Chapter III
Countable Fragments of $L_{\infty\omega}$

In this chapter the student is introduced to the infinitary logic $L_{\infty\omega}$ and its countable fragments. The reason for treating infinitary logic so early in the book is two-fold. In the first place it offers a nice application of the very notion of admissible set, since the fragments of $L_{\infty\omega}$ most like ordinary logic are those given by countable admissible sets. More important, however, is the powerful tool that infinitary logic gives us in our study of admissible sets. The results from model theory presented in this chapter are all chosen because of their applicability to the theory of admissible sets and generalized recursion theory.

1. Formalizing Syntax and Semantics in KPU

In § I.3 we formalized informal notions of mathematics in KPU, notions like "function", "natural number", and "ordinal". In this section we do the same thing for informal notions of logic, notions like "language", "structure", "formula".

In this section we work in KPU but we suppose that among the atomic predicates of our metalanguage L^* are the following:

$$\mathsf{Relation\text{-}symbol}\ (x),$$
$$\mathsf{Function\text{-}symbol}\ (x),$$
$$\mathsf{Constant\text{-}symbol}\ (x),$$
$$\mathsf{Variable}\ (x)$$

and among the operation symbols of our metalanguage are two unary ones:

$$\mathsf{v} \text{ and } \# .$$

We use $\mathsf{r}, \mathsf{r}_1, \ldots$ to vary over objects x satisfying $\mathsf{Relation\text{-}symbol}\ (x)$. Similarly $\mathsf{h}, \mathsf{h}_1, \ldots$ for function symbols and $\mathsf{c}, \mathsf{c}_1, \mathsf{d}, \ldots$ for constant symbols. We also assume that among the constant symbols of our *metalanguage* L^* are

$$\neg,\ \bigwedge,\ \bigvee,\ \forall,\ \exists,\ \equiv .$$

These twelve symbols may be part of our original metalanguage L* or they may be defined symbols introduced into KPU as in § I.5. In applications, the latter is almost always the case.

We assume the following *axioms on syntax:*

(1) An axiom asserting that the classes of variables, function symbols, relation symbols, contant symbols are all disjoint, and that none of the six constants displayed above are in any of these classes.

(2) An axiom on variables which asserts, writing v_α for $\mathsf{v}(\alpha)$,

$$\alpha \neq \beta \implies v_\alpha \neq v_\beta,$$
$$\mathsf{Variable}\,(x) \iff \exists\alpha\,(x = v_\alpha)$$

(3) An axiom on $\#$, which tells us the "arity" of relation and function symbols:

if x is a relation or function symbol then $\#(x)$ is a positive natural number.

A set L is a *language* if L is a *set* of relation, function, and constant symbols.

The predicates "t is a term" and "t is a term of L" are defined by recursion on TC(t):

1.1 Definition. t is a *term* (*of* L)$\leftrightarrow t$ is a variable, or t is a constant symbol (in L), or $t = \langle \mathsf{h}, y\rangle$ where h is a function symbol (in L), $y = \langle y_1, \ldots, y_{\#(h)}\rangle$ and each y_i is a term (of L).

These two definitions ("t is a term", and "t is a term of L") are of the type permitted by I.6.6 so they define Δ predicates. (The only sticky point comes in checking that the predicates $P(y,n)$ iff "y is a sequence of length n" and $Q(y,n,x,i)$ iff "$P(y,n)$ and $1 \leqslant i \leqslant n$ and x is the i^{th} term in the sequence y" are Δ predicates. This also follows from I.6.6 by recursion on n. For example, $P(y,n)$ iff n is a natural number $\geqslant 1$ and, if $n > 1$ then there exist $z_1, z_2 \in \mathrm{TC}(y)$ such that $y = \langle z_1, z_2\rangle$ and $P(z_2, n-1)$.)

1.2 Definition. An *atomic formula* (of L) is a set of one of the following forms:

(i) $\langle \equiv, t_1, t_2\rangle$ where t_1, t_2 are terms (of L); we write $(t_1 \equiv t_2)$ or even $(t_1 = t_2)$.

(ii) $\langle \mathsf{r}, t_1, \ldots, t_n\rangle$ where r is a relation symbol (in L), $n = \#(\mathsf{r})$ and $t_1, \ldots, t_n$ are terms (of L); we write $\mathsf{r}(t_1, \ldots, t_n)$ for $\langle \mathsf{r}, t_1, \ldots, t_n\rangle$.

1.3 Definition. A set φ is a *finite formula* (of L) iff

φ is an atomic formula (of L), or
φ is $\langle \neg, \psi\rangle$ and ψ is a finite formula (of L), or
φ is $\langle \bigwedge, \{\psi, \theta\}\rangle$ or $\langle \bigvee, \{\psi, \theta\}\rangle$ where ψ, θ are finite formulas (of L), or
φ is $\langle \exists, v, \psi\rangle$ or $\langle \forall, v, \varphi\rangle$ where v is a variable and ψ is a finite formula (of L).

We write $\neg\psi$ for $\langle\neg,\psi\rangle$, $\psi\wedge\theta$ for $\bigwedge\{\psi,\theta\}$ and $\exists v\psi$ for $\langle\exists,v,\psi\rangle$; similarly for $\vee,\forall$. We use the usual abbreviations like $\varphi\rightarrow\psi$ for $((\neg\varphi)\vee\psi)$. All of the above predicates are Δ predicates, the last again by I.6.6.

1.4 Proposition. *If* Infinity *is true then for any language* L *there is a set*

$$\mathsf{L}_{\omega\omega}=\{\varphi\,|\,\varphi \textit{ a finite formula of } \mathsf{L} \textit{ with only variables of the form } v_n \textit{ occurring in } \varphi, n<\omega\}.$$

Proof. We first show that

$$\mathit{Terms}=\{t\,|\,t \text{ a term of } \mathsf{L} \text{ with only variables of the form } v_n \text{ in } t\}$$

is a set. Define

$$\mathit{Terms}(0)=\{\mathsf{c}\in\mathsf{L}\,|\,\mathsf{c} \text{ a constant symbol}\}\cup\{\mathsf{v}^{(n)}\,|\,n<\omega\},$$

$$\mathit{Terms}(n+1)=\{\langle\mathsf{h},t_1,\dots,t_k\rangle\,|\,h\in\mathsf{L},\ \mathsf{h} \text{ a constant symbol}, k=\#(\mathsf{h}),\ t_1,\dots,t_k\in\mathit{Terms}(n)\}\cup\mathit{Terms}(n).$$

by induction on n. This makes sense if ω exists, by replacement for $\mathit{Terms}(0)$, as does

$$\mathit{Terms}=\bigcup_{n<\omega}\mathit{Terms}(n).$$

A similar proof shows that $\mathsf{L}_{\omega\omega}$ is a set. $\square$

For the past twenty years, and more, logicians have been working to find manageable strengthenings of $\mathsf{L}_{\omega\omega}$. It has turned out that languages with expressions of infinite length are one of the best lines of attack. These languages allow us to form expressions like the following

$$\forall x\bigvee_{n<\omega}[x\equiv\underbrace{\mathsf{h}(\mathsf{h}(\dots(\mathsf{h}}_{n}(\mathsf{c})\dots)]$$

which says that every element is of the form $c, h(c), h(h(c))$, etc.; or

$$\forall x\bigvee_{\varphi\in\mathsf{L}_{\omega\omega}}[\varphi(x)\wedge\exists!y\,\varphi(y/x)]$$

which says that every x is definable by some finite formula; or $\varphi_\alpha(x)$ defined by recursion on α by:

$$\varphi_0(v_0) \text{ is } \forall y\neg(y<v_0),$$

$$\varphi_\alpha(v_\alpha) \text{ is } \forall y\,(y<v_\alpha\leftrightarrow\bigvee_{\beta<\alpha}\varphi_\beta(y/v_\beta)).$$

Then $\varphi_\alpha(x)$ is going to be true in a linearly ordered structure iff the predecessors of x have order type exactly α.

1.5 Definition. A set φ is an *infinitary formula* if one of the following hold:

φ is a finite formula,
φ is $\neg\psi$ where ψ is an infinitary formula,
φ is $\exists v\psi$ or $\forall v\psi$ where v is a variable and ψ is an infinitary formula,
φ is $\langle\bigwedge,\Phi\rangle$ or φ is $\langle\bigvee,\Phi\rangle$ where Φ is a nonempty set of infinitary formulas.

Again this definition is justified by I.6.6. We write

$$\bigwedge\Phi \quad \text{for} \quad \langle\bigwedge,\Phi\rangle,$$
$$\bigvee\Phi \quad \text{for} \quad \langle\bigvee,\Phi\rangle;$$

$\bigwedge\Phi$ is called the *conjunction* of the formulas in Φ, $\bigvee\Phi$ the *disjunction.* The notion of infinitary formula of a language L is defined in a parallel way.

We assume that the reader can carry out all the syntactic definitions (free and bound variable, substitution of a term t for a free variable in φ, for example) only noting that substitution must be defined by recursion over TC(φ). We denote the result of substituting t for v in φ by $\varphi(t/v)$. A *sentence* is a formula with no free variables.

We define the set sub(φ) of *subformulas* of φ by recursion over TC as follows:

$$\begin{aligned} \mathrm{sub}(\varphi) &= \{\varphi\} && \text{if } \varphi \text{ is atomic} \\ &= \{\varphi\} \cup \mathrm{sub}(\psi) && \text{if } \varphi \text{ is } \neg\psi,\ \exists v\psi \text{ or } \forall v\psi \\ &= \{\varphi\} \cup \textstyle\bigcup_{\psi\in\Phi} \mathrm{sub}(\psi) && \text{if } \varphi \text{ is } \bigwedge\Phi \text{ or } \bigvee\Phi. \end{aligned}$$

1.6 Lemma. *If φ has a finite number of free variables so does any $\psi \in \mathrm{sub}(\varphi)$. In particular, if ψ is a subformula of some sentence then ψ has a finite number of free variables.*

Lemma 1.6 is proved by a routine induction on formulas, and motivates the following definition.

A *proper infinitary formula* is one with only a finite number of free variables. The notion of "proper infinitary formula" is a Δ notion, since

$$\begin{aligned} \varphi \text{ is proper iff } &\{v \mid v \text{ a free variable in } \varphi\} \text{ is finite} \\ \text{iff } &\{\alpha \mid v_\alpha \text{ is free in } \varphi\} \text{ is finite,} \end{aligned}$$

and the notion "a is a finite set of ordinals" is Δ by Exercise I.7.6. Since we will only be discussing proper infinitary formulas, we might just as well drop the adjective "proper" once and for all. We use the symbol $\mathsf{L}_{\infty\omega}$ to denote the class of all (proper) infinitary formulas of L.

1.7 Definition. A *structure* $\mathfrak{M}$ for a language L is a pair $\mathfrak{M} = \langle M, f\rangle$ such that, writing $x^{\mathfrak{M}}$ for $f(x)$ we have:

(i) M is a nonempty set,
(ii) f is a function with $\mathrm{dom}(f)=\mathsf{L}$,
(iii) $\mathsf{r}\in\mathsf{L}$ implies $\mathsf{r}^{\mathfrak{M}}$ is a subset of $M^{\#(\mathsf{r})}$,
(iv) $\mathsf{h}\in\mathsf{L}$ implies $\mathsf{h}^{\mathfrak{M}}$ is a function with domain $M^{\#(\mathsf{h})}$ and range $\subseteq M$,
(v) if $\mathsf{c}\in\mathsf{L}$ then $\mathsf{c}^{\mathfrak{M}}\in M$.

This too is a Δ predicate of $\mathfrak{M}$ and L.

An *assignment* in $\mathfrak{M}$ is a function s with $\mathrm{dom}(s)$ a finite set of variables and $\mathrm{rng}(s)\subseteq M$. Given a structure $\mathfrak{M}$ for L, a term t of L and an assignment s in $\mathfrak{M}$ with the variables of t contained in $\mathrm{dom}(s)$, we let $t^{\mathfrak{M}}(s)$ be the value of t in $\mathfrak{M}$ at s. This is defined by recursion:

$$\begin{aligned} t^{\mathfrak{M}}(s) &= \mathsf{c}^{\mathfrak{M}} && \text{if } t \text{ is the constant symbol } \mathsf{c} \\ &= s(v) && \text{if } t \text{ is the variable } v \\ &= \mathsf{h}^{\mathfrak{M}}(t_1^{\mathfrak{M}}(s),\dots,t_k^{\mathfrak{M}}(s)) && \text{if } t \text{ is } \mathsf{h}(t_1,\dots,t_k). \end{aligned}$$

Our next goal is to formalize the notion of satisfaction:

$$\mathfrak{M}\models\varphi[s]$$

where $\mathfrak{M}$ is a structure for L, φ is a formula of L and s is an assignment to the free variables of φ. In order to make the definition fit into the form of definition by recursion available to us, we have to be a little awkward. Since there is no set of all variables, there can't be a set of all assignments to $\mathfrak{M}$. There is, however, a Σ operation G such that for all L, all L structures $\mathfrak{M}$ and all $\varphi\in\mathsf{L}_{\infty\omega}$

$$G(\mathfrak{M},\varphi)=\{s\mid s \text{ an assignment in } \mathfrak{M} \text{ with } \mathrm{dom}(s)=\text{free variables of } \varphi\}.$$

We outline the definition of G in Exercise 1.11. It is then a routine matter to define

$$\mathrm{Sat}_{\mathsf{L}}(\mathfrak{M},\varphi)=\{s\in G(\mathfrak{M},\varphi)\mid \mathfrak{M}\models\varphi[s]\}$$

for languages L, structures $\mathfrak{M}$ for L and formulas φ of $\mathsf{L}_{\infty\omega}$, by recursion over $\mathrm{TC}(\varphi)$. We give some of the clauses of this recursive definition:

$$\mathrm{Sat}_{\mathsf{L}}(\mathfrak{M},\mathsf{r}(t_1,t_2))=\{s\in G(\mathfrak{M},\mathsf{r}(t_1,t_2))\mid \langle t_1^{\mathfrak{M}}(s),t_2^{\mathfrak{M}}(s)\rangle\in\mathsf{r}^{\mathfrak{M}}\},$$

$$\mathrm{Sat}_{\mathsf{L}}(\mathfrak{M},\neg\varphi)=\{s\in G(\mathfrak{M},\neg\varphi)\mid s\notin\mathrm{Sat}_{\mathsf{L}}(\mathfrak{M},\varphi)\},$$

$$\mathrm{Sat}_{\mathsf{L}}(\mathfrak{M},\textstyle\bigwedge\Phi)=\{s\in G(\mathfrak{M},\bigwedge\Phi)\mid \text{for all } \varphi\in\Phi,\ s\restriction\text{Free-Var}(\varphi)\in\mathrm{Sat}_{\mathsf{L}}(\mathfrak{M},\varphi)\},$$

$$\begin{aligned} \mathrm{Sat}_{\mathsf{L}}(\mathfrak{M},\exists v\,\varphi) &= \{s\in G(\mathfrak{M},\exists v\,\varphi)\mid \text{for some } x\in M,\ s\cup\{\langle v,x\rangle\}\in\mathrm{Sat}_{\mathsf{L}}(\mathfrak{M},\varphi)\} \\ &\qquad \text{if } v \text{ is free in } \varphi \\ &= \mathrm{Sat}_{\mathsf{L}}(\mathfrak{M},\varphi) \quad \text{if } v \text{ is not free in } \varphi. \end{aligned}$$

We can now define, for L-structures $\mathfrak{M}$, formulas φ of L, and assignments s, the predicate $\mathfrak{M}\models\varphi[s]$ by

$$\mathfrak{M}\models\varphi[s] \quad \text{iff} \quad s\upharpoonright\text{Free-Var}(\varphi)\in\text{Sat}_{\mathsf{L}}(\mathfrak{M},\varphi).$$

For sentences φ we have $\mathfrak{M}\models\varphi$ if the empty function 0 is in $\text{Sat}_{\mathsf{L}}(\mathfrak{M},\varphi)$. Since $\text{Sat}_{\mathsf{L}}(\mathfrak{M},\varphi)$ is a Σ operation, by I.6.4, we see that $\mathfrak{M}\models\varphi[s]$ *is a* Δ *predicate of* $\mathfrak{M},\varphi,s$ and the suppressed L. (Note that L can be recovered from $\mathfrak{M}$; $\mathsf{L}=\text{dom}(2^{\text{nd}}\mathfrak{M})$.) If the free variables of φ are among $v_1,\ldots,v_n$ we write

$$\mathfrak{M}\models\varphi[a_1,\ldots,a_n]$$

for $\mathfrak{M}\models\varphi[s]$ where $s=\{\langle v_1,a_1\rangle,\ldots,\langle v_n,a_n\rangle\}$.

