Appendix
Nonstandard Compactness Arguments and the Admissible Cover

One of the subjects we have not touched on in this book is applications of infinitary logic to constructing models of set theory and the relationship between compactness and forcing arguments. At one time we planned to include a chapter on these matters, but the book developed along other lines.

In this appendix we present one example of such a result because it leads very naturally to *the admissible cover* of a model $\mathfrak{M}$ of set theory. We want to treat this admissible set for two reasons. In the first place, it gives an example of an admissible set with urelements which has no counterpart in the theory without urelements, and it is as different from $\mathbb{HYP}_{\mathfrak{M}}$ as possible. Secondly, we promised (in Barwise [1974]) to present the details of the construction of this admissible set in this book.

1. Compactness Arguments over Standard Models of Set Theory

Let $\mathbb{A}=\langle A,\in\rangle$ be a countable transitive model of ZF. Then $\mathbb{A}$ is an admissible set and, moreover, $(\mathbb{A},R)$ is admissible for every definable relation R. We can therefore apply Completeness and Compactness to $\mathsf{L}_{\mathbb{A}}$ or $\mathsf{L}_{(\mathbb{A},R)}$, for any such R. There are many interesting results to be obtained in this way; we present one here and refer the reader to Barwise [1971], Barwise [1974], Friedman [1973], Krivine-MacAloon [1973], Suzuki-Wilmers [1973], and Wilmers [1973] for other examples. We also refer the reader to Keisler [1973] for connections with forcing.

The axiom $\mathrm{V}=\mathrm{L}$ asserts that every set is constructible.

1.1 Theorem. *Let* $\mathbb{A}$ *be a countable transitive model of* ZF. *There is an end extension* $\mathfrak{B}=\langle B,E\rangle$ *of* $\mathbb{A}$ *which is a model of* $\mathrm{ZF}+\mathrm{V}=\mathrm{L}$.

Proof. Let T be the theory of $\mathsf{L}_{\mathbb{A}}$ containing:

ZF.

The Infinitary diagram of $\mathbb{A}$.

We need to see that $T\cup\{\mathrm{V}=\mathrm{L}\}$ has a model. If not, then

$$T\models \mathrm{V}\neq\mathrm{L}$$

so

$$T \vdash \mathrm{V} \neq \mathrm{L}$$

by the Extended Completeness Theorem of § III.5. Thus $\mathbb{A}$ is a model of the Σ_1 sentence expressing:

(1) $\exists \Phi\, \exists p\, [p$ is a proof of $(\bigwedge \Phi) \to (\mathrm{V} \neq \mathrm{L})$ where $\forall x \in \Phi\, (x \in \mathrm{ZF}$ or x is a member of the infinitary diagram)$]$.

This Σ_1 sentence contains no parameters. Now let $\alpha = o(\mathbb{A})$ and let $\mathbb{A}_0 = \mathrm{L}(\alpha)$. Then $\mathbb{A}_0$ is a model of $\mathrm{ZF} + \mathrm{V} = \mathrm{L}$ (it is the constructible sets in the model $\mathbb{A}$ of ZF) and, interpreting Shoenfield's Lemma (Theorem V.8.1) in $\mathbb{A}$, we have: Any Σ_1 sentence true in $\mathbb{A}$ is true in $\mathbb{A}_0$.

Thus the sentence (1) is also true in $\mathbb{A}_0$. But this means that there is some subset T_0 of the infinitary diagram of $\mathbb{A}_0$ such that

$$T_0 + \mathrm{ZF} \vdash \mathrm{V} \neq \mathrm{L}$$

which is ridiculous since $\mathbb{A}_0$ itself is a model of $T_0 + \mathrm{ZF} + \mathrm{V} = \mathrm{L}$. ▯

There are a number of extensions of the above which will strike the reader; most of these are covered by the version contained in Theorem 3.1 of Barwise [1971]. What is not so obvious is how to extend the result from standard models of set theory to nonstandard models. For if $\mathfrak{A} = \langle A, E \rangle$ is a nonstandard model of ZF then we have no guarantee that a "proof" in the sense of $\mathfrak{A}$ proves anything at all. What we need is a new admissible set intimately related to $\mathfrak{A}$ which will allow us to carry out the above, and similar, proofs.

What is even less obvious is how to generalize results like Theorem 1.1 to the uncountable. There are uncountable models of ZFC with no end extension satisfying $\mathrm{V} = \mathrm{L}$, assuming of course that ZFC is consistent. Is there an uncountable generalization of Theorem 1.1, involving consideration like $h_\Sigma(\mathbb{A})$, which explains more satisfactorily why the result holds in the countable case? The same question applies to all the results in Barwise [1971] and Barwise [1974].

2. *The Admissible Cover and its Properties*

In this section we will be considering models of set theory as basic structures over which we build admissible sets. Thus we denote such structures by $\mathfrak{M} = \langle M, E \rangle$ where E is binary. Recall, for $x \in \mathfrak{M}$, the definition

$$x_E = \{ y \in M \mid yEx \}.$$

Let L contain only the relation symbol E; let $\mathsf{L}^* = \mathsf{L}(\in, \mathsf{F})$ where F is a unary function symbol. Let (†) be the axiom of L^* given by

(†) $$\forall p, x\, [x\, \mathsf{E}\, p \leftrightarrow x \in \mathsf{F}(p)] \wedge \forall a\, [\mathsf{F}(a) = 0].$$

An admissible set (for L^*), say $\mathbb{A}_{\mathfrak{M}}=(\mathfrak{M}; A, \in, F)$, is a *cover* of $\mathfrak{M}$ if $\mathbb{A}_{\mathfrak{M}}$ is a model of (†). That is, $\mathbb{A}_{\mathfrak{M}}$ is a cover of $\mathfrak{M}$ iff

$$F(x)=x_E \quad \text{for} \quad x \in M,$$

$$F(x)=0 \quad \text{for} \quad x \in A.$$

The point of the definition is pretty obvious, assuming that we are working in an admissible set $\mathbb{A}_{\mathfrak{M}}$ with $\mathfrak{M} \notin \mathbb{A}_{\mathfrak{M}}$. A quantifier like $\forall x\,(x \mathsf{E} y \rightarrow \ldots)$ is a bounded quantifier in the sense of L but it is not bounded, in general, in L^*. Using the axiom (†) however, it becomes equivalent to the bounded quantifier $\forall x \in \mathsf{F}(y)\,(\ldots)$.

