sup(x)$, pick one α-branch $b(x, q)$ of $\mathbf{T} \upharpoonright \alpha$ as in the claim, and set

$$T_\alpha = \{\bigcup b(x, q) \mid x \in T \upharpoonright \alpha \wedge q \in \mathbb{Q} \wedge q > \sup(x)\}.$$

It is easily seen that $\mathbf{T} \upharpoonright (\alpha + 1)$ satisfies $P(\alpha + 1)$. In particular, T_α is countable because both $T \upharpoonright \alpha$ and $\mathbb{Q}$ are countable.

That completes the construction of $\mathbf{T}$. Since $\mathbf{T} \upharpoonright \alpha$ satisfies $P(\alpha)$ for all $\alpha < \omega_1$, $\mathbf{T}$ is an ω_1-tree, and so we are done. □

Related to the notion of an Aronszajn tree is that of a *Souslin tree*. This is defined to be an ω_1-tree having no uncountable antichain. (We shall see later that Souslin trees are closely connected with the Souslin Problem.) As the following result shows, Souslin trees are just special kinds of Aronszajn trees.

1.2 Theorem. *Every Souslin tree is an Aronszajn tree.*

Proof. Let $\mathbf{T}$ be a Souslin tree. Let b be any branch of $\mathbf{T}$. We show that b must be countable. Since $\mathbf{T}$ is normal, for each $x \in b$ we can pick an element $x^* \in T$ such that $x <_T x^*$, $ht(x^*) = ht(x) + 1$, and $x^* \notin b$. It is easily seen that $\{x^* \mid x \in b\}$ is an antichain of $\mathbf{T}$. But if $x, y \in b$ are such that $x \neq y$, then $x^* \neq y^*$. So as $\mathbf{T}$ has no uncountable antichain, b must be countable. □

The above proof made use of the normality requirements on a Souslin tree. These are rather strong conditions, since they tend to point in the opposite direction to the Aronszajn and Souslin requirements of no uncountable branches or antichains. In the case of Aronszajn trees, the somewhat "paradoxical" situation arose (in 1.1) that essential use was made of normality requirements in order to construct an Aronszajn tree. But in the case of Souslin trees, the full normality requirements turn out to be a burden as far as construction of such trees in connection with the Souslin Problem is concerned. The next lemma shows that this burden is easily shed.

1.3 Lemma. (i) *Let* $\mathbf{T}$ *be an* (ω_1, ω_1)*-tree with unique limits, having no uncountable branch. Then there is a subset* T^* *of* T *such that, under the induced ordering,* T^* *is an Aronszajn tree.*

(ii) *Let* $\mathbf{T}$ *be an* (ω_1, ω_1)*-tree with unique limits, having no uncountable branch and no uncountable antichain. Then there is a subset* T^* *of* T *such that, under the induced ordering,* T^* *is a Souslin tree.*

Proof. (i) Since T_0 is countable, we can find an element x_0 of T_0 such that T' is uncountable, where we set

$$T' = \{x \in T \mid x_0 \leqslant_T x\}.$$

Let T'' be the set of all members of T' which have extensions on all higher levels of T'. It is easily seen that each member of T'' has extensions on all higher levels of T'' itself. It follows that for every point $x \in T''$ there are points $y, z \in T''$ such that $x <_T y$, $x <_T z$, and y and z are incomparable in **T**. (Otherwise the extensions of x would form an uncountable branch of **T**.) Hence we can define a function $f\colon \omega_1 \to \omega_1$ by the following recursion:

$$\begin{aligned} f(0) &= 0; \\ f(\alpha+1) &= \text{the least } \beta > f(\alpha) \text{ such that for all } x \in T''_{f(\alpha)} \text{ there are } y, z \in T''_\beta \\ &\quad\ \text{such that } x <_T y,\ x <_T z, \text{ and } y \neq z; \\ f(\lambda) &= \sup\nolimits_{\nu<\lambda} f(\nu), \quad \text{if } \lim(\lambda). \end{aligned}$$

Set

$$T^* = \bigcup_{\alpha<\omega_1} T''_{f(\alpha)}.$$

It is easily checked that T^* is as required.

(ii) The above proof works in this case also. □

Notice that unique limits played no role in the above proof. We could have omitted this requirement from all definitions and results, but it is common to include it, and we shall always do so.

Our next result indicates our usage of the phrase "Souslin tree".

1.4 Theorem. *Souslin's Hypothesis is equivalent to the non-existence of a Souslin tree.*

Proof. Assume first that there is a Souslin tree. We construct a counterexample to SH, i.e. a densely ordered set having the Souslin Property but no countable dense subset.

Let **T** be a Souslin tree. By replacing **T** by its restriction to the limit levels of **T**, if necessary, we may assume that each member of **T** has infinitely many successors on the next level of **T**. For each non-zero $\alpha < \omega_1$, let $<_\alpha$ be a linear ordering of T_α, isomorphic to the rationals, so that the set of all successors on $T_{\alpha+1}$ of any element of T_α is ordered as the rationals by $<_{\alpha+1}$. Let X be the set of all maximal branches of **T**, and define a linear ordering on X by setting $b <_X d$ iff $b(\alpha) <_\alpha d(\alpha)$, where α is the least ordinal such that $b \cap T_\alpha \neq d \cap T_\alpha$ and $b(\alpha)$ denotes the unique element of $b \cap T_\alpha$, $d(\alpha)$ the unique element of $d \cap T_\alpha$. Clearly, $\langle X, \leqslant_X\rangle$ is a densely ordered set of cardinality 2^ω.

We show that X has the Souslin Property. Let I be any interval of X, say $I = (b, d)$. Choose α minimal so that $b(\alpha) \neq d(\alpha)$. Pick $x_I \in T_\alpha$ so that $b(\alpha) <_\alpha x_I <_\alpha d(\alpha)$. Let $e(I)$ be a maximal branch of **T** containing x_I. Thus $e(I) \in I$. Suppose now that I and J are disjoint intervals of X. Then $e(I) \notin J$ and $e(J) \notin I$, so x_I and x_J must be incomparable in **T**. Since **T** has no uncountable antichains, it follows that any pairwise disjoint collection of intervals of X must be countable.

We complete the proof of this half of the theorem by showing that X has no countable dense subset. Let A be any countable subset of X. For each pair b, d of distinct elements of A, let $\alpha(b, d)$ be the least ordinal α such that $b(\alpha) \neq d(\alpha)$. Let

$$\gamma = \sup\{\alpha(b, d) \mid b, d \in A \;\&\; b \neq d\}.$$

Since A is countable, $\gamma < \omega_1$. Let $w \in T_\gamma$ and choose $x, y, z \in T_{\gamma+1}$ so that $w <_T x, y, z$ and $x <_{\gamma+1} y <_{\gamma+1} z$. Let b_x be a maximal branch of **T** containing x, and choose b_y, b_z similarly. If A were dense in X, we could find $d, d' \in A$ such that $b_x <_X d <_X b_y$ and $b_y <_X d' <_X b_z$. But since b_x, b_y, b_z all contain w, we would have $\alpha(d, d') > \gamma$, contrary to the choice of γ. Hence A cannot be dense in X.

Thus X is a counterexample to SH.

We now assume that SH is false and construct a tree satisfying the hypotheses of 1.3 (ii), which by virtue of 1.3 (ii) at once implies the existence of a Souslin tree.

By the failure of SH, let X be a densely ordered set with the Souslin Property but no countable dense subset. By recursion on the levels we define a *partition tree* $\mathbf{T} = \langle T, \supseteq \rangle$ of X, elements of which are non-empty "intervals" of X. To commence, we set $T_0 = \{X\}$.

Suppose we have defined T_α. For every $I \in T_\alpha$ of cardinality greater than 1, choose an interior point $x(I)$ of I. (Since X is densely ordered, if I has at least two elements, such a point always exists.) Let

$$I_0 = \{y \in I \mid y <_X x(I)\}$$
$$I_1 = \{y \in I \mid x(I) \leqslant_X y\}.$$

Set

$$T_{\alpha+1} = \{I_0 \mid I \in T_\alpha \wedge |I| > 1\} \cup \{I_1 \mid I \in T_\alpha \wedge |I| > 1\}.$$

Now suppose that $\lim(\alpha)$ and T_β has been defined for all $\beta < \alpha$. In this case, set

$$T_\alpha = \{\bigcap b \mid b \text{ is an } \alpha\text{-branch of } \mathbf{T} \restriction \alpha \text{ such that } |\bigcap b| > 1\}.$$

That defines **T**. Let θ be the least ordinal such that $T_\theta = \emptyset$. We shall show that $\theta = \omega_1$ and that **T** satisfies the hypotheses of 1.3 (ii). It is clear that T has unique limits. We show first that **T** has no uncountable branch (so that, in particular, $\theta \leqslant \omega_1$).

Suppose that B were an uncountable branch of T. Let $(I_\alpha \mid \alpha < \omega_1)$ be the canonical enumeration of the first ω_1 elements of B. Set

$$A_0 = \{\alpha < \omega_1 \mid (\forall y \in I_{\alpha+1})(y <_X x(I_\alpha))\},$$
$$A_1 = \{\alpha < \omega_1 \mid (\forall y \in I_{\alpha+1})(x(I_\alpha) \leqslant_X y)\}.$$

Thus A_0 and A_1 constitute a disjoint partition of ω_1. Hence at least one of A_0, A_1 is uncountable. Suppose, for the sake of argument, that A_0 were uncountable.

(The other case is handled similarly.) For $\alpha \in A_0$, let J_α be the X-interval

$$J_\alpha = (x(I_\beta), x(I_\alpha)),$$

where β is the least element of A_0 above α. Now, if $\alpha \in A_0$ and $\alpha < \beta$, we have $x(I_\beta) <_X x(I_\alpha)$. Hence $\{J_\alpha \mid \alpha \in A_0\}$ is an uncountable set of pairwise disjoint intervals of X, which is impossible. Thus **T** has no uncountable branch.

