Chapter IV
Elementary Results on $\mathbb{HYP}_{\mathfrak{M}}$

We have seen, in Chapter III, how admissible sets provide a tool for the study of infinitary logic by giving rise to those countable fragments which are especially well-behaved. In this chapter we begin the study of $\mathbb{HYP}_{\mathfrak{M}}$ by means of the logical tools developed in Chapter III.

1. On Set Existence

Given $\mathfrak{M}$ we form the universe of sets $\mathbb{V}_{\mathfrak{M}}$ on $\mathfrak{M}$ and speak glibly about arbitrary sets $a \in \mathbb{V}_{\mathfrak{M}}$. In practice, however, one seldom considers the impalpable sets of extremely high rank. There is even a feeling that these sets have a weaker claim to existence than the sets one normally encounters. Without becoming too philosophical, we want to touch here on the question: If we assume $\mathfrak{M}$ as given, to the existence of what sets are we more or less firmly committed?

$\mathbb{HYP}_{\mathfrak{M}}$ is the intersection of all models $\mathfrak{A}_{\mathfrak{M}}$ of KPU$^+$ and is an admissible set above $\mathfrak{M}$. There appears to be a certain *ad hoc* feature to $\mathbb{HYP}_{\mathfrak{M}}$, however, since it might depend on the exact axioms of KPU$^+$ in a sensitive way. You would expect that if you took a stronger theory than KPU$^+$ (say throw in Power, or Infinity or Full Separation) that more sets from $\mathbb{V}_{\mathfrak{M}}$ would occur in all models of this stronger theory. That, for $\mathfrak{M}$ countable, this cannot happen, lends considerable weight to the contension that $\mathbb{HYP}_{\mathfrak{M}}$ is here to stay.

Of the two results which follow, the second implies the first. We present them in the opposite order for expository and historical reasons.

A set $S \subseteq \mathfrak{M}$ is *internal* for $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, E, \ldots)$ if there is an $a \in A$ such that $S = a_E = \{x \in \mathfrak{A}_{\mathfrak{M}} \mid x E a\}$.

1.1 Theorem. *Let $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ be a countable structure for* L. *Let T be a consistent theory (finitary or infinitary) which is Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$ and which has a model of the form $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, E, \ldots)$. Let $S \subseteq M$ be such that S is internal for every such model of T. Then $S \in \mathbb{HYP}_{\mathfrak{M}}$.*

Proof. The proof is a routine application of Completeness and Omitting Types. Given the above assumptions we see that there can be no model $\mathfrak{A}_{\mathfrak{M}}$ of

$$T' + \forall v \bigvee \Phi(v)$$

where T' is T plus

(1) $$\forall v\,[\mathsf{U}(v) \to \bigvee_{p \in M} v = \overline{\mathsf{p}}]$$
Diagram $(\mathfrak{M})$

and Φ is the set of formulas

$$\{\overline{\mathsf{p}} \notin v \mid p \in S\} \cup \{\overline{\mathsf{p}} \in v \mid p \notin S\},$$

for then S would not be internal for $\mathfrak{A}_{\mathfrak{M}}$. The formulas in T' and in $\Phi(v)$ are members of the admissible fragment $\mathsf{L}^*_{\mathbb{A}}$ of $\mathsf{L}^*_{\infty\omega}$ where $\mathbb{A} = \mathbb{HYP}_{\mathfrak{M}} = (\mathfrak{M}; A, \in)$, and where we have introduced $\overline{\mathsf{p}}$ by some convention like $\overline{\mathsf{p}} = \langle 0, p \rangle$. By the Omitting Types Theorem there is a formula $\sigma(v)$ of $\mathsf{L}^*_{\mathbb{A}}$ such that $T' + \exists v\, \sigma(v)$ is consistent but such that:

$$T' \models \forall v\,[\sigma(v) \to \overline{\mathsf{p}} \in v], \quad \text{for all} \quad p \in S;$$

$$T' \models \forall v\,[\sigma(v) \to \overline{\mathsf{p}} \notin v], \quad \text{for all} \quad p \notin S.$$

But then

$$S = \{p \in \mathfrak{M} \mid T' \models \forall v\,(\sigma(v) \to \overline{\mathsf{p}} \in v)\}$$

so S is Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$ by the Extended Completeness Theorem for $\mathsf{L}^*_{\mathbb{A}}$. Similarly $\neg S$ is Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$ so S is Δ_1 on $\mathbb{HYP}_{\mathfrak{M}}$. Thus $S \in \mathbb{HYP}_{\mathfrak{M}}$ by Δ_1 Separation. ☐

Before stating our next result we need a more sophisticated notion of what it means for a set $a \in \mathbb{V}_{\mathfrak{M}}$ to be *internal* for $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, E, \ldots)$.

1.2 Definition. A set $a \in \mathbb{V}_{\mathfrak{M}}$ is *internal for* $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, E, \ldots)$ if $a \in \mathscr{W}\!\mathscr{f}(\mathfrak{M}; A, E)$, where we again identify $\mathscr{W}\!\mathscr{f}(\mathfrak{M}; A, E)$ with its transitive collapse.

Note that for $a \subseteq M$ this is equivalent to the existence of an $x \in A$ with $a = x_E$. Also notice that if a is internal and $b \in a$ then b is internal.

1.3 Theorem. *Let $\mathfrak{M}$ be countable and let $a \in \mathbb{V}_{\mathfrak{M}}$ be a set which is internal for every model*

$$\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, E, \ldots)$$

of some consistent theory T, finitary or not, formulated in $\mathsf{L}^* = \mathsf{L}(\in, \ldots)$, $\mathrm{KPU}^+ \subseteq T$. If T is Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$, then $a \in \mathbb{HYP}_{\mathfrak{M}}$.

Proof. We prove the theorem by $\in$-induction. By the comment above, if a is internal for every model $\mathfrak{A}_{\mathfrak{M}}$ of T, so is every $b \in a$. By $\in$-induction, each of these b is in $\mathbb{HYP}_{\mathfrak{M}}$. That is, $a \subseteq \mathbb{HYP}_{\mathfrak{M}}$. A routine modification of the proof of 1.1 shows that a is Δ_1 on $\mathbb{HYP}_{\mathfrak{M}}$. If we can prove that $a \subseteq L(\mathfrak{M}, \beta)$ for some $\beta < o(\mathbb{HYP}_{\mathfrak{M}})$ then, by Δ_1 separation, $a \in \mathbb{HYP}_{\mathfrak{M}}$. Assume, on the contrary, that

$$O(\mathfrak{M}) = \text{the least ordinal } \beta \text{ such that } a \subseteq L(\mathfrak{M}, \beta).$$

In any model $\mathfrak{A}_{\mathfrak{M}}$ of T there would be a unique ordinal x such that

$$\mathfrak{A}_{\mathfrak{M}} \models \text{"}x = \text{least ordinal } \beta \text{ such that } a \subseteq L(\mathfrak{M}, \beta)\text{"}.$$

By Σ Reflection in $\mathfrak{A}_{\mathfrak{M}}$ and, by the absoluteness of $L(\cdot,\cdot)$, this x must be $O(\mathfrak{M})$. Hence $T +$ the following theory pins down $O(\mathfrak{M})$, contrary to Corollary III.7.4.

Diagram $(\mathfrak{M})$,

$\forall x\, [\mathsf{U}(x) \to \bigvee_{p \in M} x = \overline{\mathsf{p}}]$,

"$<$ *is the order type of the* $\in$*-precedessors of* c",

(2) "c *is the first ordinal such that* $L(\mathfrak{M}, \mathsf{c})$ *is admissible*" (if $\alpha > \omega$)

or

(3) "c *is the first limit ordinal*" (if $\alpha = \omega$).

This theory is formulated in $L(\in, \ldots, <, \mathsf{c}, \overline{\mathsf{p}})_{p \in M}$. (The reason for the two cases is that we do not yet know how to write "x is admissible" by a finite formula.) We can write (2) as

$$\forall x\, [x < \mathsf{c} \to \bigvee_{\varphi \in \mathbf{KPU}} \neg \varphi^{L(\mathfrak{M}, x)}]. \quad \square$$

Thus we see that no matter how we strengthen KPU^+ to an axiomitizable theory T, we cannot assure that any set in $\mathbb{V}_{\mathfrak{M}} - \mathbb{HYP}_{\mathfrak{M}}$ should be internal to every model $\mathfrak{A}_{\mathfrak{M}}$ of T.

One could consider $\mathbb{HYP}_{\mathfrak{M}}$ as a new structure $\mathfrak{N}$ and form $\mathbb{HYP}_{\mathfrak{N}}$ but it is more natural, and essentially equivalent, to procced differently.

1.4 Definition. Let $\mathbb{A}_{\mathfrak{M}} = (\mathfrak{M}; A, \in)$ be transitive in $\mathbb{V}_{\mathfrak{M}}$. Then $\mathbb{HYP}(\mathbb{A}_{\mathfrak{M}})$ is the structure $(\mathfrak{M}; B, \in)$ where

$$B = \bigcap \{B' \mid (M \cup A) \in B', (\mathfrak{M}; B', \in) \text{ admissible}\}.$$

We consider $\mathbb{HYP}_{\mathfrak{M}}$ as a special case of $\mathbb{HYP}(\mathbb{A}_{\mathfrak{M}})$.

1.5—1.9 Exercises

1.5. Show that $\mathbb{HYP}(\mathbb{A}_{\mathfrak{M}})$ is admissible.

1.6. Show that every element $a \in \mathbb{HYP}(\mathbb{A}_{\mathfrak{M}})$ has a good Σ_1 definition with parameters from $M \cup A \cup \{M, A\}$.

1.7. Show that the obvious generalizations of 1.1 and 1.3 are true.

1.8. Let $\mathcal{N}=\langle\omega,+,\cdot,0\rangle$ and let $X\subseteq\omega$. Show that there is a $T\subseteq L_{\omega\omega}$, $\mathrm{KPU}^+\subseteq T$ such that X is in every model $\mathfrak{A}_{\mathcal{N}}$ of T. This shows that the condition that T be Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$ is necessary in 1.1 and 1.3.

1.9. Show that the hypothesis $\mathrm{KPU}^+\subseteq T$ can be dropped from Theorem 1.3. [Hint: add a new $\in$-symbol and a function symbol used to denote an $\in$-isomorphism.]

1.10 Notes. Theorem 1.1 is a modern version of the Gandy-Kreisel-Tait Theorem: For any consistent Π_1^1 T set of axioms for second order number theory, if $a\subseteq\omega$ is internal to every model of T, then a is hyperarithmetic.

Theorem 1.3 was announced by Barwise in Barwise-Gandy-Moschovakis [1971]. The part of it contained in Theorem 1.1 is due independently to Grilliot [1972]. The improvement in 1.9 is due to Ville [1974].

2. *Defining* Π_1^1 *and* Σ_1^1 *Predicates*

Let $\mathfrak{M}=\langle M,R_1,\ldots,R_l\rangle$ be a fixed infinite structure for a language L. An n-ary relation S on $\mathfrak{M}$ is Π_1^1 *on* $\mathfrak{M}$ if it can be defined by a second order formula of the form

$$S(p_1,\ldots,p_n)\quad\text{iff}\quad\forall\mathsf{T}_1,\ldots,\forall\mathsf{T}_k\,\varphi(p_1,\ldots,p_n,\mathsf{T}_1,\ldots,\mathsf{T}_k),$$

where φ is a first order formula of $\mathsf{L}(\mathsf{T}_1,\ldots,\mathsf{T}_k)$, possibly containing parameters from $\mathfrak{M}$. More formally we should write this as: for all $p_1,\ldots,p_n\in M$, $S(p_1,\ldots,p_n)$ holds iff for all relations $T_1,\ldots,T_k$ on $\mathfrak{M}$,

$$(\mathfrak{M},T_1,\ldots,T_k)\models\varphi[p_1,\ldots,p_n].$$

The negation of a Π_1^1 relation is called Σ_1^1 *on* $\mathfrak{M}$. Thus S is Σ_1^1 iff it can be defined by

$$S(\vec{p})\quad\text{iff}\quad\exists\mathsf{T}_1,\ldots,\exists\mathsf{T}_k\,\psi(\vec{p},\mathsf{T}_1,\ldots,\mathsf{T}_k)$$

for some first order ψ. If S is both Π_1^1 and Σ_1^1 on $\mathfrak{M}$ then S is said to be Δ_1^1 *on* $\mathfrak{M}$.

This section is primarily concerned with techniques that can be used to show that predicates are Π_1^1 or Σ_1^1 on $\mathfrak{M}$. The reason for discussing this material can be seen by glancing at the next section.

2.1 Examples. (i) *If* $\mathcal{N}=\langle\omega,0,+,\cdot\rangle$, *then a set is* Δ_1^1 *over* $\mathcal{N}$ *iff it is hyperarithmetic.* (This is the classical Souslin-Kleene theorem. See, e.g., Shoenfield [1967].)

(ii) *If* $\mathfrak{N}=\langle N,0,+,\cdot\rangle$ *is a nonstandard model of arithmetic then the standard integers form a* Π_1^1 *set but not, in general, a* Δ_1^1 *set:*

$$x \quad \text{is standard iff} \quad \forall S\,[S(0)\wedge\forall y\,(S(y)\to S(y+1))\to S(x)].$$

(iii) *If* $\mathfrak{M}=\langle G,0,+\rangle$ *is an abelian group then the torsion part* T *of* G, the set of elements of G of finite order, *is* Π_1^1 *on* G*:*

$$x\in T \quad \text{iff} \quad \forall S\,[S(x)\wedge\forall y\,(S(y)\to S(x+y))\to S(0)].$$

(iv) *If* $\mathfrak{M}=\langle G,0,+\rangle$ *is an abelian group then the largest divisible subgroup* D *of* G *is* Σ_1^1, but this time it is not so obvious.

$$x\in D \quad \text{iff} \quad \exists H\,[H \text{ a subgroup } \wedge H \text{ divisible } \wedge H(x)]$$

but the clause "H is divisible", meaning

for all integers n, $\forall y\in H\ \exists z\in H,\ nz=y$

cannot be expressed by a single first order sentence. It is still possible, though, to write D out as a Σ_1^1 predicate. The student should try this before going on in order to appreciate the machinery developed below. ▯

The last example is just the tip of an iceberg. In writing out Π_1^1 predicates we frequently discover that we would like to use an extended first order formula as defined in § II.2. (In writing out the Σ_1^1 predicate in 2.1 (iv) we need the co-extended predicate "H is divisible".) It turns out we can allow ourselves this freedom without changing the class of Π_1^1 predicates.

2.2 Definition. (i) An *extended* Π_1^1 *predicate over* $\mathfrak{M}$ is a predicate $\mathsf{S}(p_1,\ldots,p_i, S_1,\ldots,S_m, a_1,\ldots,a_j, P_1,\ldots,P_n)$ defined by

$$(\mathfrak{M},S_1,\ldots,S_m;\ \mathbb{HF}_{\mathfrak{M}},\in,P_1,\ldots,P_n)\models\forall\mathsf{T}_1,\ldots,\forall\mathsf{T}_m\ \forall\mathsf{Q}_1,\ldots,\forall\mathsf{Q}\ \varphi(\vec{p},\vec{a},\vec{\mathsf{S}},\vec{\mathsf{T}},\vec{\mathsf{P}},\vec{\mathsf{Q}}),$$

for some extended first order formula φ which may have parameters in it from $M\cup\mathbb{HF}_{\mathfrak{M}}$. (We use S, T for relations over M; P, Q for relations over $M\cup\mathbb{HF}_{\mathfrak{M}}$.)

(ii) S is *co-extended* Σ_1^1 if it is in the dual class; that is, if it can be defined by

$$(\mathfrak{M},\vec{S};\ \mathbb{HF}_{\mathfrak{M}},\in,\vec{P})\models\exists\vec{\mathsf{T}}\ \exists\vec{\mathsf{Q}}\ \varphi(\vec{p},\vec{a},\vec{\mathsf{S}},\vec{\mathsf{T}},\vec{\mathsf{P}},\vec{\mathsf{Q}})$$

where φ is co-extended.