Given structures $\mathfrak{M},\mathfrak{N}$ for a language L we write

$$\mathfrak{M}\equiv\mathfrak{N}\,(\mathsf{L}_{\omega\omega}) \quad \text{if, for all finite sentences } \varphi,\quad \mathfrak{M}\models\varphi \text{ iff } \mathfrak{N}\models\varphi; \quad \text{and}$$

$$\mathfrak{M}\equiv\mathfrak{N}\,(\mathsf{L}_{\infty\omega}) \quad \text{if, for all sentences } \varphi \text{ of } \mathsf{L}_{\infty\omega},\quad \mathfrak{M}\models\varphi \text{ iff } \mathfrak{N}\models\varphi.$$

As written both of these are Π_1 predicates of $\mathfrak{M},\mathfrak{N}$ (and L). By Proposition 1.1, $\mathsf{L}_{\omega\omega}$ is a set if Infinity holds; in fact the operation which takes L to the set $\mathsf{L}_{\omega\omega}$ is a Σ operation on L. Thus, in the presence of the axiom of infinity we can rewrite $\mathfrak{M}\equiv\mathfrak{N}\,(\mathsf{L}_{\omega\omega})$ to see that it is an absolute predicate of $\mathfrak{M},\mathfrak{N}$ and L. In the presence of Σ_1 Separation $\mathfrak{M}\equiv\mathfrak{N}\,(\mathsf{L}_{\infty\omega})$ also becomes absolute, but for entirely different reasons. More on that in Chapter VII.

1.8—1.12 Exercises. Work in KPU

1.8. Show that a^n ($=a\times a\times\cdots\times a$, n-times) is a Σ operation of a and n.

1.9. Prove that the predicate Q, defined in the parenthetical remark following Definition 1.1, is indeed a Δ predicate.

1.10. Show that the operation $a\mapsto\text{card}(a)$, defined on finite sets a of variables, is a Σ operation. [Use the collapsing lemma.]

1.11. Using 1.10 and the Σ operation

$$S(a,n)=\{b\subseteq a\mid \text{card}(b)=n\},$$

show that the operation G used above is a Σ operation.

1.12. Write out the few remaining details needed for the definition of $\text{Sat}_{\mathsf{L}}(\mathfrak{M},\varphi)$.

2. Consistency Properties

There is a very general method for constructing models which has evolved into what Keisler [1971] calls the "Model Existence Theorem". We will prove this theorem here in KPU + Infinity. To prove it in this weak metatheory we must be a little more careful than usual. Among notions which are equivalent in ZF we must choose those which avoid unnecessary uses of Power and Choice. This explains why our presentation must diverge in minor ways from Keisler's.

The collection of infinitary formulas of a language L never forms a set but we must usually deal with a set of formulas. Hence the next definition. We repeat, for emphasis, that *we work in* KPU + Infinity *in this section.*

2.1 Definition. Let L be a language. A *fragment* of $\mathsf{L}_{\infty\omega}$ is a set L_A of infinitary formulas and variables such that

(i) every finite formula of $\mathsf{L}_{\omega\omega}$ is in L_A,
(ii) if $\varphi\in\mathsf{L}_A$ then every subformula and variable of φ is in L_A,
(iii) if $\varphi(v)\in\mathsf{L}_A$ and t is a term of L all of whose variables lie in L_A then $\varphi(t/v)$ is in L_A, and
(iv) if φ,ψ,v are in L_A so are

$$\neg\varphi,\ \sim\varphi,\ \exists v\,\varphi,\ \forall v\,\varphi,\ \varphi\vee\psi,\ \varphi\wedge\psi\,.$$

At this stage, the subscript A serves merely as an index. It will serve a more useful purpose later.

We have used an undefined notion in 2.1, a silly technical device $\sim\varphi$. It is defined by:

$$\begin{array}{lll} \sim\varphi & \text{is} & \neg\varphi \quad \text{if } \varphi \text{ is atomic,}\\ \sim(\neg\varphi) & \text{is} & \varphi,\\ \sim(\bigwedge\Phi) & \text{is} & \bigvee_{\varphi\in\Phi}\neg\varphi,\\ \sim(\bigvee\Phi) & \text{is} & \bigwedge_{\varphi\in\Phi}\neg\varphi,\\ \sim(\exists v\,\varphi) & \text{is} & \forall v\,\neg\varphi,\\ \sim(\forall v\,\varphi) & \text{is} & \exists v\,\neg\varphi\,. \end{array}$$

We see that $\sim$ has an explicit Σ definition, not a recursive one, since $\sim$ does not occur on the right hand side of the above. Note that $\sim\varphi$ is logically equivalent to $\neg\varphi$. (Keisler uses $\varphi\neg$ for our $\sim\varphi$.)

Let K be a language and $C=\{\mathsf{c}_n\colon n<\omega\}$ a countable set of constant symbols not in K. We keep K, C and $\mathsf{L}=\mathsf{K}\cup C$ fixed for the rest of this section.

If K_A is a fragment of $\mathsf{K}_{\infty\omega}$ then there is a natural fragment $\mathsf{L}_A=\mathsf{K}_A(C)$ of $\mathsf{L}_{\infty\omega}$ associated with it; namely the set of all formulas of the form $\varphi(\mathsf{c}_{i_1},\ldots,\mathsf{c}_{i_n})$ which result by replacing a finite number of free variables by constants from C. Fix these fragments K_A and $\mathsf{L}_A=\mathsf{K}_A(C)$ for the rest of this section. A term t of L_A is *basic* if it is in C or if it's of the form $\mathsf{h}(\mathsf{c}_{i_1},\ldots,\mathsf{c}_{i_n})$ for $\mathsf{h}\in\mathsf{K}$ and the c_i's in C.

Next comes the cumbersome but crucial definition.

2.2 Definition. A *consistency property* for L_A is a set S of sets s such that each $s \in S$ is a set of sentences of L_A and such that all the following hold for every $s \in S$:

(C0) (*Triviality rule*) $0 \in S$; if $s \subseteq s' \in S$ then $s \cup \{\varphi\} \in S$ for each $\varphi \in s'$.
(C1) (*Consistency rule*) If $\varphi \in s$ then $\neg\varphi \notin s$.
(C2) ($\neg$-*rule*) If $\neg\varphi \in s$ then $s \cup \{\sim\varphi\} \in S$.
(C3) ($\bigwedge$-*rule*) If $\bigwedge\Phi \in s$ then for all $\varphi \in \Phi$, $s \cup \{\varphi\} \in S$.
(C4) ($\forall$-*rule*) If $(\forall v\, \varphi(v)) \in s$ then for each $\mathsf{c} \in C$, $s \cup \{\varphi(\mathsf{c}/v)\} \in S$.
(C5) ($\bigvee$-*rule*) If $\bigvee\Phi \in s$ then for some $\varphi \in \Phi$, $s \cup \{\varphi\} \in S$.
(C6) ($\exists$-*rule*) If $(\exists v\, \varphi(v)) \in s$ then for some $\mathsf{c} \in C$, $s \cup \{\varphi(\mathsf{c}/v)\} \in S$.
(C7) (*Equality rules*). Let t be a basic term of L_A and $\mathsf{c}, \mathsf{d} \in C$.

i) If $(\mathsf{c} \equiv \mathsf{d}) \in s$ then $s \cup \{(\mathsf{d} \equiv \mathsf{c})\} \in S$.
ii) If $\varphi(t), (\mathsf{c} \equiv t) \in s$ then $s \cup \{\varphi(\mathsf{c})\} \in S$.
iii) For some $\mathsf{e} \in C$, $s \cup \{\mathsf{e} \equiv t\} \in S$.

The rule (C0) was not included in Keisler's definition. It really is a triviality though.

2.3 Lemma. *If S satisfies all of* 2.2 *except* (C0) *then there is a smallest consistency property $S' \supseteq S$.*

Proof. Define

$$f(0) = S \cup \{0\},$$
$$f(n+1) = f(n) \cup \{s \cup \{\varphi\} \mid s \in f(n) \wedge \exists s' \in f(n)[s \subseteq s' \wedge \varphi \in s']\},$$
$$S' = \bigcup_{n<\omega} f(n).$$

This is easily seen to be a consistency property. If $S \subseteq S''$ and S'' is a consistency property then $f(n) \subseteq S''$ by induction on n. ▯

2.4 Lemma. *Let S be a consistency property, $s \in S$.*

i) $\varphi, (\varphi \to \psi) \in s$ *implies* $s \cup \{\psi\} \in S$.
ii) $\mathsf{c} \in C$ *implies* $s \cup \{(\mathsf{c} \equiv \mathsf{c})\} \in S$.
iii) $\mathsf{c}, \mathsf{d}, \mathsf{e} \in C$, $(\mathsf{c} \equiv \mathsf{d}) \in s$, $(\mathsf{d} \equiv \mathsf{e}) \in s$ *implies* $s \cup \{\mathsf{c} \equiv \mathsf{e}\} \in S$.

Proof. These are all similar. Assume $\varphi, (\varphi \to \psi) \in s$. Since $(\varphi \to \psi)$ is really $(\neg\varphi \vee \psi)$ we see by (C5) that either $s \cup \{\neg\varphi\} \in S$ or $s \cup \{\psi\} \in S$. The first possibility is ruled out by (C1). Next assume the hypothesis of (iii). By (C7i) we have $s' = s \cup \{\mathsf{e} \equiv \mathsf{d}\} \in S$. By (C7ii) we have $s' \cup \{\mathsf{c} \equiv \mathsf{e}\} \in S$ so, by (C0), $s \cup \{\mathsf{c} \equiv \mathsf{e}\} \in S$. To prove (ii) let $\mathsf{c} \in C$. By (C7iii) there is an $\mathsf{e} \in C$ with $s \cup \{\mathsf{c} \equiv \mathsf{e}\} \in S$. By (C7i), $s \cup \{\mathsf{c} \equiv \mathsf{e}, \mathsf{e} \equiv \mathsf{c}\} \in S$. Applying (iii) we have $s \cup \{\mathsf{c} \equiv \mathsf{e}, \mathsf{e} \equiv \mathsf{c}, \mathsf{c} \equiv \mathsf{c}\} \in S$, so, by (C0), $s \cup \{\mathsf{c} \equiv \mathsf{c}\} \in S$. ▯

The point of Definition 2.2 is that it exactly isolates the principles needed to carry out the "Henkin argument". To be more specific, it allows us to prove the Model Existence Theorem. A structure $\mathfrak{M}$ for L is a *canonical structure* if every element of $\mathfrak{M}$ is of the form $\mathsf{c}^{\mathfrak{M}}$ for some $\mathsf{c} \in C$.

2.5 Model Existence Theorem. *Let L_A be a countable fragment and let S be a consistency property for L_A. For every $s \in S$ there is a canonical structure $\mathfrak{M}$ for L such that $\mathfrak{M}$ is a model of s, i.e., for every $\varphi \in s$, $\mathfrak{M} \models \varphi$.*

Proof. We can't be quite as free wheeling as Keisler [1971, p. 13] since we have a rather limited metatheory. We have already taken care of most of the difficulties, though, by careful choice of definitions and by the wording of the theorem. Since L_A is countable we can enumerate its sentences:

$$\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots \quad (n<\omega);$$

and the terms occurring in L_A:

$$t_0, t_1, \ldots, t_n, \ldots \quad (n<\omega).$$

We shall construct a sequence

$$s_0 \subseteq s_1 \subseteq \cdots \subseteq s_n \subseteq \cdots$$

of elements of S as follows. We take s_0 to be the s of the theorem. Given s_n we define s_{n+1} by adding on one, two, or three sentences of L_A.

Step 1. Find the first constant symbol c of C, in the list of terms, such that $s_n \cup \{\mathsf{c} \equiv t_n\} \in S$ and let $s'_n = s_n \cup \{\mathsf{c} \equiv t_n\}$.

Step 2. If $s'_n \cup \{\varphi_n\} \notin S$ let $s_{n+1} = s'_n$. If $s'_n \cup \{\varphi_n\} \in S$ then let $s''_n = s'_n \cup \{\varphi_n\}$.

There now are three distinct cases to consider, depending on the principal connective in φ_n.

Step 3. If φ_n does not begin with $\exists$ or $\bigvee$ let $s_{n+1} = s''_n$.

Step 4. If φ_n is $\exists v \psi$ then find the first $\mathsf{c} \in C$ in the list (2), by (C6), such that $s''_n \cup \{\psi(\mathsf{c}/v)\} \in S$ and let s_{n+1} be this element of S.