Moreover, **T** has no uncountable antichain. Essentially this is because incomparability in **T** means disjointness as "intervals" in X. For suppose $\{I_\alpha \mid \alpha < \omega_1\}$ were an uncountable antichain of **T**. Then for each $\alpha < \omega_1$ we could choose $x_\alpha, y_\alpha \in I_\alpha$, $x_\alpha <_X y_\alpha$, whence $\{(x_\alpha, y_\alpha) \mid \alpha < \omega_1\}$ would be an uncountable set of pairwise disjoint intervals of X, which is impossible.

Since **T** has no uncountable antichains, each level of **T** must be countable. If we can show that T is uncountable, we shall thus be able to conclude that **T** is an (ω_1, ω_1)-tree and be done. But it follows easily from the construction of **T** that the set $\{x(I) \mid I \in T\}$ is dense in X. (Roughly speaking, this is because we keep on "splitting" intervals of X until it is not possible to go any further.) So, as X has no countable dense subset, we see that T is indeed uncountable. □

1.4 enables us to prove that SH fails if we assume $V = L$.

1.5 Theorem. *Assume $V = L$. Then there is a Souslin tree.*

Proof. We construct an ω_1-tree, **T**, by recursion on the levels. The elements of T_α will be sequences from ${}^\alpha 2$, and the ordering of **T** will be sequence extension (= set-theoretic inclusion). We carry out the construction so that at each stage $\alpha < \omega_1$, $\mathbf{T} \upharpoonright \alpha$ is a normal (α, ω_1)-tree. This will ensure that **T** is an ω_1-tree, so the only problem will be to ensure that **T** has no uncountable antichains.

To commence, set

$$T_0 = \{\emptyset\}.$$

The definition of $T_{\alpha+1}$ is dictated by the normality requirements. If $\mathbf{T} \upharpoonright \alpha + 1$ is defined, we set

$$T_{\alpha+1} = \{s \frown \langle i \rangle \mid s \in T_\alpha \wedge i = 0, 1\}.$$

If $\mathbf{T} \upharpoonright \alpha + 1$ is a normal $(\alpha + 1, \omega_1)$-tree, then $\mathbf{T} \upharpoonright \alpha + 2$ is clearly a normal $(\alpha + 2, \omega_1)$-tree.

There remains the definition of T_α when α is a limit ordinal and $\mathbf{T} \upharpoonright \alpha$ has been defined. Notice first that if $\mathbf{T} \upharpoonright \beta$ is a normal (β, ω_1)-tree for all $\beta < \alpha$, $\mathbf{T} \upharpoonright \alpha$ will be a normal (α, ω_1)-tree. Now, if $s \in {}^\alpha 2$ is to be a member of T_α, $\{s \upharpoonright \beta \mid \beta < \alpha\}$ will have to be an α-branch of $\mathbf{T} \upharpoonright \alpha$. Hence for some collection, B_α, of α-branches of $\mathbf{T} \upharpoonright \alpha$ we shall have

$$T_\alpha = \{\bigcup b \mid b \in B_\alpha\}.$$

What properties must the set B_α have? Certainly it must be countable. And to preserve normality requirements, each element of $T \upharpoonright \alpha$ must be a member of some branch in B_α. Since trees consisting of sequences as in this case necessarily have unique limits, these two conditions on B_α suffice to ensure that $\mathbf{T} \upharpoonright \alpha + 1$ will be a normal $(\alpha + 1, \omega_1)$-tree. So we are left with choosing B_α to ensure that $\mathbf{T}$ will be a Souslin tree. How can we do this? Well, the final tree, $\mathbf{T}$, will be a subset of $\bigcup_{\alpha < \omega_1} {}^\alpha 2$ of cardinality ω_1. By GCH, the set T will have ω_2 many uncountable subsets. We must choose the collections B_α so that none of these uncountable subsets of T is an antichain of $\mathbf{T}$. To see how this might be achieved, suppose that in fact there were an uncountable antichain in $\mathbf{T}$. Then there would be a maximal uncountable antichain, A. For each $\alpha < \omega_1$, $A \cap (T \upharpoonright \alpha)$ is an antichain in $\mathbf{T} \upharpoonright \alpha$. Let

$$C = C_A = \{\alpha \in \omega_1 \mid \lim(\alpha) \wedge A \cap (T \upharpoonright \alpha) \text{ is a } \textit{maximal} \text{ antichain in } \mathbf{T} \upharpoonright \alpha\}.$$

The set C is club in ω_1. Closure is immediate, of course. To prove the unboundedness of C in ω_1, given $\alpha_0 < \omega_1$, define $\alpha_n < \omega_1$ recursively by setting α_{n+1} to be the least ordinal $\gamma > \alpha_n$ such that each element of $T \upharpoonright \alpha_n$ is comparable with some member of $A \cap (T \upharpoonright \gamma)$, in which case it is easily seen that $\alpha = \bigcup_{n<\omega} \alpha_n \in C$. Suppose now that we can somehow choose the sets B_α so that for each maximal uncountable antichain A of $\mathbf{T}$, there is an $\alpha \in C_A$ for which the definition of T_α prevents the addition of any elements to $\mathbf{T}$ which are incomparable with all of the elements of $A \cap (T \upharpoonright \alpha)$. This would then ensure that in fact there are *no* uncountable antichains in $\mathbf{T}$. (The above discussion would be a proof by contradiction of this fact.) Now, constructing T_α so that some specific maximal antichain $A \cap (T \upharpoonright \alpha)$ of $\mathbf{T} \upharpoonright \alpha$ does not "grow" in $\mathbf{T}$ (at any subsequent stage) is easy. Define B_α so that each element of B_α contains a member of $A \cap (T \upharpoonright \alpha)$. Since $A \cap (T \upharpoonright \alpha)$ is a *maximal* antichain in $\mathbf{T} \upharpoonright \alpha$, each element of $T \upharpoonright \alpha$ is comparable with some member of $A \cap (T \upharpoonright \alpha)$, so constructing B_α with this property causes no difficulties, and will ensure that every element of T_α extends a member of $A \cap (T \upharpoonright \alpha)$, and hence that any element of $\mathbf{T}$ of height greater than α will have to extend an element of $A \cap (T \upharpoonright \alpha)$. Our problem now reduces to one of cardinalities. In constructing $\mathbf{T}$ there are ω_1 limit stages α where we can "kill off" maximal antichains $A \cap (T \upharpoonright \alpha)$ of $\mathbf{T} \upharpoonright \alpha$ in the above sense. But there are ω_2 many potential sets A. So we must somehow deal with ω_2 possibilities in ω_1 steps. This is where we use $V = L$.

Suppose then that we are at stage α, where $\lim(\alpha)$ and $\mathbf{T} \upharpoonright \alpha$ has been defined. Let A_α be the $<_L$-least maximal antichain of $\mathbf{T} \upharpoonright \alpha$ with the property that the set

$$\{\gamma < \alpha \mid A_\alpha \cap T_\gamma \neq \emptyset\}$$

is unbounded in α. (Such a set always exists, as is easily seen.) For each $x \in T \upharpoonright \alpha$, let b_x be the $<_L$-least α-branch of $\mathbf{T} \upharpoonright \alpha$ such that $x \in b_x$ and $b_x \cap A_\alpha \neq \emptyset$. Since A_α is a maximal antichain of $\mathbf{T} \upharpoonright \alpha$, b_x is always defined. Let

$$B_\alpha = \{b_x \mid x \in T \upharpoonright \alpha\}.$$

Set

$$T_\alpha = \{\textstyle\bigcup b \mid b \in B_\alpha\}.$$

That completes the definition of **T**. We must check that **T** has no uncountable antichain. Suppose, on the contrary, that it did. Let A be the $<_L$-least maximal uncountable antichain of **T**. Now, all of the sets involved in the above definition of **T** are members of L_{ω_2}, so we could in fact carry out the construction of **T** within the set L_{ω_2}. Thus **T** is a definable element of L_{ω_2}. Hence A is also a definable element of L_{ω_2}. Moreover, we clearly have (by a trivial absoluteness observation):

$$\models_{L_{\omega_2}} \text{"}A \text{ is the } <_L\text{-least maximal antichain of } \mathbf{T} \text{ such that the set } \{\gamma \in \omega_1 \mid A \cap T_\gamma \neq \emptyset\} \text{ is unbounded in } \omega_1\text{"}.$$

Let M be the smallest elementary submodel of L_{ω_2}. By II.5.11, $M \cap L_{\omega_1}$ is transitive and of the form L_α for some $\alpha < \omega_1$. (M is, of course, countable.) Since **T** and A are definable in L_{ω_2}, they are elements of M. We have

$$T \cap M = T \restriction \alpha.$$

To see this, suppose first that $\beta < \alpha$. Then there is a surjection $f: \omega \to T_\beta$. Hence there is such a surjection in M. But $\omega \subseteq M$, so it follows that $T_\beta = f''\omega \subseteq M$. Thus $T \restriction \alpha \subseteq M$. Again, if $x \in T \cap M$, then (again because $M \prec L_{\omega_2}$) $ht(x) \in M$, so $ht(x) < \alpha$, so $x \in T \restriction \alpha$. We also have

$$A \cap M = A \cap (T \restriction \alpha).$$

(This is an immediate consequence of the previous equality.) So, if we let

$$\pi: M \cong L_\beta$$

(by the Condensation Lemma) we have:

$$\pi \restriction L_\alpha = \mathrm{id} \restriction L_\alpha, \quad \pi(\omega_1) = \alpha, \quad \pi(\mathbf{T}) = \mathbf{T} \restriction \alpha, \quad \pi(A) = A \cap (T \restriction \alpha).$$

(These are all easy consequences of the properties of the collapsing isomorphism. Such considerations will occur often in our later development.) Thus, by elementary substructure and isomorphism, we have:

$$\models_{L_\beta} \text{"}(A \cap T \restriction \alpha) \text{ is the } <_L\text{-least maximal antichain of } \mathbf{T} \restriction \alpha \text{ such that the set } \{\gamma \in \alpha \mid (A \cap T \restriction \alpha) \cap (\mathbf{T} \restriction \alpha)_\gamma \neq \emptyset\} \text{ is unbounded in } \alpha\text{"}.$$

By elementary absoluteness considerations, this clearly implies that $A \cap T \restriction \alpha$ really is the $<_L$-least maximal antichain of $\mathbf{T} \restriction \alpha$ such that the set $\{\gamma < \alpha \mid (A \cap T \restriction \alpha) \cap T_\gamma \neq \emptyset\}$ is unbounded in α. Hence

$$A \cap T \restriction \alpha = A_\alpha.$$

But then by the construction of T_α, every element of **T** of height greater than or equal to α is comparable with some element of $A \cap T \restriction \alpha$. This contradicts the fact that A is an uncountable antichain of **T**. Hence **T** must be a Souslin tree, and we are done. □

In section 3 we shall analyse the use of $V = L$ in the above proof.