Thus extended Π_1^1 predicates over $\mathfrak{M}$ are not really predicates over $\mathfrak{M}$; they are predicates of points in $\mathfrak{M}$, relations on $\mathfrak{M}$, sets in $\mathbb{HF}_{\mathfrak{M}}$ and relations on $\mathbb{HF}_{\mathfrak{M}}$. They are important as a tool for showing predicates over $\mathfrak{M}$ are Π_1^1. For example, in 2.1 (iv), it is clear that D is co-extended Σ_1^1, so that D is Σ_1^1 over G by 2.8 below.

2.3 Lemma. *If* $\mathsf{S}_1, \mathsf{S}_2$ *are extended* Π_1^1 *(respectively co-extended* Σ_1^1*) so are* $(\mathsf{S}_1 \vee \mathsf{S}_2)$ *and* $(\mathsf{S}_1 \wedge \mathsf{S}_2)$.

Proof. For example,

$$\forall \mathsf{T}\, \psi(\_\_, \mathsf{T}) \wedge \forall \mathsf{T}'\, \forall \mathsf{Q}\, \theta(\_\_, \mathsf{T}', \mathsf{Q})$$

is equivalent to

$$\forall \mathsf{T}\, \forall \mathsf{T}'\, \forall \mathsf{Q}\, [\psi(\_\_, \mathsf{T}) \wedge \theta(\_\_, \mathsf{T}'\, \mathsf{Q})]$$

as long as we first make sure T and T′ are distinct symbols. The part inside the brackets is still extended first order. ∎

2.4 Lemma. *If* S *is extended* Π_1^1 *(respectively, co-extended* Σ_1^1*) then* $\neg$**S** *is co-extended* Σ_1^1 *(respectively, extended* Π_1^1*).* ∎

2.5 Lemma. *If* $\mathsf{S} = \mathsf{S}(p_1, \ldots, p_i, \_\_)$ *is extended* Π_1^1 *(co-extended* Σ_1^1*) then so are*

$$\mathsf{S}_1(p_1, \ldots, p_{i-1}, \_\_) \quad \textit{iff} \quad \forall p_i\, \mathsf{S}(p_1, \ldots, p_{i-1}, p_i, \_\_),$$

$$\mathsf{S}_2(p_1, \ldots, p_{i-1}, \_\_) \quad \textit{iff} \quad \exists p_i\, \mathsf{S}(p_1, \ldots, p_{i-1}, p_i, \_\_).$$

Proof. It is hard to see the extended Π_1^1 case directly, but we can prove the co-extended Σ_1^1 case and then apply 2.4. If

$$\mathsf{S}(\vec{p}, \_\_) \quad \text{iff} \quad \exists \mathsf{Q}\, \psi(\vec{p}, \_\_, \mathsf{Q})$$

then

$$\mathsf{S}_1(p_1, \ldots, p_{i-1}, \_\_) \quad \text{iff} \quad \exists \mathsf{Q}\, \exists p_i\, \psi(\vec{p}, \_\_, \mathsf{Q})$$

and

$$\mathsf{S}_2(p_1, \ldots, p_{i-1}, \_\_) \quad \text{iff} \quad \forall p_i\, \exists \mathsf{Q}\, \psi(p_1, \ldots, p_i, \_\_, \mathsf{Q})$$

$$\text{iff} \quad \exists \mathsf{Q}'\, \forall p_i\, \psi(p_1, \ldots, p_i, \_\_, \mathsf{Q}'(\ldots, p_i))$$

where the notation indicates that we have replaced the n-ary relation $\mathsf{Q}(t_1, \ldots, t_n)$ by the new $n+1$-ary $\mathsf{Q}'(t_1, \ldots, t_n, p_i)$ throughout ψ. ∎

2.6 Lemma. *If* $\mathsf{S} = \mathsf{S}(a_1, \ldots, a_j, \_\_)$ *is extended* Π_1^1 *then*

$$\mathsf{S}_1(a_1, \ldots, a_{j-1}, \_\_) \quad \textit{iff} \quad \exists a_j\, \mathsf{S}(a_1, \ldots, a_{j-1}, a_j, \_\_)$$

is extended Π_1^1. *If* S *is co-extended* Σ_1^1 *then*

$$\mathsf{S}_2(a_1, \ldots, a_{j-1}, \_\_) \quad \textit{iff} \quad \forall a_j\, \mathsf{S}(a_1, \ldots, a_{j-1}, a_j, \_\_)$$

is co-extended Σ_1^1.

Proof. Again we do the extended Σ_1^1 case and then apply 2.4. The proof is just like the "hard" half of 2.5. Note that the easy half does not go through! ∎

2.7 Lemma. *If* $\mathbf{S}=\mathbf{S}(\vec{p},S_1,\ldots,S_m,\vec{a},P_1,\ldots,P_n)$ *is extended* Π_1^1 *then so are*

$$\forall S_m\, \mathbf{S}(\_\_ S_m \_\_) \quad \text{and} \quad \forall P_n\, \mathbf{S}(\_\_, P_n).$$

If $\mathbf{S}$ *is co-extended* Σ_1^1 *then so are*

$$\exists S_m\, \mathbf{S}(\_\_, S_m, \_\_) \quad \text{and} \quad \exists P_n\, \mathbf{S}(\_\_, P_n). \quad \square$$

2.8 Proposition. *If* $S=S(p_1,\ldots,p_i)$ *is extended* Π_1^1 *(co-extended* Σ_1^1*) and is really a predicate over* $\mathfrak{M}$*; i.e.* $S\subseteq M^i$*, then* S *is* Π_1^1 *over* $\mathfrak{M}$ *(*Σ_1^1 *over* $\mathfrak{M}$*).*

Proof. It suffices to prove one of these and take negations, so we prove the Σ_1^1 case. Typically S has a definition of the form

$$S(\vec{p}) \quad \text{iff} \quad \exists \vec{T}\, \exists \vec{Q}\, \varphi(\vec{p},\vec{q},\vec{a},\vec{T},\vec{Q})$$

where $\vec{a}$ are some parameters from $\mathbb{HF}_{\mathfrak{M}}$, $\vec{q}\in\mathfrak{M}$, and φ is co-extended. The quantifiers $\exists T_i$ can alway be treated as quantifiers over relations on $\mathbb{HF}_{\mathfrak{M}}$, since we can always say in φ that T_i is a relation of urelements, so we restrict ourselves to

$$S(\vec{p}) \quad \text{iff} \quad \exists Q\, \varphi(\vec{p},q,a,Q)$$

where φ is co-extended. First we need to get rid of the parameter a. But every $a\in\mathbb{HF}_{\mathfrak{M}}$ can be defined over $\mathbb{HF}_{\mathfrak{M}}$ by some extended formula $\psi(x,q_1,\ldots,q_r)$ so

$$S(\vec{p}) \quad \text{iff} \quad \forall x\, [\psi(x,q_1,\ldots,q_r)\rightarrow \exists Q\, \varphi(\vec{p},q,x,Q)]$$

and the right hand side, by the above rules, is extended Σ_1^1. We are therefore down to the case

$$S(p) \quad \text{iff} \quad \exists Q\, \varphi(p,q,Q)$$

where Q is, say, 3-ary and φ is co-extended. Now the following are equivalent, where ψ is the conjunction of the axioms of extensionality, pair and union and the empty set axiom:

$$S(p),$$
$$\mathbb{HF}_{\mathfrak{M}}\models \exists \mathrm{Q}\, \varphi(p,q,\mathrm{Q}),$$
$$(\mathbb{HF}_{\mathfrak{M}},Q)\models \varphi(p,q,\mathrm{Q}), \quad \text{for some } Q,$$
$$(\mathfrak{A}_{\mathfrak{M}},Q)\models \varphi(p,q,\mathrm{Q}), \quad \text{for some } (\mathfrak{A}_{\mathfrak{M}},Q) \text{ with } \mathbb{HF}_{\mathfrak{M}}\subseteq_{\text{end}}\mathfrak{A}_{\mathfrak{M}},$$
$$(\mathfrak{A}_{\mathfrak{M}},Q)\models \varphi(p,q,\mathrm{Q}), \quad \text{for some } (\mathfrak{A}_{\mathfrak{M}},Q) \text{ with } \mathfrak{A}_{\mathfrak{M}}\models\psi.$$

The structure $\mathfrak{A}_{\mathfrak{M}}$ can be have the same cardinality as $\mathfrak{M}$ in the last two lines since $\mathfrak{M}$ is infinite. The equivalence of the third and fourth lines follows from the fact that φ is co-extended so it drops down from $\mathfrak{A}_{\mathfrak{M}}$ to $\mathbb{HF}_{\mathfrak{M}}$ by II.2.8. The equivalence of the fourth and last lines in a consequence of the fact that $\mathfrak{A}_{\mathfrak{M}}$ must be isomorphic to an end extension of $\mathbb{HF}_{\mathfrak{M}}$ if $\mathfrak{A}_{\mathfrak{M}}$ is a model of the axioms mentioned. The last line can be rewritten as a Σ_1^1 relation on $\mathfrak{M}$ without much trouble. Let's assume that $\mathfrak{M} = \langle M, R \rangle$ with R binary, to simplify things. We introduce a lot of new relation symbols and define $\mathbf{S}_1(M', R', A, E, F, Q)$ by

$$M' \subseteq M,$$

$$R' \subseteq M' \times M',$$

$$A \subseteq M, \ A \cap M' = 0,$$

$$E \subseteq (M' \cup A) \times A,$$

$$F \subseteq M \times M',$$

"*F establishes an isomorphism between* $\langle M, R \rangle$ *and* $\langle M', R' \rangle$",

$$Q \subseteq (M' \cup A)^3.$$

Thus $\mathbf{S}_1$ insures that $(\langle M', R' \rangle; A, E, Q)$ is isomorphic to an $(\mathfrak{A}_{\mathfrak{M}}, Q)$. Let $\mathbf{S}_2(M', A, E)$ assert that this structure satisfies *Extensionality*, *Pair*, *Union* and *Empty set;* e. g. *Pair* can be expressed by

$$\forall x \, \forall y \, [\mathsf{A}(x) \vee \mathsf{M}'(x)) \wedge (\mathsf{A}(y) \vee \mathsf{M}'(y)) \\ \rightarrow \exists z (\mathsf{A}(z) \wedge \forall w \, [w \, \mathsf{E} z \leftrightarrow w = x \vee w = y])].$$

Both $\mathbf{S}_1, \mathbf{S}_2$ can be defined by first order sentences over $\mathfrak{M}$ in the additional symbols. Finally, we let $\varphi'(x,y)$ result from $\varphi(x,y)$ by rewritting it in terms of the structure $(\langle M', R' \rangle, A, E, Q)$. For example $\in$ is replaced by E throughout. Then we have

$S(p)$ iff there are M', R', A, E, F, Q such that

$$\mathbf{S}_1(M', R', A, E, F, Q),$$

$$\mathbf{S}_2(M', A, E) \quad \text{and}$$

$$\exists p' \, \exists q' (F(p,p') \wedge F(q,q') \wedge \varphi'(p',q'))$$

which makes S Σ_1^1 on $\mathfrak{M}$. □

2.9 Examples. (i) It is worthwhile going back to look at some of the examples in 2.1. In 2.1 (ii) and 2.1 (iii) the Π_1^1 predicates are actually extended first order. For example, in 2.1 (iii),

$$x \text{ is torsion iff } \mathbb{HF}_{\mathfrak{M}} \models \exists n (nx = 0)$$

where nx is defined by recursion in $\mathbb{HF}_{\mathfrak{M}}$ just as usual:

$$0x = 0,$$

$$(n+1)x = nx + x$$

where the 0 and + on the right hand side are the group 0 and group addition. In 2.1(iv), D is not co-extended but it is co-extended Σ_1^1, hence Σ_1^1 by 2.8.

(ii) Another example that will come up later is where $\mathfrak{M} = \langle M, \sim \rangle$ with $\sim$ an equivalence relation. Define

$$x < y \quad \text{iff} \quad \text{card}(x/\sim) < \text{card}(y/\sim).$$

This relation is Π_1^1. (This is so simple that the above machinery is of little use.) *If each equivalence class is finite then* $<$ *is also* Σ_1^1*:*

$$\neg(x<y) \quad \text{iff} \quad \mathbb{HF}_{\mathfrak{M}} \models \exists a\, \exists b\, (a = x/\sim \wedge b = y/\sim \wedge \text{card}(b) \leqslant \text{card}(a)),$$

which is extended first order so $\neg(x<y)$ is Π_1^1 so $x<y$ is Σ_1^1. ◻

Let $\mathbf{S}(\vec{p}, \vec{S})$ be a predicate of i-tuples $\vec{p}$ from $\mathfrak{M}$ and m-tuples $\vec{S}$ of relations over $\mathfrak{M}$. $\mathbf{S}$ is Π_1^1 on $\mathfrak{M}$ if there is a $\varphi(\vec{p}, \vec{S}, \vec{T})$ such that

$$\mathbf{S}(\vec{p}, \vec{S}) \quad \text{iff} \quad (\mathfrak{M}, \vec{S}) \models \forall \mathsf{T}_1, \ldots, \forall \mathsf{T}_m\, \varphi(\vec{p}, \vec{\mathsf{S}}, \vec{\mathsf{T}}).$$

Some authors refer to such predicates as second order Π_1^1 predicates. The proof of 2.8 may be modified in an obvious way to yield a little more.

2.10 Proposition. *If* $\mathbf{S}(\vec{p}, \vec{S})$ *is extended* Π_1^1 *then* $\mathbf{S}$ *is* Π_1^1 *on* $\mathfrak{M}$.

Proof. The extra relations $\vec{S}$ ride along for free. ◻

Probably the most familiar example of a Δ_1^1 non-elementary set over $\mathcal{N}$ is the set of (Gödel numbers of) true sentences of arithmetic. This kind of example is very important. It is contained in the following proposition. Here K is some finite language which is coded up in $\mathbb{HF}$. To keep the notation (barely) manageable, we restrict the statement of the propositions to the case where K has one binary symbol r.

2.11 Proposition. *Define a predicate* $\mathbf{S}(N, R, \varphi, s)$ *by the conjunction:*

(i) $N \subseteq M$; $R \subseteq N \times N$; $\varphi, s \in \mathbb{HF}_{\mathfrak{M}}$;

(ii) φ *is a formula of* $\mathsf{K}_{\omega\omega}$, s *is a function with* $\text{dom}(s) \supseteq \text{free variables}(\varphi)$;

(iii) $\forall x \in \text{rng}(s) N(x)$;

(iv) $\langle N, R \rangle \models \varphi[s]$.

Then $\mathbf{S}$ *is both extended* Π_1^1 *and co-extended* Σ_1^1.