In this way every formula φ of L translates into a formula $\hat{\varphi}$ of L^* with the properties:

if φ is Δ_0 (resp. Σ_1) is L then $\hat{\varphi}$ is Δ_0 (resp. Σ_1) in L^*

$$\mathrm{KPU}+(\dagger) \vdash \forall p_1, \ldots, p_n\,[\varphi(p_1, \ldots, p_n) \leftrightarrow \hat{\varphi}(p_1, \ldots, p_n)].$$

We use these remarks below without comment.

There are many admissible sets which cover a given structure $\mathfrak{M}$. For example, if $\mathbb{A}_{\mathfrak{M}}=(\mathfrak{M}; A, \in)$ is admissible *above* $\mathfrak{M}$ (in the sense of $\mathsf{L}(\in)$) then we can define an $\mathbb{A}_{\mathfrak{M}}$-recursive F by

$$F(x)=\{y \in M \mid yEx\}, \quad x \in M,$$

$$F(x)=0, \qquad\qquad x \notin M,$$

and then $(\mathbb{A}_{\mathfrak{M}}, F)$ will be admissible in the sense of $\mathsf{L}(\in, \mathsf{F})$ and will cover $\mathfrak{M}$. These admissible sets are not tied closely enough to the intended interpretation of $\mathfrak{M}$ for the applications we have in mind; they are too big with too many subsets of $\mathfrak{M}$. What we would like would be an admissible set $\mathbb{A}_{\mathfrak{M}}$ which covers $\mathfrak{M}$ and whose only sets of urelements are the sets of the form p_E for $p \in \mathfrak{M}$.

2.1 Definition. Let $\mathfrak{M}=\langle M, E\rangle$ be an L-structure and let $\mathbb{C}\mathrm{ov}_{\mathfrak{M}}$ be the intersection of all admissible sets which cover $\mathfrak{M}$. More precisely,

$$\mathbb{C}\mathrm{ov}_{\mathfrak{M}}=(\mathfrak{M}; A, \in, F)$$

where:

$$A=\bigcap\{B \mid (\mathfrak{M}; B, \in, F) \text{ is admissible and covers } \mathfrak{M}\}.$$

$$F(p)=p_E \quad \text{for} \quad p \in M.$$

$$F(a)=0 \quad \text{for} \quad a \in A.$$

2.2 Theorem. *If* $\mathfrak{M}$ *is a model of* KP *then* $\mathbb{C}\mathrm{ov}_{\mathfrak{M}}$ *is admissible.* $\mathbb{C}\mathrm{ov}_{\mathfrak{M}}$ *is called the* admissible cover *of* $\mathfrak{M}$.

Proof. Deferred to § 3. □

If we proved this theorem right now, the proof would look complicated and *ad hoc*. What we shall do instead is to develop further properties of the admissible cover in this section until, by the end of the section, we will know almost exactly what $\mathfrak{Cov}_{\mathfrak{M}}$ looks like. This should make the proofs (in § 3) easier to follow.

The next property of the admissible cover suggests the main step in the proof of Theorem 2.2 and shows us that $\mathfrak{Cov}_{\mathfrak{M}}$ really lives in $\mathfrak{M}$. (The corollaries of Theorem 2.3 are easier to understand than 2.3 at a first reading.)

2.3 Theorem. *Let $\mathfrak{M}=\langle M,E\rangle$ be a model of* KP. *There is a single valued notation system p projecting $\mathfrak{Cov}_{\mathfrak{M}}$ into $\mathfrak{M}$ satisfying the following equations (where we use $\dot{x}$ for the unique y such that $p(x)=\{y\}$, where $0,1$ denote the first two ordinals in the sense of $\mathfrak{M}$ and where $\langle\,,\rangle$ is the ordered pair operation as defined in $\mathfrak{M}$):*

(i) *For $x\in M$,*

$$\dot{x}=\langle 0,x\rangle$$

(ii) *for $a\in\mathfrak{Cov}_{\mathfrak{M}}$, there is a $y\in M$ such that*

$$\dot{a}=\langle 1,y\rangle$$

and $y_E=\{\dot{x}\mid x\in a\}$.

Proof. Deferred to § 3, 3.1—3.7. ☐

Call a set $a\subseteq\mathfrak{M}$ of urelements $\mathfrak{M}$-*finite* if $a=x_E$ for some $x\in\mathfrak{M}$.

2.4 Corollary. *Let $\mathfrak{M}\models$KP and let $a\subseteq\mathfrak{M}$. Then a is $\mathfrak{M}$-finite iff $a\in\mathfrak{Cov}_{\mathfrak{M}}$. Hence for any $a\in\mathfrak{Cov}_{\mathfrak{M}}$, the support of a is $\mathfrak{M}$-finite. In particular, $M\notin\mathfrak{Cov}_{\mathfrak{M}}$.*

Proof. Let $a\subseteq\mathfrak{M}$, $a\in\mathfrak{Cov}_{\mathfrak{M}}$. Using the notation from 2.3,

$$\dot{a}=\langle 1,y\rangle$$

where $y_E=\{\dot{x}:x\in a\}$. But $a\subseteq\mathfrak{M}$ so $\dot{x}=\langle 0,x\rangle$ for all $x\in a$. Then we can define, inside the model $\mathfrak{M}$, the following set by Σ Replacement, remembering that $\mathfrak{M}\models$KP:

$$z=\{x\mid\langle 0,x\rangle Ey\}$$

and then $z_E=a$. The converse is trivial. ☐

Corollary 2.4 is very useful in compactness arguments involving $\mathfrak{Cov}_{\mathfrak{M}}$, for it tells us that if $T_0\in\mathfrak{Cov}_{\mathfrak{M}}$ is a set of infinitary sentences, then the set

$$\{x\in M:\mathsf{x}\ \text{is mentioned in}\ T_0\}$$

is $\mathfrak{M}$-finite. Recall that x is the constant symbol used to denote x. ☐

We can use the projection from 2.3 to identify the pure sets in $\mathbb{C}\mathrm{ov}_{\mathfrak{M}}$ and the ordinals of $\mathbb{C}\mathrm{ov}_{\mathfrak{M}}$.