2. The Kurepa Hypothesis

We have seen that there are ω_1-trees with no ω_1-branches. And by making simple modifications to an Aronszajn tree it is possible to construct ω_1-trees with exactly κ many ω_1-branches, where κ is any of the cardinals $1, 2, 3, \ldots, n, \ldots, \omega, \omega_1$. Now, any ω_1-tree is a set of cardinality ω_1, so the maximum possible number of branches is 2^{ω_1}. Thus, if we assume GCH, no ω_1-tree can have more than ω_2 many ω_1-branches. A natural question is whether in fact there are *any* ω_1-trees which have ω_2 many ω_1-branches. This turns out to be related to an old question of D. Kurepa concerning the Generalised Continuum Hypothesis (see later), and as a result, an ω_1-tree with ω_2 (or more) ω_1-branches is called a *Kurepa tree.*

In ZFC, or even in ZFC + GCH, it is not possible to decide whether or not Kurepa trees exist. The sharpest results are these:

(I) If ZF is consistent, so too is the theory

ZFC + GCH + "*there is a Kurepa tree*".

(II) If the theory

ZFC + "*there is an inaccessible cardinal*"

is consistent, so too is the theory

ZFC + GCH + "*there are no Kurepa trees*".

(III) If the theory

ZFC + "*there are no Kurepa trees*"

is consistent, so too is the theory

ZFC + "*there is an inaccessible cardinal*".

Hence the non-existence of Kurepa trees is closely bound up with the notion of inaccessible cardinals. We shall prove that if $V = L$, there is a Kurepa tree. But before we do this, we relate the notion of Kurepa trees to the problem of Kurepa, mentioned earlier.

The *Kurepa Hypothesis* (*KH*) is the assertion that there is a family $\mathscr{F} \subseteq \mathscr{P}(\omega_1)$ of cardinality ω_2 such that for all $\alpha < \omega_1$, the set

$$\mathscr{F} \upharpoonright \alpha = \{x \cap \alpha \mid x \in \mathscr{F}\}$$

is countable. The following lemma is due to Kurepa himself.

2.1 Lemma. *The Kurepa Hypothesis is equivalent to the existence of a Kurepa tree.*

Proof. Suppose first that there is a Kurepa tree, **T**. We may clearly assume that $\mathbf{T} = \langle \omega_1, \leqslant_T \rangle$ and that $\alpha <_T \beta$ implies $\alpha < \beta$. Let $\mathscr{F}$ be the set of all ω_1-branches of **T**. It is immediately clear that $\mathscr{F}$ satisfies *KH*.

Conversely, let $\mathscr{F} \subseteq \mathscr{P}(\omega_1)$ satisfy *KH*. For each $x \in \mathscr{F}$, define a function $f_x \colon \omega_1 \to \mathscr{P}(\omega_1)$ by setting

$$f_x(\alpha) = x \cap \alpha .$$

Let

$$T = \{ f_x \upharpoonright \alpha \mid x \in \mathscr{F} \wedge \alpha < \omega_1 \} .$$

For $g_1, g_2 \in T$, say $g_1 <_T g_2$ iff $g_1 \subset g_2$. It is clear that $\mathbf{T} = \langle T, \leqslant_T \rangle$ is a tree such that $T_\alpha \subseteq {}^\alpha\mathscr{P}(\omega_1)$. Since $\mathscr{F} \upharpoonright \alpha$ is countable for each $\alpha < \omega_1$, each level of **T** is countable. Hence **T** is an (ω_1, ω_1)-tree. For each $x \in \mathscr{F}$, the set

$$b_x = \{ f_x \upharpoonright \alpha \mid \alpha < \omega_1 \}$$

is an ω_1-branch of **T**, and if $x \neq y$ then $b_x \neq b_y$. Hence **T** has ω_2 many ω_1-branches. Hence we shall be done if we can show that **T** is normal. Well, it is easily seen that **T** satisfies all of the normality requirements except possibly the requirement that each element of **T** has at least two immediate successors. But this is easily achieved: simply add two copies of an Aronszajn tree above each point of **T**. The resulting tree will then be a Kurepa tree. □

2.2 Theorem. *Assume $V = L$. Then there is a Kurepa tree.*

Proof. We verify *KH*, rather than construct a Kurepa tree directly, as this turns out to be marginally simpler (because there is less to check).

Using II.5.4 and II.5.10, we can define a function $f \colon \omega_1 \to \omega_1$ by letting $f(\alpha)$ be the least ordinal $\gamma > \alpha$ such that $L_\gamma \prec L_{\omega_1}$. Notice that $L_{f(\alpha)}$ will be a model of the theory ZF^- (= ZF minus the Power Set Axiom). (As is often the case in such situations, we are being a little sloppy here. As formulated, ZF^- will be a theory in LST, and we have no concept of a model for an LST-theory. We can avoid this sloppiness either by formulating a "copy" of the theory ZF^- in the language $\mathscr{L}$, or else defining within set theory the notion of "a model of ZF^-" in an entirely semantic fashion, just as we defined the notions of amenable sets and admissible sets to provide us with the notions of "models" of the theories *BS* and *KP*, respectively (Chapter I). What matters to us is that, working inside $L_{f(\alpha)}$, we can carry out any construction which can be carried out in ZF without use being made of the Power Set Axiom.)

Define $\mathscr{F} \subseteq \mathscr{P}(\omega_1)$ by:

$$\mathscr{F} = \{ x \subseteq \omega_1 \mid (\forall \alpha < \omega_1)(x \cap \alpha \in L_{f(\alpha)}) \} .$$

For any $\alpha < \omega_1$, $\mathscr{F} \upharpoonright \alpha \subseteq L_{f(\alpha)}$, so certainly $|\mathscr{F} \upharpoonright \alpha| \leqslant \omega$. What we must show, in order to prove that $\mathscr{F}$ satisfies *KH*, is that $|\mathscr{F}| = \omega_2$. Intuitively, this is because, although countable, $f(\alpha)$ is "much larger" than α (in the sense that $L_{f(\alpha)}$ is a "partial universe" as far as the theory ZF^- is concerned).

We shall assume that $|\mathscr{F}| \neq \omega_2$ and work for a contradiction. By this assumption, $\mathscr{F}$ has an ω_1-enumeration (not necessarily one-one). Let $X = (x_\alpha \mid \alpha < \omega_1)$ be the $<_L$-least ω_1-enumeration of $\mathscr{F}$. Notice that the function f is definable in L_{ω_2} (because the definition of f given above only involves sets in L_{ω_2}), whence both $\mathscr{F}$ and X are definable in L_{ω_2}.

By recursion, we define elementary submodels $N_\nu \prec L_{\omega_2}$ for $\nu < \omega_1$ as follows:

$$N_0 = \text{the smallest } N \prec L_{\omega_2};$$

$$N_{\nu+1} = \text{the smallest } N \prec L_{\omega_2} \text{ such that } N_\nu \cup \{N_\nu\} \subseteq N;$$

$$N_\delta = \bigcup_{\nu<\delta} N_\nu, \quad \text{if } \lim(\delta).$$

By II.5.11, $N_\nu \cap \omega_1$ is transitive for each $\nu < \omega_1$. Let $\alpha_\nu = N_\nu \cap \omega_1$. Now, by a simple induction, we see that each N_ν is countable, so each α_ν is a countable ordinal. Moreover, since $N_\nu \in N_{\nu+1} \prec L_{\omega_2}$, we have $\alpha_\nu = N_\nu \cap \omega_1 \in N_{\nu+1}$, so $\alpha_\nu < \alpha_{\nu+1}$. Hence $(\alpha_\nu \mid \nu < \omega_1)$ is a normal sequence in ω_1. (Continuity follows from the continuity of the sequence $(N_\nu \mid \nu < \omega_1)$, of course.) Set

$$x = \{\alpha_\nu \mid \nu < \omega_1 \wedge \alpha_\nu \notin x_\nu\}.$$

For each $\nu < \omega_1$, $x \neq x_\nu$, so $x \notin \mathscr{F}$. We obtain our contradiction by showing that $x \cap \alpha \in L_{f(\alpha)}$ for all $\alpha < \omega_1$.