Proof. There is no trouble with (i)—(iii) since (i), (ii) are Δ_1 on $\mathbb{HF}_{\mathfrak{M}}$ and (iii) is both extended and co-extended first order. The work comes in with (iv). Note, however, that if this particular $\mathbf{S}$ is co-extended Σ_1^1 then it is also extended Π_1^1 since

$$\mathbf{S}(N, R, \varphi, s) \quad \text{iff} \quad (\text{i}) \wedge (\text{ii}) \wedge (\text{iii}) \wedge \exists x\, [x = \langle \neg, \varphi \rangle \wedge \neg \mathbf{S}(N, R, x, s)]$$

and the right hand side is extended Π_1^1 by the various lemmas above. We prove that $\mathbf{S}$ is co-extended Σ_1^1 by introducing another binary relation symbol Sat and finding a co-extended first order $\mathbf{S}^*(N,R,Sat)$ such that for N,R,φ,s satisfying (i)—(iii),

$$\langle N,R\rangle \models \varphi[s] \quad \text{iff} \quad \exists\, Sat\,[\mathbf{S}^*(N,R,Sat) \wedge Sat(\varphi,s)]\,.$$

To write out $\mathbf{S}^*$ we use $s(p/v_i)$ for

$$s\restriction(\mathrm{dom}(s)-\{v_i\})\cup\{\langle v_i,p\rangle\}\,,$$

this being a Δ_1 operation of s, p and v_i. Now define $\mathbf{S}^*(N,R,Sat)$ by

$$\forall\varphi\,\forall s\,[(\mathrm{i})\wedge(\mathrm{ii})\wedge(\mathrm{iii})\rightarrow$$

if φ is atomic, say $\mathsf{r}(v_i,v_j)$, then $R(s(v_i),s(v_j))\leftrightarrow Sat(\varphi,s)$,

if φ is $\langle\wedge,\{\psi,\theta\}\rangle$ then $Sat(\varphi,s)\leftrightarrow Sat(\psi,s)\wedge Sat(\theta,s)$,

if φ is $\langle\neg,\psi\rangle$ then $Sat(\varphi,s)\leftrightarrow\neg Sat(\psi,s)$,

if φ is $\langle\exists,v_i,\psi\rangle$ then $Sat(\varphi,s)\leftrightarrow\exists p\,[N(p)\wedge Sat(\psi,s(p/v_i))]$

with similar clauses for equality, $\bigvee$, $\forall$. Note that the only unbounded existential quantifier comes from the last clause and that quantifier is over urelements so $\mathbf{S}^*$ is co-extended first order. It clearly has the properties needed to finish our proof. ▯

3. Π_1^1 and Δ_1^1 on Countable Structures

We continue to consider a fixed infinite structure $\mathfrak{M}=\langle M,R_1,\ldots,R_l\rangle$. Our goal here is to show that if $\mathfrak{M}$ is countable then the Δ_1^1 relations over $\mathfrak{M}$ are exactly those relations in $\mathbb{HYP}_{\mathfrak{M}}$. In view of II.5, this shows that the Δ_1^1 relations over $\mathfrak{M}$ are exactly those which are constructible from $\mathfrak{M}$ by the time you come to the first $\mathfrak{M}$-admissible ordinal.

We split the result in half to isolate the role of countability.

3.1 Theorem. *Let $\mathfrak{M}$ be countable. If S is a Π_1^1 relation on $\mathfrak{M}$ then S is Σ_1 on* $\mathbb{HYP}_{\mathfrak{M}}$.

Proof. Consider the language $\mathsf{L}\cup\{\mathsf{P}\}$ as coded in $\mathbb{HYP}_{\mathfrak{M}}$ with $\overline{\mathsf{p}}$ a distinct constant symbol for each $p\in M$. Suppose $S(p)$ iff $\mathfrak{M}\models\forall\mathsf{P}\;\varphi(p,q,\mathsf{P})$. Then $S(p)$ holds iff $(\sigma\rightarrow\varphi(\overline{\mathsf{p}},\overline{\mathsf{q}},\mathsf{P}))$ is valid, where σ is the conjunction of the diagram of $\mathfrak{M}$ and $\forall x\bigvee_{p\in M}(x=\overline{\mathsf{p}})$.

Thus $S(p)$ holds iff the following is true in $\mathbb{HYP}_{\mathfrak{M}}$:

$$\vdash(\sigma\to\varphi(\overline{\mathsf{p}},\overline{\mathsf{q}},\mathsf{P}))$$

by the Completeness Theorem for countable, admissible fragments. Thus S is Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$. □

3.2 Corollary. *Let $\mathfrak{M}$ be countable. If S is Δ_1^1 on $\mathfrak{M}$ then $S\in\mathbb{HYP}_{\mathfrak{M}}$.*

Proof. Immediate from 3.1 and Δ_1 separation in $\mathbb{HYP}_{\mathfrak{M}}$. □

The converse does not need the countability assumption.

3.3 Theorem. *Let S be a relation on $\mathfrak{M}$. If S is Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$ then S is Π_1^1 on $\mathfrak{M}$.*

3.4 Corollary. *If a relation S on $\mathfrak{M}$ is in $\mathbb{HYP}_{\mathfrak{M}}$ then S is Δ_1^1 on $\mathfrak{M}$.*

Proof. If $S\in\mathbb{HYP}_{\mathfrak{M}}$ then S and $\neg S$ are Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$. (Remember that parameters from $\mathbb{HYP}_{\mathfrak{M}}$ are allowed in Σ_1 definitions.) □

Proof of 3.3. Let $S(p)$ be Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$. By Proposition II.8.8 we can find a Σ_1 formula $\varphi(x,q,M)$ such that the following are equivalent:

$S(p)$,

$\mathbb{HYP}_{\mathfrak{M}}\models\varphi[p,q,M]$,

(1) $\mathfrak{A}_{\mathfrak{M}}\models\varphi[p,q,M]$ for every model $\mathfrak{A}_{\mathfrak{M}}$ of KPU^+ (of cardinality card(M)).

The last line is true with or without the parenthetical phrase since card(M) $=$card$(\mathbb{HYP}_{\mathfrak{M}})$. Now code up the language $\mathsf{L}(\in)$ in $\mathbb{HF}$. Call the resultung set K, $\mathsf{K}\in\mathbb{HF}$. Let kpu$^+$ be the set of codes of KPU^+ and let $\varphi=\varphi(v_1,v_2,v_3)$ denote the code of itself. Thus $\varphi\in\mathbb{HF}$ and kpu$^+$ is a Δ_1 subset ot $\mathbb{HF}$ by Theorem II.2.3. Our plan is to rewrite (1) as a Π_1^1 relation over $\mathfrak{M}$ with the aid of 2.10 and 2.8. Again we simplify notation by assuming $\mathfrak{M}=\langle M,R\rangle$ with R binary. Now (1) is equivalent to:

For all M,R,F and all A,E,

(2) *if* $\langle M',R'\rangle\overset{F}{\cong}\langle M,R\rangle$
and $(\langle M',R'\rangle;A,E)$ *is a structure of the appropriate kind, and*

(3) *if* $\langle M',R',A,E\rangle\models\mathrm{kpu}^+$,

(4) *then for some* p',q',m, $\langle M',R',A,E\rangle\models\varphi(p',q',m)$ *where* $F(p)=p'$, $F(q)=q'$ *and* $m\in A$ *is such that* $x\,Em\leftrightarrow M'(x)$ *for all* x.

Let $\mathbf{S}_1(M',R',A,E,F)$ be just as in the proof of 2.8 so that $\mathbf{S}_1$ is first order in the symbols and $\mathbf{S}_1$ expresses line (2). Let $\mathbf{S}_2(M',R',A,E)$ hold if

$$\forall\psi\,[\psi\in\mathrm{kpu}^+\to(M',R',A,E)\models\psi].$$

$\mathbf{S}_2$ expresses (3) and is co-extended Σ_1^1 by 2.11, 2.6 and other lemmas. (It is not necessarily extended Π_1^1, though, due to the $\forall\psi$ in front.) Line (4) can be written in extended Π_1^1 form by 2.11. This makes $S(p)$ of the form

$$\forall M', R', A, E, F\, [\mathbf{S}_1 \wedge \mathbf{S}_2 \rightarrow \mathbf{S}_3]$$

where $\mathbf{S}_1$ is first order, $\mathbf{S}_2$ is co-extended Σ_1^1 and $\mathbf{S}_3$ extended Π_1^1 so S is extended Π_1^1 and hence Π_1^1 by 2.8. ◻

3.5 Corollary. *For any structure $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$, countable or not, the relations S on $\mathfrak{M}$ in $\mathbb{HYP}_M$ are exactly the relations definable over $\mathfrak{M}$ by some formula $\varphi(v_1, \ldots, v_n, q_1, \ldots, q_m)$ of the admissible fragment $\mathsf{L}_{\mathbb{A}}$ where $\mathbb{A} = \mathbb{HYP}_{\mathfrak{M}}$.*

Proof. If S is defined by

$$S(p_1, \ldots, p_n) \quad \text{iff} \quad \mathfrak{M} \models \varphi(p_1, \ldots, p_n, q_1, \ldots, q_m)$$

where $\varphi \in \mathbb{HYP}_{\mathfrak{M}}$ then S is Δ_1 since $\models$ is Δ_1. Thus $S \in \mathbb{HYP}_{\mathfrak{M}}$ by Δ_1 separation.

To prove the converse, first assume $\mathfrak{M}$ is countable. Since $S \in \mathbb{HYP}_{\mathfrak{M}}$ we can write

$$S(\vec{p}) \quad \text{iff} \quad \mathfrak{M} \models \forall \mathsf{T}\, \varphi(\mathsf{T}, \vec{p})$$
$$\quad\quad \text{iff} \quad \mathfrak{M} \models \exists \mathsf{T}'\, \psi(\mathsf{T}', \vec{p})$$

for some first order formulas φ, ψ possibly with constants $\bar{\mathsf{q}}_1, \ldots, \bar{\mathsf{q}}_m$. We may assume T, T' are distinct symbols. Let σ be the sentence

$$\bigwedge \text{Diagram}(\mathfrak{M}) \wedge \forall x \bigvee_{p \in M} x = \bar{\mathsf{p}}\,.$$

The sentence

$$\forall v_1, \ldots, v_n\, [(\sigma \wedge \psi(\mathsf{T}', v_1, \ldots, v_n)) \rightarrow \varphi(\mathsf{T}, v_1, \ldots, v_n)]$$

is logically valid since for any T' on $\mathfrak{M}$, $(\mathfrak{M}, T') \models \psi(p_1, \ldots, p_n)$ implies $S(p_1, \ldots, p_n)$, which in turn implies $(\mathfrak{M}, T) \models \varphi(p_1, \ldots, p_n)$ for any T on $\mathfrak{M}$. By the Interpolation Theorem of III.6.1 there is a formula $\theta(v_1, \ldots, v_n) \in \mathbb{HYP}_{\mathfrak{M}}$, θ involving only the symbols of L and any constants $\bar{\mathsf{q}}$ in φ such that both

$$\sigma \wedge \psi(\mathsf{T}', v_1, \ldots, v_n) \rightarrow \theta(v_1, \ldots, v_n) \quad \text{and}$$
$$\theta(v_1, \ldots, v_n) \rightarrow \varphi(\mathsf{T}, v_1, \ldots, v_n)$$

are valid. But then

$$S(p_1, \ldots, p_n) \quad \text{iff} \quad \mathfrak{M} \models \theta[p_1, \ldots, p_n]\,.$$

Thus the result holds if $\mathfrak{M}$ is countable.

To prove the result for uncountable $\mathfrak{M}$ we apply the Lévy Absoluteness Principle of II.9. The theorem to be proved can be written out as

$$\forall \mathfrak{M}\,\forall S\,[S\subseteq M^n \wedge S\in \mathbb{HYP}_{\mathfrak{M}} \to \exists \theta\in \mathbb{HYP}_{\mathfrak{M}}(\forall \vec{p}\in M(S(\vec{p})\leftrightarrow \mathfrak{M}\models\theta(\vec{p})))]$$

so we need to see that the part within brackets can be written as a Π predicate in ZFC. Recalling that $\mathbb{HYP}_{\mathfrak{M}}=\mathrm{L}(\alpha)_{\mathfrak{M}}$ for the first α to make $\mathrm{L}(\alpha)_{\mathfrak{M}}$ admissible, we can rewrite it as

$$S\subseteq M^n \wedge \forall\alpha[\mathrm{L}(\alpha)_M\models \mathrm{KPU}^+ \wedge \forall\beta<\alpha(\mathrm{L}(\beta)_{\mathfrak{M}}\not\models \mathrm{KPU}^+)$$
$$\wedge\, S\in \mathrm{L}(\alpha)_{\mathfrak{M}} \to \exists\theta(v_1,\ldots,v_n)\in \mathrm{L}(\alpha)_{\mathfrak{M}}(\forall \vec{p}\in M^n, \vec{p}\in S\leftrightarrow \mathfrak{M}\models\theta[\vec{p}])]$$

The part within brackets here is clearly Δ_1 since $\models$ is Δ_1. Thus the theorem is a Π sentence and so it suffices to prove it for countable structures $\mathfrak{M}$. □

There are useful second order generalizations of the above theorems. For example, generalizing 3.1 we get the following result.

3.6 Theorem. *Let $\mathrm{S}(\vec{p},\vec{S})$ be a Π_1^1 predicate on a countable structure $\mathfrak{M}$. For every admissible set $\mathbb{A}$ with $M\in\mathbb{A}$, $S\cap\mathbb{A}$ is Σ_1 on $\mathbb{A}$. The Σ_1 definition is independent of $\mathbb{A}$.*

Proof. If $\mathrm{S}(\vec{p},S)$ holds iff $(\mathfrak{M},S)\models\forall \mathsf{T}\,\varphi(\vec{p},\mathsf{S},\mathsf{T})$, then $\mathrm{S}(\vec{p},S)$ holds iff $(\sigma(\mathsf{S})\to\varphi(\overline{\mathsf{p}},\mathsf{S},T))$ is valid, where $\sigma(\mathsf{S})$ is

$$\bigwedge \mathrm{diagram}(\mathfrak{M},S)\wedge \forall x \bigvee_{p\in M}(x=\overline{\mathsf{p}}).$$

This is a countable sentence of $\mathsf{L}_{\infty\omega}$ so the proof given in 3.1 carries over. □

The second order generalization of 3.3 is not quite the converse of 3.6.

3.7 Theorem. *Let $\mathrm{S}=\mathrm{S}(\vec{p},\vec{S})$ be a second order predicate on $\mathfrak{M}$ which is a Σ_1 subset of $\mathbb{HYP}_{\mathfrak{M}}$. Then $\mathbf{S}$ is Π_1^1 on $\mathfrak{M}$.*

Proof. A simple modification of the proof of 3.3 suffices. Line (1) becomes

(1′) $\quad (\mathfrak{A}_{\mathfrak{M}},S)\models\varphi[p,q,\mathsf{S},M]$, for every model $\mathfrak{A}_{\mathfrak{M}}$ of KPU^+ and every S

which results in a modification of (4) to

(4′) *then for some* p',q',m,s, $\langle M',R',A,E\rangle\models\varphi(p',q',s,m,)$ *where*

$$F(p)=p',\ F(q)=q',\ A(m)\wedge\forall x\,[xEm\leftrightarrow M'(x)]\wedge A(s),$$
$$\forall x\,[S(x)\leftrightarrow\exists y(F(x)=y\wedge yEs)]. \quad □$$

3.8 Corollary. *The set* $\mathbf{S}$ *defined by*

$$\mathbf{S}=\{S\subseteq M^n\colon S\in\mathbb{HYP}_{\mathfrak{M}}\}$$

is Π_1^1 on $\mathfrak{M}$ (as a second order predicate).

Proof. $\mathbf{S}$ is Δ_0 on $\mathbb{HYP}_{\mathfrak{M}}$ since

$$x \in \mathbf{S} \quad \text{iff} \quad \mathbb{HYP}_{\mathfrak{M}} \models \text{“}x \textit{ is a subset of } M\text{”}$$

so $\mathbf{S}$ is Π_1^1 on $\mathfrak{M}$ by 3.7. Note, however, that 3.7 will not allow us to conclude that $\neg\mathbf{S}$ is Π_1^1 on $\mathfrak{M}$ since $\neg\mathbf{S}$ is not a subset of $\mathbb{HYP}_{\mathfrak{M}}$; far from it. □

3.9 Example. Let us return to consider nonstandard models of arithmetic. We showed in § 3 that the set of standard integers in a nonstandard model $\mathfrak{M}=\langle M,0,+,\cdot\rangle$ is Π_1^1 on $\mathfrak{M}$. Sometimes it is Σ_1^1 hence Δ_1^1, sometimes not. Recall that $\mathcal{N}=\langle \omega,0,+,x\rangle$.

i) For an $\mathfrak{M}$ where the set of standard integers *is* Σ_1^1 let $\mathfrak{M}$ be a minimal elementary extension of $\mathcal{N}$: i.e., $\mathcal{N}\prec\mathfrak{M}$ but $\mathcal{N}\prec\mathfrak{N}\prec\mathfrak{M}$ implies $\mathcal{N}=\mathfrak{N}$ or $\mathfrak{N}=\mathfrak{M}$. Such $\mathfrak{M}$ exist by results of Gaifman [1970]. In such an $\mathfrak{M}$ we can define, for $x\in\mathfrak{M}$,

x is standard iff $\exists M_0$ [M_0 is the universe of a proper elementary submodel of $\mathfrak{M}$ and $M_0(x)$].