Step 5. If φ_n is $\bigvee \Phi$ then use (C5) to find the least $\psi \in \Phi$, least in the list (1), such that $s''_n \cup \{\psi\} \in S$ and let s_{n+1} be this element of S.

Now let $s_\omega = \bigcup_{n<\omega} s_n$. The rest of the proof is exactly as in Keisler [1971]. We define an equivalence relation on C by

$$\mathsf{c} \approx \mathsf{d} \quad \text{iff} \quad (\mathsf{c} \equiv \mathsf{d}) \in s_\omega$$

and let $M=\{\mathsf{c}/\approx:\mathsf{c}\in C\}$. By (C7), if $\varphi(\mathsf{c}_1,\ldots,\mathsf{c}_n)\in s_\omega$ and $\mathsf{c}_i\approx\mathsf{d}_i$ then $\varphi(\mathsf{d}_1,\ldots,\mathsf{d}_n)\in s_\omega$. This tells us how to interpret the relation and function symbols of L:

$$\langle\mathsf{c}_1/\approx,\ldots,\mathsf{c}_n/\approx\rangle\in\mathsf{r}^{\mathfrak{M}}\quad\text{iff}\quad\mathsf{r}(\mathsf{c}_1,\ldots,\mathsf{c}_n)\in s_\omega,$$

$$\mathsf{h}^{\mathfrak{M}}(\mathsf{c}_1/\approx,\ldots,\mathsf{c}_n/\approx)\ \text{ is }\ \mathsf{d}/\approx\ \text{ for that }\mathsf{d}\text{ such that }\ (\mathsf{h}(\mathsf{c}_1,\ldots,\mathsf{c}_n)\equiv\mathsf{d})\in s_\omega.$$

A simple proof by induction on formulas of L_A shows that $\mathfrak{M}\models\varphi$ for every $\varphi\in s_\omega$. One uses the properties (C0)—(C7) of course. □

2.6 Extended Model Existence Theorem. *Let L_A and S be as in the model existence theorem. If T is a set of sentences of L_A such that*

$$s\in S,\ \varphi\in T\quad\textit{implies}\quad s\cup\{\varphi\}\in S,$$

then for any $s\in S$, $T\cup s$ has a canonical model.

Proof. Let $S'=\{T\cup s: s\in S\}$. While S' is not a consistency property, it almost is one. It satisfies (C1)—(C7) so we can apply Lemma 2.3 to get a consistency property $S''\supset S'$. Apply the Model Existence Theorem to $(T\cup s)\in S''$. □

Note, in passing, that canonical structures for countable fragments are countable structures.

2.7—2.8 Exercises

2.7. Prove, in KPU + Infinity, that if φ is an infinitary sentence with sub(φ) countable then there is a countable fragment L_A with $\varphi\in\mathsf{L}_A$.

2.8. Use the Model Existence Theorem to show that if L_A is a countable fragment and if $\varphi\in\mathsf{L}_A$ has a model then it has a countable model.

2.9 Notes. A history of the Model Existence Theorem can be found in the Preface and Lecture 3 of Keisler [1971].

The completeness and compactness theorems of § 5 cannot be proved in KPU + Infinity. The main reason for working in KPU + Infinity in this section is that it allows us to pinpoint the exact place where stronger principles are needed in these other theorems.

3. $\mathfrak{M}$-*Logic and the Omitting Types Theorem*

As a first, and important, application of the Model Existence Theorem, we prove a general version of the Henkin-Orey ω-Completeness Theorem. We then use the same proof to obtain the Omitting Types Theorem for countable fragments.

Let $\mathfrak{M}=\langle M,f\rangle$ be a structure for a countable language L and let L^+ be some language containing L, a unary relation symbol $\bar{\mathsf{M}}$, and for each $m\in M$ a constant symbol $\bar{\mathsf{m}}$, and possibly other symbols.

3.1 Definition. An $\mathfrak{M}$*-structure for* L^+ is a structure $\mathfrak{N}=\langle N,g\rangle$ satisfying

i) The interpretation of $\bar{\mathsf{M}}$ in $\mathfrak{N}$ is M;
ii) The interpretation of $\bar{\mathsf{m}}$ in $\mathfrak{N}$ is m, for all $m\in M$; and
iii) $\mathfrak{M}$ is a substructure of $\langle N,g\restriction\mathsf{L}\rangle$.

3.2 Examples. i) ω*-logic*: Let $M=\omega=\{0,1,\ldots\}$ and $\mathfrak{M}=\langle M\rangle$. In this case $\mathfrak{M}$-logic is usually called ω*-logic*. Thus, in ω-logic one adds a symbol $\bar{\omega}(x)$, and constant symbols $\bar{0},\bar{1},\bar{2},\ldots$ An ω-model is a structure $\mathfrak{N}$ with $\bar{\omega}^{\mathfrak{N}}=\omega$ and $\bar{n}^{\mathfrak{N}}=n$. To study ω-logic one adds the ω*-rule* to the usual rules of proof:

If you can prove $\varphi(\bar{n}/v_0)$ *for each* $n<\omega$ *then conclude*

$$\forall v_0[\bar{\omega}(v_0)\rightarrow\varphi(v_0)]\,.$$

The $\mathfrak{M}$-rule below is the natural generalization of this.

ii) Let $\mathfrak{M}=\langle M,\text{---}\rangle$ be a fixed structure for L and let $\mathsf{L}^*=\mathsf{L}(\in,\ldots)$ be as usual. Treat L^* as a single sorted language with a symbol U for the collection of urelements (as we have done from time to time). Among structures $\mathfrak{A}_{\mathfrak{N}}$ for L^* we want to single out those with $\mathfrak{N}=\mathfrak{M}$. Let $\bar{\mathsf{M}}$ be U and add a constant symbol $\bar{\mathsf{p}}$ for each $p\in M$. Let $\mathsf{L}^+=\mathsf{L}^*\cup\{\bar{\mathsf{p}}\mid p\in M\}$. An $\mathfrak{M}$-structure for L^+ has, by 3.1, the form

$$\langle M\cup A, M,\text{---},A,E,\ldots\rangle$$

(with $\bar{\mathsf{p}}$ interpreted by p for $p\in M$), for some $A,E,\ldots$. If it is a model of that part of KPU contained in the definition of L^*-structure (cf. I.2.6) then we can write it as $(\mathfrak{M};A,E,\ldots)$. In particular, *any* $\mathfrak{M}$*-structure for* L^+ *which is a model of* KPU *has the form* $\mathfrak{A}_{\mathfrak{M}}=(\mathfrak{M};A,E,\ldots)$, with $\bar{\mathsf{p}}$ interpreted by p for each $p\in M$.

These two examples are the most important ones but what we do in this section is entirely general.

Thus let $\mathfrak{M}$ be a structure for L and let L^+ be as described prior to Definition 3.1. We wish to find a set of axioms and rules which generate the finite sentences of L^+ which hold in all $\mathfrak{M}$-structures or, more generally, in all models of some theory T which are $\mathfrak{M}$-structures. We can do this only if $\mathfrak{M}$ is countable. Given a set of sentences T we write $T\models_{\mathfrak{M}}\varphi$ if φ holds in all $\mathfrak{M}$-structures which are models of T and $\models_{\mathfrak{M}}\varphi$ if φ is true in all $\mathfrak{M}$-structures.

The results of this section must be carried out in a set theory a little more powerful than KPU. Our metatheory is discussed in the notes for this section.

3.3 Axioms for $\mathfrak{M}$-logic. Let L and L^+ be as above.

i) $\bar{\mathsf{M}}(\bar{\mathsf{m}})$ is an axiom of $\mathfrak{M}$-logic, for all m in $\mathfrak{M}$.

ii) Every atomic or negated atomic sentence of $\mathsf{L}\cup\{\bar{\mathsf{m}}: m\in\mathfrak{M}\}$ true in $\mathfrak{M}$ is an axiom of $\mathfrak{M}$-logic.

iii) The usual axioms for $\mathsf{L}^+_{\omega\omega}$ are all axioms of $\mathfrak{M}$-logic.

3.4 Definition. Let T be a set of finitary sentences of L^+. A finite formula φ is a *consequence* of T by the $\mathfrak{M}$-rule, written

$$T \vdash_{\mathfrak{M}} \varphi\,,$$

if φ is in the smallest set of formulas containing T and the axioms of $\mathfrak{M}$-logic and closed under the following rules:

i) (*Modus ponens*) If $T \vdash_{\mathfrak{M}} \varphi$ and $T \vdash_{\mathfrak{M}} (\varphi \to \psi)$ then $T \vdash_{\mathfrak{M}} \psi$.

ii) (*Generalization*) If $T \vdash_{\mathfrak{M}} (\varphi \to \psi(v_n))$ and v_n not free in φ then $T \vdash_{\mathfrak{M}} (\varphi \to \forall v_n \psi(v_n))$.

iii) ($\mathfrak{M}$-*rule*) If $T \vdash_{\mathfrak{M}} \varphi(\overline{\mathsf{m}}/v_0)$ for every $m \in \mathfrak{M}$ then $T \vdash_{\mathfrak{M}} \forall v_0 (\overline{\mathsf{M}}(v_0) \to \varphi(v_0))$.

A sentence is provable by the $\mathfrak{M}$-rule, written $\vdash_{\mathfrak{M}} \varphi$, if $T \vdash_{\mathfrak{M}} \varphi$ for $T = 0$.

Notice that we have made no mention of the phrase "proof by the $\mathfrak{M}$-rule". Instead we gave an inductive definition of $T \vdash_{\mathfrak{M}} \varphi$ directly. A straightforward proof by induction shows that

$$T \vdash_{\mathfrak{M}} \varphi(v_1, \ldots, v_n) \quad \text{implies} \quad T \models_{\mathfrak{M}} \forall v_1, \ldots, \forall v_n \varphi(v_1, \ldots, v_n)\,.$$

If $\mathfrak{M}$ and L^+ are countable, then the converse holds. It is known as the ω-Completeness, or $\mathfrak{M}$-Completeness, Theorem.

A set of sentences T is *consistent in* $\mathfrak{M}$-*logic* if

$$T \vdash_{\mathfrak{M}} (\varphi \wedge \neg\varphi)$$

is false for all formulas φ of L^+. (Note the $\vdash_{\mathfrak{M}}$ as opposed to $\models_{\mathfrak{M}}$.)

3.5 $\mathfrak{M}$-Completeness Theorem. *Let* L^+ *and* $\mathfrak{M}$ *be countable and let* T *be a set of finitary sentences of* L^+. *If* φ *is a finite sentence of* L^+ *then*

$$T \vdash_{\mathfrak{M}} \varphi \quad \textit{iff} \quad T \models_{\mathfrak{M}} \varphi\,.$$

Proof. We can assume L^+ has a countable set $\{\mathsf{c}_n : n < \omega\}$ of constant symbols not in $\mathsf{L} \cup \{\overline{\mathsf{m}} : m \in \mathfrak{M}\}$ and not mentioned in T for otherwise we simply enlarge L^+ a little more. This enlargement would not enlarge the set of theorems of $\mathfrak{M}$-logic of the original L^+. Suppose φ is not a theorem of T in $\mathfrak{M}$-logic. Our goal is to construct, via the Model Existence Theorem, an $\mathfrak{M}$-structure $\mathfrak{N}$ which is a model of T and $\neg\varphi$.

Let L_A^+ be a countable fragment of $\mathsf{L}_{\infty\omega}^+$ with the sentence

$$\forall v_0 \bigvee_{m \in M} [\neg \overline{\mathsf{M}}(v_0) \vee v_0 \equiv \overline{\mathsf{m}}]$$

in L_A^+. Let S consist of all sets s of the form

$$s_0 \cup s_1$$

where s_0 is a set of finitary sentences of L^+ such that $T \cup s_0$ is consistent in $\mathfrak{M}$-logic, and s_1 is a finite set of infinitary sentences of the form

$$(1) \qquad \bigvee_{m \in M} [\neg \bar{\mathsf{M}}(\mathsf{c}_n) \vee \mathsf{c}_n \equiv \bar{\mathsf{m}}].$$

Note that $s = \{\neg\varphi\} \in S$ by hypothesis. The only nontrivial step in showing that S is a consistency property is to show that S satisfies the $\bigvee$-rule, (C5). Suppose $\bigvee\Phi \in s \in S$ where $s = s_0 \cup s_1$ is partitioned as above. If $\bigvee\Phi$ is in s_0 then it is just a binary disjunction $\psi \vee \theta$. If neither ψ nor θ were consistent in $\mathfrak{M}$-logic with $T \cup s_0$ then $\neg\psi \wedge \neg\theta$ would be a consequence of $T \cup s_0$ in $\mathfrak{M}$-logic, hence $T_0 \cup s_0$ would not be consistent, since $\psi \vee \theta \in s_0$. If $\bigvee\Phi \in s_1$ then it is of the form (1) for some $n < \omega$. We need to show that either

$$(2) \qquad T \cup s_0 \cup \{\neg \bar{\mathsf{M}}(\mathsf{c}_n)\}$$

or one of

$$(3)_m \qquad T \cup s_0 \cup \{(\mathsf{c}_n \equiv \bar{\mathsf{m}})\}$$

for some $m \in M$, is consistent in $\mathfrak{M}$-logic so that one of

$$s \cup \{\neg \bar{\mathsf{M}}(\mathsf{c}_n)\}, \qquad s \cup \{\mathsf{c}_n \equiv \bar{\mathsf{m}}\}$$

is in S. Suppose that none of the $(3)_m$ are consistent in $\mathfrak{M}$-logic. Write $\psi(\mathsf{c}_n)$ for the conjunction of s_0. It follows that

$$T \vdash_{\mathfrak{M}} \neg\psi(\bar{\mathsf{m}})$$

for each $m \in M$, so, by the $\mathfrak{M}$-rule

$$T \vdash_{\mathfrak{M}} \forall v_0 [\bar{\mathsf{M}}(v_0) \rightarrow \neg\psi(v_0)]$$

and hence

$$T \vdash_{\mathfrak{M}} \bar{\mathsf{M}}(\mathsf{c}_n) \rightarrow \neg\psi(\mathsf{c}_n).$$

Taking the contrapositive we get

$$T \vdash_{\mathfrak{M}} \bigwedge s_0 \rightarrow \neg \bar{\mathsf{M}}(\mathsf{c}_n)$$

so $\neg \bar{\mathsf{M}}(\mathsf{c}_n)$ is a consequence of $T \cup s_0$ in $\mathfrak{M}$-logic, so (2) is consistent in $\mathfrak{M}$-logic. This completes the proof of (C5). Let $T' = T +$ all sentences of the form (1). Then if $s \in S$ and $\psi \in T'$ then $s \cup \{\psi\} \in S$. Thus by the Extended Model Existence Theorem, $T' \cup \{s\}$ has a canonical model whenever $s \in S$. But a canonical model of all of (1) is isomorphic to an $\mathfrak{M}$-structure so every $s \in S$ is true in some $\mathfrak{M}$-structure, in particular $s = \{\neg\varphi\}$. □

3.6 Corollary. *If $\mathfrak{M}$ and L^+ are countable then a theory T of L^+ has an $\mathfrak{M}$-structure for a model iff T is consistent in $\mathfrak{M}$-logic.* □

3.7 Corollary. *If* $\mathfrak{M}$ *and* L^+ *are countable and* φ *is a sentence of* L^+ *then* $\vdash_{\mathfrak{M}} \varphi$ *iff* $\models_{\mathfrak{M}} \varphi$. ☐

In applications in this book, L^+ will usually be as given in Example 3.2(ii) and T will usually be KPU or KPU$^+$.