2.5 Corollary. *Let $\mathfrak{M} \models \mathrm{KP}$. Let $\mathbb{A}_0$ be the transitive set isomorphic to $\mathcal{W}\!f(\mathfrak{M})$. The pure sets in $\mathbb{C}\mathrm{ov}_{\mathfrak{M}}$ are exactly the sets in $\mathbb{A}_0$. In particular, $o(\mathbb{C}\mathrm{ov}_{\mathfrak{M}}) = o(\mathbb{A}_0)$.*

Proof. Since $\mathbb{A}_0$ is admissible (by the Truncation Lemma) it is closed under TC so it suffices to prove that every transitive set $a \in \mathbb{A}_0$ is in $\mathbb{C}\mathrm{ov}_{\mathfrak{M}}$ in order to prove $\mathbb{A}_0 \subseteq \mathbb{C}\mathrm{ov}_{\mathfrak{M}}$, since $\mathbb{C}\mathrm{ov}_{\mathfrak{M}}$ is transitive. Let $a \in \mathbb{A}_0$ be transitive and let

$$\langle a, \in \rangle \cong \langle x, E \upharpoonright x_E \rangle$$

where $x \in \mathcal{W}\!f(\mathfrak{M})$. Since $\mathbb{C}\mathrm{ov}_{\mathfrak{M}}$ is admissible, by 2.2, we can apply Theorem V.3.1 in $\mathbb{C}\mathrm{ov}_{\mathfrak{M}}$ to see that $a \in \mathbb{C}\mathrm{ov}_{\mathfrak{M}}$. To prove the other inclusion define the following function by recursion in $\mathfrak{M}$ (more precisely, define it by Σ Recursion in KP and interpret the result in $\mathfrak{M}$):

$$\langle 0, x \rangle' = x$$
$$\langle 1, x \rangle' = \{ y' \mid yEx \} .$$

(It is only the second clause which is relevant here but we'll use $'$ again later.) Let $\eta : \langle \mathcal{W}\!f(\mathfrak{M}), E \rangle \cong \langle A_0, \in \rangle$ and consider the following diagram, where $D_0 = \{ \dot{a} \mid a \text{ a pure set in } \mathbb{C}\mathrm{ov}_{\mathfrak{M}} \}$:

$$\begin{tikzcd}
\text{Pure part}(\mathbb{C}\mathrm{ov}_{\mathfrak{M}}) \arrow[r, "\cdot"] \arrow[d, dashed, "\mathrm{Id}"'] & D_0 \subseteq M \arrow[d, "'"] \\
\mathbb{A}_0 & \mathcal{W}\!f(\mathfrak{M}) \arrow[l, "\eta"']
\end{tikzcd}$$

We claim that, for every pure set $a \in \mathbb{C}\mathrm{ov}_{\mathfrak{M}}$, $(\dot{a})' \in \mathcal{W}\!f(\mathfrak{M})$ and $\eta((\dot{a})') = a$, which will conclude 2.5. The proof is by induction on $\in$. First, $\dot{a} = \langle 1, x \rangle$ where $x_E = \{ \dot{b} : b \in a \}$. But then $(\dot{a})' = z$ where

$$z_E = \{ y' \mid yEx \}$$
$$= \{ (\dot{b})' \mid b \in a \} .$$

Thus $(\dot{a})'_E \subseteq \mathcal{W}\!f(\mathfrak{M})$ by part of the induction hypothesis, and hence $(\dot{a})' \in \mathcal{W}\!f(\mathfrak{M})$. Computing $\eta((\dot{a})')$ we get

$$\eta((\dot{a})') = \{ \eta(y') \mid yEz \}$$
$$= \{ \eta((\dot{b})') \mid b \in a \} .$$

The other part of the induction hypothesis states that $\eta((\dot{b})')=b$ for $b\in a$ so we get

$$\begin{aligned}\eta((\dot{a})') &= \{b\,|\,b\in a\}\\ &= a\,. \quad \square\end{aligned}$$

Using 2.4 and 2.5 we can give a picture of $\mathfrak{Cov}_{\mathfrak{M}}$. The dotted line in $\mathfrak{M}$ is the level at which it becomes nonstandard (if it is nonstandard).

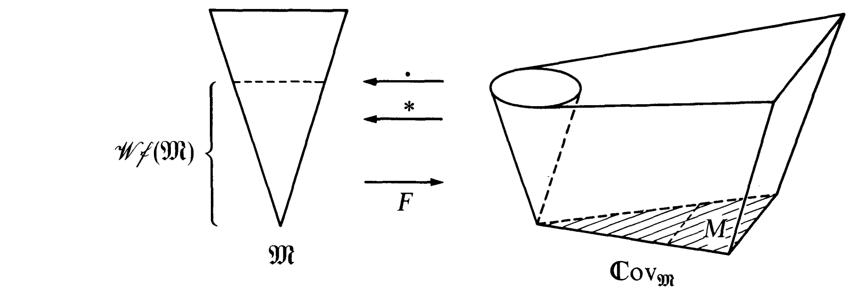

Fig. 2A. A model $\mathfrak{M}$ of set theory next to its admissible cover

The projection given in 2.3 is *ad hoc* in that we could have used others. The next function, by contrast, is canonical.

Let $\mathbb{A}_{\mathfrak{M}}=(\mathfrak{M};A,\in,F)$ be admissible and a cover of $\mathfrak{M}$. A function * is an *$\in$-retraction of* $\mathbb{A}_{\mathfrak{M}}$ *onto* $\mathfrak{M}$ if x^* is defined for every $x\in\mathbb{A}_{\mathfrak{M}}$ and satisfies the following equations:

$$(1)\begin{cases} p^*=p & \text{for } \; p\in\mathfrak{M}\\ (a^*)_E=\{b^*\,|\,b\in a\} & \text{for all } \; a\in\mathbb{A}_{\mathfrak{M}}\,. \end{cases}$$

We can use the projection given by Theorem 2.3 to prove the following characterization of $\mathfrak{Cov}_{\mathfrak{M}}$.