Fix $\alpha < \omega_1$ arbitrarily. We prove that $x \cap \alpha \in L_{f(\alpha)}$. Let η be the largest limit ordinal such that $\alpha_\eta \leqslant \alpha$. (If no such η exists, then $x \cap \alpha$ is finite and hence $x \cap \alpha \in L_{f(\alpha)}$.) Since $x \cap \alpha$ differs from $x \cap \alpha_\eta$ by only a finite amount, and since $L_{f(\alpha)}$ is amenable, it clearly suffices to prove that $x \cap \alpha_\eta \in L_{f(\alpha)}$. But $\alpha_\eta \leqslant \alpha$ and f is clearly monotone, so it suffices to prove that $x \cap \alpha_\eta \in L_{f(\alpha_\eta)}$. Hence we may assume that $\alpha = \alpha_\eta$, where $\lim(\eta)$.

Now, we have

$$x \cap \alpha = \{\alpha_\nu \mid \nu < \eta \wedge \alpha_\nu \notin x_\nu \cap \alpha\},$$

so as $L_{f(\alpha)}$ is a model of ZF^- we shall be done if we can show that

$$(\alpha_\nu \mid \nu < \eta), \quad (x_\nu \cap \alpha \mid \nu < \eta) \in L_{f(\alpha)}.$$

Let

$$\pi\colon N_\eta \cong L_\beta.$$

Clearly,

$$\pi \upharpoonright L_\alpha = \mathrm{id} \upharpoonright L_\alpha, \quad \pi(\omega_1) = \alpha, \quad \pi(X) = (x_\nu \cap \alpha \mid \nu < \alpha).$$

In particular,

$$(x_\nu \cap \alpha \mid \nu < \alpha) \in L_\beta.$$

So as $\eta \leqslant \alpha$,

$$(x_\nu \cap \alpha \mid \nu < \eta) \in L_\beta.$$

Now, $\alpha \in L_{f(\alpha)} \prec L_{\omega_1}$, so

$$\models_{L_{f(\alpha)}} \text{"}\alpha \text{ is countable"}.$$

But since $\pi(\omega_1) = \alpha$,

$$\alpha = \omega_1^{L_1}.$$

Hence,

$$\beta < f(\alpha).$$

Thus

$$(x_\nu \cap \alpha \mid \nu < \eta) \in L_{f(\alpha)}.$$

It remains only to prove that $(\alpha_\nu \mid \nu < \eta) \in L_{f(\alpha)}$.
For each $\nu < \eta$, let

$$\pi_\nu \colon N_\nu \cong L_{\beta(\nu)}.$$

Then,

$$\pi_\nu \upharpoonright L_{\alpha_\nu} = \mathrm{id} \upharpoonright L_{\alpha_\nu}, \qquad \pi_\nu(\omega_1) = \alpha_\nu.$$

Since $\alpha_\nu = \omega_1^{L_{\beta(\nu)}}$, the sequence $(\alpha_\nu \mid \nu < \eta)$ is definable from the sequence $(\beta(\nu) \mid \nu < \eta)$ in ZF^-, so we shall be done if we can prove that

$$(\beta(\nu) \mid \nu < \eta) \in L_{f(\alpha)}.$$

Well, we proved above that $\beta < f(\alpha)$, so certainly $\beta \in L_{f(\alpha)}$. Moreover, $L_{f(\alpha)}$ is a model of ZF^-. So, working inside $L_{f(\alpha)}$ we can define a sequence $(N'_\nu \mid \nu < \eta')$ of elementary submodels of L_β (for some η') as follows:

$$N'_0 = \text{the smallest } N \prec L_\beta;$$
$$N'_{\nu+1} = \text{the smallest } N \prec L_\beta \text{ such that } N'_\nu \cup \{N'_\nu\} \subseteq N;$$
$$N'_\delta = \bigcup_{\nu<\delta} N'_\nu, \qquad \text{if } \lim(\delta).$$

(The ordinal η' is the largest $\eta' \leqslant \eta$ for which the above construction is possible: in a moment we shall see that in fact $\eta' = \eta$.) Still inside $L_{f(\alpha)}$, let

$$\pi'_\nu \colon N'_\nu \cong L_{\beta'(\nu)} \qquad (\nu < \eta').$$

Thus

$$(\beta'(\nu) \mid \nu < \eta') \in L_{f(\alpha)}.$$

Now recall the dfinition of the original sequence $(N_\nu \mid \nu < \omega_1)$. Since $\nu < \eta$ implies $N_\nu \prec N_\eta \prec L_{\omega_2}$, in the definition of the initial part $(N_\nu \mid \nu < \eta)$ of this sequence we could equally well use N_η in place of L_{ω_2}. That is to say, for $\nu < \eta$ we have:

$$N_0 = \text{the smallest } N \prec N_\eta;$$
$$N_{\nu+1} = \text{the smallest } N \prec N_\eta \text{ such that } N_\nu \cup \{N_\nu\} \subseteq N;$$
$$N_\delta = \bigcup_{\nu<\delta} N_\nu, \quad \text{if } \lim(\delta).$$

Now.

$$\pi\colon N_\eta \cong L_\beta,$$

so an easy induction argument shows that for each $\nu < \eta$,

$$(\pi \restriction N_\nu)\colon N_\nu \cong N'_\nu.$$

(The successor step uses II.5.3.) Hence $\eta' = \eta$ and for each $\nu < \eta$, the structures N_ν and N'_ν have the same transitive collapse, i.e.

$$\nu < \eta \to \beta(\nu) = \beta'(\nu).$$

Thus

$$(\beta(\nu) \mid \nu < \eta) \in L_{f(\alpha)},$$

and we are done. □

3. Some Combinatorial Principles Related to the Previous Constructions

Both for later use and for independent interest, we shall analyse the use of the condensation lemma in the two previous constructions using $V = L$. We begin with the construction of a Souslin tree (1.5). If we try to eliminate the use of the elementary substructure argument of 1.5, we see that what we need is the following:

There should be a sequence $(A_\alpha \mid \alpha < \omega_1)$ such that $A_\alpha \subseteq T \restriction \alpha$, with the property that whenever $A \subseteq T$, then for any club set $C \subseteq \omega_1$ there is an $\alpha \in C$ such that $A \cap (T \restriction \alpha) = A_\alpha$.

For then, given an uncountable maximal antichain $A \subseteq \mathbf{T}$, we take

$$C = \{\alpha \in \omega_1 \mid A \cap (T \restriction \alpha) \text{ is a maximal antichain of } \mathbf{T} \restriction \alpha\}$$

and find an $\alpha \in C$ for which $A \cap (T \restriction \alpha) = A_\alpha$.

The problem with the above approach is that the sequence $(A_\alpha \mid \alpha < \omega_1)$ is too closely bound up with the tree, **T**, which we are trying to define. And until T_α has been defined, we do not know which members of ${}^\alpha 2$ will lie in **T**, of course. However, this problem is easily overcome. By taking the elements of **T** to be countable binary sequences as we did, we fixed in advance the *ordering* of **T** (namely $\subseteq$), and concentrated all our efforts upon choosing the correct subset of $\bigcup_{\alpha<\omega_1} {}^\alpha 2$ for the *domain* of **T**. An alternative approach is to fix in advance the *domain* of **T**, say the set ω_1, and to define the *ordering*, $<_T$, by recursion. Thus we can commence by setting $T_0 = \{0\}$, and if $\mathbf{T} \upharpoonright (\alpha + 1)$ is defined, then for each $x \in T_\alpha$ we can pick the first two unused ordinals in ω_1 and appoint them as successors to x in $T_{\alpha+1}$ (subject to some well-ordering of T_α). For limit ordinals α, if $\mathbf{T} \upharpoonright \alpha$ is defined, we use the next ω unused ordinals to provide extensions in T_α of each member of a suitably chosen countable collection, B_α, of α-branches of $\mathbf{T} \upharpoonright \alpha$. Analysis of the proof in this form leads to the following combinatorial principle:

There should be a sequence $(S_\alpha \mid \alpha < \omega_1)$ such that $S_\alpha \subseteq \alpha$ and for each $X \subseteq \omega_1$ and each club $C \subseteq \omega_1$ there is an $\alpha \in C$ such that $X \cap \alpha = S_\alpha$.

(See 3.2 below for a construction of a Souslin tree using this principle.)

The above principle implicitly involves the classical set-theoretic concept of a stationary set, which we now consider briefly.

A subset, E, of a limit ordinal λ is said to be *stationary* in λ iff E has a non-empty intersection with every club subset of λ.

It is immediate that stationary sets are unbounded. They need not be club, since the result of removing one (limit) point from any stationary set is a stationary set, of course. If κ is an uncountable, regular cardinal, every club set $C \subseteq \kappa$ is stationary (by I.6.1), and in this case the property of being stationary lies strictly between the properties of being club and of being unbounded. For example, in the case of ω_2, the set $\{\alpha + 1 \mid \alpha \in \omega_2\}$ is unbounded in ω_2 but not stationary, whilst the set $\{\alpha \in \omega_2 \mid \mathrm{cf}(\alpha) = \omega\}$ is stationary in ω_2 but not club. A classical result of Ulam (which we do not prove here) states that if $E \subseteq \omega_1$ is stationary, there are disjoint stationary sets $E_\nu \subseteq \omega_1$, for $\nu < \omega_1$, such that $E = \bigcup_{\nu<\omega_1} E_\nu$.

Stationary sets are closely connected with "regressive functions". If λ is an ordinal and $E \subseteq \lambda$, a function $f\colon E \to \lambda$ is said to be *regressive* iff, for each non-zero $\alpha \in E, f(\alpha) < \alpha$.

3.1 Theorem (Fodor's Theorem). *Let κ be an uncountable regular cardinal, and let $E \subseteq \kappa$ be stationary. If $f\colon E \to \kappa$ is regressive, then for some $\beta \in \kappa$, the set*

$$\{\alpha \in E \mid f(\alpha) = \beta\}$$

is stationary in κ.