The fraternal twin of the ω-Completeness Theorem is the so-called Omitting Types Theorem, a result which helps us construct models which "omit" elements not satisfying certain infinite disjunctions.

3.8 Omitting Types Theorem. *Let* L_A *be a countable fragment of* $\mathsf{L}_{\infty\omega}$, *and let* T *be a set of sentences of* L_A *which has a model. For each* n *let* Φ_n *be a set of formulas of* L_A *with free variables among* $v_1, \ldots, v_{k_n}$. *Assume that for each* n *and each formula* $\psi(v_1, \ldots, v_{k_n})$ *of* L_A: *if*

$$T + \exists v_1, \ldots, v_{k_n} \psi$$

has a model, so does

$$T + \exists v_1, \ldots, v_{k_n} (\psi \wedge \varphi)$$

for some $\varphi(v_1, \ldots, v_{k_n}) \in \Phi_n$. *Given this hypothesis, there is a model* $\mathfrak{M}$ *of* T *such that for each* $n < \omega$,

$$\mathfrak{M} \models \forall v_1, \ldots, v_{k_n} \bigvee_{\varphi \in \Phi_n} \varphi(v_1, \ldots, v_{k_n}).$$

Proof. A simple modification of the proof of the $\mathfrak{M}$-Completeness Theorem suffices. We first expand L to a language $\mathsf{L}' = \mathsf{L} \cup \{\mathsf{c}_n : n < \omega\}$. Let L_B be a countable fragment containing L_A and each of the sentences

$$\forall v_1, \ldots, v_{k_n} \bigvee_{\varphi \in \Phi_n} \varphi(v_1, \ldots, v_{k_n})$$

and let $\mathsf{L}'_A, \mathsf{L}'_B$ be the natural fragment of $\mathsf{L}'_{\infty\omega}$ associated with L_A and L_B as in § 2. Let S consist of all finite sets s of the form

$$s_0 \cup s_1$$

where s_0 is a finite set of sentences of L'_A with $T \cup s_0$ having a model and where s_1 is a finite set of sentences of the form

$$(1) \qquad \bigvee_{\varphi \in \Phi_n} \varphi(\mathsf{c}_{i_1}/v_1, \ldots, \mathsf{c}_{i_{k_n}}/v_{k_n}).$$

The proof that S is a consistency property is just like the proof in 3.5 except we use "has a model" for "consistent in $\mathfrak{M}$-logic". If $\varphi \in T$ or φ is of the form (1) then for each $s \in S$, $s \cup \{\varphi\} \in S$ so there is a canonical model $\mathfrak{M}$ of T and each of (1). The student who has trouble filling in the details is referred to Lecture 11 of Keisler [1971]. ☐

3.9 Exercise. Show how the $\mathfrak{M}$-completeness theorem can be derived from the Omitting Types Theorem.

3.10 Notes. The ω-Completeness Theorem goes back to Henkin [1954], [1957] and Orey [1956]. The extension of the Omitting Types Theorem to arbitrary countable fragments is due to Keisler [1971].

We cannot carry out the $\mathfrak{M}$-Completeness Theorem in KPU or KPU + Infinity. The problems arises from the definition of "the set of consequences f of T by the $\mathfrak{M}$-rule". This inductive definition would need something like Σ_1 Separation to justify it by proving that there *is* a smallest set of the kind described in 3.4. In Chapter VI we will step back and look at such inductive definitions.

4. A Weak Completeness Theorem for Countable Fragments

Let L_A be a fragment of $L_{\infty\omega}$. A sentence φ of L_A is *valid*, written $\models\varphi$, if

$$\mathfrak{M}\models\varphi$$

for every structure $\mathfrak{M}$ for L. We would like to prove a generalization of the ordinary completeness theorem for L by showing that

$$\models\varphi \quad \text{iff} \quad \exists P\,[P \textit{ is a proof of } \varphi]$$

for some notion of "proof". For such a result to be of any use there must be something "effective" about the notion of proof (otherwise we could take as proofs all valid sentences) and there should be a relation between L_A and the "size" of proofs of sentences in L_A.

We approach the notion of "proof" in a tentative fashion so that we can see exactly what it is that forces us to consider admissible fragments for the eventual result, Theorem 5.5.

After the brief respite of § 3 we return to use KPU + Infinity as our metatheory in this section.

4.1 Definition. Let L_A be a fragment of L. A set Γ of formulas of L_A is a *validity property* for L_A if Γ contains (A 1)—(A 7) below, is closed under (R 1)—(R 3), and does not contain $\varphi\wedge\neg\varphi$, for any $\varphi\in L_A$.

(A 1) Any instance of a tautology of finitary propositional logic.
(A 2) $(\neg\varphi)\leftrightarrow(\sim\varphi)$.
(A 3) $\bigwedge\Phi\rightarrow\varphi$, if $\varphi\in\Phi$.
(A 4) $v_\alpha=v_\alpha$.
(A 5) $v_\alpha=v_\beta\rightarrow v_\beta=v_\alpha$.
(A 6) $\forall v\,\varphi(v)\rightarrow\varphi(t/v)$, t any term free for v in $\varphi(v)$.
(A 7) $\varphi(v)\wedge v=t\rightarrow\varphi(t/v)$, t any term free for v in $\varphi(v)$.

(R 1) (*Modus Ponens*). If φ and $(\varphi\to\psi)$ are in Γ so is ψ.
(R 2) (*Generalization*). If $(\varphi\to\psi(v))$ is in Γ and v is not free in φ then $(\varphi\to\forall v\,\psi(v))$ is in Γ.
(R 3) (*Conjunction*). If $\bigwedge\Phi\in\mathsf{L}_A$ and $(\psi\to\varphi)$ is in Γ for each $\varphi\in\Phi$, then $(\psi\to\bigwedge\Phi)$ is in Γ.

All formulas in the above are assumed to be elements of L_A.

4.2 Example. (i) *Let $\mathfrak{M}$ be a structure for L and let $\Gamma_{\mathfrak{M}}$ be the set of all $\varphi(v_1,\ldots,v_n)\in\mathsf{L}_A$ such that*

$$\mathfrak{M}\models\forall v_1,\ldots,v_n\,\varphi(v_1,\ldots,v_n).$$

$\Gamma_{\mathfrak{M}}$ is a set by Δ_1 Separation. It is clearly a validity property.
(ii) *If $\mathscr{X}$ is a set of validity properties then*

$$\bigcap\mathscr{X}=\bigcap\{\Gamma:\Gamma\in\mathscr{X}\}$$

is a validity property.

As one might guess from the way the definition and examples were given, we cannot prove (in our current metatheory KPU + Infinity) that there is a smallest validity property, though it is very instructive to try. It is also useful to think of the members of Γ as the "provable" formulas in the next lemma.

Let us fix for the rest of this section, a fragment L_A, a set $C=\{\mathsf{c}_n: n<\omega\}$ of new constant symbols, and let $\mathsf{K}=\mathsf{L}\cup C$ and $\mathsf{K}_A=\mathsf{L}_A(C)$ be the natural fragment of $\mathsf{K}_{\infty\omega}$ associated with L_A: $\varphi\in K_A$ iff there is a $\psi\in\mathsf{L}_A$ such that φ results from ψ by replacing some free variables by constant symbols throughout,

$$\varphi=\psi(\mathsf{c}_{i_1}/v_{\alpha_1},\ldots,\mathsf{c}_{i_k}/v_{\alpha_k}).$$

We say that φ is a *free substitution instance* of ψ. We have purposely interchanged the roles of K and L from § 2.

The following proposition allows us to apply the Extended Model Existence Theorem.

4.3 Proposition. *Let Γ_0 be a validity property for L_A and let Γ be the set of all free substitution instances of formulas in Γ_0.*
Define

$$S=\{s\mid s \text{ a finite set of } \mathsf{K}_A\text{-sentences, } (\neg\bigwedge s)\notin\Gamma\}$$

Then:
(i) *S is a consistency property for K_A.*
(ii) *if $\varphi\in\Gamma$, $s\in S$, then $s\cup\{\varphi\}\in S$.*

Proof. We first observe that Γ is a validity property for K_A. Two examples should suffice.

(A 7) Consider a sentence of K_A of the form

(1) $$\varphi(\mathsf{c}_1,\ldots,\mathsf{c}_n,v)\wedge v=t\rightarrow\varphi(\mathsf{c}_1,\ldots,\mathsf{c}_n,t/v),$$

where t also may contain $\mathsf{c}_1,\ldots,\mathsf{c}_n$. Now this is a free substitution of some formula of the form

(2) $$\varphi(w_1,\ldots,w_n,v)\wedge v=t'\rightarrow\varphi(w_1,\ldots,w_n,t'/v)$$

where $t=t'(\mathsf{c}_1/w_1,\ldots,\mathsf{c}_n/w_n)$. By (A 7) for Γ_0, (2) is in Γ_0 so (1) is in Γ.

(R 3) Suppose $(\psi\rightarrow\bigwedge\Phi)\in\mathsf{K}_A$ and that $(\psi\rightarrow\varphi)\in\Gamma$ for each $\varphi\in\Phi$. We need to see that $(\psi\rightarrow\bigwedge\Phi)\in\Gamma$. Now $(\psi\rightarrow\bigwedge\Phi)$ is a free substitution instance of some formula in L_A, say it is of the form

$$\psi(\mathsf{c}_1/v_1)\rightarrow\bigwedge\{\varphi(\mathsf{c}_1/v_1)\mid\varphi(v_1)\in\Phi_0\}.$$

Since $(\psi(\mathsf{c}_1)\rightarrow\varphi(\mathsf{c}_1))\in\Gamma$ it is a free substitution instance of, say,

$$\psi(\mathsf{c}_1/v_\beta)\rightarrow\varphi(\mathsf{c}_1/v_\beta)$$

where $\psi(v_\beta)\rightarrow\varphi(v_\beta)\in\Gamma_0$. But then using (R 1), (R 2) and (A 6) for Γ_0 we get $(\psi(v_1)\rightarrow\varphi(v_1))\in\Gamma_0$, for each $\varphi(v_1)\in\Phi_0$. By (R 3) for Γ_0, $\psi(v_1)\rightarrow\bigwedge\{\varphi(v_1)\mid\varphi(v_1)\in\Phi_0\}$ is in Γ_0, and hence $\psi\rightarrow\bigwedge\Phi$ is in Γ. We leave the other clauses to the student, and assume Γ is a validity property for K_A. The verification of (ii) is entirely routine, for suppose $\varphi\in\Gamma$, $s\in S$ but $s\cup\{\varphi\}\notin S$. Then $\neg(\bigwedge s\wedge\varphi)\in\Gamma$, so by (A 1) and (R 1), $(\varphi\rightarrow\neg\bigwedge s)$. But then, by (R 1), $\neg\bigwedge s\in\Gamma$, so $s\notin S$, which is a contradiction. The various cases in the verification that S is a consistency property are similar, with one slight twist for (C 6). Suppose $\exists v\,\varphi(v)\in s\in S$ but that for each $\mathsf{c}\in C$, $s\cup\{\varphi(\mathsf{c}/v)\}\notin S$. Hence $\bigwedge s\rightarrow\neg\varphi(\mathsf{c}/v)$ is in Γ for each $\mathsf{c}\in C$ and, in particular, for some c not appearing in S. Since c does not appear in S,

$$\bigwedge s\rightarrow\neg\varphi(v)$$

is also a free substitution instance of something in Γ_0, so it is in Γ, and hence, so are all the following:

$\bigwedge s\rightarrow\forall v\neg\varphi(v)$ (by (R 2)),

$\bigwedge s\rightarrow\neg\exists v\,\varphi(v)$ (by (R 1), (A 1), (A 2)),

$\neg\bigwedge s$ (by (A 1), (R 1))

which contradicts $s\in S$. The other clauses are left to the student. ▯

4.4 Definition. A sentence φ of L_A is a *theorem of* L_A if φ is in every validity property Γ for L_A.

A word of warning: The predicate

$$\varphi \text{ is a theorem of } \mathsf{L}_A$$

is Π_1 in KPU but not in general Δ_1 in KPU. Thus we cannot assert (in KPU) that there is a set of all theorems of L_A.

4.5 Weak Completeness Theorem for Countable Fragments. *Let L_A be a countable fragment. A sentence φ of L_A is valid iff it is a theorem of L_A.*

Proof. Assume φ is a theorem of L_A. Let $\mathfrak{M}$ be any model and let $\Gamma_{\mathfrak{M}}$ be as in Example 4.2 (i). $\Gamma_{\mathfrak{M}}$ is a validity property for L_A so $\varphi \in \Gamma_{\mathfrak{M}}$, i. e. $\mathfrak{M} \models \varphi$.

Now assume φ is not a theorem of L_A. Hence there is a validity property Γ_0 with $\varphi \notin \Gamma_0$. Let $\Gamma_0 \subseteq \Gamma$ and S be as in 4.3. Then $\{\neg\varphi\} \in S$ by 4.3 (ii) and S is a consistency property so $\neg\varphi$ has a model, by the Model Existence Theorem. □

The word "weak" in Theorem 4.5 is there because we still have no notions of proof compatable with L_A such that

(3) $\models \varphi$ iff $\exists P\,[P \textit{ is a proof of } \varphi]$.

All we have managed to do so far is replace one Π_1 notion ($\models \varphi$) with another Π_1 notion (φ is a theorem of L_A). We want a Δ_1 notion of proof so that line (3) gives a Σ_1 form for $\models \varphi$, and we want a proof of φ to be essentially the same "size" as the members of L_A.