2.6 Corollary. *Let $\mathfrak{M}\models$ KP. $\mathfrak{Cov}_{\mathfrak{M}}$ has an $\in$-retraction into $\mathfrak{M}$ and it is the only admissible set covering $\mathfrak{M}$ which has such an $\in$-retraction.*

Proof. The proof is an elaboration of the proof of Theorem 2.5. It is clear that any admissible set $\mathbb{A}_{\mathfrak{M}}$ covering $\mathfrak{M}$ has a function * satisfying (1), simply by the second recursion theorem for KPU:

$$\begin{aligned} x^*=y \quad \text{iff} \quad & (x \text{ is an urelement} \wedge y=x)\vee\\ & (x \text{ is a set and } \; F(y)=\{b^*\,|\,b\in x\})\,. \end{aligned}$$

The problem is that x^* won't usually be defined for all x. Let us first show that for $\mathbb{A}_{\mathfrak{M}}=\mathfrak{Cov}_{\mathfrak{M}}$, x^* *is* defined for all x. Define ′ just as in the proof of 2.5. We claim that for all $x \in \mathbb{C}\mathrm{ov}_{\mathfrak{M}}$,

$$(\dot{x})' \text{ is defined}$$
$$(\dot{p})' = p \quad \text{for} \quad p \in M$$
$$((\dot{a})')_E = \{(\dot{x})' \mid x \in a\} \quad \text{for} \quad a \in M\,.$$

This is proved by induction just as in 2.5 and shows that x^* is defined for all x since $x^* = (\dot{x})'$. This proves that $\mathbb{C}\mathrm{ov}_{\mathfrak{M}}$ has an $\in$-retraction onto $\mathfrak{M}$. Let $\mathbb{A}_{\mathfrak{M}}$ be any other cover

$$(\mathfrak{M}; A, \in, F)$$

which has a totally defined $\in$-retraction $*$. Let D be the domain (in the peculiar sense of Definition V.5.1; that is $D = \mathrm{rng}(\dot{\ }))$ of the notation system of Theorem 2.3 and let

$$|p| = \text{the unique } x \text{ such that } \quad \dot{x} = p$$

for $p \in D$. Thus $|\ |$ maps D onto $\mathbb{C}\mathrm{ov}_{\mathfrak{M}}$. Define an $\mathbb{A}_{\mathfrak{M}}$-recursive function f from $\mathbb{A}_{\mathfrak{M}}$ into $\mathfrak{M}$ using $*$:

$$f(p) = \langle 0, p \rangle$$
$$f(a) = \langle 1, \{f(b) \colon b \in a\}^* \rangle\,.$$

See Fig. 2B at this point.

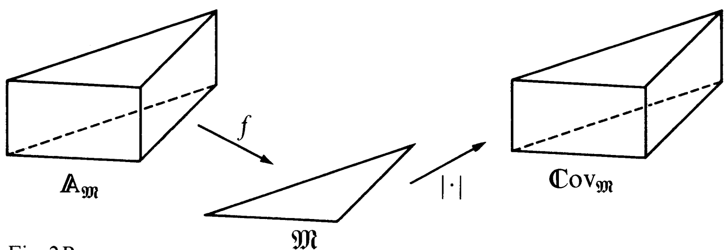

Fig. 2B.

A simple proof by induction on $\in$ shows that $f(x) \in D$ and $|f(x)| = x$, for all $x \in \mathbb{A}_{\mathfrak{M}}$. Thus $\mathbb{A}_{\mathfrak{M}} \subseteq \mathbb{C}\mathrm{ov}_{\mathfrak{M}}$ so $\mathbb{C}\mathrm{ov}_{\mathfrak{M}} = \mathbb{A}_{\mathfrak{M}}$ since $\mathbb{C}\mathrm{ov}_{\mathfrak{M}}$ is the smallest admissible set covering $\mathfrak{M}$. $\square$

The $\in$-retraction $*$ of $\mathbb{C}\mathrm{ov}_{\mathfrak{M}}$ onto $\mathfrak{M}$ is not one-one, of course, since $(a^*)^* = a^*$ but $a^* \neq a$, for any set $a \in \mathbb{C}\mathrm{ov}_{\mathfrak{M}}$. Otherwise, though, it is far more natural and less *ad hoc* than the projection of Theorem 2.3. We saw in the proof of 2.6 how to reconstruct the projection from $*$.

Also note that $*$ is $\mathbb{C}\mathrm{ov}_{\mathfrak{M}}$-recursive.

For applications of $\mathbb{C}\mathrm{ov}_{\mathfrak{M}}$ we need two more properties of $\mathbb{C}\mathrm{ov}_{\mathfrak{M}}$. The first tells us what Σ_1 on $\mathbb{C}\mathrm{ov}_{\mathfrak{M}}$ means in term of $\mathfrak{M}$.

2.7 Theorem. *Let* $\mathfrak{M} \models \mathrm{KP}$. *A relation* S *on* $\mathfrak{M}$ *is* Σ_1 *on* $\mathfrak{Cov}_{\mathfrak{M}}$ *iff* S *is* Σ_+ *inductive on* $\mathfrak{M}$; *that is, iff* S *is a section of* I_φ *where* $\varphi = \varphi(v_1, \ldots, v_n, \mathsf{R}_+)$ *is some* Σ *inductive definition (in the language* $\mathsf{L}(\mathsf{R})$*) interpreted over* $\mathfrak{M}$.

Proof. Deferred to 3.9. ▯

The last property we need relates the admissible covers of two different models $\mathfrak{M}, \mathfrak{N}$. Let $\mathfrak{M} = \langle M, E \rangle$, $\mathfrak{N} = \langle N, F \rangle$ where $\mathfrak{M} \subseteq \mathfrak{N}$. Note that $\mathfrak{M} \subseteq_{\mathrm{end}} \mathfrak{N}$ if $\mathfrak{Cov}_{\mathfrak{M}} \subseteq \mathfrak{Cov}_{\mathfrak{N}}$. If $\mathfrak{M}, \mathfrak{N} \models \mathrm{KP}$ and $\mathfrak{M} \subseteq_{\mathrm{end}} \mathfrak{N}$ then $\mathfrak{Cov}_{\mathfrak{M}} \subseteq \mathfrak{Cov}_{\mathfrak{N}}$, as the construction in § 3 makes translucent.

2.8 Theorem. *Let* $\mathfrak{M}, \mathfrak{N} \models \mathrm{KP}$, $\mathfrak{M} \subseteq_{\mathrm{end}} \mathfrak{N}$. *Then*

$$\mathfrak{M} \prec_1 \mathfrak{N}$$

if and only if

$$\mathfrak{Cov}_{\mathfrak{M}} \prec_1 \mathfrak{Cov}_{\mathfrak{N}} .$$

Proof. The translation $\varphi \to \hat{\varphi}$ defined at the beginning of this section makes the ($\Leftarrow$) half of this theorem immediate. The converse follows from the considerations of the next section. ▯

3. *An Interpretation of* KPU *in* KP

The proofs of the theorems of § 2 all involve interpreting the theory KPU of $\mathsf{L}(\in, \mathsf{F})$ in the theory KP of L, in the sense of § II.4, and then applying this interpretation to models $\mathfrak{M}$ of KP.