This is extended Σ_1^1 by 3.10, hence Σ_1^1 by 3.8.

ii) For an $\mathfrak{M}\succ\mathcal{N}$ where the set of standard integers is not Δ_1^1 hence not Σ_1^1, choose a countable $\mathfrak{M}$ with $O(\mathfrak{M})=\omega$ (by II.8.7). The subsets of $\mathfrak{M}$ in $\mathbb{HYP}_{\mathfrak{M}}$ are exactly the first order definable sets (by II.6.7) so the set of standard integers are not in $\mathbb{HYP}_{\mathfrak{M}}$ and hence, by the results of this section, they are not Δ_1^1 on $\mathfrak{M}$. In fact, we see that for countable M, the set of standard integers is Δ_1^1 on $\mathfrak{M}$ iff $O(\mathfrak{M})>\omega$. We will return to this example later. □

3.10—3.12 Exercises

3.10. Let $\mathfrak{M}$ be countable and let $\mathbf{S}_1(p,P)$, $\mathbf{S}_2(p,P)$ be predicates of $p\in M$, $P\subseteq M^2$, each Σ_1^1 on $\mathfrak{M}$. Assume that no pair (p,P) satisfies both $\mathbf{S}_1$ and $\mathbf{S}_2$. Show that there is a Δ_1^1 predicate $\mathbf{S}(p,P)$ containing $\mathbf{S}_1$ but disjoint from $\mathbf{S}_2$. [Copy the proof of 3.5 to find a $\theta(p,P)$ in $\mathsf{L}_{\mathbb{A}}$ such that $\mathbf{S}(p,P)$ iff $(\mathfrak{M},P)\models\theta(p,P)$ and then show that $\mathbf{S}$ is Δ_1^1.]

3.11. Recall Example 2.1 (iv). Let $\alpha>\omega$ be any countable admissible ordinal. Let p be any prime. Show that there is a countable p-group G with length $(G)=\alpha$ such that G has a proper divisible subgroup but none in $\mathbb{HYP}_G$. For such a G the largest divisible subgroup of G is thus Σ_1^1 but not Π_1^1. [Use the $\overline{YY}$-Compactness Theorem.]

3.12. Generalize the results of this section to show, for $\mathbb{A}_{\mathfrak{M}}$ transitive, $\mathbb{HF}_{\mathfrak{M}}\subseteq\mathbb{A}_{\mathfrak{M}}$:

i) If S is a relation on $\mathbb{A}_{\mathfrak{M}}$ and S is Σ_1 on $\mathbb{HYP}(\mathbb{A}_{\mathfrak{M}})$ then S is Π_1^1 on $\mathbb{A}_{\mathfrak{M}}$.

ii) If $\mathbb{A}_{\mathfrak{M}}$ is countable then the converse of i) holds.

3.13 Notes. Kripke and Platek proved that a subset X of $\mathbb{HF}$ is Π_1^1 over $\mathbb{HF}$ iff X is Σ_1 over $\mathbb{HYP}(\mathbb{HF})$ and hence that X is Δ_1^1 over $\mathbb{HF}$ iff $X\in\mathbb{HYP}(\mathbb{HF})$. This was generalized in Barwise-Gandy-Moschovakis [1971] by replacing $\mathbb{HF}$ by any countable transitive set A closed under pairs. It is clear from the proof given there that Theorem 3.1 holds. It came as somewhat of a surprise that its converse, Theorem 3.3, holds without any coding assumptions about the structure $\mathfrak{M}$, since the inductive definability approach (discussed in Chapter VI) does not work in this complete generality.

4. Perfect Set Results

In this section we give a more sophisticated example of the interplay of model theory and recursion theory showing how each subject can shed light on the other and how logic on admissible sets sheds light on both. The results themselves will not be used in the remainder of the book.

The following, a classical result on hyperarithmetic sets, is the effective version (due to Harrison) of an even older result in descriptive set theory.

4.1 Theorem. *If* $\mathbf{S} \subseteq \mathrm{Power}(\omega)$ *is* Σ_1^1 *on* $\mathcal{N} = \langle \omega, 0, +, \cdot \rangle$ *and* $\mathrm{card}(\mathbf{S}) < 2^{\aleph_0}$ *then* $\mathbf{S}$ *is a set of hyperarithmetic sets.*

Compare this with two results from model theory. The first is due to Kueker [1968].

4.2 Theorem. *Let* $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ *be a countable structure for a language* L *and let* P *be an n-ary relation on* M. *If the set*

$$\mathbf{S} = \{Q \mid (\mathfrak{M}, P) \cong (\mathfrak{M}, Q)\}$$

has $\mathrm{card}(\mathbf{S}) < 2^{\aleph_0}$ *then*

$$P = \{(x_1, \ldots, x_n) \mid \mathfrak{M} \models \varphi[x_1, \ldots, x_n, q_1, \ldots, q_m]\}$$

for some formula φ *of* $\mathsf{L}_{\omega_1\omega}$ *and some* $q_1, \ldots, q_m \in \mathfrak{M}$.

(A formula φ is in $\mathsf{L}_{\omega_1\omega}$ if it is in $\mathsf{L}_{\mathbb{A}}$ for some countable fragment $\mathsf{L}_{\mathbb{A}}$ of $\mathsf{L}_{\infty\omega}$.)

The next result is a theorem of Chang [1964], Makkai [1964], and Reyes [1968]. Chang and Makkai had a stronger hypothesis.

4.3 Theorem. *Let* $\varphi(\mathsf{P})$ *be a finitary sentence of* $\mathsf{L} \cup \{\mathsf{P}\}$. *Assume that for each countable model* $\mathfrak{M}$ *there are fewer than* $2^{\aleph_0}$ *relations* P *such that*

$$(\mathfrak{M}, P) \models \varphi(\mathsf{P}).$$

Then there are finitary formulas $\psi_1(\vec{x}, y_1, \ldots, y_k), \ldots, \psi_m(\vec{x}, y_1, \ldots, y_{k_m})$ *of* $\mathsf{L}_{\omega\omega}$ *such that for every model* $(\mathfrak{M}, P)$ *of* $\varphi(\mathsf{P})$, *there is an* i, $1 \leqslant i \leqslant m$, *and* $q_1, \ldots, q_{k_i} \in \mathfrak{M}$ *such that*

$$P = \{(x_1, \ldots, x_n) \mid \mathfrak{M} \models \psi_i[\vec{x}, q_1, \ldots, q_{k_i}]\}.$$

The conclusion of 4.3 can be restated as: *the sentence*

$$\varphi(\mathsf{P}) \to \bigvee_{1 \leqslant i \leqslant m} \exists y_1, \ldots, y_{k_i} \forall \vec{x} [\mathsf{P}(\vec{x}) \leftrightarrow \psi_i(\vec{x}, y_1, \ldots, y_{k_i})]$$

is logically valid.

These three results, while incomparable, are obviously quite similar. They all begin with the assumption that a certain definable or Σ_1^1 class $\mathbf{S}$ has fewer than $2^{\aleph_0}$ elements and conclude that each element of $\mathbf{S}$ is definable in some way. We want to show these results are more than merely analogous, that they are in fact shadows of a single definability result about logic on admissible sets. First, though, we prove a generalization of 4.1, because the proof is relevant to our general result.

4.4 Theorem. *Let* $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ *be a countable structure and let* $\mathbf{S}$ *be a second order* Σ_1^1 *predicate on* $\mathfrak{M}$. *If* $\mathrm{card}(\mathbf{S}) < 2^{\aleph_0}$ *then* $\mathbf{S} \subseteq \mathbb{HYP}_{\mathfrak{M}}$ *(and hence* $\mathbf{S}$ *is countable).*

Proof. After a trick the result falls right out of III.8.2. Assume $\mathbf{S} \nsubseteq \mathbb{HYP}_{\mathfrak{M}}$. Then by 3.8 (and this is the trick), $\mathbf{S}_0 = \mathbf{S} - \mathbb{HYP}_{\mathfrak{M}}$ is Σ_1^1 and non-empty. We prove that $\mathbf{S}_0$ (and hence $\mathbf{S}$) has cardinality $2^{\aleph_0}$. Let us handle the case where $\mathbf{S}_0$ is a predicate of one relation:

$$\mathbf{S}_0(S) \text{ iff } (\mathfrak{M}, S) \models \exists \mathsf{T}\, \varphi(\mathsf{S}, \mathsf{T}).$$

Let $\mathsf{L}' = \mathsf{L} \cup \{\bar{\mathsf{p}} : p \in M\} \cup \{\mathsf{S}\}$, $\mathsf{K} = \mathsf{L}' \cup \{\mathsf{T}\}$ and let $\mathsf{L}'_{\mathbb{A}}, \mathsf{K}_{\mathbb{A}}$ be the countable admissible fragments given by $\mathbb{HYP}_{\mathfrak{M}}$. If σ is

$$\mathrm{Diagram}(\mathfrak{M}) \wedge \forall x \bigvee_{p \in M} (x = \bar{\mathsf{p}})$$

then $\sigma \wedge \varphi(\mathsf{S}, \mathsf{T})$ is in $\mathsf{K}_{\mathbb{A}}$. We claim that σ can have no model which is decidable for $\mathsf{L}'_{\mathbb{A}}$. Such a model would be isomorphic to some structure of the form $(\mathfrak{M}, S, T)$, where S is Δ_1 on $\mathbb{HYP}_{\mathfrak{M}}$ and hence $S \in \mathbb{HYP}_{\mathfrak{M}}$, whereas $(\mathfrak{M}, S, T) \models \varphi(\mathsf{S}, \mathsf{T})$, implies $S \in \mathbf{S}_0$. Thus the result follows from III.8.2. ▯

We now turn to consider the relationship between 4.2 and 4.4. If we apply 4.4 to the situation described in Theorem 4.2 we learn that if there are $< 2^{\aleph_0} Q$'s with $(\mathfrak{M}, P) \cong (\mathfrak{M}, Q)$, then each of these is Δ_1^1 on $(\mathfrak{M}, P)$ which (while interesting and not obvious from 4.2) says nothing about the original P. There are examples $(\mathfrak{M}, P)$ satisfying 4.2 but where $P \notin \mathbb{HYP}_{\mathfrak{M}}$, i.e., is not Δ_1^1 on $\mathfrak{M}$, which rules out one possible strengthing of 4.4 that would yield 4.2. To find the correct generalization of 4.2, 4.3 and 4.4 we need a new definition.

4.5 Definition. A Σ_1^1 sentence of an admissible fragment $\mathsf{L}_{\mathbb{A}}$ is a second order infinitary sentence of the form

$$\exists \mathcal{Q}\, \varphi$$

where $\mathcal{Q}$ is a set of symbols of L, $\mathcal{Q} \in \mathbb{A}$, and $\varphi \in \mathsf{L}_{\mathbb{A}}$.

If $\mathcal{Q}$ is finite, the requirement $\mathcal{Q} \in \mathbb{A}$ is automatically true, and we could write

$$\exists \vec{\mathsf{Q}}\, \varphi(\vec{\mathsf{Q}})$$

or

$$\exists \mathsf{Q}_1, \ldots, \exists \mathsf{Q}_n\, \varphi(\mathsf{Q}_1, \ldots, \mathsf{Q}_n)\,.$$

In the infinite case, however, we should not think of $\mathcal{Q}$ as being a well ordered sequence of symbols. Note that even though we have written $\mathcal{Q}$, the definition actually permits function and constant symbols to occur in $\mathcal{Q}$ as well as relations symbols.

The following result has 4.2—4.4 as consequences. For ordinary (as opposed to Σ_1^1) sentences of $\mathsf{L}_{\mathbb{A}}$ it is due to Makkai [1973]. For 4.4, though, it is the Σ_1^1 version which matters. The proof is a minor variation on Makkai's theme, the Interpolation Theorem taking the part formerly played by Beth's theorem.

4.6 Theorem. *Let $\exists \mathcal{Q}\, \varphi(\mathsf{P}, \mathcal{Q})$ be a Σ_1^1 sentence of the countable admissible fragment $\mathsf{L}_{\mathbb{A}}$. If for each countable structure $\mathfrak{M}$ there are less than $2^{\aleph_0}$ relations P such that*

$$(\mathfrak{M}, P) \models \exists \mathcal{Q}\, \varphi(\mathsf{P}, \mathcal{Q})$$

then there is a sentence σ of $\mathsf{L}_{\mathbb{A}}$ of the form

$$\bigvee_{i \in I} \exists y_1, \ldots, \exists y_{m_i} \forall x_1, \ldots, \forall x_n [\mathsf{P}(x_1, \ldots, x_n) \leftrightarrow \psi_i(x_1, \ldots, x_n, y_1, \ldots, y_{m_i})]$$

which is a logical consequence of $\varphi(\mathsf{P}, \mathcal{Q})$, where each ψ_i contains only symbols of L not in $\mathcal{Q} \cup \{\mathsf{P}\}$.

The converse is obvious. In fact, the conclusion implies that every such P is in any admissible set containing $\mathfrak{M}$ and φ so there are $\leqslant \aleph_0$ such P.

Note that Theorem 4.3 is the special case of Theorem 4.6 where $\mathsf{L}_{\mathbb{A}}$ is $\mathsf{L}_{\omega\omega}$ and where the Q's do not occur in $\varphi(\mathsf{P}, \mathcal{Q})$.

Before attempting to prove 4.6 it is good to get some idea of what it says by applying it to prove 4.2 and strengthen 4.4.

4.7 Corollary. *Under the assumption of Theorem 4.4 there is an $\mathrm{S}' \in \mathbb{HYP}_{\mathfrak{M}}$ such that $\mathrm{S} \subseteq \mathrm{S}'$.*

Proof. Suppose $P \in \mathrm{S}$ iff $(\mathfrak{M}, P) \models \exists \mathsf{Q}\, \varphi_0(\mathsf{P}, \mathsf{Q})$. Let φ be the conjunction of $\varphi_0(\mathsf{P}, \mathsf{Q})$, diagram $(\mathfrak{M})$ and $\forall x \bigvee_{p \in M} (x = \bar{\mathsf{p}})$. The hypothesis of 4.6 is satisfied so let σ be as in the conclusion of 4.6, σ of the form

$$\bigvee_{i \in I} \exists y_1, \ldots, \exists y_{m_i} \forall x_1, \ldots, \forall x_n [\mathsf{P}(x_1, \ldots, x_n) \leftrightarrow \psi_i(x_1, \ldots, x_n, y_1, \ldots, y_{m_i})]\,,$$

where each ψ_i is in the language $\mathsf{L} \cup \{\bar{\mathsf{p}} \mid \in M\}$. For each $i \in I$ and $q_1, \ldots, q_{m_i} \in M$ let

$$P_{i,\vec{q}} = \{(x_1, \ldots, x_n) \mid \mathfrak{M} \models \psi_i[x_1, \ldots, x_n, q_1, \ldots, q_m]\}\,.$$

Each $P_{i,\vec{q}} \in \mathbb{HYP}_{\mathfrak{M}}$ by Δ_1 Separation and, as an operation of i and $\vec{q}$, $P_{i,\vec{q}}$ is a Σ operation in $\mathbb{HYP}_{\mathfrak{M}}$ so we may form the set

$$\mathbf{S}' = \{P_{i,\vec{q}} \mid i \in I, \vec{q} \in M\} \in \mathbb{HYP}_{\mathfrak{M}}$$

by Σ Replacement and $\mathbf{S} \subseteq \mathbf{S}'$. ◻

4.8 Theorem. *Let $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ be a countable recursively saturated structure (i.e. $o(\mathbb{HYP}_{\mathfrak{M}}) = \omega$). Let $\mathbf{S}$ be a second order Σ_1^1 predicate with* $\mathrm{card}(\mathbf{S}) < 2^{\aleph_0}$, *say $\mathbf{S} \subseteq \mathrm{Power}(M^n)$. There is a finite set of finitary formulas*

$$\psi_1(\vec{x}, y_1, \ldots, y_{m_1}), \ldots, \psi_k(\vec{x}, y_1, \ldots, y_{m_k})$$

of $\mathsf{L}_{\omega\omega}$ such that for each $S \in \mathbf{S}$ there is an i, $1 \leqslant i \leqslant k$, and elements $q_1, \ldots, q_{m_i}$ of $\mathfrak{M}$ so that S is defined by

$$S(\vec{x}) \quad \text{iff} \quad \mathfrak{M} \models \psi_i[\vec{x}, q_1, \ldots, q_{m_i}].$$

Proof. Using 4.7 choose $\mathbf{S}'$ so that $\mathbf{S}' \subseteq \mathrm{Power}(M^n)$ and

$$\mathbf{S} \subseteq \mathbf{S}' \in \mathbb{HYP}_{\mathfrak{M}}.$$

Since $o(\mathbb{HYP}_{\mathfrak{M}}) = \omega$ we have, by II.7.3,

$$\forall S \in \mathbf{S}' \, \exists \psi \, \exists \vec{q}$$

[ψ is a formula of $\mathsf{L}_{\omega\omega}$, $\vec{q}$ is an m-tuple of elements of M (where the free variables of ψ are among $v_1, \ldots, v_{n+m}$) so that for all $x_1, \ldots, x_n \in M$:

$$\langle x_1, \ldots, x_n \rangle \in S \quad \text{iff} \quad \mathfrak{M} \models \psi[x_1, \ldots, x_n, q_1, \ldots, q_m]].$$

Since L is finite we can assume $\mathsf{L}_{\omega\omega}$ is coded up on $\mathbb{HF}$. By Σ Collection in $\mathbb{HYP}_{\mathfrak{M}}$ there is a finite set Φ of formulas such that each ψ can be chosen in Φ. ◻

4.9 Example. *Let $\mathcal{N} = \langle \omega, 0, +, \cdot \rangle$ and let $\mathfrak{M}$ be a countable recursively saturated elementary extension of $\mathcal{N}$. Then there are $2^{\aleph_0}$ distinct $\mathfrak{M}_0$ such that*

(i) $\mathfrak{M}_0 \prec \mathfrak{M}$, *and*

(ii) $\mathfrak{M}_0$ *is an initial segment of* $\mathfrak{M}$.