4.6 Exercise. Define: *φ is a theorem of T* iff φ is in every validity property containing T as a subset. Show that if $T \subseteq \mathsf{L}_A$ where L_A is countable, then φ is a theorem of T iff every model of T is a model of φ.

4.7 Note. The weak completeness theorem is one form of the Karp Completeness Theorem for $\mathsf{L}_{\omega_1\omega}$.

5. Completeness and Compactness for Countable Admissible Fragments

In this section we prove the completeness theorem alluded to in the previous section. We have reduced the task to finding a suitable notion of L_A-*proof* to go along with the notion of theorem of L_A introduced in Definition 4.4.

The first notion of proof one thinks of in this setting is: an L_A-proof is a well-ordered sequence

$$\varphi_0, \varphi_1, \ldots, \varphi_\beta$$

such that each φ_α, for $\alpha \leqslant \beta$, is either an axiom (A 1)—(A 7) of L_A or is a consequence of earlier φ_γ's $(\gamma < \alpha)$ by one of the rules (R 1), (R 2) or (R 3). We can of course prove (given a strong enough metatheory) that φ is a theorem of L_A iff there is such a sequence with $\varphi_\beta = \varphi$. This notion of proof is too restrictive, however, and does not have the nice properties we need for applications. We need a notion which does not have the axiom of choice built into its very definition. There are many ways of doing this. We simply choose one.

5.1 Definition. An ordered pair P is an *infinitary proof* iff one of the following holds:

(A 1)—(A 7) $P = \langle n, \varphi \rangle$ where $1 \leqslant n \leqslant 7$ and φ is an axiom of $\mathsf{L}_{\infty\omega}$ by An of Definition 4.1.

(R 1) $P = \langle f, \psi \rangle$ where f is a function, $\mathrm{dom}(f) = \{0, 1\}$, $f(0)$ is an infinitary proof P_0 with $2^{\mathrm{nd}} P_0$ of the form $(\varphi \to \psi)$, and $f(1)$ is an infinitary proof P_1 with $2^{\mathrm{nd}} P_1 = \varphi$.

(R 2) $P = \langle P_0, (\varphi \to \forall v_\alpha \psi(v_\alpha)) \rangle$ where P_0 is an infinitary proof with $2^{\mathrm{nd}} P_0$ of the form $(\varphi \to \psi(v_\alpha))$ where v_α is not free in φ.

(R 3) $P = \langle f, (\psi \to \bigwedge \Phi) \rangle$ where f is a function with domain Φ such that for each $\varphi \in \Phi$, $f(\varphi)$ is a nonempty set of infinitary proofs, and for each $P_0 \in f(\varphi)$, $2^{\mathrm{nd}} P_0 = (\psi \to \varphi)$.

If P is an infinitary proof and $\varphi = 2^{\mathrm{nd}} P$ then P is said to be a *proof of* φ. See Fig. 5A, B, C.

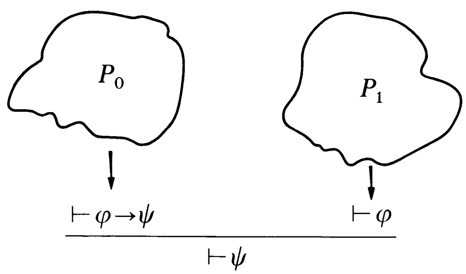


Fig. 5A. A proof P ending with an application of R1

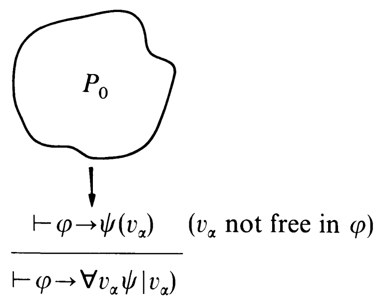


Fig. 5B. A proof P ending with an application of R2

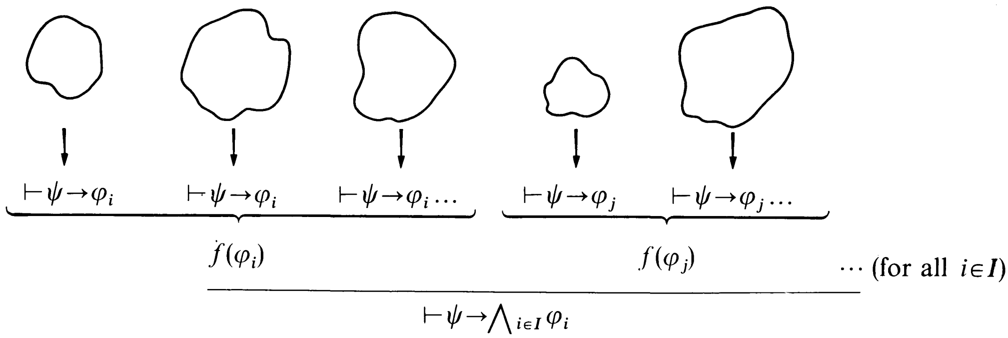


Fig. 5C. A proof P ending with an application of R3

Definition 5.1 can be given in KPU by recursion over TC(P) and consequently results in a Δ_1 predicate in KPU. Consequently,

$$\exists P\,[P \textit{ is an infinitary proof of } \varphi]$$

is a Σ_1 predicate of φ.

We now leave our weak metatheory and step out into the universe $\mathbb{V}_M$ of all sets on M (which is of course a "model" of KPU). It is the interplay between this universe and admissible sets which is of interest.

Let $\mathbb{A}=\mathbb{A}_{\mathfrak{M}}$ be an admissible set with constants $\bigwedge, \bigvee, \neg, \exists, \forall, \equiv$, and the predicates and functions (v, $\#$) mentioned in § 1 satisfying the axioms on syntax found there. We can interpret the results of § 1 and Definition 5.1 in either $\mathbb{A}_{\mathfrak{M}}$ or $\mathbb{V}_{\mathfrak{M}}$. As long as we are dealing with Δ_1 notions, we know that the results are the same, by absoluteness. For example, we have, for $\varphi\in\mathbb{A}_{\mathfrak{M}}$

"φ is an infinitary formula"

true in $\mathbb{A}_{\mathfrak{M}}$ iff it is true (in $\mathbb{V}_{\mathfrak{M}}$). If, moreover, $\mathfrak{N}\in\mathbb{A}_{\mathfrak{M}}$ then $\mathfrak{N}\models\varphi$ holds in $\mathbb{A}_{\mathfrak{M}}$ iff it is true (in $\mathbb{V}_{\mathfrak{M}}$) and, if $P\in\mathbb{A}_{\mathfrak{M}}$ then

"*P is an infinitary proof of* φ"

is true interpreted in $\mathbb{A}_{\mathfrak{M}}$ iff it is true (in $\mathbb{V}_{\mathfrak{M}}$).

5.2 Definition. (i) If $\mathbb{A}=\mathbb{A}_{\mathfrak{M}}=(\mathfrak{M}; A,\in,\ldots)$ is admissible and L is a language which is Δ_1 on $\mathbb{A}$ then

$$\begin{aligned}\mathsf{L}_{\mathbb{A}}&=\{\varphi\in\mathbb{A}\mid\varphi \text{ is an infinitary formula of } \mathsf{L}_{\infty\omega}\}\\&=\{\varphi\in\mathbb{A}\mid\mathbb{A}\models\text{“}\varphi \text{ is an infinitary formula of } \mathsf{L}_{\infty\omega}\text{”}\}\end{aligned}$$

is called the *admissible fragment* of $\mathsf{L}_{\infty\omega}$ given by $\mathbb{A}$.

(ii) If $P\in\mathbb{A}$ and P is an infinitary proof then P is said to be an $\mathsf{L}_{\mathbb{A}}$-*proof*.

It is a trivial matter to check that an admissible fragment $L_{\mathbb{A}}$ really is a fragment in the sense of Definition 2.1.

5.3 Theorem. *Let $L_{\mathbb{A}}$ be an admissible fragment of $L_{\infty\omega}$ and let φ be a sentence of $L_{\mathbb{A}}$. The following are equivalent:*

(i) $\exists P$ [*P is an $L_{\mathbb{A}}$-proof of φ*],

(ii) $\exists P$ [*P is an infinitary proof of φ*],

(iii) *φ is a theorem of $L_{\mathbb{A}}$, i.e., φ is in every validity property for $L_{\mathbb{A}}$.*

(Warning: One can*not* in general add a (iv) asserting that φ is in every validity property *which is an element of* $\mathbb{A}$. This (iv) is usually much weaker than (iii).)

Proof. (i)$\Rightarrow$(ii) is trivial.

(ii)$\Rightarrow$(iii). Let Γ be any validity property for $L_{\mathbb{A}}$. A routine proof by induction on TC(P) shows that if

$$P \text{ is a proof of } \varphi$$

then $\varphi \in \Gamma$ since Γ contains (A 1)—(A 7) and is closed under (R 1)—(R 3).

(iii)$\Rightarrow$(i). We need to show that the set

$$\Gamma = \{\psi \in L_{\mathbb{A}} \mid \exists P \in \mathbb{A}\ (P \textit{ a proof of } \varphi)\}$$

is a validity property for $L_{\mathbb{A}}$, for then $\varphi \in \Gamma$ since φ is in all validity properties. We need to see, then, that Γ contains (A 1)—(A 7) and is closed under (R 1)—(R 3). The first part is obvious so let us check (R 1) and (R 3), (R 2) being similar to (R 1).

(R 1) Suppose φ, $(\varphi \to \psi) \in \Gamma$. There are $P_0, P_1 \in \mathbb{A}$ with

$$2^{\text{nd}} P_0 = (\varphi \to \psi), \qquad 2^{\text{nd}} P_1 = \varphi.$$

Let $f(0) = P_0$, $f(1) = P_1$. Then $P = \langle f, \psi \rangle \in \mathbb{A}$ and P is a proof of ψ, hence $\psi \in \Gamma$.

(R 3) This is where admissibility and our careful choice of the notion of proof come into play. Suppose $(\psi \to \bigwedge \Phi) \in L_{\mathbb{A}}$ and that $(\psi \to \varphi) \in \Gamma$ for all $\varphi \in \Phi$. Thus, for each $\varphi \in \Phi$ there is a $P \in \mathbb{A}$ such that P is a proof of $(\psi \to \varphi)$. Apply strong Σ Replacement in $\mathbb{A}$ to get a function $f \in \mathbb{A}$, $\text{dom}(f) = \Phi$ so that for each $\varphi \in \Phi$:

$$f(\varphi) \neq 0, \quad \text{and}$$

$$\text{if } P \in f(\varphi) \text{ then } P \text{ is a proof of } (\psi \to \varphi).$$

Then $\langle f, (\psi \to \bigwedge \Phi) \rangle \in \mathbb{A}$ and it is a proof of $(\psi \to \bigwedge \Phi)$. Thus $(\psi \to \bigwedge \Phi)$ is in Γ. $\square$

5.4 Corollary. *If $L_{\mathbb{A}}$ is an admissible fragment then the set of theorems of $L_{\mathbb{A}}$ is a Σ_1 subset of $\mathbb{A}$. Moreover, the Σ_1 definition has no parameters in it and is independent of $\mathbb{A}$.*

Proof. The Σ_1 formula is

$$\exists P\ [P \text{ is a proof of } \varphi]. \quad \square$$

Let us write

$$\vdash \varphi$$

for the Σ_1 formula

$$\exists P\ [P \text{ is a proof of } \varphi].$$

By combining 5.3 with the Weak Completeness Theorem of 4.5, we obtain the desired result.

5.5 Barwise Completeness Theorem. *Let* $\mathsf{L}_{\mathbb{A}}$ *be a countable admissible fragment. Then for all* $\varphi \in \mathsf{L}_{\mathbb{A}}$, *the following are equivalent:*
(i) $\models \varphi$,
(ii) $\vdash \varphi$,
(iii) $\mathbb{A}$ *satisfies* $\vdash \varphi$.
Thus the set of valid sentences of $\mathsf{L}_{\mathbb{A}}$ *is* Σ_1 *on* $\mathbb{A}$. $\square$

It is this completeness theorem which accounts for the tractable nature of countable, admissible fragments. It is used to prove many of the results in this book. Before going on though, we pause to point out one thing that the theorem most emphatically does *not* say, but which is sometimes mistaken for the conclusion of the theorem. It does not say that the sentence

$$\models \varphi \quad \text{iff} \quad \vdash \varphi$$

is true *in* the countable admissible set $\mathbb{A}$. This, together with 5.5, would imply that if $\varphi \in \mathsf{L}_{\mathbb{A}}$ and φ has a model then φ has a model $\mathfrak{N}$, $\mathfrak{N} \in \mathbb{A}$. This is false for most $\mathbb{A}$. See Exercises 5.11—5.14.

Now we turn to our first application of the completeness theorem.

5.6 Barwise Compactness Theorem. *Let* $\mathsf{L}_{\mathbb{A}}$ *be a countable admissible fragment of* $\mathsf{L}_{\infty\omega}$. *Let* T *be a set of sentences of* $\mathsf{L}_{\mathbb{A}}$ *which is* Σ_1 *on* $\mathbb{A}$. *If every* $T_0 \subseteq T$ *which is an element of* $\mathbb{A}$ *has a model, then* T *has a model.*

Proof. Expand L to $\mathsf{K} = \mathsf{L} \cup \{\mathsf{c}_n \mid n < \omega\}$ as usual but do it so that K remains Δ_1 on $\mathbb{A}$. Let $\mathsf{K}_{\mathbb{A}} = \mathsf{L}_{\mathbb{A}}(C)$ be the usual fragment of $\mathsf{L}_{\mathbb{A}}$ associated with K. Thus $\mathsf{K}_{\mathbb{A}}$ is the set of all sentences of $\mathsf{K}_{\infty\omega}$ which are elements of $\mathbb{A}$ and have only a finite number of c's in them. We use the Model Existence Theorem for $\mathsf{K}_{\mathbb{A}}$.