The interpretation is the one suggested by the projection of $\mathfrak{Cov}_{\mathfrak{M}}$ into $\mathfrak{M}$ which we want to construct to prove Theorem 2.3:

$$\dot{p} = \langle 0, p \rangle , \quad \dot{a} = \langle 1, y \rangle$$

where

$$y_E = \{\dot{x} \mid xEa\} .$$

3.1 The Interpretation *I*. We are dealing with two separate set theories, KP formulated in L with E as a membership symbol and $\mathrm{KPU} + (\dagger)$ formulated in $\mathsf{L}(\in, \mathsf{F})$ with $\in$ as the membership symbol, so this must make things a bit confusing no matter what we do. In this subsection we want to work axiomatically within KP so we use $\in$ for membership when we really ougth to use E, just because it seems the lesser of two evils. We use the usual notation for symbols defined in KP, symbols like $0, 1, \langle x, y \rangle, \mathsf{OP}$ (for ordered pair).

Define predicates within KP by the following:

$$\begin{aligned}
\mathsf{N}(x) &\leftrightarrow \exists y(x=\langle 0,y\rangle)\\
&\leftrightarrow \mathsf{OP}(x)\wedge \mathsf{1^{st}}(x)=0\\
x\mathsf{E}'y &\leftrightarrow \mathsf{N}(x)\wedge \mathsf{N}(y)\wedge(\mathsf{2^{nd}}(x)\in \mathsf{2^{nd}}(y))\\
\mathsf{Set}(x) &\leftrightarrow \exists y[x=\langle 1,y\rangle\wedge \forall z\in y(\mathsf{N}(z)\vee \mathsf{Set}(z))]\\
z\mathscr{E}x &\leftrightarrow \exists y[x=\langle 1,y\rangle\wedge z\in y]\\
&\leftrightarrow \mathsf{OP}(x)\wedge \mathsf{1^{st}}(x)=1\wedge z\in \mathsf{2^{nd}}(x)\\
\mathsf{F}'(x) &= \langle 1,\{\langle 0,y\rangle : y\in \mathsf{2^{nd}}(x)\}\rangle .
\end{aligned}$$

The predicates $\mathsf{N}, \mathsf{E}', \mathscr{E}$ and F' are defined by Δ_0 formulas. The predicate Set is defined, using the second recursion theorem, by a Σ_1 formula. We use these to define our interpretation as follows, where $\mathsf{L}^* = \mathsf{L}(\in, \mathsf{F})$ is considered as a one-sorted language with relation symbols U (for urelement), S (for set)

Symbol of L^*	*Interpretation in* KP *under* I
$\forall x$	$\forall x(\mathsf{N}(x)\vee \mathsf{Set}(x)\rightarrow\ldots)$
$=$	$=$
$\mathsf{U}(x)$	$\mathsf{N}(x)$
$\mathsf{S}(x)$	$\mathsf{Set}(x)$
$x\mathsf{E}y$	$x\mathsf{E}'y$
$x\in y$	$x\mathscr{E}y$
$\mathsf{F}(x)$	$\mathsf{F}'(x)$

3.2 Lemma. *I is an interpretation of* KPU + (†) *in* KP. *That is, for each axiom φ of* KPU + (†), *φ^I is a theorem of* KP.

Proof. We run quickly through the axioms, beginning with (†). The interpretation of (†) reads

$$\forall x\,\forall y[\mathsf{N}(x)\wedge \mathsf{N}(y)\rightarrow(x\mathsf{E}'y\leftrightarrow x\mathscr{E}\mathsf{F}'(y))] .$$

So suppose $\mathsf{N}(x)\wedge \mathsf{N}(y)$. Let $x=\langle 0,x_0\rangle$, $y=\langle 0,y_0\rangle$. Then the following are equivalent:

$$x\mathsf{E}'y ,$$

$$x_0\in y_0 ,$$

$$x\mathscr{E}\mathsf{F}'(y) .$$

Extensionality: The interpretation of Extensionality asserts that if $\mathsf{Set}(x)$ and $\mathsf{Set}(y)$ and

$$\forall z[(\mathsf{N}(z)\vee \mathsf{Set}(z))\rightarrow(z\mathscr{E}x\leftrightarrow z\mathscr{E}y)]$$

then $x=y$. Assume the three hypotheses. Let $x=\langle 1,u\rangle$, $y=\langle 1,v\rangle$. Then $z\mathscr{E}x$ iff $z\in u$, $z\mathscr{E}y$ iff $z\in v$. Since every $z\in u\cup v$ satisfies $\mathsf{N}(z)\vee\mathsf{Set}(z)$, $u=v$ and hence $x=y$.

Foundation: Suppose there is an x such that

$$\mathsf{Set}(x)\wedge\varphi^I(x).$$

Choose such an x of least possible rank. Then since $y\mathscr{E}z\to rk(y)<rk(z)$, we have

$$\forall z[\mathsf{Set}(z)\wedge z\mathscr{E}x\to\neg\varphi^I(z)].$$

Pair: Suppose $\mathsf{N}(x)\vee\mathsf{Set}(x)$ and $\mathsf{N}(y)\vee\mathsf{Set}(y)$. Let

$$z=\langle 1,\{x,y\}\rangle.$$

Then $\mathsf{Set}(z)\wedge(u\mathscr{E}z\leftrightarrow(u=x\vee u=y))$.

Union: Suppose $\mathsf{Set}(x)$. Let

$$y_0=\{z\mid\exists u\mathscr{E}x(z\mathscr{E}u)\}$$

by Δ_0 Separation and let $y=\langle 1,y_0\rangle$.