Proof. Let

$$\mathbf{S} = \{M_0 \subseteq M \mid M_0 \text{ is the universe of an } \mathfrak{M}_0 \text{ with (i) and (ii)}\}.$$

The techniques of § 2 show that $\mathbf{S}$ is Σ_1^1 on $\mathfrak{M}$. Suppose, toward a contradiction, that $\mathrm{card}(\mathbf{S}) < 2^{\aleph_0}$. Then since $\omega \in \mathbf{S}$, there is a formula $\psi(x, q_1, \ldots, q_m)$ with parameters from $\mathfrak{M}$ such that

$$\omega = \{x \mid \mathfrak{M} \models \psi[x, q_1, \ldots, q_m]\}$$

which is a contradiction. ▯

Before turning to the proof of Theorem 4.6, we show how 4.8 can be used to strengthen the Chang-Makkai-Reyes Theorem (4.3). The result is interesting because of the light it sheds on the usual proofs of this theorem by means of saturated (or special) models.

4.10 Corollary. *Let $\varphi(\mathsf{P}, \mathsf{Q})$ be a finitary sentence such that for each* recursively saturated *countable model $\mathfrak{M}$, there are less than $2^{\aleph_0}$ different P with*

$$(\mathfrak{M}, P) \models \exists \mathsf{Q}\, \varphi(\mathsf{P}, \mathsf{Q}).$$

Then there is a finite list of finitary formulas $\psi_1(\vec{x}, \vec{y}), \ldots, \psi_m(\vec{x}, \vec{y})$ such that

$$\models \varphi(\mathsf{P}, \mathsf{Q}) \rightarrow \bigvee_{1 \leqslant i \leqslant m} \exists \vec{y}\, \forall \vec{x} [\mathsf{P}(\vec{x}) \leftrightarrow \psi(\vec{x}, \vec{y})].$$

Proof. Suppose that the hypothesis holds but that the conclusion falls. Let T be the theory

$$\varphi(\mathsf{P}, \mathsf{Q})$$

$$\neg \exists \vec{y}\, \forall \vec{x} [\mathsf{P}(\vec{x}) \leftrightarrow \psi(\vec{x}, \vec{y})], \quad \text{for all} \quad \psi \in \mathsf{L}_{\omega\omega}.$$

By the ordinary compactness theorem, this theory is consistent. By Theorem II.8.8, it has a countable recursively saturated model $(\mathfrak{M}, P)$. But this structure $\mathfrak{M}$ has $< 2^{\aleph_0} P'$ such that $(\mathfrak{M}, P') \models \exists \mathsf{Q}\, \varphi(\mathsf{P}, \mathsf{Q})$ so, by 4.8, each of these P' (in particular the original P) is definable, contradicting the fact that $(\mathfrak{M}, P)$ is a model of T. ▯

4.11. Proof of 4.2 from 4.6. We must cheat a bit by quoting a result, Scott's Theorem, from Chapter VII. Let $\mathfrak{M}, P, \mathbf{S}$ be given as in 4.2 and suppose that $\operatorname{card}(\mathbf{S}) < 2^{\aleph_0}$. Let $\varphi(\mathsf{P})$ be the Scott sentence of $(\mathfrak{M}, P)$ so that for all countable structures $(\mathfrak{M}', P')$,

$$(\mathfrak{M}', P') \models \varphi(\mathsf{P}) \text{ iff } (\mathfrak{M}, P) \cong (\mathfrak{M}', P').$$

(The sentence $\varphi(\mathsf{P})$ involves only constants from $\mathsf{L} \cup \{\mathsf{P}\}$.) Thus there are, for each model $\mathfrak{M}'$, fewer than $2^{\aleph_0} P'$ such that

$$(\mathfrak{M}', P') \models \varphi(\mathsf{P}).$$

From 4.6 we get a $\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ such that for some $q_1, \ldots, q_m \in M$

$$(\mathfrak{M}, P) \models \forall \vec{x}[\mathsf{P}(\vec{x}) \leftrightarrow \psi(x_1, \ldots, x_n, q_1, \ldots, q_m)].$$

which yields the conclusion of 4.2. ▯

Having no excuse for further procrastination, we begin the proof of 4.6.

4.12. Proof of 4.6. Since 4.6 implies 4.4 we expect to use considerations similar to those used in proving 4.4, that is, the method of § III.8. The chief difference is that instead of constructing $2^{\aleph_0}$ distinct models $\mathfrak{M}$ we need one model with $2^{\aleph_0}$ distinct P such that

$$(\mathfrak{M}, P) \models \exists \mathcal{Q}\, \varphi(\mathsf{P}, \mathcal{Q}).$$

This accounts for the complications in the proof below. We prove the contrapositive, so suppose $\varphi(\mathsf{P}, \mathcal{Q})$ does not have any sentence of the desired form as a logical consequence. Let us simplify matters by assuming that $\mathcal{Q}$ has only one relation symbol Q and, further, that P is unary. The proof will make it clear that these assumptions do not really matter. Let

$$\mathsf{L}^0 = \mathsf{L} - \{\mathsf{P}, \mathsf{Q}\}, \quad C = \{\mathsf{c}_n \mid n < \omega\}, \quad \mathsf{K}^0 = \mathsf{L}^0 \cup C, \quad \mathsf{K} = \mathsf{L} \cup C.$$

Call a set s of sentences of $\mathsf{K}_{\mathbb{A}}$ *special* if the following conditions are fulfilled, conditions (D1)—(D7) coming from (C1)—(C7) of III.2.2 respectively.

(D1) If $\varphi \in s$ then $\neg\varphi \notin s$.
(D2) If $\neg\varphi \in s$ then $\sim\varphi \in s$.
(D3) If $\bigwedge \Phi \in s$ then $\varphi \in s$ for all $\varphi \in \Phi$.
(D4) If $\forall v \varphi(v) \in s$ then $\varphi(\mathsf{c}) \in s$ for all $\mathsf{c} \in C$.
(D5) If $\bigvee \Phi \in s$ then $\varphi \in s$ for some $\varphi \in \Phi$.
(D6) If $\exists v \varphi(v) \in s$ then for some $\mathsf{c} \in C$, $\varphi(\mathsf{c}) \in s$.
(D7) If t is a basic term of $\mathsf{L}_{\mathbb{A}}$ and $\mathsf{c}, \mathsf{d} \in C$ then: if $(\mathsf{c} \equiv \mathsf{d}) \in s$ then $(\mathsf{d} \equiv \mathsf{c}) \in s$; if $\varphi(t)$, $(\mathsf{c} \equiv t) \in s$ then $\varphi(\mathsf{c}) \in s$; for some $\mathsf{e} \in C$, $(\mathsf{e} \equiv t) \in s$.
(D8) If $\varphi \in \mathsf{K}^0_{\mathbb{A}}$ then $\varphi \in s$ or $\neg\varphi \in s$.

In the proof of the Model Existence Theorem we first constructed a set s_ω satisfying (D1)—(D7) and then showed that any set s satisfying (D1)—(D7) gave rise to a unique canonical model $\mathfrak{M}$ by the conditions

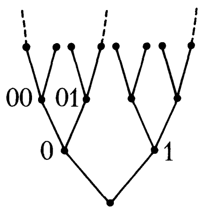
$$\mathfrak{M} \models \mathsf{R}(\mathsf{c}_1, \ldots, \mathsf{c}_n) \text{ iff } \mathsf{R}(\mathsf{c}_1, \ldots, \mathsf{c}_n) \in s.$$

Furthermore, this model was a model of each $\varphi \in s$. We shall use these facts here. Note that if a consistency property S has the property

$$\text{(C8) if } s \in S \text{ and } \varphi \in \mathsf{K}^0_{\mathbb{A}} \text{ then } s \cup \{\varphi\} \in S \text{ or } s \cup \{\neg\varphi\} \in S$$

then the resulting s_ω will satisfy (D8) and hence will be a special set of sentences.

Now recall the notation from § III.8:



d is a typical node on the tree; $d0$ extends d by putting a 0 on right end; $d1$ a 1; and b is a typical branch.

The *level* of a node is just its length as a sequence. The plan for the proof is to attach a finite set s_d of sentences of $\mathsf{K}_{\mathbb{A}}$ to each node d of the tree in a way that insures the following conditions:

(1) $\{\varphi(\mathsf{P},\mathsf{Q})\}$ *is placed at the bottom of the tree; i.e.,* $s_{<>}=\{\varphi(\mathsf{P},\mathsf{Q})\}$.

(2) *If b is any branch and* $s^b=\bigcup\{s_d \mid d \text{ a node on } b\}$ *then s^b is a special set of sentences of* $\mathsf{K}_{\mathbb{A}}$.

(3) *Any two sets s_d and $s_{d'}$ on the tree are consistent with respect to the sentences of* $\mathsf{K}^0_{\mathbb{A}}$; that is, if $\varphi\in\mathsf{K}^0_{\mathbb{A}}$ and $\varphi\in s_d$ then $(\neg\varphi)\notin s_{d'}$.

(4) *Distinct branches through the tree are inconsistent with respect to the symbol* P; that is, if b_1, b_2 split at d then there is a constant symbol c so that $\mathsf{P}(\mathsf{c})$ is in s_{d0}, but $\neg\mathsf{P}(\mathsf{c})$ is in s_{d1}.

Suppose we contrive to fulfill (1)—(4). The canonical model determined by a branch b through the tree will have the form $(\mathfrak{M}^b,P^b,Q^b)$ with $\varphi(\mathsf{P},\mathsf{Q})$ true by (1), (2) and the above remarks on special sets. Furthermore, $\mathfrak{M}^{b_1}=\mathfrak{M}^{b_2}$ for all branches b_1, b_2. For if $\mathsf{R}\in\mathsf{L}^0$ and $\mathsf{R}(\mathsf{c}_1,\mathsf{c}_2)$ holds in $\mathfrak{M}^{b_1}$ then $\mathsf{R}(\mathsf{c}_1,\mathsf{c}_2)\in s_d$ for some d on b_1 but then $\neg\mathsf{R}(\mathsf{c}_1,\mathsf{c}_2)$ is never put into any $s_{d'}$ on b_2, by (3), so $\mathsf{R}(\mathsf{c}_1,\mathsf{c}_2)$ is in some $s_{d'}$ on b_2 by (D8) so $\mathsf{R}(\mathsf{c}_1,\mathsf{c}_2)$ holds in $\mathfrak{M}^{b_2}$. Finally, if b_1, b_2 are distinct branches then $P^{b_1}\neq P^{b_2}$ by (4). In other words we have one model $\mathfrak{M}$ with $2^{\aleph_0}$ distinct P each satisfying

$$(\mathfrak{M},P)\models\exists\mathsf{Q}\,\varphi(\mathsf{P},\mathsf{Q})$$

and so we will have proved our theorem. Satisfying (1)—(4), though, is not so trivial.

In order ultimately to satisfy condition (4), we would like to have a symbol P^b for each branch b thru the tree, but this would make our language uncountable. Instead we introduce new relation symbols P^d, Q^d for each node d on the tree.

We think of P^d as our original P with a ghostly superscript d just barely visible. Our original P, Q are P^d, Q^d where d is the empty sequence, $d = < >$. We denote this expanded language by K^g and the admissible fragment by K^g_A. As usual we consider only formulas with finite many c's and, this time, only finitely many different P^{d}'s ans Q^{d}'s. A finite set s of sentences of K^g_A is *g-consistent* if all the nodes occuring as ghostly superscripts in s lie on some branch (e.g., P^{010} and Q^{01010} could both occur in s but P^{010} and Q^{011} could not). If s is g-consistent then $\hat{s}$ is the result of increasing all superscripts in s to the longest one appearing in s. E. g., if 010 and 01010 are the only superscripts in s then $\hat{s}$ has all P^{010} and Q^{010} replaced by P^{01010} and Q^{01010}. We define a giant consistency machine $\mathbf{S}$ by $\{s_1, \ldots, s_n\} \in \mathbf{S}$ iff $s_1, \ldots, s_n$ are each finite, g-consistent, and $\hat{s}_1 \cup \cdots \cup \hat{s}_n$ does not imply any sentence of $\mathsf{K}^g_{\mathbb{A}}$ of the form

$$(*) \qquad \bigvee_{1 \leq i \leq n} \bigvee_{\psi \in \Psi_i} [\exists \vec{y}\, \forall x\, \mathsf{P}^{d_i}(x) \leftrightarrow \psi(x, \vec{y})]$$

where each $\psi \in \mathsf{L}^0_{\mathbb{A}}$ and d_i is the longest node in s_i. (Note that if $\{s_1, \ldots, s_n\} \in \mathbf{S}$ then $\hat{s}_1 \cup \cdots \cup \hat{s}_n$ is consistent which will give us (3) above.) Our hypothesis insures us that

$$(5) \quad \{\{\varphi(\mathsf{P}, \mathsf{Q})\}\} \in \mathbf{S}.$$

While $\mathbf{S}$ is not really a consistency property, it generates many of them.

(6) *If* $\{s_1, \ldots, s_n, s_{n+1}\} \in \mathbf{S}$ *then*

$$S = \{s \mid \{s_1, \ldots, s_n, s\} \in \mathbf{S}\}$$

is a consistency property satisfying (C8) *above.*

Most of the clauses are routine. Let us check (C5) and (C8).

(C5) Suppose $\bigvee \Theta \in s \in S$, but that for each $\theta \in \Theta$, $s \cup \{\theta\} \notin S$ so that

$$\{s_1, \ldots, s_n, s \cup \{\theta\}\} \notin \mathbf{S}.$$

Since s is g-consistent so is $s \cup \{\theta\}$ so the problem comes from (*). We must have, for each $\theta \in \Theta$, some σ_θ of the form (*) such that

$$\hat{s}_1 \cup \cdots \cup \hat{s}_n \cup \widehat{s \cup \{\theta\}} \vdash \sigma_\theta.$$

Now, just as in the proof of the interpolation theorem, we can assume the σ_θ is given as a function of θ, a function in our admissible set (σ_θ will be the disjunction of the σ's given by strong Σ replacement). But then $\sigma = \bigvee_{\theta \in \Theta} \sigma_\theta$ is again of the form (*), once you rearrange it a bit, and

$$\hat{s}_1 \cup \cdots \cup \hat{s}_n \cup \hat{s} \vdash \sigma,$$

a contradiction.