Let S be the set of all finite sets s of sentences of $\mathsf{K}_{\mathbb{A}}$ such that for all $T_0 \subseteq T$ with $T_0 \in \mathbb{A}$,

$$T_0 \cup s \quad \text{has a model.}$$

Note that if $s \in S$ and $\varphi \in T$ then $s \cup \{\varphi\} \in S$ so we are all set to apply the Extended Model Existence Theorem to get a model of T once we show that S is a consistence property. As usual it is (C 5) that causes the problems. So suppose $\bigvee \Phi \in s \in S$ but that for each $\varphi \in \Phi$, $s \cup \{\varphi\} \notin S$. Thus, for each $\varphi \in \Phi$ there is a $T_0 \subseteq T$, $T_0 \in \mathbb{A}$ such that

$$T_0 \cup s \cup \{\varphi\} \quad \text{has no model.}$$

Let $\theta(x)$ be the Σ_1 definition of T on $\mathbb{A}$. The following Σ sentence is true in $\mathbb{A}$ by the Completeness Theorem for $\mathsf{K}_{\mathbb{A}}$:

$$(1) \qquad \forall \varphi \in \Phi \, \exists T_0 \, [\forall \psi \in T_0 \, \theta(\psi) \wedge \vdash (\bigwedge (T_0 \cup s) \rightarrow \neg \varphi)] .$$

By Σ Reflection there is a set $a \in \mathbb{A}$ such that (1) holds relativized to a. We can assume a is transitive by I.4.2. Let

$$T_1 = \{\psi \in a \mid \theta^{(a)}(\psi)\} .$$

by Δ_0 Separation. Then $T_1 \in \mathbb{A}$, $T_1 \subseteq T$ by I.4.2 and, for each $\varphi \in \Phi$,

$$\models (\bigwedge (T_1 \cup s) \rightarrow \neg \varphi)$$

since there is some $T_0 \subseteq T_1$ with

$$\models \bigwedge (T_0 \cup s) \rightarrow \neg \varphi .$$

But then $s \cup T_1$ can have no model since $\bigvee \Phi \in s$. This contradicts the assumption that $s \in S$. $\square$

Combining Completeness and Compactness we obtain the following extension of the Completeness Theorem. We use $T \models \varphi$ to indicate that every model of T is a model of φ.

5.7 Extended Completeness Theorem. *Let $\mathsf{L}_{\mathbb{A}}$ be a countable admissible fragment. Let T be a set of sentences of $L_{\mathbb{A}}$ which is Σ_1 on $\mathbb{A}$. The set*

$$\{\varphi \in \mathsf{L}_{\mathbb{A}} : T \models \varphi\}$$

is Σ_1 on $\mathbb{A}$.

Proof. If φ is a sentence of L_A then

$$T \models \varphi \quad \text{iff} \quad \exists T_0 \in \mathbb{A} \, [T_0 \subseteq T \wedge T_0 \models \varphi]$$

by the Barwise Compactness Theorem (applied to $T \cup \{\neg \varphi\}$) so $T \models \varphi$ iff the following is true in $\mathbb{A}$, where $\theta(x)$ defines T,

$$\exists T_0 [\forall \psi \in T_0 \, \theta(\psi) \wedge \vdash \bigwedge T_0 \rightarrow \varphi]$$

which gives a Σ definition of $T \models \varphi$. Note that it depends only on the definition θ of T, not on $\mathbb{A}$, and that is has only the same parameters occuring in it that occur in θ. ☐

One peculiar instance of the Compactness Theorem deserves special mention because it comes up frequently. It applies, for example, to $\mathbb{A} = \mathbb{HYP}_{\mathfrak{M}}$ when $\mathfrak{M}$ is recursively saturated.

5.8 Theorem. *Let $\mathbb{A} = \mathbb{A}_{\mathfrak{M}}$ be a countable admissible set with $o(\mathbb{A}_{\mathfrak{M}}) = \omega$. Let T, T' be theories of $\mathsf{L}_{\mathbb{A}}$ which are Σ_1 on $\mathbb{A}_{\mathfrak{M}}$ such that every $\varphi \in T$ is a pure set. (Hence T is a set of finitary sentences.) If for each finite $T_0 \subseteq T$,*

$$T_0 \cup T' \quad \textit{has a model,}$$

then $T \cup T'$ has a model.

Proof. If $T \cup T'$ has no model then, by the Compactness Theorem, $\models \neg \bigwedge \Phi$, for some $\Phi \in \mathbb{A}$, $\Phi \subseteq T \cup T'$. Now, if we write θ_1, θ_2 for the Σ_1 definitions of T and T', respectively, then we have $\forall x \in \Phi [\theta_1(x) \vee \theta_2(x)]$, so by Σ Reflection there is an $a \in \mathbb{A}$ such that $\forall x \in \Phi [\theta_1^{(a)}(x) \vee \theta_2^{(a)}(x)]$.

Thus, if we use Δ_0 Separation to form

$$\Phi_1 = \{x \in \Phi : \theta_1^{(a)}(x)\},$$
$$\Phi_2 = \{x \in \Phi : \theta_2^{(a)}(x)\}$$

then $\Phi = \Phi_1 \cup \Phi_2$, $\Phi_1 \subseteq T$, $\Phi_2 \subseteq T'$ and $\Phi_1 \cup \Phi_2$ has no model. But Φ_1 is a set of pure sets, hence a pure set, hence finite since $o(\mathbb{A}) = \omega$. ☐

There is a question that often comes up. Let $\mathbb{A}_{\mathfrak{M}} = \mathbb{HF}_{\mathfrak{M}}$. To what extent does the Compactness Theorem 5.6 give us the full compactness theorem for $\mathsf{L}_{\mathbb{A}} = \mathsf{L}_{\omega\omega}$? In other words, how does the requirement that T be Σ_1 on $\mathbb{A}_{\mathfrak{M}}$ affect us. If $\mathbb{HF}_{\mathfrak{M}}$ is countable (i.e. if $\mathfrak{M}$ is countable) then 5.6 gives us the full compactness for $\mathsf{L}_{\omega\omega}$. For let T be any theory of $\mathsf{L}_{\omega\omega}$. T is Σ_1 on $(\mathfrak{M}; \mathbb{HF}_M, \in, T)$ which is admissible by Theorem II.2.1 so we can apply 5.6 to this admissible set.

5.9 – 5.14 Exercises

5.9. Define "P is a proof from axioms in T" parallel to Definition 5.1. You must build in a Σ_1 definition of T. Use this to prove the Extended Completeness Theorem and Compactness Theorem in one fell swoop, without using the Completeness Theorem. [You will need to use Exercise 4.6.]

5.10. Let $\mathsf{L}_{\mathbb{A}}$ be an admissible fragment. Show that if P is an $\mathsf{L}_{\mathbb{A}}$-proof then all formulas in the proof P are $\mathsf{L}_{\mathbb{A}}$-formulas.

5.11. Let $\mathbb{A}$, $\mathbb{B}$ be admissible sets with $\mathbb{A} \in \mathbb{B}$ and

$$\text{“}\mathbb{A} \text{ is countable”}$$

true in $\mathbb{B}$. Let $\mathsf{L}_{\mathbb{A}}$ be an admissible fragment given by $\mathbb{A}$. Show that

$$\mathbb{B} \models \text{“}\varphi \text{ is valid”}$$

iff

$$\mathbb{A} \models \exists P\ [P \text{ is a proof of } \varphi].$$

Conclude that if $\varphi \in \mathsf{L}_{\mathbb{A}}$ has a model then it has a model in $\mathbb{B}$.

5.12. Let $\mathbb{A}$ be admissible and satisfy the following:

(i) (locally countable). $\mathbb{A} \models \forall a$ (a is countable)

(ii) (recursively inaccessible). $\forall a \in \mathbb{A}$ there is an admissible $\mathbb{B} \in \mathbb{A}$ with $a \in \mathbb{B}$. Show that for any sentence φ of $\mathsf{L}_{\mathbb{A}}$

$$(\models \varphi) \leftrightarrow (\vdash \varphi)$$

holds in $\mathbb{A}$. Show that $\mathbb{HC}_{\mathfrak{M}}$ is locally countable and recursively inaccessible. (Of course, for $\mathbb{HC}_{\mathfrak{M}}$ the conclusion is trivial since every countable structure for a countable language is isomorphic to a structure in $\mathbb{HC}_{\mathfrak{M}}$.)

5.13. Let $\mathbb{A}$ be a countable transitive model of ZFC (or enough of it to insure that $\mathbb{A}$ is admissible and prove that $\aleph_1$ exists). Let α be the ordinal of $\mathbb{A}$ which satisfies

$$\mathbb{A} \models \text{“}\alpha \text{ is the first uncountable ordinal”}.$$

Write a sentence φ of $\mathsf{L}_{\mathbb{A}}$ which asserts that α is countable. Thus φ has a model but does not have a model $\mathfrak{N} \in \mathbb{A}$. In other words, $\neg\varphi$ is valid in the sense of $\mathbb{A}$ but it is not provable since φ does indeed have a model.

5.14. Let α be the first admissible ordinal $>\omega$. Let $\mathbb{A} = L(\alpha)$. Unlike the $\mathbb{A}$ in 5.13, this $\mathbb{A}$ is a model of

$$\forall a\ [a \text{ is countable}].$$

Find a sentence φ of $\mathsf{L}_{\mathbb{A}}$ which has a model but none in $\mathbb{A}$.

5.15 Notes. The completeness and compactness theorems of this section are due to Barwise [1967] and appeared in Barwise [1969]. The terminology "Barwise Completeness Theorem" and "Barwise Compactness Theorem" have become so standard that it would be false modesty (and confusing) to give them some other name here.

The observation that these theorems go through unchanged in the presence of urelements was first made in Barwise [1973], though it was really clear all along. The odd 5.8 first appears in Barwise [1973].

6. The Interpolation Theorem

The Interpolation Theorem is one of the results which holds for countable admissible fragments but not for arbitrary countable fragments; the proof requires the Completeness Theorem of § 5.

6.1 Theorem. *Let $\mathsf{L}_{\mathbb{A}}$ be a countable admissible fragment, φ, ψ sentences of $\mathsf{L}_{\mathbb{A}}$ such that*

$$\models \varphi \rightarrow \psi .$$

There is a sentence θ of $\mathsf{L}_{\mathbb{A}}$ whose relation, function and constant symbols are common to those of both φ and ψ such that

$$\models \varphi \rightarrow \theta \quad \text{and} \quad \models \theta \rightarrow \psi .$$

Note. Equality is not treated as a relation symbol. It may appear in θ while appearing in only one of φ, ψ.

Proof. Let L^0 be the set of those symbols occurring in φ and let L^1 be the set of those symbols occurring in ψ. Let C be a countable set of new constant symbols coded as a Δ_1 subset of $\mathbb{A}$ and let $\mathsf{L}_{\mathbb{A}}(C)$ be the set of free substitution instances of formulas of $\mathsf{L}_{\mathbb{A}}$ by a finite number of symbols from C. Define $\mathsf{L}^0_{\mathbb{A}}(C)$ and $\mathsf{L}^1_{\mathbb{A}}(C)$ similarly. We define the consistency property S to be the set of all finite sets s of $\mathsf{L}_{\mathbb{A}}(C)$ which can be written as a union $s = s_0 \cup s_1$ satisfying the following conditions:

(1) s_0 is a set of sentences of $\mathsf{L}^0_{\mathbb{A}}(C)$, and similarly for s_1;

(2) If $\theta_0, \theta_1 \in \mathsf{L}^0_{\mathbb{A}}(C) \cap \mathsf{L}^1_{\mathbb{A}}(C)$ are such that $s_0 \models \theta_0$ and $s_1 \models \theta_1$, then the sentence $\theta_0 \wedge \theta_1$ has a model.

The verification that S is indeed a consistency property is routine (indeed, it is just like the $\mathsf{L}_{\omega_1\omega}$ case in Keisler [1971]) except for the $\bigvee$-rule, (C5). It is in the verification of this rule that we need the Completeness Theorem for $\mathsf{L}_{\mathbb{A}}(C)$. So suppose $s = s_0 \cup s_1$ is as above and that $\bigvee \Phi \in s$. Since the two cases are symmetric, we may assume that $\bigvee \Phi \in s_0$. We want to prove that for some $\sigma \in \Phi$,

$$s_0 \cup \{\sigma\} \cup s_1 \in S .$$

Suppose this is not the case. Then for every $\sigma \in \Phi$, there is a pair θ_0, θ_1 such that

(3) $\models \bigwedge s_0 \wedge \sigma \rightarrow \theta_0$, $\models \bigwedge s_1 \rightarrow \theta_1$, and $\models \neg(\theta_0 \wedge \theta_1)$, and the constants from C in θ_0 and θ_1 are in $s_0 \cup s_1$.

Let us indulge in a little wishful thinking and suppose that there are functions f, g which are *elements of our admissible set* $\mathbb{A}$ with $\mathrm{dom}(f) = \mathrm{dom}(g) = \Phi$ such that, for each $\sigma \in \Phi$, $\langle f(\sigma), g(\sigma) \rangle$ is a pair $\langle \theta_0, \theta_1 \rangle$ satisfying (3). Then we can let

$$\theta_0' = \bigvee \{f(\sigma) \mid \sigma \in \Phi\}, \quad \theta_1' = \bigwedge \{g(\sigma) \mid \sigma \in \Phi\} .$$

By Σ Replacement, θ_0' and θ_1' are elements of $\mathbb{A}$ so they are both sentences of the languages $L^0_{\mathbb{A}}(C)$ and $L^1_{\mathbb{A}}(C)$. Furthermore, $s_0 \models \theta_0'$, $s_1 \models \theta_1'$ and $\models \neg(\theta_0' \wedge \theta_1')$, which contradicts $s_0 \cup s_1 \in S$. But what about our bit of wishful thinking? At first it seems exactly that, since we have no choice principle holding in $\mathbb{A}$. Once again is it Strong Σ Replacement which which comes to the rescue. By the Completeness Theorem for $L_{\mathbb{A}}(C)$, line (3) can be expressed by a Σ_1 formula. By Strong Σ Replacement there is a function $h \in \mathbb{A}$ with $\mathrm{dom}(h) = \Phi$ such that, for each $\sigma \in \Phi$, $h(\sigma)$ is a nonempty set of pairs $\langle \theta_0, \theta_1 \rangle$ satisfying (3). Define f and g by

$$f(\sigma) = \bigwedge \{1^{\mathrm{st}}\, h(\sigma) \mid \sigma \in \Phi\},$$
$$g(\sigma) = \bigwedge \{2^{\mathrm{nd}}\, h(\sigma) \mid \sigma \in \Phi\}.$$

Then $f, g \in \mathbb{A}$ and our wish has come true since $\theta_0 = f(\sigma)$ and $\theta_1 = g(\sigma)$ also satisfy (3). Thus S is a consistency property.

The conclusion of the theorem now follows easily from the observation that $\{\varphi, \neg\psi\} \notin S$. Just quantify out the finite number of new constant symbols in the sentence $\theta(c_1, \ldots, c_n)$ $(= \theta_0$ in the notation used above):

$$\models \varphi \to \forall v_1, \ldots, v_n\, \theta(v_1, \ldots, v_n),$$
$$\models \forall v_1, \ldots, v_n\, \theta(v_1, \ldots, v_n) \to \psi. \quad \square$$

We could use the interpolation theorem for $L_{\mathbb{A}}$ to prove Beth's Theorem for $L_{\mathbb{A}}$, but we will not be needing this result.