Proof. Suppose that, on the contrary, for each $\beta \in \kappa$ the set

$$\{\alpha \in E \mid f(\alpha) = \beta\}$$

is not stationary in κ. Then for each $\beta \in \kappa$ we can find a club set $C_\beta \subseteq \kappa$ such that

$$\alpha \in C_\beta \cap E \to f(\alpha) \neq \beta .$$

Let

$$C = \{\alpha \in \kappa \mid \alpha \in \bigcap_{\beta < \alpha} C_\beta\} .$$

This set, C, is called the *diagonal intersection* of the sets C_β, $\beta < \kappa$. It is not hard to see that C is club in κ. Hence, as E is stationary in κ we can find a non-zero ordinal $\alpha \in C \cap E$. For $\beta < \alpha$, we have $\alpha \in C_\beta$, so $f(\alpha) \neq \beta$. (Since $\alpha \in E$, $f(\alpha)$ is defined, of course.) Thus $f(\alpha) \geqslant \alpha$. But this is absurd, since f is regressive. The theorem is proved. □

As an easy exercise, the reader might like to prove that if $E \subseteq \kappa$ is not stationary, there is a regressive function on E which is not constant on any unbounded set. Thus stationary sets may be characterised as those unbounded sets E such that all regressive functions on E are constant on an unbounded subset of E.

In terms of stationary sets, our previous combinatorial principle can be expressed as follows:

> *There is a sequence $(S_\alpha \mid \alpha < \omega_1)$ such that $S_\alpha \subseteq \alpha$, with the property that whenever $X \subseteq \omega_1$, the set $\{\alpha \in \omega_1 \mid X \cap \alpha = S_\alpha\}$ is stationary in ω_1.*

In turns out that this combinatorial principle has many applications, and thus deserves a name. Following Jensen, who discovered it, we call it $\diamondsuit$ (i.e. "diamond").

By amending the argument of 1.5 we prove:

3.2 Theorem. *$\diamondsuit$ implies the existence of a Souslin tree.*

Proof. Assume $\diamondsuit$, and let $(S_\alpha \mid \alpha < \omega_1)$ be a $\diamondsuit$-sequence as described above. By recursion on the levels we construct a Souslin tree, $\mathbf{T}$, with domain ω_1. The elements of $T \upharpoonright \omega$ will be the finite ordinals, and for infinite α the elements of T_α will be the oridinals in the set

$$\{\xi \mid \omega\alpha \leqslant \xi < \omega\alpha + \omega\} .$$

We shall carry out the construction so that for each $\alpha < \omega_1$, $\mathbf{T} \upharpoonright \alpha$ is a normal (α, ω_1)-tree.

Set $T_0 = \{0\}$. If $n \in \omega$ and $\mathbf{T} \upharpoonright n + 1$ is defined, define $\mathbf{T} \upharpoonright n + 2$ by taking the elements of T_n in turn, for each one picking the next two unused finite ordinals to be its successors in T_{n+1}. If $\alpha \geqslant \omega$ and $\mathbf{T} \upharpoonright \alpha + 1$ is defined, define $\mathbf{T} \upharpoonright \alpha + 2$ by using the ordinals in the set $\{\xi \mid \omega\alpha \leqslant \xi < \omega\alpha + \omega\}$ to provide each element of T_α with two successors on $T_{\alpha+1}$. Since T_α is countable, this is easily arranged. There remains the case where $\lim(\alpha)$ and $\mathbf{T} \upharpoonright \alpha$ is defined. By the normality of $\mathbf{T} \upharpoonright \alpha$, for each $x \in T \upharpoonright \alpha$ we can pick an α-branch b_x of $\mathbf{T} \upharpoonright \alpha$ containing x. The exact choice of b_x is unimportant except when S_α is a maximal antichain of $\mathbf{T} \upharpoonright \alpha$, in which case we ensure that $b_x \cap S_\alpha \neq \emptyset$, which is easy to do by virtue of the maximality of the antichain S_α in $\mathbf{T} \upharpoonright \alpha$. The ordinals in the set $\{\xi \mid \omega\alpha \leqslant \xi < \omega a + \omega\}$ are then used to provide one-point extensions in T_α of each of the (countably many) branches b_x, $x \in T \upharpoonright \alpha$.

The above construction clearly provides us with an ω_1-tree, $\mathbf{T}$. We need to check that $\mathbf{T}$ is Souslin. It suffices to show that every *maximal* antichain of $\mathbf{T}$ is countable. Let $X \subseteq \omega_1$ be a maximal antichain of $\mathbf{T}$. Set

$$C = \{\alpha \in \omega_1 \mid \omega\alpha = \alpha \wedge X \cap \alpha \text{ is a maximal antichain of } \mathbf{T} \upharpoonright \alpha\}.$$

Now, if $\omega\alpha = \alpha$, then $T \upharpoonright \alpha = T \cap \alpha$, so $X \cap \alpha$ is certainly an antichain of $\mathbf{T} \upharpoonright \alpha$. It is easy to see that C is club in ω_1 now. (The argument was given in 1.5.) So by $\diamondsuit$, we can pick an $\alpha \in C$ so that $X \cap \alpha = S_\alpha$. By the construction of T_α, every element of T_α lies above an element of $X \cap \alpha$. Hence $X \cap \alpha$ is a maximal antichain in $\mathbf{T}$. Thus $X = X \cap \alpha$, which means that X is countable, as required. □

Notice that $\diamondsuit$ implies CH: for if $(S_\alpha \mid \alpha < \omega_1)$ is a $\diamondsuit$-sequence, then for each set $X \subseteq \omega$ there is an ordinal α such that $X = X \cap \alpha = S_\alpha$. In fact $\diamondsuit$ can be regarded as a sort of "super-CH". This is highlighted by the following fact, whose proof is left as an exercise (see Exercise 3). $\diamondsuit$ is equivalent to the existence of a sequence $(S_\alpha \mid \alpha < \omega_1)$ such that $S_\alpha \subseteq \alpha$ for each α and, whenever $X \subseteq \omega_1$ there is *at least one* infinite ordinal α such that $X \cap \alpha = S_\alpha$. CH, on the other hand, is equivalent to the existence of a sequence $(S_\alpha \mid \alpha < \omega_1)$ such that $S_\alpha \subseteq \alpha$ for each α and, whenever $X \subseteq \omega_1$, then for all $\alpha < \omega_1$ there is a $\beta < \omega_1$ such that $X \cap \alpha = S_\beta$.

The following result completes our analysis of the proof of 1.5.

3.3 Theorem. *Assume $V = L$. Then $\diamondsuit$ is valid.*

Proof. By recursion on α we define sets $S_\alpha \subseteq \alpha$, $C_\alpha \subseteq \alpha$ for each $\alpha < \omega_1$.

To commence, we set

$$S_0 = C_0 = \emptyset.$$

If S_α, C_α are defined, set

$$S_{\alpha+1} = C_{\alpha+1} = \alpha + 1.$$

Finally, suppose $\lim(\alpha)$ and S_γ, C_γ are defined for all $\gamma < \alpha$. Let (S_α, C_α) be the $<_L$-least pair of subsets of α such that:

(i) C_α is club in α;
(ii) $(\forall \gamma \in C_\alpha)(S_\alpha \cap \gamma \neq S_\gamma)$,

providing that such sets exist, and set

$$S_\alpha = C_\alpha = \alpha,$$

otherwise.

Notice that, by the above definition, the sequence $((S_\alpha, C_\alpha) \mid \alpha < \omega_1)$ is definable in L_{ω_2}. We show that the sequence $(S_\alpha \mid \alpha < \omega_1)$ satisfies $\diamondsuit$.

Suppose that $(S_\alpha | \alpha < \omega_1)$ were not a $\diamond$-sequence. Then for some set $S \subseteq \omega_1$, the set

$$\{\alpha \in \omega_1 | S \cap \alpha = S_\alpha\}$$

would fail to be stationary in ω_1, so there would be a club set $C \subseteq \omega_1$ such that

$$(\forall \gamma \in C)(S \cap \gamma \neq S_\gamma).$$

Let (S, C) be the $<_L$-least pair of such sets S, C. Notice that this definition will define (S, C) in L_{ω_2}.

Let $X \prec L_{\omega_2}$ be countable, and let $\pi: X \cong L_\beta$. By II.5.11, $X \cap L_{\omega_1}$ is transitive. Let $\alpha = X \cap \omega_1$. Then

$$\pi \restriction L_\alpha = \text{id} \restriction L_\alpha \quad \text{and} \quad \pi(\omega_1) = \alpha.$$

Moreover, as is easily checked (cf. similar arguments in 1.5)

$$\pi(S) = S \cap \alpha, \quad \pi(C) = C \cap \alpha, \quad \pi((S_\gamma | \gamma < \omega_1)) = (S_\gamma | \gamma < \alpha),$$
$$\pi((C_\gamma | \gamma < \omega_1)) = (C_\gamma | \gamma < \alpha).$$

Now, by elementary absoluteness considerations, we have

$\models_{L_{\omega_2}}$ "(S, C) is the $<_L$-least pair of subset of ω_1 such that C is club in ω_1 and $(\forall \gamma \in C)(S \cap \gamma \neq S_\gamma)$".

So, as $\pi^{-1}: L_\beta \prec L_{\omega_2}$,

$\models_{L_\beta}$ "$(S \cap \alpha, C \cap \alpha)$ is the $<_L$-least pair of subsets of α such that $C \cap \alpha$ is club in α and $(\forall \gamma \in C \cap \alpha)((S \cap \alpha) \cap \gamma \neq S_\gamma)$".

Thus, by another simple absoluteness observation (together with II.3.4(i)), we see that $(S \cap \alpha, C \cap \alpha)$ really is the $<_L$-least pair of such subsets of α. But by definition, this means that $S_\alpha = S \cap \alpha$ and $C_\alpha = C \cap \alpha$.