(C8) Suppose $\varphi(\mathsf{c}_1,\dots,\mathsf{c}_n)\in\mathsf{K}^0_{\mathbb{A}}$, that $s\in S$ but neither $s\cup\{\varphi(\mathsf{c}_1,\dots,\mathsf{c}_n)\}$ nor $s\cup\{\neg\varphi(\mathsf{c}_1,\dots,\mathsf{c}_n)\}\in S$. Then there are sentences σ_1, σ_2 of the form (*) such that

$$\hat{s}_1\cup\cdots\cup\hat{s}_n\cup\hat{s}\vdash\varphi(\mathsf{c}_1,\dots,\mathsf{c}_n)\rightarrow\sigma_1$$
$$\hat{s}_1\cup\cdots\cup\hat{s}_n\cup\hat{s}\vdash\neg\varphi(\mathsf{c}_1,\dots,\mathsf{c}_n)\rightarrow\sigma_2$$

but then

$$\hat{s}_1\cup\cdots\cup\hat{s}_n\cup\hat{s}\vdash\sigma_1\vee\sigma_2$$

and $\sigma_1\vee\sigma_2$ is equivalent to a sentence of the form (*).

We now come to the crucial step which will yield (4) above.

(7) *If* $\{s_1,\dots,s_n\}\in\mathbf{S}$, *if d is the longest node in* s_n, *if* $d0, d1$ *do not occur in* $s_1\cup\cdots\cup s_n$, *and if* c *is a constant symbol not in* $s_1\cup\cdots\cup s_n$ *then*

$$\{s_1,\dots,s_{n-1},s_n\cup\{\mathsf{P}^{d0}(\mathsf{c})\},s_n\cup\{\neg\mathsf{P}^{d1}(\mathsf{c})\}\}$$

is in **S**.

We use the Interpolation Theorem for $\mathsf{K}_{\mathbb{A}}$ to prove (7). We invite the student to try the case $n=1$ for himself before going on. We do the case $n=2$ because it exhibits the problems that arise in general. Now, if (7) fails, the trouble cannot arise from g-consistency since

$$s_1, s_2\cup\{\mathsf{P}^{d0}(\mathsf{c})\},\quad s_2\cup\{\mathsf{P}^{d1}(\mathsf{c})\}$$

are g-consistent so it must be that there are sentences σ_1, σ_2, σ_3 where σ_i is of the form

$$\bigvee_{\psi\in\Psi_i}\exists\vec{y}\,\forall x\,[\mathsf{P}_i(x)\leftrightarrow\psi(x,\vec{y})]$$

(where P_1 is the symbol P^d in $\hat{s}_1$, P_2 is P^{d0}, P_3 is P^{d1}), such that

(8) $$\hat{s}_1\cup\widehat{s_2\cup\{\mathsf{P}^{d0}(\mathsf{c})\}}\cup\widehat{s_2\cup\{\neg\mathsf{P}^{d1}(\mathsf{c})\}}\vdash\sigma_1\vee\sigma_2\vee\sigma_3.$$

We show that (8) implies $\{s_1,s_2\}\notin\mathbf{S}$ by finding a sentence σ of the form (*) such that

$$\hat{s}_1\cup\hat{s}_2\vdash\sigma.$$

Rewrite (8) as follows:

$$[s_1(\mathsf{P}_1,\mathsf{Q}_1)\wedge\neg\sigma_1(\mathsf{P}_1)\wedge s_2(\mathsf{P}^{d0},\mathsf{Q}^{d0})\wedge\neg\sigma_2(\mathsf{P}^{d0})\wedge\mathsf{P}^{d0}(\mathsf{c})]$$
$$\rightarrow[s_2(\mathsf{P}^{d1},\mathsf{Q}^{d1})\wedge\neg\sigma_3(\mathsf{P}^{d1})\rightarrow\mathsf{P}^{d1}(\mathsf{c})]$$

where $s_2(\mathsf{P}^{d0},\mathsf{Q}^{d0})$ indicates the result of replacing P^d by P^{d0} in $\hat{s}_2$. Notice that the only symbols on both sides of the implication sign are in K^0. By the Interpolation Theorem there is a $\psi(\mathsf{c},\mathsf{c}_1,\ldots,\mathsf{c}_m)$ which is an interpolant. We may write this as:

$$\hat{s}_1(\mathsf{P}_1,\mathsf{Q}_1)\wedge\hat{s}_2(\mathsf{P}^{d0},\mathsf{Q}^{d0})\wedge\mathsf{P}^{d0}(\mathsf{c})\to\sigma_1(\mathsf{P}_1)\vee\sigma_2(\mathsf{P}^{d0})\vee\psi(\mathsf{c},\mathsf{c}_1,\ldots,\mathsf{c}_m),\text{ and}$$
$$\hat{s}_2(\mathsf{P}^{d1},\mathsf{Q}^{d1})\wedge\psi(\mathsf{c},\mathsf{c}_1,\ldots,\mathsf{c}_m)\to\mathsf{P}^{d1}(\mathsf{c})\vee\sigma_3(\mathsf{P}^{d1}).$$

Now replace P^{d0}, Q^{d0} by P^{d}, Q^{d} in the top line, P^{d1}, Q^{d1} by P^{d}, Q^{d} in the second line. We obtain

$$\hat{s}_1\cup\hat{s}_2\to\sigma_1(\mathsf{P}_1)\vee\sigma_2(\mathsf{P}^d)\vee\sigma_3(\mathsf{P}^d)\vee[\mathsf{P}^d(\mathsf{c})\leftrightarrow\psi(\mathsf{c},\mathsf{c}_1,\ldots,\mathsf{c}_m)].$$

Since c does not occur in $\hat{s}_1\cup\hat{s}_2$ we get

$$\hat{s}_1\cup\hat{s}_2\vdash\sigma_1(\mathsf{P}_1)\vee\sigma_2(\mathsf{P}^d)\vee\sigma_3(\mathsf{P}^d)\vee\exists y_1,\ldots,\exists y_m\,\forall x\,[\mathsf{P}^d(x)\leftrightarrow\psi(x,y_1,\ldots,y_m)]$$

and hence $\{s_1,s_2\}\notin\mathbf{S}$.

Now we are ready to decorate our tree. List the sentences of $\mathsf{K}^g_{\mathbb{A}}$ as a sequence

$$\varphi_0,\varphi_1,\ldots,\varphi_n,\ldots$$

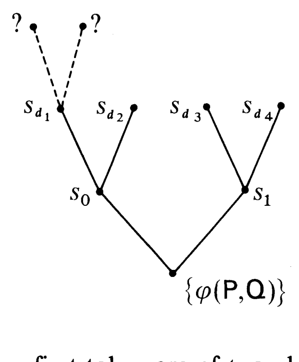
in such a way that any node d appearing in φ_n is of level $\leqslant n$. List the terms occuring in $\mathsf{L}_{\mathbb{A}}$:

$$t_0,t_1,\ldots,t_n,\ldots.$$

We work our way up the tree as follows. Place $\{\varphi(\mathsf{P},\mathsf{Q})\}$ at $<\,>$. Assume we have placed sets s_d at every node d of level n so that d is the longest node in s_d and the set

$$\{s_d\,|\,d\text{ a node at level }n\}$$

is in our consistency machine $\mathbf{S}$.



Given s_{d_1} we first take care of t_n and φ_n (if φ_n happens to be g-consistent with s_{d_1}) as in the proof of the Model Existence Theorem, using (6), giving us some

$$\{s', s_{d_2}, s_{d_3}, s_{d_4}\} \in \mathbf{S}.$$

We then apply (7) to get

$$\{s' \cup \{\mathsf{P}^{d0}(\mathsf{c})\}, s' \cup \{\neg \mathsf{P}^{d1}(\mathsf{c})\}, s_2, s_3, s_4\} \in \mathbf{S}$$

and we let $s_{d_1 0} = s' \cup \{\mathsf{P}^{d0}(\mathsf{c})\}$, $s_{d_1 1} = s' \cup \{\neg \mathsf{P}^{d1}(\mathsf{c})\}$. In this way we work our way along level $n+1$ and on up the tree. We see that any finite set of nodes on the tree is in **S**. This takes care of (3) since, otherwise, they would certainly imply a formula of the form (*). Now that there is a set at each node, let the superscripts vanish and you will discover we have satisfied (1), (2), (3) and (4), proving our theorem. ▯

4.13 — 4.17 Exercises

4.13. Show that Example 4.9 is not true without the assumption $o(\mathbb{HYP}_{\mathfrak{M}}) = \omega$. [Let $\mathfrak{M}$ be a minimal elementary extension of $\mathcal{N} = \langle \omega, 0, +, \cdot \rangle$].

4.14. Let $\mathfrak{M} = \langle M, 0, +, \cdot \rangle$ be a countable nonstandard model of Peano arithmetic with $o(\mathbb{HYP}_{\mathfrak{M}}) = \omega$. Show that there are $2^{\aleph_0}$ initial segments of $\mathfrak{M}$ which are models of Peano arithmetic.

4.15. Improve 4.14 to get $2^{\aleph_0}$ initial submodels of $\mathfrak{M}$ which are isomorphic to $\mathfrak{M}$. [Hint: Use a theorem of Friedman [1973] to the effect that every countable nonstandard model of Peano arithmetic is isomorphic to some initial segment of itself.]

4.16. Use 4.4 to show that if a countable abelian group G has $< 2^{\aleph_0}$ divisible subgroups then they are all in $\mathbb{HYP}_G$ and hence there are at most $\aleph_0$ of them. Give a direct group theoretic proof of this fact.

4.17. Extend Theorem 4.6 from simple sentences to Σ_1 theories. Similarly extend the applications of 4.6 given above.

4.18 Notes. The results of this section are called perfect set results because one always ends up constructing, by a tree argument, a perfect set of objects, perfect in the topological sense.

5. Recursively Saturated Structures

Having discovered several interesting facts about structures $\mathfrak{M}$ with $O(\mathfrak{M}) = \omega$, we take time in this section to relate this condition on $\mathbb{HYP}_{\mathfrak{M}}$ to more traditional notions.

Recall that a structure $\mathfrak{M}=\langle M,R_1,\ldots,R_l\rangle$ for L is $\aleph_0$-saturated if for every $k<\omega$ and every set $\Phi(x,v_1,\ldots,v_k)$ of formulas of $L_{\omega\omega}$ with free variables among $x,v_1,\ldots,v_k$ the following infinitary sentence is true in $\mathfrak{M}$:

$$\forall v_1,\ldots,v_k\,[(\bigwedge_{\Phi_0\in S_\omega(\Phi)} \exists x \bigwedge \Phi_0(x,v_1,\ldots,v_k))\to \exists x \bigwedge \Phi(x,v_1,\ldots,v_k)]$$

where $S_\omega(\Phi)$ is the set of all finite subsets of Φ.

5.1 Definition. The structure $\mathfrak{M}=\langle M,R_1,\ldots,R_l\rangle$ for L is *recursively saturated* if the above holds for all $k<\omega$ and all recursive sets $\Phi(x,v_1,\ldots,v_k)$ of $L_{\omega\omega}$.

Just as in the case of $\aleph_0$-saturated we have the following lemma.

5.2 Lemma. *Let $\mathfrak{M}$ be recursively saturated and let $\Phi(x_1,\ldots,x_n,v_1,\ldots,v_k)$ be a recursive set of formulas with free variables as indicated. The following infinitary sentence is true in $\mathfrak{M}$:*

$$\forall v_1,\ldots,v_k\,[(\bigwedge_{\Phi_0\in S_\omega(\Phi)} \exists x_1,\ldots,x_n \bigwedge \Phi_0)\to \exists x_1,\ldots,x_n \bigwedge \Phi].$$

Proof. The proof is by induction on n, the case $n=1$ being the hypothesis. It clearly suffices to prove the result for Φ satisfying the condition

$$\Phi_0\in S_\omega(\Phi) \quad \text{implies} \quad \bigwedge \Phi_0\in\Phi,$$

since we could close Φ under finite conjunctions. Let $\Psi(x_1,\ldots,x_n,v_1,\ldots,v_k)$ be the set of all formulas

$$\exists x_{n+1}\,\varphi(x_1,\ldots,x_n,x_{n+1},v_1,\ldots,v_k)$$

for $\varphi\in\Phi$. Suppose that $q_1,\ldots,q_k\in\mathfrak{M}$ are such that

$$\mathfrak{M}\models \exists x_1,\ldots,x_{n+1} \bigwedge \Phi_0(\vec{x},\vec{q})$$

for all $\Phi_0\in\Phi$. By the induction hypothesis, there are $p_1,\ldots,p_n\in\mathfrak{M}$ such that

$$\mathfrak{M}\models \bigwedge \Psi(p_1,\ldots,p_n,q_1,\ldots,q_k)$$

and hence

$$\mathfrak{M}\models \exists x_{n+1} \bigwedge \Phi_0(p_1,\ldots,p_n,x,q_1,\ldots,q_k)$$

for all $\Phi_0\in S_\omega(\Phi)$, since every such $\exists x_{n+1}\bigwedge\Phi_0$ is in Ψ. But then since $\mathfrak{M}$ is recursively saturated there is a $p_{n+1}\in\mathfrak{M}$ such that

$$\mathfrak{M}\models \Phi(p_1,\ldots,p_{n+1},q_1,\ldots,q_k). \quad \square$$

The principal link between recursively saturated structures and admissible sets is the following theorem of John Schlipf.

5.3 Theorem. *Let* $\mathfrak{M}=\langle M, R_1,\ldots,R_l\rangle$ *be a structure for* L. $\mathfrak{M}$ *is recursively saturated iff* $O(\mathfrak{M})=\omega$.

Proof. We prove the easy half first. Suppose that $o(\mathbb{HYP}_{\mathfrak{M}})=\omega$. Let $\Phi(v, w_1,\ldots,w_k)$ be a recursive set of formulas of $\mathsf{L}_{\omega\omega}$. We may consider Φ as a Δ_1 subset of $\mathbb{HF}$ by II.2.3. Since $\mathbb{HF}$ is Δ_1 on every admissible set, Φ is also Δ_1 on $\mathbb{HYP}_{\mathfrak{M}}$. Let $\vec{q}=q_1,\ldots,q_k\in\mathfrak{M}$ be such that

$$\mathfrak{M}\models\neg\exists v \bigwedge \Phi(v,q_1,\ldots,q_k).$$

We need to find a finite subset Φ_0 of Φ such that

$$\mathfrak{M}\models\neg\exists v \bigwedge \Phi_0(v,q_1,\ldots,q_k).$$

Now, since

$$\forall p\in M\ \exists\varphi\ [\varphi\in\Phi\wedge\mathfrak{M}\models\neg\varphi[p,\vec{q}]]$$

we have, by strong Σ Collection, a set b such that

(1) $\forall p\in M\ \exists\varphi\in b\ [\varphi\in\Phi\wedge\mathfrak{M}\models\neg\varphi[p,\vec{q}]]$

and

(2) $\forall\varphi\in b\ \exists p\in M\ [\varphi\in\Phi\wedge\mathfrak{M}\models\neg\varphi(p,\vec{q}]]$.

From (2) we see that $b\subseteq\Phi$ so let $\Phi_0=b$. Φ_0 is finite since it is in $\mathbb{HYP}_{\mathfrak{M}}$, is a set of pure sets, and has finite rank. From (1) we see that $\Phi_0(v,\vec{q})$ is not satisfiable on $\mathfrak{M}$.