6.2–6.6 Exercises

6.2 (Hard). Let I_1, I_2 be interpretations of a language L^0 in consistent infinitary theories, T_1, T_2 formulated in languages L^1, L^2 respectively, and suppose that there are no two models $\mathfrak{M}_1, \mathfrak{M}_2$ of T_1, T_2 respectively such that $\mathfrak{M}_1^{-I_1} = \mathfrak{M}_2^{-I_2}$.

If L^0, L^1, L^2 are Δ_1 on the countable admissible $\mathbb{A}$, T_1, T_2 are Σ_1 theories of $L^1_{\mathbb{A}}$ and $L^2_{\mathbb{A}}$ and the interpretation I_1, I_2 are Σ_1 functions on $\mathbb{A}$ then there is a sentence φ of $L^0_{\mathbb{A}}$ such that

$$\mathfrak{M}_1 \models T_1 \text{ implies } \mathfrak{M}_1^{-I_1} \models \varphi, \quad \text{for all } L^1\text{-structures } \mathfrak{M}_1;$$
$$\mathfrak{M}_2 \models T_2 \text{ implies } \mathfrak{M}_2^{-I_2} \models \neg\varphi, \quad \text{for all } L^2\text{-structures } \mathfrak{M}_2.$$

6.3. Let $L_{\mathbb{A}}$, $K_{\mathbb{A}}$ be countable admissible fragments and let I be an interpretation of L into a theory T of $K_{\mathbb{A}}$, T and I being Σ_1 on $\mathbb{A}$. Let φ be a sentence of $K_{\mathbb{A}}$ such that for all models $\mathfrak{N}_1, \mathfrak{N}_2$ of T with $\mathfrak{N}_1^{-I} = \mathfrak{N}_2^{-I}$, $\mathfrak{N}_1 \models \varphi$ iff $\mathfrak{N}_2 \models \varphi$, Then there is a $\psi \in L_{\mathbb{A}}$ such that $T \vdash (\varphi \leftrightarrow \psi^I)$.

6.4. Let I be an interpretation of a complete theory T of $L_{\omega\omega}$ in an incomplete theory T_1 of $K_{\omega\omega}$. Show that if $\varphi \in K_{\omega\omega}$ is not decided by T_1 then there are models $\mathfrak{M}, \mathfrak{N}$ of T with $\mathfrak{M} \models \varphi$, $\mathfrak{N} \models \neg\varphi$ and $\mathfrak{M}^{-I} = \mathfrak{N}^{-I}$.

6.5. Show that there are models $\mathfrak{N}_1=\langle N,+,x_1\rangle$, $\mathfrak{N}_2=\langle N,+,x_2\rangle$ of Peano arithmetic, with the same integers and addition, but $\mathfrak{N}_1\not\equiv\mathfrak{N}_2$.

6.6. Show that there are models $\mathfrak{M}=\langle M,E\rangle$, $\mathfrak{N}=\langle N,F\rangle$ of ZF+V=L with the same ordinals, $\langle \mathrm{Ord}^{\mathfrak{M}},E\restriction \mathrm{Ord}^{\mathfrak{M}}\rangle=\langle \mathrm{Ord}^{\mathfrak{N}},F\restriction \mathrm{Ord}^{\mathfrak{N}}\rangle$, but with different sets of hereditarily finite sets. [Use Gödel's Incompleteness Theorem, 6.4 and the fact that every true sentence about the ordinals (with $<$) is provable in ZF.]

6.7 Notes. The interpolation theorem for $\mathsf{L}_{\omega\omega}$ is due to Craig [1957]. For the full $\mathsf{L}_{\omega_1\omega}$ it is due to Lopez-Escobar [1965]. Theorem 6.1 is due to Barwise [1969]. Exercise 6.2, a generalization of 6.1, is useful in abstract logic and is due to Barwise [1973]. References for the other exercises can be found there.

7. *Definable Well-Orderings*

In this section we prove a model theoretic result which will have applications to $\mathbb{HYP}_{\mathfrak{M}}$ in Chapter IV. The basic question is: What ordinals can be define in an admissible fragment? We solve this problem here for countable fragments. The uncountable case is taken up in Chapters VII and VIII.

7.1 Example. Define $\theta_\alpha(x)$ by recursion over α as follows:

$$\theta_0(x) \quad \text{is} \quad \forall y\neg(y<x),$$
$$\theta_\alpha(x) \quad \text{is} \quad \forall y\,(y<x\leftrightarrow \textstyle\bigvee_{\beta<\alpha}\theta_\beta(y/x)).$$

Let $\mathfrak{M}=\langle M,<\rangle$ be a linear ordering. A simple proof by induction shows that if $\mathfrak{M}\models\theta_\alpha[x]$ then $\{y\in\mathfrak{M}\mid y<x\}$ is well ordered and has order type α. Hence $\mathfrak{M}\models\forall x\bigvee_{\beta<\alpha}\theta_\beta(x)$ iff $\mathfrak{M}$ is well ordered and has order type $\leqslant\alpha$. Thus, if we set σ_α equal to

$$\forall x \textstyle\bigvee_{\beta<\alpha}\theta_\beta(x)\wedge\bigwedge_{\beta<\alpha}\exists x\,\theta_\beta(x)$$

then $\mathfrak{M}\models\sigma_\alpha$ iff $\mathfrak{M}$ has order type exactly α. □

The formulas from Example 7.1 were defined by recursion on α so the definition can be phrased as a Σ recursion in KPU. Hence, if α is in an admissible set $\mathbb{A}$ then $\varphi_\alpha(x)$, $\sigma_\alpha\in\mathsf{L}_{\mathbb{A}}$ (as long as the symbol $<$ is in L and $\mathbb{A}$).

7.2 Definition. Let L have a binary symbol $<$ and possible other symbols. A sentence $\varphi(<)$ of an admissible fragment $\mathsf{L}_{\mathbb{A}}$ *pins down the ordinal* α if

(i) $\mathfrak{N}\models\varphi$ implies $<^{\mathfrak{N}}$ is a well-ordering of its field.

(ii) φ has a model $\mathfrak{N}$ with $<^{\mathfrak{N}}$ of order type α.

A theory T of $\mathsf{L}_{\mathbb{A}}$ pins down α if $\bigwedge T$ pins down α.

The example above shows that every ordinal in the admissible set A can be pinned down by a sentence of $L_{\mathbb{A}}$; in fact,

$$(\text{“}< \textit{ is a linear ordering”}) \wedge \sigma_\alpha$$

has only models of order type α. For countable admissible fragments, no other ordinal can be pinned down, as we show below.

For uncountable admissible fragments one can often pin down an ordinal $\alpha > o(\mathbb{A})$ (though one cannot give an explicit definition like σ_α above). The least ordinal which cannot be pinned down plays a key role in the model theory of uncountable fragments. We will go into this further in Chapter VII.

Note that if $\varphi(<_1)$ pins down α then $\psi(<_1, <, \mathsf{f})$ defined by

$$\varphi(<_1) \wedge \text{“}\mathsf{f}\textit{ maps } < \textit{ into } <_1 \textit{ in an order preserving fashion”},$$

as a sentence about $<$, pins down all ordinals $\leqslant \alpha$. Thus we can always work with sentences which pin down an initial segment of the ordinals. We will use this implicitly in the proof of the next theorem and several times in Chapter VII.

7.3 Theorem. *Let $L_{\mathbb{A}}$ be a countable admissible fragment, $\varphi(\prec)$ a sentence which pins down ordinals. There is an ordinal α in the admissible set $\mathbb{A}$ such that every ordinal pinned down by φ is less than α.*

Proof. Suppose, to prove the contrapositive, that for every $\alpha \in \mathbb{A}$, $\varphi(\prec)$ has a model $\mathfrak{N}$ with $\prec^{\mathfrak{N}}$ of order type α. We prove that $\varphi(\prec)$ has a model $\mathfrak{N}$ where $\prec^{\mathfrak{N}}$ is not well ordered. It is instructive to split into cases, though not really necessary.

Case 1. $o(\mathbb{A}) = \omega$. If $\mathbb{A} = \mathbb{HF}_{\mathfrak{M}}$ then $\varphi(\prec)$ is just a finitary sentence and the result is well known to follow from the compactness theorem. But even if $\mathbb{A} \neq \mathbb{HF}_{\mathfrak{M}}$, the proof goes through. Let $<$, $\mathsf{c}_0, \mathsf{c}_1, \ldots$ be new symbols in the pure part of $\mathbb{A}$ and let T be the theory

$$\mathsf{c}_{n+1} \prec \mathsf{c}_n \ (n = 0, 1, 2, \ldots).$$

Let ψ be

$$\varphi(\prec) \wedge \text{“}\mathsf{f}\textit{ maps } < \textit{ order preserving into } \prec\text{”}.$$

We need only show $T + \psi$ has a model. Since T is a set of pure sets and $o(\mathbb{A}) = \omega$ we need (by 5.8) only see that ψ is consistent with every finite subset of T, which is obvious.

Case 2. $o(\mathbb{A}) > \omega$. The basic idea is the same, but we must work a little harder. Let $a_0 = \mathrm{TC}(\varphi)$. We may assume that $\mathbb{A}$ is the smallest admissible set with a_0 as an element. Introduce the following new symbols into L: $\in$, unary U, S, N for urelement, set and member of N respectively, a function symbol f, a constant c and, for each $x \in \mathbb{A}$ a constant $\overline{\mathsf{x}}$. Let T be the following set of sentences of $L_{\mathbb{A}}$:

(0) "$\mathsf{U}, \mathsf{S}, \mathsf{N}$ *are disjoint and their union is everything*",
(1) φ^{N}, the relativization of $\varphi(\prec)$ to N,
(2) KPU formulated in terms of $\mathsf{U}, \mathsf{S}, \in$,
(3) diagram $(\mathbb{A})$,
(4) $\forall x[x \in \bar{\mathsf{a}} \to \bigvee_{b \in a} x \equiv \bar{\mathsf{b}}]$, for each $a \in \mathbb{A}$,
(5) "$\bar{\beta} \in \mathsf{c} \wedge \mathsf{c}$ *is an ordinal*", for each $\beta < o(\mathbb{A})$,
(6) $\forall x \leqslant \mathsf{c} \bigvee_{\psi \in \mathrm{KPU}} \neg \psi^{\mathrm{L}(a_0, x)}$,
(7) "f *maps the* $\in$ *predecessors of* c *into* $\prec$ *so that*

$$x < y \Rightarrow \mathsf{f}(x) \prec \mathsf{f}(y)\text{''}.$$

This theory is Σ_1 (in fact Δ_0) on $\mathbb{A}$ so we can apply the Compactness Theorem. If $T_0 \subseteq T$, $T_0 \in \mathbb{A}$ then T_0 will have a model of the form

$$(\mathbb{A}, \beta, \mathfrak{N}, f)$$

where $\beta \in \mathbb{A}$, f maps the ordinals $< \beta$ into $\prec^{\mathfrak{N}}$, and $\mathfrak{N} \models \varphi$. Let

$$(\mathfrak{B}, c, \mathfrak{N}, f)$$

be a model of the whole theory T. By (1), $\mathfrak{N} \models \varphi$. By (2)—(5), $\mathbb{A} \subseteq_{\text{end}} \mathfrak{B}$, but, by (6), c is not in the well-founded part of $\mathfrak{B}$. Hence, by (7), $\prec^{\mathfrak{N}}$ cannot be a well-ordering. ▯

7.4 Corollary. *Let* $\mathsf{L}_{\mathbb{A}}$ *be a countable admissible fragment,* T *a* Σ_1 *theory of* $\mathsf{L}_{\mathbb{A}}$ *which pins down ordinals. There is an* $\alpha < o(\mathbb{A})$ *such that every ordinal pinned down by* T *is less than* α.

Proof. If not, then every $T_0 \subseteq T$, $T_0 \in \mathbb{A}$ would be consistent with

$$\{\mathsf{c}_{n+1} \prec \mathsf{c}_n : n = 0, 1, 2, \ldots\}$$

by 7.3 and hence T would also be consistent with this set by the Barwise Compactness Theorem. ▯

We can use Theorem 7.4 to prove a general version of a theorem of Friedman on models of set theories $T \supseteq \mathrm{KP}$. Note that if $<$ is a linear ordering then, by the definition in II.8.3, $\mathscr{W}\!f(<)$ is the largest well-ordered initial segment of $<$. We can identify $\mathscr{W}\!f(<)$ with an ordinal without confusion. In Theorem 7.5, L is a language containing a binary symbol $<$ among its symbols.

7.5 Theorem. *Let* $\mathsf{L}_{\mathbb{A}}$ *be a countable admissible fragment of* $\mathsf{L}_{\infty\omega}$ *and let* $\alpha = o(\mathbb{A})$. *Let* T *be a* Σ_1 *theory of* $\mathsf{L}_{\mathbb{A}}$ *such that:*

i) $T \models$ "$<$ *is a linear ordering*";
ii) *for each* $\beta < \alpha$, T *has a model* $\mathfrak{M}$ *with*

$$\mathscr{W}\!f(<^{\mathfrak{M}}) \geqslant \beta.$$

Then T *has a model* M *with* $\mathscr{W}\!f(<^{\mathfrak{M}}) = \alpha$.