Now, as we saw above,

$\models_{L_\beta}$ "$C \cap \alpha$ is unbounded in α".

Thus $C \cap \alpha$ really must be unbounded in α. But C is closed in ω_1. Hence $\alpha \in C$. But this implies that $S \cap \alpha \neq S_\alpha$, so we have a contradiction. The proof is complete. □

A natural strenghtening of $\diamond$ would be the following: there is a sequence $(S_\alpha | \alpha < \omega_1)$ such that $S_\alpha \subseteq \alpha$ for each α and whenever $X \subseteq \omega_1$ there is a *club* set $C \subseteq \omega_1$ such that $X \cap \alpha = S_\alpha$ for all $\alpha \in C$. However, it is an easy exercise to show that this is impossible. But by modifying the formulation of $\diamond$ a little, we can obtain an equivalent statement which can be strengthened in the above fashion.

Let $\diamondsuit'$ mean the following assertion:

> *There is a sequence $(T_\alpha \mid \alpha < \omega_1)$ such that for each α, T_α is a countable subset of $\mathscr{P}(\alpha)$, with the property that whenever $X \subseteq \omega_1$, the set $\{\alpha \in \omega_1 \mid X \cap \alpha \in T_\alpha\}$ is stationary in ω_1.*

Clearly, $\diamondsuit'$ is a consequence of $\diamondsuit$: if $(S_\alpha \mid \alpha < \omega_1)$ is a $\diamondsuit$-sequence, then $(T_\alpha \mid \alpha < \omega_1)$ is a $\diamondsuit'$-sequence, where we set $T_\alpha = \{S_\alpha\}$ for all $\alpha < \omega_1$. In fact, $\diamondsuit'$ and $\diamondsuit$ are equivalent, as we now show.

3.4 Lemma. *$\diamondsuit'$ and $\diamondsuit$ are equivalent.*

Proof. Let $(T_\alpha \mid \alpha < \omega_1)$ be a $\diamondsuit'$-sequence. We first of all use $(T_\alpha \mid \alpha < \omega_1)$ in order to construct a "$\diamondsuit'$-sequence" on $\omega_1 \times \omega$. That is, we define a sequence $(U_\alpha \mid \alpha < \omega_1)$ such that U_α is a countable subset of $\mathscr{P}(\alpha \times \omega)$ and for each set $X \subseteq \omega_1 \times \omega$, the set

$$\{\alpha \in \omega_1 \mid X \cap (\alpha \times \omega) \in U_\alpha\}$$

is stationary in ω_1.

To this end, choose a bijection

$$j\colon \omega_1 \leftrightarrow \omega_1 \times \omega$$

so that for all limit $\alpha < \omega_1$,

$$(j \restriction \alpha)\colon \alpha \leftrightarrow \alpha \times \omega .$$

For instance, using the fact that any ordinal in ω_1 has a unique expression of the form

$$\delta + 2^m \cdot (2n + 1) - 1 ,$$

where δ is either 0 or else a limit ordinal, and where $m, n \in \omega$, we can set

$$j(\delta + 2^m \cdot (2n + 1) - 1) = (\delta + m, n) .$$

For each $\alpha < \omega_1$, now, set

$$U_\alpha = \begin{cases} \{j'' U \mid U \in T_\alpha\}, & \text{if } \lim(\alpha), \\ \emptyset, & \text{otherwise}. \end{cases}$$

It is easily checked that $(U_\alpha \mid \alpha < \omega_1)$ has the desired properties.

Now let $(U_\alpha^n \mid n < \omega)$ enumerate U_α, for each $\alpha < \omega_1$. Thus $U_\alpha^n \subseteq \alpha \times \omega$ and whenever $X \subseteq \omega_1 \times \omega$ there is a stationary set $E \subseteq \omega_1$ such that for every $\alpha \in E$ there is an $n \in \omega$ such that $X \cap (\alpha \times \omega) = U_\alpha^n$. Now, in general, the n here, for which $X \cap (\alpha \times \omega) = U_\alpha^n$, will depend upon α. But as we shall show below, this is not always the case.

Claim. If $X \subseteq \omega_1 \times \omega$, there is a stationary set $F \subseteq \omega_1$ such that for some *fixed* $n \in \omega$, $X \cap (\alpha \times \omega) = U_\alpha^n$ for all $\alpha \in F$.

To see this, let $X \subseteq \omega_1 \times \omega$ be given. Choose $E \subseteq \omega_1$ stationary so that

$$\alpha \in E \to (\exists n \in \omega)[X \cap (\alpha \times \omega) = U_\alpha^n].$$

Define f: $E \to \omega$ by setting $f(n) = 0$ for $n \in E \cap \omega$, and letting $f(\alpha)$ be the least n such that $X \cap (\alpha \times \omega) = U_\alpha^n$, otherwise. Since f is regressive, Fodor's Theorem (3.1) tells us that for some $n \in \omega$, the set

$$F = \{\alpha \in E \mid f(\alpha) = n\}$$

is stationary in ω_1. Clearly, F is a claimed.

For each $n < \omega$ and each $\alpha < \omega_1$, now, set

$$S_\alpha^n = \{v \in \alpha \mid (v, n) \in U_\alpha^n\}.$$

We show that for some $n \in \omega$, $(S_\alpha^n \mid \alpha < \omega_1)$ is a $\diamondsuit$-sequence. Well, suppose otherwise. Thus for each $n \in \omega$ we can find a set $X_n \subseteq \omega_1$ and a club set $C_n \subseteq \omega_1$ such that

$$\alpha \in C_n \to X_n \cap \alpha \neq S_\alpha^n.$$

Set

$$X = \bigcup_{n<\omega} (X_n \times \{n\}),$$

$$C = \bigcap_{n<\omega} C_n.$$

Then C is club in ω_1 and for all $n < \omega$,

$$\alpha \in C \to X \cap (\alpha \times \omega) \neq U_\alpha^n.$$

This contradicts our earlier claim, and completes the proof. □

The following principle, known as $\diamondsuit^*$ ("diamond-star") is an obvious strengthening of $\diamondsuit'$.

> $\diamondsuit^*$: *there is a sequence* $(S_\alpha \mid \alpha < \omega_1)$ *such that* S_α *is a countable subset of* $\mathscr{P}(\alpha)$ *and for any* $X \subseteq \omega_1$ *there is a club set* $C \subseteq \omega_1$ *such that* $X \cap \alpha \in S_\alpha$ *for all* $\alpha \in C$.

It is clear that $\diamondsuit^*$ implies $\diamondsuit'$ (hence $\diamondsuit$). And it is known that $\diamondsuit^*$ does not follow from $\diamondsuit$. The next theorem provides us with an alternative proof of $\diamondsuit$ from $V = L$.

3.5 Theorem. *Assume $V = L$. Then $\diamondsuit^*$ is true.*

Proof. Define a function f: $\omega \to \omega_1$ by setting

$$f(\alpha) = \text{the least } \gamma > \alpha \text{ such that } \models_{L_\gamma} \text{"}\alpha \text{ is countable"}.$$

Let

$$S_\alpha = \mathscr{P}(\alpha) \cap L_{f(\alpha)}, \quad \alpha < \omega_1.$$

We show that $(S_\alpha | \alpha < \omega_1)$ is a $\diamond^*$-sequence. Since each S_α is clearly a countable subset of $\mathscr{P}(\alpha)$, what we must prove is that if $X \subseteq \omega_1$ is given, there is a club set $C \subseteq \omega_1$ such that $X \cap \alpha \in S_\alpha$ for all $\alpha \in C$.

By recursion, we define a sequence of elementary submodels

$$N_\nu \prec L_{\omega_2}, \quad \nu < \omega_1.$$

Let

$$N_0 = \text{the smallest } N \prec L_{\omega_2} \text{ such that } X \in N;$$
$$N_{\nu+1} = \text{the smallest } N \prec L_{\omega_2} \text{ such that } N_\nu \cup \{N_\nu\} \subseteq N;$$
$$N_\delta = \bigcup_{\nu<\delta} N_\nu, \quad \text{if } \lim(\delta).$$

By II.5.11 we can define $\alpha_\nu \in \omega_1$ by

$$\alpha_\nu = N_\nu \cap \omega_1.$$

Clearly, the set $C = \{\alpha_\nu | \nu < \omega_1\}$ is club in ω_1. We show that $X \cap \alpha \in S_\alpha$ for each $\alpha \in C$. Let $\nu < \omega_1$ be given. Let

$$\pi\colon N_\nu \cong L_\beta.$$

Then,

$$\pi \restriction \alpha_\nu = \mathrm{id} \restriction \alpha_\nu, \quad \pi(\omega_1) = \alpha_\nu, \quad \pi(X) = X \cap \alpha_\nu.$$

In particular,

$$X \cap \alpha_\nu \in L_\beta.$$

But

$$\models_{L_{f(\alpha_\nu)}} \text{“}\alpha_\nu \text{ is countable”},$$

whereas

$$\alpha_\nu = \omega_1^{L_\beta}.$$

Hence $\beta < f(\alpha_\nu)$ and we see that

$$X \cap \alpha_\nu \in L_{f(\alpha_\nu)}.$$

Thus $X \cap \alpha_\nu \in S_{\alpha_\nu}$ and we are done. □

We turn now to our analysis of the construction of a Kurepa tree from $V = L$ (2.2). The essential combinatorial property of L used here is the following generalisation of $\diamond^*$ known as $\diamond^+$ (“diamond-plus”):

> $\diamond^+$: *there is a sequence $(S_\alpha | \alpha < \omega_1)$ such that S_α is a countable subset of $\mathscr{P}(\alpha)$ and whenever $X \subseteq \omega_1$ there is a club set $C \subseteq \omega_1$ such that for all $\alpha \in C$, both $X \cap \alpha \in S_a$ and $C \cap \alpha \in S_\alpha$.*

It is clear that $\diamondsuit^*$ is just an apparently weaker version of $\diamondsuit^+$. In fact $\diamondsuit^+$ is a real strengthening of $\diamondsuit^*$. In particular, $\diamondsuit^*$ does not imply the existence of a Kurepa tree, whereas $\diamondsuit^+$ does, as we show below.