To prove the other half of the theorem, let $\mathfrak{M}$ be recursively saturated. We need to prove that $\mathrm{L}(\mathfrak{M},\omega)$ is admissible; i.e., that it satisfies Δ_0 Collection. Call a set $a\in\mathrm{L}(\mathfrak{M},\omega)$ *simple* if there is a single term $\mathscr{F}(v_1,\ldots,v_{k+1})$ built up from $\mathscr{F}_1,\ldots,\mathscr{F}_N$, $\mathscr{D}$ such that each $x\in a$ is of the form

$$x=\mathscr{F}(p_1,\ldots,p_k,M)$$

for some $p_1,\ldots,p_k\in M$. Assume, for the moment, that we have established (3) and (4):

(3) *Every* $a\in\mathrm{L}(\mathfrak{M},\omega)$ *is the union of a finite number of simple sets;*

(4) *If* $z\in\mathrm{L}(\mathfrak{M},\omega)$ *and if a simple, then* $\mathrm{L}(\mathfrak{M},\omega)$ *satisfies*

$$\forall x\in a\ \exists y\ \varphi(x,y,z)\rightarrow\exists b\ \forall x\in a\ \exists y\in b\ \varphi(x,y,z)$$

for all Δ_0 *formulas* $\varphi(x,y,z)$.

Assuming this, let $\varphi(x,y,z)$ be a Δ_0 formula such that $L(\mathfrak{M},\omega)$ satisfies

$$\forall x \in a\, \exists y\, \varphi(x,y,z).$$

Write $a = a_1 \cup \cdots \cup a_m$ where each a_i is simple. Since

$$\forall x \in a_i\, \exists y\, \varphi(x,y,z)$$

holds in $L(\mathfrak{M},\omega)$ there are sets $b_1,\ldots,b_m$ in $L(\mathfrak{M},\omega)$ such that

$$\forall x \in a_i\, \exists y \in b_i\, \varphi(x,y,z).$$

But then let $b = b_1 \cup \cdots \cup b_m$. Then $b \in L(\mathfrak{M},\omega)$ and

$$\forall x \in a\, \exists y \in b\, \varphi(x,y,z)$$

so $L(\mathfrak{M},\omega)$ satisfies Δ_0 Collection.

To prove (3) note that in the proof of II.7.7 we showed that for each n there are a finite number of terms $\mathscr{F}^1,\ldots,\mathscr{F}^m$ such that each $x \in L(\mathfrak{M},n)$ is of the form

$$x = \mathscr{F}^i(\vec{p},M)$$

for some $i \leqslant m$ and some $\vec{p} \in M$. If $a \in L(\mathfrak{M},n)$ then $a \subseteq L(\mathfrak{M},n)$. Define, by Δ_0 Separation, sets $a_1,\ldots,a_m$ by

$$a_i = \{x \in a \mid \exists \vec{p} \in M\; x = \mathscr{F}^i(\vec{p},M)\}.$$

Then $a = a_1 \cup \cdots \cup a_m$.

Finally we prove (4). Let φ be given. By II.7.7 and II.7.6 we may assume that the only parameters in φ are M and some $\vec{q} \in M$. Given the simple set a let $\mathscr{F}^0(v_1,\ldots,v_{n+1})$ be as given in the definition of simple. Let $a = \mathscr{F}^1(r_1,\ldots,r_k,M)$ for some $r_1,\ldots,r_k \in M$. Rather than prove (4) we prove its contrapositive. Let ψ be $\neg\varphi$, so that we want to verify that $L(\mathfrak{M},\omega)$ is a model of

$$\forall b\, \exists x \in a\, \forall y \in b\, \psi(x,y,\vec{q},M) \rightarrow \exists x \in a\, \forall y\, \psi(x,y,\vec{q},M).$$

Assume the hypothesis. In particular we have, for each $m < \omega$,

$$(5)_m \quad \exists x \in a\, \forall y \in L(M,m)\, \varphi(x,y,\vec{q},M)$$

which becomes

$$(6)_m \quad \exists p_1,\ldots,p_n \in M\, [\mathscr{F}^0(\vec{p},M) \in \mathscr{F}^1(\vec{r},M) \wedge \forall y \in L(M,m)\, \psi(x,y,\vec{q},M)].$$

This is a Δ_0 formula of $\vec{p},\vec{q},\vec{r}$ so, by the effective version of II.7.8, we can find a formula $\psi_m(\vec{p},\vec{q},\vec{r})$ of $L_{\omega\omega}$ equivalent to it. Note that by (5) we have

$$\mathfrak{M} \models \forall v_1,\ldots,v_n\, [\psi_m(v_1,\ldots,v_n,\vec{q},\vec{r}) \rightarrow \psi_{m'}(v_1,\ldots,v_n,\vec{q},\vec{r})],$$

whenever $m \geq m'$. By $(6)_m$ we see that

$$\Phi = \{\psi_m(v_1,\dots,v_n,\vec{q},\vec{r}) \mid m < \omega\}$$

is finitely satsfiable. Since it is clearly a recursive set (by the exercises at the end of II.7) and $\mathfrak{M}$ is recursively saturated there are $p_1,\dots,p_n \in \mathfrak{M}$ so that

$$\mathfrak{M} \models \psi_m(\vec{p},\vec{q},\vec{r})$$

for all $m < \omega$. Thus for this $\vec{p}$, we have, setting $x = \mathscr{F}^0(\vec{p},M)$, $x \in a$, and for all $m < \omega$,

$$\forall y \in \mathrm{L}(M,m)\, \psi(x,y,\vec{q},M)$$

and hence

$$\forall y \in \mathrm{L}(M,\omega)\, \psi(x,y,\vec{q},M)$$

as desired. ▯

Schlipf discovered 5.3 by generalizing the results 5.4 and 5.7 below.

5.4 Corollary. *If* $\mathfrak{M} = \langle M, R_1,\dots,R_l \rangle$ *is* $\aleph_0$*-saturated then* $O(\mathfrak{M}) = \omega$. ▯

5.5 Corollary. *If* $\mathfrak{M} = \langle M, R_1,\dots,R_l \rangle$ *is recursively saturated and* $\Phi(x,v_1,\dots,v_k)$ *is any set of formulas of* $\mathsf{L}_{\omega\omega}$ *which is* Σ_1 *on* $\mathbb{HYP}_{\mathfrak{M}}$ *then* $\mathfrak{M}$ *satisfies:*

$$\forall v_1,\dots,v_k \left[\left(\bigwedge\nolimits_{\Phi_0 \in S_\omega(\Phi)} \exists x \bigwedge \Phi_0 \right) \rightarrow \exists x \bigwedge \Phi \right].$$

Proof. The proof that $o(\mathbb{HYP}_{\mathfrak{M}}) = \omega$ implies $\mathfrak{M}$ is recursively saturated actually proves this stronger result. ▯

5.6 Corollary. *For every infinite* $\mathfrak{M} = \langle M, R_1,\dots,R_l \rangle$ *there is a proper elementary extension* $\mathfrak{N}$ *of* $\mathfrak{M}$ *of the same cardinality such that* $\mathfrak{N}$ *is recursively saturated.*

Proof. Immediate from 5.3 and II.8.6. ▯

The above corollary shows a contrast between the notions of recursively saturated and $\aleph_0$-saturated structures since there is no countable $\aleph_0$-saturated elementary extension of $\mathcal{N} = \langle \omega, 0, +, \cdot \rangle$. Of course one could also prove 5.6 by a more standard model theoretic argument using elementary chains.

The following result shows that 5.3 can be improved for countable structures. It shows that if $\mathfrak{M}$ is countable and $o(\mathbb{HYP}_{\mathfrak{M}}) = \omega$ then $\mathfrak{M}$ is saturated for certain sets of Σ_1^1 formulas.

5.7 Theorem. *Let* $\mathfrak{M} = \langle M, R_1,\dots,R_l \rangle$ *be a countable structure for* L *with* $O(\mathfrak{M}) = \omega$. *Let* $\mathsf{K} = \mathsf{L} \cup \{\mathsf{S}_1,\dots,\mathsf{S}_m\}$ *and let* $\Phi(x_1,\dots,x_n,v_1,\dots,v_k,\mathsf{S}_1,\dots,\mathsf{S}_m)$ *be a* *set of formulas of* $\mathsf{K}_{\omega\omega}$ *which is* Σ_1 *on* $\mathbb{HYP}_{\mathfrak{M}}$. *The following infinitary second order sentence holds in* $\mathfrak{M}$:

$$\forall v_1,\ldots,v_k[(\bigwedge_{\Phi_0\in S_\omega(\Phi)}\exists \mathsf{S}_1,\ldots,\mathsf{S}_m\,\exists x_1,\ldots,x_n\bigwedge\Phi_0(\vec{x},\vec{v},\mathsf{S}))\to\exists\mathsf{S}_1,\ldots,\mathsf{S}_m\,\exists x_1,\ldots,x_n\bigwedge\Phi].$$

Proof. We use Theorem III.5.8. Let $q_1,\ldots,q_k\in M$ be given so that

$$\mathfrak{M}\models\exists\mathsf{S}_1,\ldots,\mathsf{S}_m\,\exists x_1,\ldots,x_n\bigwedge\Phi_0(\vec{x},\mathsf{S},q_1,\ldots,q_k)$$

for all $\Phi_0\in S_\omega(\Phi)$. We can assume that $\mathsf{K}\cup\{\mathsf{c}_1,\ldots,\mathsf{c}_n,\mathsf{d}_1,\ldots,\mathsf{d}_k\}$ is coded up on $\mathbb{HF}$. Let T be the theory

$$\Phi(\mathsf{c}_1,\ldots,\mathsf{c}_n,\mathsf{d}_1,\ldots,\mathsf{d}_k,\mathsf{S}_1,\ldots,\mathsf{S}_m).$$

Introduce symbols $\bar{\mathsf{p}}$ for $p\in M$ as usual and let $T'=\{\psi\}$ be the conjunction of

$$\bigwedge\text{Diagram}(\mathfrak{M})$$
$$\forall x\bigvee_{p\in M}x=\bar{\mathsf{p}}$$
$$\mathsf{d}_1=\bar{\mathsf{q}}_1,\ldots,\mathsf{d}_k=\bar{\mathsf{q}}_k.$$

For every finite subset T_0 of T, $T_0\cup T'$ has a model, so $T\cup T'$ has a model. This model is isomorphic to some

$$(\mathfrak{M},S_1,\ldots,S_m,p_1,\ldots,p_n,q_1,\ldots,q_k)$$

with

$$(\mathfrak{M},S_1,\ldots,S_m)\models\Phi[p_1,\ldots,p_n,q_1,\ldots,q_k].\quad\square$$

5.8—5.14 Exercises

5.8. Show that every recursively saturated structure is ω-homogeneous.

5.9. Suppose $\mathfrak{M}$ is uncountable. Show that $o(\mathbb{HYP}_{\mathfrak{M}})=\omega$ iff for all relations $T_1,\ldots,T_k$ on $\mathfrak{M}$ there is a countable recursively saturated structure $\mathfrak{N}$ with

$$(\mathfrak{N},T_1{\restriction}N,\ldots,T_k{\restriction}N)\prec(\mathfrak{M},T_1,\ldots,T_k).$$

5.10. Show that the predicate

"$\mathfrak{M}$ is recursively saturated"

is absolute (Δ_1) for models of KPU + Infinity but that the predicate

"$\mathfrak{M}$ is $\aleph_0$-saturated"

cannot be expressed by a Σ formula.

5.11 (J. Schlipf, J.-P. Ressayre). Let α be an admissible ordinal and let $\mathbb{A} = L(\alpha)$. Let L be a language with a finite number of symbols. A structure $\mathfrak{M}$ for L is α-recursively saturated iff for every set $\Phi(x, v_1, \ldots, v_k)$ of sentences of $\mathsf{L}_{\mathbb{A}}$ which is Δ_1 on A the following sentence holds in $\mathfrak{M}$:

$$\forall v_1, \ldots, v_k \left[\left(\bigwedge\nolimits_{\Phi_0 \in S_{\mathbb{A}}(\Phi)} \exists x \bigwedge \Phi_0(x, v_1, \ldots, v_k) \right) \to \exists x \bigwedge \Phi(x, v_1, \ldots, v_k) \right]$$

where $S_{\mathbb{A}}(\Phi) = \{\Phi_0 \subseteq \Phi \mid \Phi_0 \in \mathbb{A}\}$.

(i) Prove that if $L(\alpha)_{\mathfrak{M}}$ is admissible then $\mathfrak{M}$ is α-recursively saturated.

(ii) Prove that $O(\mathfrak{M})$ = the least α such that α is recursively saturated. (This result, due to J. Schlipf, strenghtens a special case of a theorem of J.-P. Ressayre. Schlipf's proof uses notions from Chapters V and VI.) Makkai has translated Ressayre's result into our setting to show that for α countable, admissible and greater than ω, $L(\alpha)_{\mathfrak{M}}$ is admissible iff $\mathfrak{M}$ is α-recursively saturated and satisfies the following condition: Suppose $\varphi_{\beta,\gamma}(v_1, \ldots, v_n)$ is an α-recursive function of β, γ. Suppose further that for some $p_1, \ldots, p_n \in \mathfrak{M}$ and some $\beta_0 < \alpha$:

$$\mathfrak{M} \models \bigwedge\nolimits_{\beta < \beta_0} \bigvee\nolimits_{\gamma < \alpha} \varphi_{\beta,\gamma}(\vec{p}).$$

Then there is a $\gamma_0 < \alpha$ such that

$$\mathfrak{M} \models \bigwedge\nolimits_{\beta < \beta_0} \bigvee\nolimits_{\gamma < \gamma_0} \varphi_{\beta,\gamma}(\vec{p}).$$

5.12. Show that $\mathfrak{M}$ is $\aleph_0$-saturated iff

(i) $o(\mathbb{HYP}_{\mathfrak{M}}) = \omega$.

(ii) for every $X \subseteq \omega$, $(\mathbb{HYP}_{\mathfrak{M}}, X)$ is admissible.

5.13. In this exercise we sketch some interesting connections between recursively saturated models of Peano arithmetic and models of nonstandard analysis. To simplify matters, we identify analysis with second order arithmetic (a standard perversion among logicians). Thus we add to the first order language of number theory new second order variables $X_1, X_2, \ldots$ and a membership symbol $\in$ which can hold between first order objects and second order objects ($(x_i \in X_j)$ is a formula but $(X_j \in x_i)$ isn't). The *axiom of induction* asserts:

$$\forall X \left[\mathbf{0} \in X \wedge \forall x (x \in X \to (x+1) \in X) \to \forall x (x \in X) \right].$$

(Warning: when working in systems weaker than the one described here it is often necessary to replace this single axiom by an axiom scheme.) The *axiom of comprehension* asserts the following, for every formula $\varphi(x, y_1, \ldots, y_k)$:

$$\forall \vec{y}\, \exists x \left[\forall x (x \in X \leftrightarrow \varphi(x, y_1, \ldots, y_k)) \right].$$

By *analysis* we mean the usual axioms of Peano arithmetic plus the axiom of induction and the axiom of comprehension. (Of course there is no need to include the first order form of induction since it follows from our second order axioms.) A *model of analysis* consists of a pair $(\mathfrak{N}, \mathscr{H})$, where $\mathscr{H}$ is a collection of subsets of the first order structure $\mathfrak{N}$, which makes all the axioms of analysis true. Any such model of analysis gives rise to a model $\mathfrak{N}$ of Peano arithmetic, but not every model of arithmetic can be expanded to a model of analysis. A *model of nonstandard analysis* is a model $(\mathfrak{N}, \mathscr{H})$ of analysis with $\mathfrak{N}$ not isomorphic to the standard model of arithmetic.

i) Prove that if $(\mathfrak{N}, \mathscr{H})$ is a model of nonstandard analysis, then $\mathfrak{N}$ is recursively saturated.

ii) Let $\mathfrak{N}$ be a nonstandard countable model of arithmetic. Let

$$\mathscr{H} = \bigcap \{\mathscr{H} \mid (\mathfrak{N}, \mathscr{H}) \models \text{analysis}\}.$$

Prove that either $\mathscr{H}$ is empty or that $\mathscr{H}$ consist of exactly the definable subsets of $\mathfrak{N}$. [This is easy from (i) and Theorem 1.1.]

5.14. Prove that there are two nonisomorphic countable recursively saturated elementary extensions of $\mathcal{N} = \langle \omega, +, x \rangle$.

5.15 Notes. It is not known whether or not there is a complete theory T in an finite language such that all models of T are recursively saturated but T is not $\aleph_0$-categorical.

6. Countable $\mathfrak{M}$-Admissible Ordinals

Since this chapter concerns the interplay of model theory and recursion theory, it seems appropriate to discuss one of the first applications of infinitary logic to the theory of admissible ordinals.