Proof. Notice that T is consistent with the set

$$\{\exists x\, \theta_\beta(x) \mid \beta < \alpha\},$$

where θ_β is as in 7.1, by the Barwise Compactness Theorem, so we may as well assume that the sentences are actually in T. This insures that any model $\mathfrak{M}$ of T has $\mathscr{W}\!\mathscr{f}(<^{\mathfrak{M}}) \geqslant \alpha$. By Theorem 7.4, T also has a model $\mathfrak{M}$ where $<^{\mathfrak{M}}$ is not well ordered. Let T' be the theory

$$T \cup \{(\mathsf{d}_{n+1} < \mathsf{d}_n) \mid n < \omega\}$$

where $\{\mathsf{d}_n \mid n < \omega\}$ is a new set of constant symbols. Thus T' is consistent. Let $\mathsf{K} = \mathsf{L} \cup \{\mathsf{d}_n \mid n < \omega\}$ and let $\mathsf{K}_{\mathbb{A}}$ be the corresponding admissible fragment. Let $\mathsf{K}^0_{\mathbb{A}}$ be the set of formulas in which at most a finite member of d_n's occur. Thus T' is a theory of $\mathsf{K}^0_{\mathbb{A}}$. We are going to use the Omitting Types Theorem for $\mathsf{K}^0_{\mathbb{A}}$ to find a model $(\mathfrak{M}, d_1, \ldots, d_n, \ldots)$ of T' and the sentence

$$\forall v\, [\text{“}v \notin \text{Field}(<)\text{”} \vee \textstyle\bigvee_{\beta<\alpha} \theta_\beta(v) \vee \bigvee_{m<\omega} (\mathsf{d}_m < v)]$$

where “$v \notin \text{Field}(<)$” stands for $\forall x\, (v \nless x \wedge x \nless v)$. Such a model $\mathfrak{M}$ must have $\mathscr{W}\!\mathscr{f}(<^{\mathfrak{M}}) = \alpha$. Suppose there were no such model. Then by the Omitting Types Theorem there is a $\sigma(v, \mathsf{d}_1, \ldots, \mathsf{d}_n) \in \mathsf{K}^0_{\mathbb{A}}$ such that $T' + \exists v\, \sigma(v, \mathsf{d}_1, \ldots, \mathsf{d}_n)$ is consistent, but such that all the following are theorems of T':

(8) $\forall v\, [\sigma(v, \vec{\mathsf{d}}) \to v \in \text{Field}(<)]$,
(9) $\forall v\, [\sigma(v, \vec{\mathsf{d}}) \to \neg\theta_\beta(v)]$,
(10) $\forall v\, [\sigma(v, \vec{\mathsf{d}}) \to v \leqslant \mathsf{d}_m]$, for all $m < \omega$.

Note that, by (10),

$$T + \exists v\, [\sigma(v, \mathsf{d}_1, \ldots, \mathsf{d}_n) \wedge v < \mathsf{d}_n < \cdots < \mathsf{d}_1]$$

is consistent. Let c be a new constant symbol and let T'' be

$$T \cup \{\sigma(\mathsf{c}, \mathsf{d}_1, \ldots, \mathsf{d}_n) \wedge \mathsf{c} < \mathsf{d}_n < \cdots < \mathsf{d}_1\}$$

which is consistent since $T \subseteq T'$. We claim that

(11) *For every model $\mathfrak{M}$ of T'', the $<^{\mathfrak{M}}$ predecessors of $c = \mathsf{c}^{\mathfrak{M}}$ are well ordered.*

If not, then there is a descending sequence below c, so we can name its members $d_{n+1}, \ldots,$ giving us

$$d_1 > \cdots > d_n > c > d_{n+1} > \cdots > \cdots$$

which makes $(\mathfrak{M}, d_1, \ldots, d_n, \ldots) \models T'$, where the element c violates (10).

From (11) and Theorem 7.4 we obtain

(12) *there is a* $\gamma<\alpha$ *such that*

$$T''\models\bigvee_{\beta<\gamma}\theta_\beta(\mathsf{c}).$$

(To see this consider $T'''=T''+$"$\prec=<\restriction$ the predecessors of c" and apply Theorem 7.4 to T''' as a theory about $\prec$.)

Thus we have

$$T\models\sigma(\mathsf{c},\mathsf{d}_1,\ldots,\mathsf{d}_n)\wedge\mathsf{c}<\mathsf{d}_n<\cdots<\mathsf{d}_1\rightarrow\bigvee_{\beta<\gamma}\theta_\beta(\mathsf{c})$$

and hence

$$T'\models\sigma(\mathsf{c},\mathsf{d}_1,\ldots,\mathsf{d}_n)\rightarrow\bigvee_{\beta<\gamma}\theta_\beta(\mathsf{c})$$

contradicting (9). ▯

7.6—7.7 Exercises

7.6 (Friedman). Assume ZF has an uncountable transitive model. Show that the order types of the ordinals in countable nonstandard ω-models of ZF are exactly the order types $\alpha(1+\eta)$ for α a countable admissible ordinal, $\alpha>\omega$, and η the order type of the rationals. [Use Theorem 7.5.]

7.7. Let α be nonadmissible. Let $\mathrm{T}=\mathrm{KP}+$"$<$ is $\in\restriction$ordinals". Show that there can be no model $\mathfrak{M}$ of T with $\alpha=\mathscr{W}\!\!\not{f}(<)$.

7.8 Notes. Theorem 7.3 is from Barwise [1969]. It refines older results of Lopez-Escobar [1966] and Morley [1965]. Theorem 7.5 is new here. It generalizes a result in Friedman [1973].

8. Another Look at Consistency Properties

There is room for a lot of creativity inside the proof of the Model Existence Theorem. Let S be a consistency property, $s_0\in S$, and recall the way we constructed a model of s_0. As we built our sequence $s_0\subseteq s_1\subset\cdots\subseteq s_n\subseteq\cdots\subseteq s_\omega$ (and in so doing built a canonical model) there was freedom in defining s_{n+1} that we didn't use. At the n^{th} stage, after defining s_n we could first enlarge s_n to some other $s_n^*\supseteq s_n$ before going on to get $s_{n+1}\supseteq s_n^*$, as long as $s_n^*\in S$. The resulting

$$s_0\subseteq s_0^*\subseteq s_1\subseteq s_1^*\subseteq\cdots\subseteq s_n\subseteq s_n^*\subseteq\cdots$$

with

$$s_\omega=\bigcup_{n<\omega}s_n=\bigcup_{n<\omega}s_n^*$$

would give rise to a canonical model $\mathfrak{M}$ of each s_n and each s_n^*. We give a modest illustration of this technique here, just to illustrate the general method. We will return to it in § IV.4.

Let $L_A \subseteq K_B$ be fragments. A theory T of K_B is *complete for* L_A if for each sentence φ of L_A, $T \models \varphi$ or $T \models \neg\varphi$, but not both. Given a structure $\mathfrak{M}$ for K we define

$$\mathrm{Th}_{L_A}(\mathfrak{M}) = \{\varphi \in L_A \mid \varphi \text{ is a sentence true in } \mathfrak{M}\}.$$

Note that this is a complete L_A-theory.

8.1 Theorem. *Let $L_A \subseteq K_B$ be countable fragments, T a consistent set of sentences of K_B such that for each sentence ψ of K_B, $T \cup \{\psi\}$ is not complete for L_A. There are $2^{\aleph_0}$ distinct L_A theories of the form $\mathrm{Th}_{L_A}(\mathfrak{M})$ for models $\mathfrak{M}$ of T.*

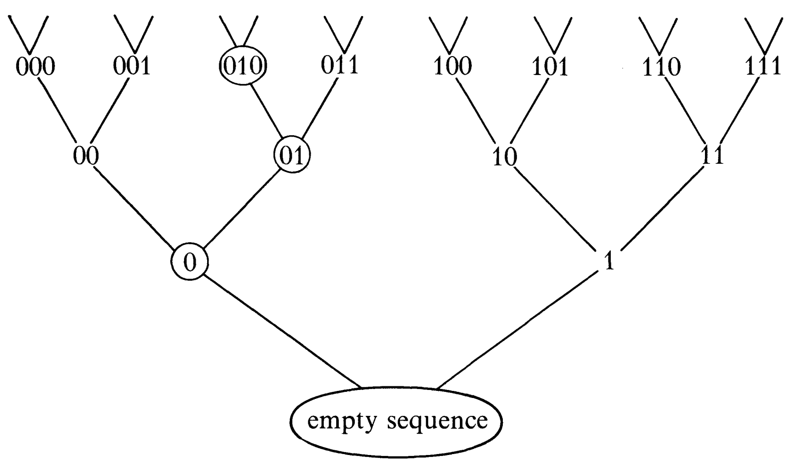
Proof. Let $K' = K \cup C = K \cup \{c_n : n < \omega\}$ and let K'_B be the set of free substitution instances of φ's in K_B. Note that there is no sentence $\psi(c_1, \ldots, c_n) \in K'_A$ such that $T + \varphi(c_1, \ldots, c_n)$ is complete for L_A, for if it were then $T + \exists v_1, \ldots, \exists v_n\, \psi(v_1, \ldots, v_n)$ would be complete for L_A. Consequently there is no finite set s of K'_A such that $T \cup s$ is complete for L_A (for otherwise we would form $\psi = \bigwedge s$). Define

$$S_0 = \{T \cup s \mid s \text{ a finite set of sentences of } K_B \text{ such that } T \cup s \text{ is consistent}\}.$$

The set S_0 obviously satisfies (C1)—(C7) so let S be the smallest consistency property containing S_0, by 2.3. To simplify notation let us suppress T altogether and write s for $T \cup s$ in what follows. We wish to construct a "tree" of members of S such that

(1) any "branch" through the tree gives us a theory T_0 of L_A consistent with T,
(2) distinct branches lead to incompatible theories, and
(3) there are $2^{\aleph_0}$ distinct branches.

Since this is our first tree argument, and since this is an important kind of argument, we give the proof in more detail than is usual. Our tree consists of all finite sequences d of 0's and 1's arranged as illustrated below.



A branch through this tree is just an infinite sequence b of 0's and 1's. (The branch $\langle 01000\ldots\rangle$ has its nodes circled in the tree drawn above.) We wish to place at each node d of length n an element $s^d \in S$ which is one of the s_n^*'s referred to in the introduction to this section. At the empty node place T and let $s_0 = T$ in the notation from the proof of the Model Existence Theorem. Before defining s_1, pick some $\varphi \in \mathsf{L}_A$ not decided by s_0, i. e.

$$s_0^* = s_0 \cup \{\varphi\} \quad \text{and} \quad s_0^{**} = s_0 \cup \{\neg\varphi\}$$

are consistent and hence in S. Let $s^d = s_0^*$ for $d = \langle 0\rangle$, $s^d = s_0^{**}$ for $d = \langle 1\rangle$.

Given d of length n we go on to find $s_{n+1} \supseteq s^d$ just as in the proof of the Model Existence Theorem. Then, given s_{n+1}, choose a $\varphi \in \mathsf{L}_A$ such that

$$s_{n+1}^* = s_{n+1} \cup \{\varphi\}, \qquad s_{n+1}^{**} = s_{n+1} \cup \{\neg\varphi\}$$

are in S, and let

$$\begin{aligned} s^{d'} &= s_{n+1}^*, \quad \text{if} \quad d' = d0 \\ &= s_{n+1}^{**}, \quad \text{if} \quad d' = d1, \end{aligned}$$

where $d0$ is the sequence d followed by 0. Now let b be any branch through the tree. The set $s^b = \bigcup\{s^d \mid d \text{ a node on } b\}$ is one of our s_ω's so it has a model $\mathfrak{M}_b$. Since $s_0 = T$, $\mathfrak{M}_b \models T$. If b_1, b_2 are distinct branches then there is a $\varphi \in \mathsf{L}_A$ such that we have put $\varphi \in s^{b_1}$, $\neg\varphi \in s^{b_2}$ at the point where b_1 and b_2 split. Thus $\mathfrak{M}_{b_1} \models \varphi$ and $\mathfrak{M}_{b_2} \models \neg\varphi$, where $\varphi \in \mathsf{L}_A$. There are $2^{\aleph_0}$ branches so the sets

$$\mathrm{Th}_{\mathsf{L}_A}(\mathfrak{M}_b)$$

form $2^{\aleph_0}$ distinct complete theories of L_A. □

We will apply the following corollary of Theorem 8.1 in Chapter IV. Let $\mathsf{L}_{\mathbb{A}}$ be an admissible fragment of $\mathsf{L}_{\infty\omega}$. A structure $\mathfrak{M}$ is *decidable* for $\mathsf{L}_{\mathbb{A}}$ if $\mathrm{Th}_{\mathsf{L}_{\mathbb{A}}}(\mathfrak{M})$ is Δ_1 on $\mathbb{A}$. The structure $\mathfrak{M}$ could be a structure for some language K properly containing L.

8.2 Corollary. *Let $\mathsf{L}_{\mathbb{A}} \subseteq \mathsf{K}_{\mathbb{A}}$ be countable admissible fragments, let T be a consistent theory of $\mathsf{K}_{\mathbb{A}}$ which is Σ_1 on the admissible set $\mathbb{A}$ such that T has no model which is decidable for $\mathsf{L}_{\mathbb{A}}$. There are $2^{\aleph_0}$ distinct theories of the form $\mathrm{Th}_{\mathsf{L}_{\mathbb{A}}}(\mathfrak{M})$ with $\mathfrak{M} \models T$.*

Proof. If there are fewer than $2^{\aleph_0}$ such sets then there is a $\psi \in \mathsf{K}_{\mathbb{A}}$ such that $T + \psi$ is complete for $\mathsf{L}_{\mathbb{A}}$. But then any model $\mathfrak{M}$ of $T + \psi$ is decidable for $\mathsf{L}_{\mathbb{A}}$ since

$$\begin{aligned} &\mathfrak{M} \models \varphi \quad \text{iff} \quad T \models \psi \to \varphi, \\ &\mathfrak{M} \not\models \varphi \quad \text{iff} \quad T \models \psi \to \neg\varphi \end{aligned}$$

which makes $\mathrm{Th}_{\mathsf{L}_{\mathbb{A}}}(\mathfrak{M})$ a Δ_1 set by the Extended Completeness Theorem. □

For $L_{\omega\omega}$ and $K_{\omega\omega}$ the theorem and its corollary are old indeed. Here the proof of the theorem is even easier since one no longer has to go back to the Model Existence Theorem but can use the Compactness Theorem for $K_{\omega\omega}$.

8.3—8.4 Exercises

8.3. Show that if $K_{\mathbb{B}} = K_{\omega\omega}$, $L_{\mathbb{A}} = L_{\omega\omega}$ then the hypothesis of Theorem 8.1 can be weakened to:

$$\forall \psi \in L_{\mathbb{A}}\ (T + \psi \text{ not complete for } L_{\mathbb{A}}).$$

Prove this directly from the Compactness Theorem for $K_{\omega\omega}$.

8.4. Let L have constant symbols $\bar{0}, \bar{1}, \ldots, \bar{n}, \ldots$ and a unary predicate P. Find a consistent theory $T = \{\varphi\}$ of a countable fragment L_A such that ψ has only $\aleph_0$ non-isomorphic models, but for each $\psi \in L_{\omega\omega}$, $T + \psi$ is not complete for $L_{\omega\omega}$. This shows that the strengthening of 8.1 carried out in 8.3 is not possible in general.

8.5 Notes. The results of this section are new here. They are suggested by, and imply, the theorem of recursion theory that any Σ_1^1 set of subsets of ω with less than $2^{\aleph_0}$ members is actually a subset of HYP. See § IV.4 for proofs of this and related results.