3.6 Theorem. *Assume $\diamondsuit^+$. Then there is a Kurepa tree.*

Proof. As in 2.2, we choose to establish the existence of a family $\mathscr{F} \subseteq \mathscr{P}(\omega_1)$ such that $|\mathscr{F}| = \omega_2$ and $|\mathscr{F} \upharpoonright \alpha| \leqslant \omega$ for all $\alpha < \omega_1$, rather than construct a Kurepa tree outright.

Let $(S_\alpha | \alpha < \omega_1)$ be a $\diamondsuit^+$-sequence. Recalling that H_{ω_1} is the set of all hereditarily countable sets, for each $\alpha < \omega_1$, let $M_\alpha \prec H_{\omega_1}$ be countable and such that

$$(\alpha + 1) \cup (\bigcup_{\beta \leqslant \alpha} S_\beta) \subseteq M_\alpha .$$

Set

$$\mathscr{F} = \{x \subseteq \omega_1 | (\forall \alpha < \omega_1)(x \cap \alpha \in M_\alpha)\} .$$

If we can prove that $|\mathscr{F}| = \omega_2$ we shall clearly be done. Suppose that, on the contrary, $|\mathscr{F}| = \omega_1$. (It is clear that $\mathscr{F}$ is at least uncountable, since $\{\alpha\} \in \mathscr{F}$ for all $\alpha < \omega_1$.) Let $(x_\nu | \nu < \omega_1)$ enumerate all unbounded members of $\mathscr{F}$. (This sequence does not have to be one-one. Hence, as we clearly have $\omega_1 \in \mathscr{F}$, the sequence does exist.) For each $\nu < \omega_1$, let

$$B_\nu = \{\alpha \in \omega_1 | \lim(\alpha) \wedge x_\nu \cap \alpha \text{ is unbounded in } \alpha\} .$$

It is easily seen that B_ν is club in ω_1. Set

$$B = \{\alpha \in \omega_1 | \lim(\alpha) \wedge (\forall \nu < \alpha)(\alpha \in B_\nu)\} .$$

It is easily seen that B is club in ω_1. (B is essentially the *diagonal intersection* of the sequence $(B_\nu | \nu < \omega_1)$, already mentioned in 3.1.) Applying $\diamondsuit^+$ to the set $B \subseteq \omega_1$ we obtain a club set $C \subseteq \omega_1$ such that

$$\alpha \in C \rightarrow B \cap \alpha, \quad C \cap \alpha \in S_\alpha .$$

Let $(\alpha_\nu | \nu < \omega_1)$ enumerate, monotonically, the club set

$$\{\alpha \in B | \alpha = \sup(C \cap \alpha)\} .$$

For $\nu < \omega_1$, set

$$\beta_\nu = \min(C - (\alpha_\nu + 1)) .$$

Thus

$$\alpha_\nu < \beta_\nu < \alpha_{\nu+1} .$$

Set

$$x = \{\beta_\nu | \nu < \omega_1\} .$$

For any $\nu < \omega_1$,

$$\nu \leqslant \alpha_\nu < \alpha_{\nu+1} \in B,$$

so $x_\nu \cap \alpha_{\nu+1}$ is unbounded in $\alpha_{\nu+1}$. But

$$x \cap \alpha_{\nu+1} = \{\beta_\tau \mid \tau \leqslant \nu\} \subseteq \beta_\nu + 1 < \alpha_{\nu+1}.$$

Hence $x \neq x_\nu$. We obtain our desired contradiction now by proving that $x \in \mathscr{F}$, i.e. that $x \cap \alpha \in M_\alpha$ for all $\alpha < \omega_1$.

If $x \cap \alpha$ is finite, then it is immediate that $x \cap \alpha \in M_\alpha$, since

$$\alpha \subseteq M_\alpha \prec H_{\omega_1}.$$

So assume $x \cap \alpha$ is infinite. Let $\beta \leqslant \alpha$ be the greatest limit point of $x \cap \alpha$. Since $x \cap \alpha$ differs from $x \cap \beta$ by at most finitely many points, and since M_α is a model of ZF^-, it suffices to prove that $x \cap \beta \in M_\alpha$. Now, β is a limit point of x and $x \subseteq C$, so as C is closed in ω_1, $\beta \in C$. Thus

$$B \cap \beta, \quad C \cap \beta \in S_\beta.$$

But $\beta \leqslant \alpha$. Hence

$$B \cap \beta, \quad C \cap \beta \in M_\alpha.$$

Let λ be such that

$$\beta = \sup_{\nu < \lambda} \beta_\nu.$$

Then

$$\{\alpha_\nu \mid \nu < \lambda\} = \{\alpha \in B \cap \beta \mid \alpha = \sup[(C \cap \beta) \cap \alpha]\}.$$

So, as

$$B \cap \beta, C \cap \beta \in M_\alpha \prec H_{\omega_1},$$

we conclude that

$$\{\alpha_\nu \mid \nu < \lambda\} \in M_\alpha.$$

But for $\nu < \lambda$,

$$\beta_\nu = \min[(C \cap \beta) - (\alpha_\nu + 1)].$$

Hence

$$x \cap \beta = \{\beta_\nu \mid \nu < \lambda\} \in M_\alpha,$$

and we are done. □

To complete our analysis of 2.2 now, we prove:

3.7 Theorem. *Assume $V = L$. Then $\diamondsuit^+$ is valid.*

Proof. As in 2.2 we may define a function f: $\omega_1 \to \omega_1$ by letting $f(\alpha)$ be the least ordinal such that

$$\alpha \in L_{f(\alpha)} \prec L_{\omega_1}.$$

Set

$$S_\alpha = \mathscr{P}(\alpha) \cap L_{f(\alpha)}.$$

Notice that f and $(S_\alpha \mid \alpha < \omega_1)$ are definable in L_{ω_2} (using the above definitions). We prove that $(S_\alpha \mid \alpha < \omega_1)$ satisfies $\diamondsuit^+$.

Suppose that $(S_\alpha \mid \alpha < \omega_1)$ did not satisfy $\diamondsuit^+$, and let X be the $<_L$-least subset of ω_1 such that for all club sets $C \subseteq \omega_1$ there is an $\alpha \in C$ such that it is not the case that both $X \cap \alpha$ and $C \cap \alpha$ lie in S_α. Notice that X is definable in L_{ω_2} by means of this definition.

By recursion, define a sequence of elementary submodels $N_\nu \prec L_{\omega_2}$, $\nu < \omega_1$, as follows:

$$N_0 = \text{the smallest } N \prec L_{\omega_2};$$

$$N_{\nu+1} = \text{the smallest } N < L_{\omega_2} \text{ such that } N_\nu \cup \{N_\nu\} \subseteq N;$$

$$N_\delta = \bigcup_{\nu<\delta} N_\nu, \qquad \text{if } \lim(\delta).$$

By II.5.11, $N \cap L_{\omega_1}$ is transitive. Set

$$\alpha_\nu = N_\nu \cap \omega_1.$$

Clearly, $(\alpha_\nu \mid \nu < \omega_1)$ is a normal sequence in ω_1. Let

$$\pi_\nu\colon N_\nu \cong L_{\beta(\nu)}.$$

Clearly,

$$\pi_\nu \restriction L_{\alpha_\nu} = \mathrm{id} \restriction L_{\alpha_\nu}, \qquad \pi_\nu(\omega_1) = \alpha_\nu, \qquad \pi_\nu(X) = X \cap \alpha_\nu.$$

Let C be the set of all limit points of the set $\{\beta(\nu) \mid \nu < \omega_1\}$. C is club in ω_1. We obtain our contradiction by showing that for all $\alpha \in C$,

$$X \cap \alpha,\ C \cap \alpha \in S_\alpha.$$

Let $\alpha \in C$ be given. For some limit ordinal $\lambda < \omega_1$,

$$\alpha = \sup_{\nu<\lambda} \beta(\nu).$$

Claim 1. $\alpha = \alpha_\lambda$.

To see this, it suffices to prove that for all $\nu < \omega_1$,

$$\alpha_\nu < \beta(\nu) < \alpha_{\nu+1}.$$

Well, clearly, $\alpha_\nu < \beta(\nu)$. But $\beta(\nu)$ is definable from N_ν since $L_{\beta(\nu)}$ is the transitive collapse of N_ν, and moreover this definition relativises to L_{ω_2} (i.e. is absolute for L_{ω_2}). So, as

$$N_\nu \in N_{\nu+1} \prec L_{\omega_2},$$

we have $\beta(\nu) \in N_{\nu+1}$. Hence $\beta(\nu) \in \alpha_{\nu+1}$, and the claim is established.

Claim 2. $\beta(\lambda) < f(\alpha)$.

To see this, note first that by definition of f,

$$\models_{L_{f(\alpha)}} \text{“}\alpha \text{ is countable”}.$$

But,

$$\alpha = \alpha_\lambda = \omega_1^{L_{\beta(\lambda)}}.$$

Hence $\beta(\lambda) < f(\alpha)$, as claimed.