Δ_0 *Separation:* Let φ be a Δ_0 formula of $\mathsf{L}(\in,\mathsf{F})$. The formula φ^I is a Δ_0 formula of L^* when L^* is expanded by the symbols N, E', E, F'. Suppose $\mathsf{Set}(x)$, say $x=\langle 1,x_0\rangle$. Let

$$y_0=\{z\in x_0\mid\varphi^I(z)\}$$

by Δ_0 Separation and let $y=\langle 1,y_0\rangle$. Then

$$z\mathscr{E}y\ \text{ iff }\ z\mathscr{E}x\wedge\varphi^I(z).$$

Δ_0 *Collection:* Suppose $\varphi(x,y)$ is Δ_0, suppose $\mathsf{Set}(a)$ and that

$$\forall x\mathscr{E}a\,\exists y[\mathsf{N}(y)\vee\mathsf{Set}(y))\wedge\varphi^I(x,y)].$$

Let $a=\langle 1,a_0\rangle$ so that the above becomes

$$\forall x\in a_0\,\exists y[(\mathsf{N}(y)\vee\mathsf{Set}(y))\wedge\varphi^I(x,y)].$$

By Σ Reflection there is a b such that

$$\forall x\in a_0\,\exists y\in b[(\mathsf{N}(y)\vee\mathsf{Set}(y))\wedge\varphi^I(x,y)]^{(b)}.$$

Let

$$b_0=\{y\in b\mid(\mathsf{N}(y)\vee\mathsf{Set}(y))^{(b)}\}$$

by Δ_0 Separation and let $b_1 = \langle 1, b_0 \rangle$. Then

$$\forall x \mathscr{E} a \, \exists y \mathscr{E} b_1 \, \varphi^I(x, y). \quad \square$$

3.3 The model $\mathfrak{M}^{-I}$. Let $\mathfrak{M} \models$ KP. Let $N, E', \mathrm{Set}, \mathscr{E}, F'$ be the predicates and function defined in $\mathfrak{M}$ by the corresponding symbols of KP. Then, letting $\mathfrak{N} = \langle N, E' \rangle$ we have

$$\begin{aligned} \mathfrak{M}^{-I} &= (\mathfrak{N}; \mathrm{Set}, \mathscr{E} \upharpoonright \mathrm{Set}, F') \\ &= \mathfrak{B}_{\mathfrak{N}}, \text{ say}. \end{aligned}$$

$\mathfrak{B}_{\mathfrak{N}}$ is a model of KPU + (†), by 3.2. The structure $\mathfrak{N}$ is isomorphic to $\mathfrak{M}$ via the map $x \mapsto \langle 0, x \rangle$. If $\mathbb{D}_{\mathfrak{M}}$ is any admissible set covering $\mathfrak{M}$ then

N, E', F' are $\mathbb{D}_{\mathfrak{M}}$-recursive, as is the isomorphism $x \mapsto \langle 0, x \rangle$.
Set, $\mathscr{E} \upharpoonright$ Set are $\mathbb{D}_{\mathfrak{M}}$-r.e.

by the remarks at the beginning of § 2.

3.4 The model $\mathscr{W}\!\mathscr{f}(\mathfrak{M}^{-I})$. Let $\mathfrak{M} \models$ KP and let $\mathfrak{B}_{\mathfrak{N}}$ be as defined in 3.3. $\mathscr{W}\!\mathscr{f}(\mathfrak{B}_{\mathfrak{N}})$ is the largest well-founded substructure of $\mathfrak{B}_{\mathfrak{N}}$, *before being identified with a transitive set this time.* Notice that $\mathscr{W}\!\mathscr{f}(\mathfrak{B}_{\mathfrak{N}})$ is closed under F' since $F'(x)$ is always a set of urelements. Thus by the Truncation Lemma, $\mathscr{W}\!\mathscr{f}(\mathfrak{B}_{\mathfrak{N}})$ is a well-founded model of KPU + (†). If $\mathbb{D}_{\mathfrak{M}}$ is admissible and covers $\mathfrak{M}$ then

N, E', F' are $\mathbb{D}_{\mathfrak{M}}$-recursive, as is the isomorphism $x \mapsto \langle 0, x \rangle$ and
$\mathrm{Set} \cap \mathscr{W}\!\mathscr{f}(\mathfrak{B}_{\mathfrak{N}}), \mathscr{E} \upharpoonright (\mathrm{Set} \cap \mathscr{W}\!\mathscr{f}(\mathfrak{B}_{\mathfrak{N}}))$ are $\mathbb{D}_{\mathfrak{M}}$-r.e.

The first follows from 3.2. The second line follows from Theorem V.3.1.

3.5 The admissible set isomorphic to $\mathscr{W}\!\mathscr{f}(\mathfrak{M}^{-I})$. Let $\mathfrak{M} \models$ KP and let

$$\mathscr{W}\!\mathscr{f}(\mathfrak{B}_{\mathfrak{N}}) \cong (\mathfrak{N}; A, \in, F') = \mathbb{A}_{\mathfrak{N}}$$

where A is transitive (in $\mathbb{V}_{\mathfrak{N}}$). By 3.4, $\mathbb{A}_{\mathfrak{N}}$ is admissible and covers $\mathfrak{N}$. Let $\mathbb{D}_{\mathfrak{M}}$ be any admissible set which covers $\mathfrak{M}$. By 3.4 and Theorem V.3.1, there is a $\mathbb{D}_{\mathfrak{M}}$-recursive isomorphism of $\mathfrak{M}$ and $\mathfrak{N}$, and A is $\mathbb{D}_{\mathfrak{M}}$-r.e.

3.6 $\mathbb{C}\mathrm{ov}_{\mathfrak{M}}$ defined. Let $\mathfrak{M} \models$ KP and let $\mathbb{A}_{\mathfrak{N}}$ be as in 3.5. The isomorphism $i: \mathfrak{N} \cong \mathfrak{M}$ extends to an isomorphism of $\mathbb{V}_{\mathfrak{N}}$ onto $\mathbb{V}_{\mathfrak{M}}$ by:

$$i(a) = \{ i(b) \mid b \in a \},$$

carrying every transitive set in $\mathbb{V}_{\mathfrak{N}}$ onto a transitive set of $\mathbb{V}_{\mathfrak{M}}$. In particular, $\mathbb{A}_{\mathfrak{N}}$ is carried over to an isomorphic admissible set over $\mathfrak{M}$, say $\mathbb{A}'_{\mathfrak{M}} =$

$$(\mathfrak{M}; A', \in, F)$$

where $A' = \{i(a) | a \in A\}$. We claim that this $\mathbb{A}'_{\mathfrak{M}}$ is the admissible cover of $\mathfrak{M}$. It clearly is admissible and covers $\mathfrak{M}$. Let $\mathbb{D}_{\mathfrak{M}}$ be admissible and cover $\mathfrak{M}$. The isomorphism i can be defined by $\in$-recursion in $\mathbb{D}_{\mathfrak{M}}$ and so $\mathbb{A}'_{\mathfrak{M}} \subseteq \mathbb{D}_{\mathfrak{M}}$. Thus $\mathbb{A}'_{\mathfrak{M}}$ is contained in every admissible set covering $\mathbb{M}$ so $\mathbb{A}'_{\mathfrak{M}} = \mathbb{C}\mathrm{ov}_{\mathfrak{M}}$. This proves Theorem 2.2.