Let $\mathcal{N} = \langle \omega, 0, +, \cdot \rangle$. Most countable admissible ordinals α (other than ω) that arise in recursion theory are of the form

$$\alpha = O((\mathcal{N}, R))$$

for some relation R on ω. The question arose: Is every countable admissible $\alpha, \alpha > \omega$, of the above form? Sacks eventually answered this in the affirmative by means of "perfect set" forcing. His proof remains unpublished since Friedman-Jensen [1968] presented a simple proof of the result by means of the Barwise Compactness Theorem. We extend this theorem as follows.

6.1 Theorem. *Let $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ be a countable infinite structure and let α be a countable ordinal. The following are equivalent:*

(i) *α is $\mathfrak{M}$-admissible;*

(ii) *for some relation S on $\mathfrak{M}$,*

$$\alpha = O(\mathfrak{M}, S);$$

(iii) *for some linear ordering* $\prec$ *of* $\mathfrak{M}$,

$$\alpha = O(\mathfrak{M}, \prec)$$

and the order type of the largest well-ordered initial segment of $\prec$ *is* α.

Proof. The implications (iii)$\Rightarrow$(ii) and (ii)$\Rightarrow$(i) are obvious. To prove (i)$\Rightarrow$(iii) we borrow a fact from Section V.3:

(1) *If* $r \subseteq a \times a$ *is a linear ordering,* r *an element of an admissible set* $\mathbb{A}$, *and if* β *is the length of a well-ordered initial segment of* r *then* $\beta \leqslant o(\mathbb{A})$.

This could be proved now, but it is easier to wait for the Second Recursion Theorem. Let α be $\mathfrak{M}$-admissible. Then there is a countable admissible set $\mathbb{A} = \mathbb{A}_{\mathfrak{M}}$ above $\mathfrak{M}$ with

$$\alpha = o(\mathbb{A}_{\mathfrak{M}})$$

by II.3.3. Let K be the language L^* plus new constant symbols c, r, and $\bar{\mathsf{x}}$ for each $\mathsf{x} \in \mathbb{A}_{\mathfrak{M}}$. Let $\mathsf{K}_{\mathbb{A}}$ be the admissible fragment of $\mathsf{K}_{\infty\omega}$ given by $\mathbb{A}_{\mathfrak{M}}$. Let T be the theory which asserts:

KPU$^+$

Diagram$(\mathbb{A}_{\mathfrak{M}})$

"$\bar{\mathsf{M}}$ *is the set of all urelements*"

$\forall v\, [v \in \bar{\mathsf{a}} \rightarrow \bigvee_{x \in a} v = \bar{\mathsf{x}}]$ (for all $a \in \mathbb{A}_{\mathfrak{M}}$),

"c *is an ordinal*"

$\mathsf{c} > \bar{\beta}$ (for all $\beta < \alpha$),

"r *is a linear ordering of* $\bar{\mathsf{M}}$ *of order type* $\in \cap (\mathsf{c} \times \mathsf{c})$".

T has a model of the form

$$(\mathfrak{M}; H(\omega_1)_{\mathfrak{M}}, \in, \alpha, r)$$

for any well-ordering r of M of order type α. By III.7.5 T has a model

$$(\mathfrak{M}; B, E, c, r)$$

with $\alpha = o\,\mathscr{W}\!f(\mathfrak{M}; B, E)$. Let $\mathbb{A}'_{\mathfrak{M}} = \mathscr{W}\!f(\mathfrak{M}; B, E)$ which is an admissible set by the Truncation Lemma. Since $r \subseteq M \times M$, $r \in \mathbb{A}'_{\mathfrak{M}}$ so $\mathbb{A}'_{\mathfrak{M}}$ is actually admissible above $(\mathfrak{M}, r)$. Hence $\alpha \geqslant o(\mathbb{HYP}_{(\mathfrak{M},r)})$. But r has an initial segment of order type α (by T) so, by (1) applied to $\mathbb{HYP}_{(\mathfrak{M},r)}$, $\alpha \leqslant o(\mathbb{HYP}_{(\mathfrak{M},r)})$. We let $\prec$ be r. ▯

6.2—6.5 Exercises

6.2. Let $(\mathfrak{M}, \prec)$ be as in 6.1(iii). Show that $\mathbb{HYP}_{(\mathfrak{M},\prec)}$ is a model of $\neg$Beta.

6.3. Prove (1) above.

6.4. Let $\mathbb{A}_{\mathfrak{M}}$ be countable, admissible above $\mathfrak{M}$ with $o(\mathbb{A}_{\mathfrak{M}})>\omega$. Find a larger admissible set $\mathbb{B}_{\mathfrak{M}}$ above $\mathfrak{M}$ with the same ordinal such that $\mathbb{B}_{\mathfrak{M}}$ is locally countable; i. e.,

$$\mathbb{B}_{\mathfrak{M}} \models \forall a \text{ ("}a\text{ is countable").}$$

[Hint: Use the $\overline{\text{YY}}$ Compactness Theorem and Theorem II.7.5.]

6.5 (Schlipf). Prove that for every countable admissible ordinal β there is an elementary extension $\mathfrak{M}$ of $\mathcal{N}=\langle\omega,0,+,x\rangle$ such that $\beta=o(\mathbb{HYP}_{\mathfrak{M}})$. [Hint: i) Show that if $\mathfrak{M}$ is not recursively saturated and the set $\{n<\omega \mid \mathfrak{M}\models\text{"}n \text{ divides } k\text{"}\}$ codes a well-ordering of ω, and if α is the length of the well-ordering, then $o(\mathbb{HYP}_{\mathfrak{M}})>\alpha$. ii) Show that if $\mathfrak{M}$ is a model of Peano arithmetic generated by a single element k, usually written $\mathfrak{M}=\mathcal{N}[k]$, then $\mathfrak{M}$ is not recursively saturated.]

6.6 Notes. Theorem 6.1 and Exercise 6.4 are just two of many results that can be proved by either forcing arguments or by compactness arguments. See the appendix for a few references. Kunen has recently removed the hypothesis of countability from 6.5.

7. *Representability in $\mathfrak{M}$-Logic*

One of our principle results in this chapter, Theorems 3.1 and 3.3, identifies the relations on $\mathfrak{M}$ which are Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$ as the Π_1^1 relations on $\mathfrak{M}$, as long as M is countable. In Chapter VI we will search for the absolute version of this result. The results of this section will be of central importance in this search.

The reader should recall the notions of representability used to characterize the r. e. and recursive sets. The following are the infinitary analogues.

7.1 Definition. Let $\mathfrak{M}$ be an L-structure, T a set of finitary sentences of L^+ which are consistent in $\mathfrak{M}$-logic, $\varphi(v_1,\ldots,v_n)$ a finitary formula of L^+ and S an n-ary relation on $\mathfrak{M}$.

i) We say that $\varphi(v_1,\ldots,v_n)$ *strongly represents* S in T by the $\mathfrak{M}$-rule if, for all $q_1,\ldots,q_n\in\mathfrak{M}$,

$$S(q_1,\ldots,q_n) \quad \text{implies} \quad T\vdash_{\mathfrak{M}}\varphi(\overline{\mathsf{q}}_1,\ldots,\overline{\mathsf{q}}_n), \quad \text{and}$$
$$\neg S(q_1,\ldots,q_n) \quad \text{implies} \quad T\vdash_{\mathfrak{M}}\neg\varphi(\overline{\mathsf{q}}_1,\ldots,\overline{\mathsf{q}}_n);$$

whereas it *weakly represents* S in T using the $\mathfrak{M}$-rule if for all $q_1,\ldots,q_n\in\mathfrak{M}$

$$S(q_1,\ldots,q_n) \quad \text{iff} \quad T\vdash_{\mathfrak{M}}\varphi(\overline{\mathsf{q}}_1,\ldots,\overline{\mathsf{q}}_n).$$

ii) We say that $\varphi(v_1,\dots,v_n)$ *invariantly defines* S in T in $\mathfrak{M}$-logic if for all $q_1,\dots,q_n \in \mathfrak{M}$

$$S(q_1,\dots,q_n) \text{ implies } T \models_{\mathfrak{M}} \varphi(\bar{q}_1,\dots,\bar{q}_n)$$

$$\neg S(q_1,\dots,q_n) \text{ implies } T \models_{\mathfrak{M}} \neg\varphi(\bar{q}_1,\dots,\bar{q}_n)$$

where as it *semi-invariantly defines* S in T in $\mathfrak{M}$-logic if for all $q_1,\dots,q_n \in \mathfrak{M}$

$$S(q_1,\dots,q_n) \text{ iff } T \models_{\mathfrak{M}} \varphi(\bar{q}_1,\dots,\bar{q}_n).$$

The following is an immediate consequence of the $\mathfrak{M}$-Completeness Theorem.

7.2 Proposition.

Strongly representable $\Rightarrow$ *invariantly definable*

weakly representable $\Rightarrow$ *semi-invariantly definable*

and, if $\mathfrak{M}$ *and* L^+ *are countable, the converses hold.* □

These are excellent examples of notions which agree in ordinary recursion theory but which diverge, yield two interesting distinct notions, in generalized recursion theory.

7.3 Theorem. *Let* $\mathfrak{M} = \langle M, R_1,\dots,R_l\rangle$ *and let* S *be a relation on* $\mathfrak{M}$.

i) *If* S *is* Σ_1 *on* $\mathbb{H}\mathrm{YP}_{\mathfrak{M}}$ *then* S *is weakly representable in* KPU$^+$ *using the* $\mathfrak{M}$*-rule.*

ii) *If* $S \in \mathbb{H}\mathrm{YP}_{\mathfrak{M}}$ *then* S *is strongly representable in* KPU$^+$ *using the* $\mathfrak{M}$*-rule.*

Proof. Our language L^+ for $\mathfrak{M}$-logic consists of $\mathsf{L} \cup \{\bar{\mathsf{p}} \mid p \in M\}$ as in III.3.2(ii). We prove the results for countable $\mathfrak{M}$. In Chapter VI we will show that the results are absolute. We prove (i) first. Choose $\varphi(x_1,\dots,x_n,p_1,\dots,p_k,M)$ as in II.8.8. We can rewrite this using the relation symbol $\bar{\mathsf{M}}$ in place of the single set M. Thus we have, for $q_1,\dots,q_n \in M$

$$S(q_1,\dots,q_n) \text{ iff } \mathrm{KPU}^+ \models_{\mathfrak{M}} \varphi(\bar{q}_1,\dots,\bar{q}_n,\bar{p}_1,\dots,\bar{p}_k,\bar{\mathsf{M}})$$

which, by 7.2, gives the desired result.

Now we prove (ii). Let us assume S is unary to simplify notation. Using II.5.15 let $\varphi(x,p_1,\dots,p_n,M)$ be a good Σ_1 definition of S so that

$$\mathbb{H}\mathrm{YP}_{\mathfrak{M}} \models \varphi[S,p_1,\dots,p_n,M]$$

and

$$\mathfrak{A}_{\mathfrak{M}} \models \exists! x\, \varphi(x,p_1,\dots,p_n,M)$$

for all models $\mathfrak{A}_{\mathfrak{M}}$ of KPU$^+$, and hence

$$\mathrm{KPU}^+ \vdash_{\mathfrak{M}} \exists! x\, \varphi(x,\bar{p}_1,\dots,\bar{p}_n,\bar{\mathsf{M}})$$

by the $\mathfrak{M}$-completeness theorem. We claim that S is strongly represented by the formula $\psi(v)$ given by

$$\exists x\,[\varphi(x,\mathsf{p}_1,\ldots,\mathsf{p}_n,\mathsf{M})\wedge v\in x].$$

If $S(q)$ holds then $\mathfrak{A}_{\mathfrak{M}}\models\psi(\mathsf{q})$ for all models $\mathfrak{A}_{\mathfrak{M}}$ of KPU$^+$ so KPU$^+\vdash_{\mathfrak{M}}\psi(\mathsf{q})$. If $\neg S(q)$ then, for any $\mathfrak{A}_{\mathfrak{M}}\models$ KPU$^+$, since $\mathfrak{A}_{\mathfrak{M}}\models\varphi(S)\wedge\exists!x\,\varphi(x)$, $\mathfrak{A}_{\mathfrak{M}}\models\neg\psi(\mathsf{q})$ and hence, KPU$^+\vdash_{\mathfrak{M}}\neg\psi(\mathsf{q})$. □

We now prove a strong converse to Theorem 7.3. The first time through this result the student should think of T as KPU$^+$ or some strong extension of it in L* given by an r. e. set of axioms.

7.4 Theorem. *Let T be a set of finitary sentences of L^+ which is Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$ and is consistent in $\mathfrak{M}$-logic. Let S be a relation on $\mathfrak{M}$.*

(i) *If S is strongly representable in T using the $\mathfrak{M}$-rule then $S\in\mathbb{HYP}_{\mathfrak{M}}$.*

(ii) *If S is weakly representable in T using the $\mathfrak{M}$-rule then S is Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$.*

Proof. First note that (ii)⇒(i) since S strongly representable implies S and $\neg S$ are weakly representable so S and $\neg S$ are Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$, so S is Δ_1 and hence $S\in\mathbb{HYP}_{\mathfrak{M}}$ by Δ Separation. We prove (ii) for the case where $\mathfrak{M}$ and L^+ are countable leaving the absoluteness of 7.4 to Chapter VI. Let $\varphi(v_1,\ldots,v_n)$ weakly represent S in T. Then we see that the following are equivalent:

$$S(q_1,\ldots,q_n),$$
$$T\vdash_{\mathfrak{M}}\varphi(\mathsf{q}_1,\ldots,\mathsf{q}_n),$$
$$T\models_{\mathfrak{M}}\varphi(\mathsf{q}_1,\ldots,\mathsf{q}_n),$$
$$T\models\psi(\mathsf{q}_1,\ldots,\mathsf{q}_n).$$

I.e., the infinitary sentence $\psi(\mathsf{q}_1,\ldots,\mathsf{q}_n)$ is a logical consequence of T, where $\psi(\mathsf{q}_1,\ldots,\mathsf{q}_n)$ is

$$\bigwedge \text{Diagram}(\mathfrak{M})\wedge\forall v\,[\mathsf{M}(v)\leftrightarrow\bigvee\nolimits_{p\in M} v\equiv\mathsf{p}]\rightarrow\varphi(\mathsf{q}_1,\ldots,\mathsf{q}_n).$$

The sentence $\psi(\mathsf{q}_1,\ldots,\mathsf{q}_n)\in\mathbb{HYP}_{\mathfrak{M}}$ and the map $(q_1,\ldots,q_n)\mapsto\psi(\mathsf{q}_1,\ldots,\mathsf{q}_n)$ is Σ_1 definable so, by the Extended Completeness Theorem, S is Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$. □

It should be obvious from the proof of 7.4 that there was no real reason to demand that T be a set of finitary sentences. It is just that we only bothered to define $\vdash_{\mathfrak{M}}$ for finite sentences. T could have been a set of sentences each in $\mathbb{HYP}_{\mathfrak{M}}$ as long as T is Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$ and the proof would go through unchanged.

One might well ask about what happens to invariant and semi-invariant definability in the uncountable case where they no longer coincide with the representability notion. They turn out to be significant classes of predicates, ones we study in Chapter VIII.

7.5 Exercise. Let $\mathfrak{M} = \langle M, R_1, \ldots, R_l \rangle$ be a structure for L. Let L^+ be as in 7.3.

i) Assume that we have added a Σ function symbol F to L^* for the operation $F(x,y) = x \cup \{y\}$ and a constant symbol $\emptyset$ for the empty set. Show that each $x \in \mathbb{HF}_{\mathfrak{M}}$ is denoted by a closed term t_x of L^+.

ii) Show that $S \subseteq \mathbb{HF}_{\mathfrak{M}}$ is Σ_1 on $\mathbb{HYP}_{\mathfrak{M}}$ iff S is weakly representable in KPU^+ using the $\mathfrak{M}$-rule.

7.6 Notes. The representability approach to the hyperarithmetic sets goes back to Grzegorczyk, Mostowski and Ryll-Nardzewski [1961].