Now, by claim 1,

$$X \cap \alpha = \pi_\lambda(X) \in L_{\beta(\lambda)},$$

so by claim 2,

$$X \cap \alpha \in L_{f(\alpha)}.$$

Thus

$$X \cap \alpha \in S_\alpha,$$

and it remains to prove that $C \cap \alpha \in S_\alpha$. It clearly suffices to prove that

$$\{\beta(\nu) \mid \nu < \lambda\} \in L_{f(\alpha)}.$$

This is proved exactly as in 2.2, so we do not repeat the details here. Our proof is complete. □

Exercises

1. *ω_1-Trees and Souslin Trees* (Section 1)

Let **T** be an ω_1-tree, **P** a totally ordered set. **T** is said to be **P**-*embeddable* iff there is an order-preserving map f: **T** → **P**. Our interest concerns the cases when **P** is either the rationals, $\mathbb{Q}$, or else the reals, $\mathbb{R}$.

1 A. Show that an ω_1-tree, **T**, is $\mathbb{Q}$-embeddable iff there are antichains A_n, $n < \omega$, of **T** such that

$$T = \bigcup_{n<\omega} A_n.$$

1 B. Show that if an ω_1-tree, **T**, is $\mathbb{R}$-embeddable, it is an Aronszajn tree but not a Souslin tree. (Hint: It is possible to utilise 1 A here.)

1 C. Construct a $\mathbb{Q}$-embeddable ω_1-tree. (The tree constructed in 1.1 *almost* suffices.) Such trees are sometimes referred to as *special* Aronszajn trees, though we shall use this name for a different notion (see Exercise IV.1).

1 D. It is known to be consistent with ZFC that every Aronszajn tree is $\mathbb{Q}$-embeddable. (See *Devlin and Johnsbråten* (1974).) Show that if $V = L$ there is a $\mathbb{R}$-embeddable Aronszajn tree which is not $\mathbb{Q}$-embeddable. (Hint: Take the elements of **T** to be countable one-one sequences of integers whose ranges are co-infinite in ω, ordered by inclusion. Construct **T** by recursion on the levels to satisfy the following condition:

> *if* $\alpha < \beta < \omega_1$ *and* $s \in T_\alpha$ *and* σ *is a finite set of integers, disjoint from* $\operatorname{ran}(s)$, *there is a* $t \in T_\beta$ *such that* $s \subset t$ *and* $\sigma \cap \operatorname{ran}(t) = \emptyset$.

Use $V = L$ to ensure that if $f\colon T \to \mathbb{Q}$ were an embedding, there would be a limit ordinal $\alpha < \omega_1$ such that for each $x \in T_\alpha$ there is a $y \in T$, $y <_T x$, such that $f(y) = f(x)$.)

2. *Kurepa Trees* (Section 2)

2 A. Assume $V = L$. Define $f\colon \omega_1 \to \omega_1$ by setting

$$f(\alpha) = \text{the least } \gamma \text{ such that } \alpha \in L_\gamma \prec L_{\omega_1}.$$

Construct an ω_1-tree as follows. The elements of T_α will be members of ${}^\alpha 2$. The ordering of **T** will be $\subset$. Let $T_0 = \{\emptyset\}$. If T_α is defined, let

$$T_{\alpha+1} = \{s \frown \langle i \rangle \mid s \in T_\alpha \wedge i = 0, 1\}.$$

If $\lim(\alpha)$ and $T \upharpoonright \alpha$ is defined, let

$$T_\alpha = \{\bigcup b \mid b \text{ is an } \alpha\text{-branch of } \mathbf{T} \upharpoonright \alpha \text{ lying in } L_{f(\alpha)}\}.$$

Prove that **T** is a Kurepa tree.

2 B. Let **T** be the Kurepa tree constructed in 2 A. Show that there is a set $U \subseteq T$ which is a Souslin tree under the induced ordering.

3. *The Combinatorial Principle* $\diamond$ (Section 3)

3 A. Let $\diamond^-$ be the following principle: there is a sequence $(S_\alpha \mid \alpha < \omega_1)$ such that S_α is a countable subset of $\mathscr{P}(\alpha)$ for each α and whenever $X \subseteq \omega_1$ there is an infinite ordinal α such that $X \cap \alpha \in S_\alpha$. Prove that $\diamond^-$ is equivalent to $\diamond$. (Hint: First show that $\diamond^-$ implies $\diamond^{-+}$, where $\diamond^{-+}$ is the same as $\diamond^-$ except that the α which is asserted to exist is required to be a limit ordinal. Now let $(S_\alpha \mid \alpha < \omega_1)$ be as in $\diamond^{-+}$. Define $j\colon \omega_1 \to \omega_1$ by $j(v) = 2v$. Set $T_\alpha = \{j^{-1}{}''x \mid x \in S_\alpha\}$. Then $(T_\alpha \mid \alpha < \omega_1)$ is a $\diamond'$-sequence. The idea is that, given a club set $C \subseteq \omega_1$ from which we must find an α

with $X \cap \alpha \in T_\alpha$, for a given $X \subseteq \omega_1$, we construct a set $Y \subseteq \omega_1$ whose intersection with the even ordinals is $j''X$ and whose intersection with the odd ordinals is a diagonalisation set ensuring that if $Y \cap \alpha \in S_\alpha$, then $\alpha \in C$.)

3 B. Show that CH is equivalent to the existence of a sequence $(S_\alpha \mid \alpha < \omega_1)$ such that S_α is a countable subset of $\mathscr{P}(\alpha)$ $(\alpha < \omega_1)$ and whenever $X \subseteq \omega_1$, then

$$(\forall \alpha)(\exists \beta)(X \cap \alpha \in S_\beta).$$

3 C. Show that $\diamondsuit$ is equivalent to the existence of a sequence $(S_\alpha \mid \alpha < \omega_1)$ and a function $f\colon \omega_1 \to \omega_1$ such that S_α is a countable subset of $\mathscr{P}(\alpha)$ for each α and whenever $X \subseteq \omega_1$ then for uncountably many $\alpha < \omega_1$.

$$(\exists \beta < f(\alpha))(X \cap \alpha \in S_\beta).$$

3 D. Let P assert the existence of a sequence $(U_\alpha \mid \alpha < \omega_1 \wedge \lim(\alpha))$ such that U_α is an increasing ω-sequence, cofinal in α, with the property that whenever $X \subseteq \omega_1$ is uncountable there is an α such that $U_\alpha \subseteq X$. Show that in the presence of CH, P is equivalent to $\diamondsuit$. (It is known that in the absence of CH, P does not necessarily imply $\diamondsuit$.) (Hint: Let $(X_\alpha \mid \alpha < \omega_1)$ enumerate all bounded subsets of ω_1 so that each set appears cofinally often. Define $S_\alpha = \bigcup \{X_\beta \cap \alpha \mid \beta \in U_\alpha\}$ to obtain a $\diamondsuit^-$-sequence.)

3 E. Show that $\diamondsuit$ implies the existence of two non-isomorphic Souslin trees.

3 F. Show that $\diamondsuit$ implies the existence of an $\mathbb{R}$-embeddable tree which is not $\mathbb{Q}$-embeddable.

3 G. Show that $\diamondsuit$ implies the existence of a family $\{A_\nu \mid \nu < \omega_2\}$ of stationary subsets of ω_1 such that the intersection of any two of them is countable.

4. $\diamondsuit$ *and* $\diamondsuit^+$ *in* $L[A]$ (Section 3)

Using the same kind of ideas employed in Exercises II.2 and II.4, we prove that $\diamondsuit$ and $\diamondsuit^+$ hold in $L[A]$, where $A \subseteq \omega_1^{L[A]}$.

4 A. Assume $V = L[A]$, where $A \subseteq \omega_1$. Prove that $\diamondsuit$ is valid. (Hint: For each limit ordinal α, let (S_α, C_α) be the $<_{L[A \cap \alpha]}$-least pair of subsets of α lying in $L[A \cap \alpha]$ such that C_α is club in α and $S_\alpha \cap \gamma \neq S_\gamma$ for all $\gamma \in C_\alpha$, whenever possible. Now argue analogously to 3.3.)

4 B. Suppose $V = L[A]$, where $A \subseteq \omega_1$. Prove that if $\omega_1^{L[A \cap \alpha]} < \omega_1$ for all $\alpha < \omega_1$, then ω_1 is inaccessible in $L[A \cap \alpha]$ for all $\alpha < \omega_1$. (Hint: If ω_1 were not inaccessible in $L[A \cap \alpha]$ for all $\alpha < \omega_1$, then for some α we would have $\omega_1 = (\theta^+)^{L[A \cap \alpha]}$. By a condensation argument, θ can be shown to be countable in some $L[A \cap \gamma]$. Then $\omega_1 = \omega_1^{L[A \cap \delta]}$ for $\delta = \max(\alpha, \gamma)$.)

4 C. Assume $V = L[A]$, where $A \subseteq \omega_1$. Prove that $\diamondsuit^+$ is valid. (Hint: Define $\delta\colon \omega_1 \to \omega_1$ by cases, depending on A. If $\omega_1 = \omega_1^{L[A \cap \alpha]}$ for some $\alpha < \omega_1$, let α_0 be the least such, and let $\delta(\alpha) = \omega_1 \cap M_\alpha$, where M_α is the smallest $M \prec L_{\omega_1}[A]$ such

that $\alpha_0, \alpha \in M$. Otherwise let $\delta(\alpha) = \omega_3^{L[A \cap \alpha]}$ (which is countable by virtue of 4 B), and set $\alpha_0 = \omega$. For $\alpha < \omega_1$ now, let $\hat{\alpha} = \max(\alpha, \alpha_0)$. Set $S_\alpha = \mathscr{P}(\alpha) \cap L_{\delta(\alpha)}[A \cap \hat{\alpha}]$. Now argue as in 3.7, except for the fact that there are now the two cases to consider instead of one.)

4 D. Prove that if there is no Kurepa tree, then ω_2 is inaccessible in L. (Hint: Use 4 C, together with an absoluteness argument concerning Kurepa trees.)