3.7 The projection. It is clear from the above construction of $\mathbb{C}\mathrm{ov}_{\mathfrak{M}}$ that every $x \in M$ is "denoted by" $\langle 0, x \rangle$ and that every $a \in \mathbb{C}\mathrm{ov}_{\mathfrak{M}}$ is denoted by

$$\langle 1, y \rangle$$

where y_E is the set of "notations for" members of a. Turning this around gives the desired projection.

We saw, early in § 2, how to translate Σ_1 formulas of L into Σ_1 formulas of L^*, using the covering function. We now see how we can translate Σ_1 formulas of L^* into "formulas" about $\mathfrak{M}$.

3.8 Translation Lemma. *Let $\exists y \varphi(x, y)$ be a Σ_1 formula of L^*, where φ is Δ_0, and let $\psi(x, z)$ be the interpretation*

$$\exists y[\mathrm{rk}(y) = z \wedge \varphi(x, y)]^I,$$

a formula of L. Let $\mathfrak{M} \models \mathrm{KP}$, let $\alpha = o(\mathbb{C}\mathrm{ov}_{\mathfrak{M}})$ and let $x \in \mathbb{C}\mathrm{ov}_{\mathfrak{M}}$. Then

$$\mathbb{C}\mathrm{ov}_{\mathfrak{M}} \models \exists y \, \varphi(x, y)$$

iff there is a $\beta < \alpha$ such that

$$\mathfrak{M} \models \psi(\dot{x}, \dot{\beta}).$$

Proof. Suppose $\mathbb{C}\mathrm{ov}_{\mathfrak{M}} \models \varphi(x, y)$. Then

$$\mathfrak{M} \models \varphi^I(\dot{x}, \dot{y}) \wedge (\mathrm{rk}(\dot{y}) = z)^I$$

for some "standard ordinal" z of $\mathfrak{M}^{-I}$. Thus, by Corollary 2.5,

$$\mathfrak{M} \models \psi(\dot{x}, \dot{\beta})$$

for some $\beta < \alpha$. The other half follows from 3.3—3.7. □

3.9 Proof of Theorem 2.7. A complete proof of Theorem 2.7 would include a proof of the following fact. The Σ_+ inductive relations on $\mathfrak{M}$ contain all Σ relations and are closed under $\wedge$, $\vee$, $\exists$ and substitution by total Σ_1 functions. This is proved just as in Exercise VI.4.18. But, given this, we have an easy proof of Theorem 2.7

from 3.8. Suppose R is Σ_1 on $\mathfrak{Cov}_{\mathfrak{M}}$, say

$$R(p) \leftrightarrow \mathfrak{Cov}_{\mathfrak{M}} \models \exists y\, \varphi(p, y)$$

where φ is Δ_0. Let $\theta(x) = \mathrm{Ord}(x)^I$ and define

$$\Gamma(U) = \{x \mid M \models \theta(x) \wedge \forall y\, Ex\, U(y)\}\,.$$

Then Γ is a Σ_+ inductive definition over $\mathfrak{M}$ and I_Γ is the set of $\{\dot{\beta} \mid \beta < \alpha = o(\mathfrak{Cov}_{\mathfrak{M}})\}$. Furthermore

$$R(p) \text{ iff } \quad \exists z \in I_\Gamma(\mathfrak{M} \models \psi(\langle 0, p\rangle, z))$$

so R is Σ_+ inductive. The other half is trivial since any Σ_+ inductive definition Γ over $\mathfrak{M}$ transforms into a Σ_+ inductive definition $\hat{\Gamma}$ over $\mathfrak{Cov}_{\mathfrak{M}}$, and then, by Gandy's Theorem, $I_{\hat{\Gamma}}$ is Σ_1 on $\mathfrak{Cov}_{\mathfrak{M}}$. $\square$

3.10 Proof of Theorem 2.8. Suppose $\mathfrak{M} \subseteq_{\mathrm{end}} \mathfrak{N}$ and $\mathfrak{M} \prec_1 \mathfrak{N}$. Since $\mathfrak{M} \subseteq_{\mathrm{end}} \mathfrak{N}$, $\mathfrak{Cov}_{\mathfrak{M}} \subseteq_{\mathrm{end}} \mathfrak{Cov}_{\mathfrak{N}}$ so any Σ predicate true in $\mathfrak{Cov}_{\mathfrak{M}}$ is true in $\mathfrak{Cov}_{\mathfrak{N}}$. In particular, the projections for $\mathfrak{Cov}_{\mathfrak{M}}$ and $\mathfrak{Cov}_{\mathfrak{N}}$ agree on $a \in \mathfrak{Cov}_{\mathfrak{M}}$, so we may write $\dot{a}$ for this projection without fear of confusion. Suppose $a \in \mathfrak{Cov}_{\mathfrak{M}}$ and

$$\mathfrak{Cov}_{\mathfrak{N}} \models \exists y\, \varphi(a, y)$$

where φ is Δ_0. Then there is a $\beta < o(\mathfrak{Cov}_{\mathfrak{M}})$ such that $\mathfrak{N}$ is a model of

$$[\exists y(\mathrm{rk}(y) = \dot{\beta} \wedge \varphi(\dot{a}, y))]^I\,,$$

by 3.8. Hence $\mathfrak{N}$ is a model of

(1) $$\exists z\,[\mathrm{Ord}(z)^I \wedge [\exists y(\mathrm{rk}(y) = z \wedge \varphi(\dot{a}, y)]^I]\,.$$

Since $\mathfrak{M} \prec_1 \mathfrak{N}$, $\mathfrak{M}$ is also a model of (1). By Lemma 3.2, $\mathfrak{M}$ is a model of (Foundation)^I so $\mathfrak{M}$ is a model of

$$\exists z\,[\mathrm{Ord}(z)^1 \wedge [\exists y(\mathrm{rk}(y) = z \wedge \varphi(\dot{a}, y)]^I \wedge$$
$$[\forall w \in z \neg \exists y(\mathrm{rk}(y) = w \wedge \varphi(\dot{a}, y))]^I]\,.$$

Pick such a "least" z. Since $\mathfrak{M} \subseteq_{\mathrm{end}} \mathfrak{N}$, this least z must be $\leqslant \dot{\beta}$ in the sense of E, so it must be a standard ordinal. That is, there must be some $\gamma < o(\mathfrak{Cov}_{\mathfrak{M}})$ such that $\dot{\gamma} = z$. Thus $\mathfrak{M}$ is a model of

$$\exists y[\mathrm{rk}(y) = \dot{\gamma} \wedge \varphi(\dot{a}, y)]$$

so, by 3.8,

$$\mathfrak{Cov}_{\mathfrak{M}} \models \exists y\, \varphi(a, y)\,.$$

Thus $\mathfrak{Cov}_{\mathfrak{M}} \prec_1 \mathfrak{Cov}_{\mathfrak{N}}$. $\square$

3.11—3.13 Exercises

3.11. Prove that a relation $S \subseteq \mathfrak{M}$ (a model of KP) is s-Π_1^1 over $\mathfrak{M}$ iff it is s-Π_1^1 over $\mathfrak{Cov}_{\mathfrak{M}}$.

3.12. Prove the following result of Aczel: Let $S \subseteq \mathfrak{M}$ (a countable model of KPU). Prove that S is s-Π_1^1 on $\mathfrak{M}$ iff S is Σ_+ inductive on $\mathfrak{M}$. [Combine 2.7, 3.9 and VII.3.1.]

3.13. Extend the construction above from models of KP to models of KPU.

4. Compactness Arguments over Nonstandard Models of Set Theory

In this final section we want to show how the admissible cover can be used to extend results from standard to nonstandard models. We give two simple examples.

We know from Theorem VII.1.3 that no countable admissible set $\mathbb{A}$ is self-definable. An equivalent statement (in view of Exercise VIII.4.19(iv)) is that if $\mathbb{A}$ is countable, admissible and

$$\mathbb{A} \models \exists \vec{\mathsf{R}}\, \varphi(\vec{\mathsf{R}})$$

for some first order sentence $\varphi(\vec{\mathsf{R}})$ (possibly involving constants from $\mathbb{A}$) then there is a *proper* end extension $\mathfrak{B}$ of $\mathbb{A}$ such that

$$\mathfrak{B} \models \exists \vec{\mathsf{R}}\, \varphi(\vec{\mathsf{R}}).$$

Phrased this way, the result holds for any countable model of KP, standard or nonstandard (or countable model of KPU by 3.13).

4.1 Theorem. *Let* $\mathfrak{M} = \langle M, E \rangle$ *be a countable model of* KP *such that*

$$\mathfrak{M} \models \exists \mathsf{R}\, \varphi(\mathsf{R})$$

for some sentence $\varphi(\mathsf{R})$. *There is a proper end extension* $\mathfrak{N}$ *of* $\mathfrak{M}$ *such that*

$$\mathfrak{N} \models \exists \mathsf{R}\, \varphi(\mathsf{R}).$$

Proof. Let $\mathbb{A} = \mathbb{A}_{\mathfrak{M}} = \mathfrak{Cov}_{\mathfrak{M}}$ and let $\mathsf{L}_{\mathbb{A}}$ be the admissible fragment given by $\mathbb{A}$. Let x be a constant symbol in $\mathbb{A}$ used to denote x, for each $x \in M$, and let T be the following Σ_1 theory of $\mathsf{L}_{\mathbb{A}}$:

$$\forall v[v \mathsf{E} \mathsf{x} \rightarrow \textstyle\bigvee_{y \in x_E} v = \mathsf{y}]$$

$$\text{diagram}(\mathfrak{M})$$

$$\varphi(\mathsf{R})$$

$$c \neq \mathsf{x} \quad (\text{all } x \in M).$$

We can form the first sentences since $\mathbb{A}$ covers $\mathfrak{M}$. We must prove that T is consistent. Since $\mathbb{A}$ is a countable admissible set, the Compactness Theorem implies that if T is not consistent, then there is a $T_0 \subseteq T$, $T_0 \in \mathbb{A}$ such that T_0 is not consistent. By Corollary 2.4,

$$\{x \in M \mid \mathsf{x} \text{ occurs in } T_0\}$$

is $\mathfrak{M}$-finite. But then there is always some $y \in M$ left over to interpret c so T_0 is consistent. □

Our final result extends Theorem 1.1 from standard to nonstandard models of set theory.

4.2 Theorem. *Let* $\mathfrak{M} = \langle M, E \rangle$ *be any countable model of* ZF. *There is an end extension* $\mathfrak{N}$ *of* $\mathfrak{M}$ *which is a model of* ZF + V = L.

Proof. Let $\mathfrak{M}_0$ be the submodel of $\mathfrak{M}$ such that

$$M_0 = \{x \in M \mid \mathfrak{M} \models \text{“}\alpha \text{ is the first stable ordinal”} \wedge x \in \mathrm{L}(\alpha)\}.$$

Then by Shoenfield's Absoluteness Lemma (see § V.8)

$$\mathfrak{M}_0 \prec_1 \mathfrak{M}.$$

Let $\mathbb{A} = \mathfrak{Cov}_{\mathfrak{M}}$, $\mathbb{A}_0 = \mathfrak{Cov}_{\mathfrak{M}_0}$, so that $\mathbb{A}_0 \prec_1 \mathbb{A}$ by Theorem 2.8. Let T be the theory of $\mathsf{L}_{\mathbb{A}}$ containing

ZF

$$\forall v[v \mathsf{E} \mathsf{x} \leftrightarrow \bigvee_{y \in x_E} v = \mathsf{y}], \quad \text{for all} \quad x \in M.$$

The proof now proceeds exactly like the proof of Theorem 1.1 except that the model of T_0 is not $\mathfrak{M}_0$ but the model $\mathfrak{M}_1$ where

$$M_1 = \{x \in M \mid \mathfrak{M} \models \text{“}x \text{ is constructible”}\}.$$

The reason for using $\mathfrak{M}_1$, rather than $\mathfrak{M}_0$, is that $\mathfrak{M}_1 \not\prec_1 \mathfrak{M}$ (parameters are not allowed in Shoenfield's Lemma) but the statement of Theorem 2.8 requires $\prec_1$. One could equally well improve 2.8. □

4.3—4.4 Exercises

4.3. Prove that both assumptions $\mathfrak{M} \models \mathrm{KP}$ and $\mathfrak{M}$ is countable are needed for Theorem 4.1.

4.4. Show that if ZF is consistent then there is an uncountable model of ZFC which has no end extension satisfying ZF + V = L.